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ENGINEERING RESEARCH INSTITUTE

ELECTRONIC DEFENSE GROUP TECHNICAL MEMORANDUM NO. 17

SUBJECT: A Minimum Transmission Loss Tschebycheff Two-Pole Matching Network

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Introduction

When dealing with bandwidths in the order of 10 to 300 megacycles, the question of matching a partially reactive load to a transmission line presents a serious problem. The special case of a reactive load that consists of a resistance shunted by a capacitor is considered in the following sections. This problem is represented schematically in Figure 1. The minimum transmission

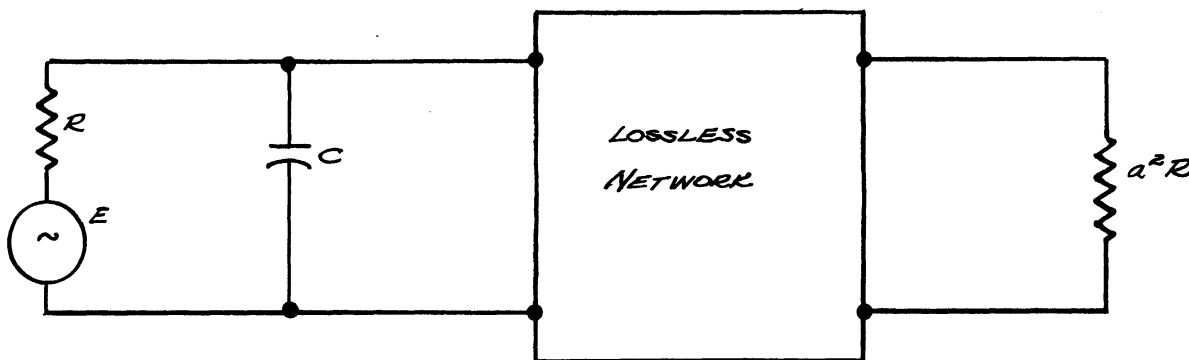


FIG. 1
MATCHING PROBLEM.

loss for this problem was derived by Bode¹ (See Appendix A). It may be expressed

¹
Bode, H. W., "Network Analysis and Feedback Amplifier Design," D. Van Nostrand, 1945.

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in db as

$$\psi_o = 10 \log_{10} \frac{\text{Exp} \frac{1}{(f_2 - f_1)RC}}{\left[\text{Exp} \frac{1}{(f_2 - f_1)RC} \right] - 1} \quad (1)$$

where:

ψ_o is the minimum transmission loss over the passband in db

f_2 is the upper cutoff frequency

f_1 is the lower cutoff frequency

$f_2 - f_1 = f_{\Delta}$ is the passband or bandwidth

R and C are indicated in Figure 1.

A plot of the minimum transmission loss versus $\omega_{\Delta}RC$ appears in Figure 2. The problem becomes one of finding the network whose response approaches this minimum transmission loss as closely as possible, with a given complexity.

Networks can be synthesized to approach this theoretical optimum as closely as desired; however, in general, the number of circuit elements increases rapidly as the theoretical limit is approached. The two pole network shown in the following sections is rather simple as far as matching networks are concerned, and hence, a desirable type of network to use in electronic circuitry. The parameters for the two pole network have been determined so as to minimize the difference between its response and the theoretical minimum transmission loss curve.

In general, it is very wasteful of gain bandwidth product to match perfectly at any frequency. It is possible to obtain considerable improvement by allowing a nominal mismatch over the band. This improvement for a two pole network can be seen in Figures 2 and 5.

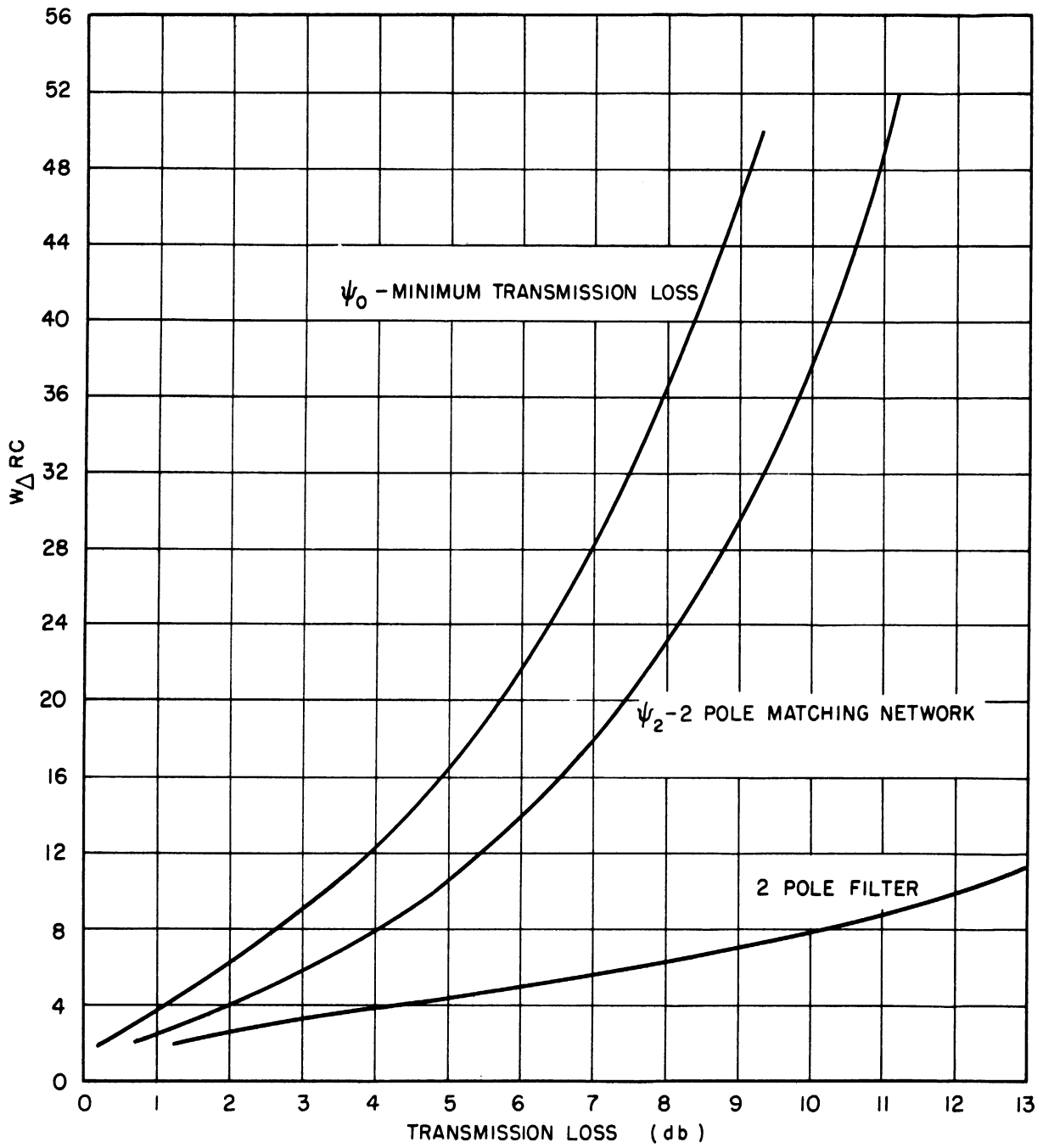


FIG. 2

$\omega_{\Delta} RC$ VS. TRANSMISSION LOSS.

1. Derivation of Equations for the Two-Pole Minimum Transmission Loss Tschebycheff Matching Network

This section outlines the derivation of a Tschebycheff matching network with a transmission loss equal to or less than a prescribed maximum within a bandwidth ω_{Δ} . The network considered is shown in Figure 3.

The frequency response of a Tschebycheff filter can be expressed as

$$\frac{P_2}{P_a} = \left| \frac{E_2}{E_1}(j\omega) \right|^2 = \frac{A^2}{1 + \epsilon T_n^2(\omega)} \quad (2)$$

where A^2 is an arbitrary constant and $T_n(\omega)$ is a Tschebycheff polynomial defined by:

$$T_n(\omega) = \cos(n \cos^{-1} \omega) \quad (3)$$

The solution to Equation 2 is given in Appendix B and results in:

$$\frac{E_2(p)}{E_1} = \frac{A}{\prod_{k=1}^n (p - p_k)} \quad (4)$$

where: $p_k = \sigma_k + j\omega_k$

$$\sigma_k = -\sin \frac{2k-1}{2n} \pi \sinh \eta$$

$$\omega_k = \cos \frac{2k-1}{2n} \pi \cosh \eta$$

$$\eta = \frac{1}{n} \sinh^{-1} \sqrt{\frac{1}{\epsilon}}$$

For a two pole Tschebycheff response ($n = 2$), equation 4 becomes

$$\frac{E_2(p)}{E_1} = \frac{A}{p^2 + p\omega_{\Delta}\sqrt{2} \sinh \eta + \frac{\omega_{\Delta}^2}{2}(2 \sinh^2 \eta + 1)} \quad (5)$$

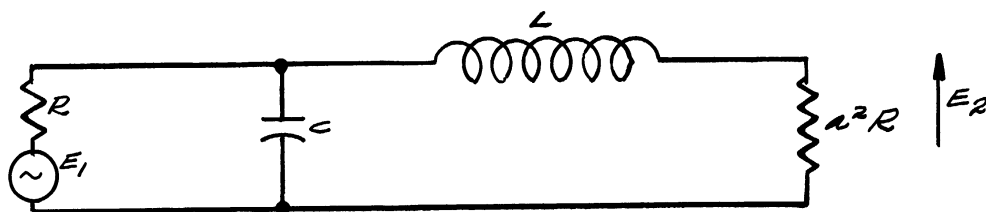


FIG. 3. LOW PASS TWO POLE NETWORK.

The response of the circuit shown in Figure 3 is given by

$$\frac{E_2}{E_1} = \frac{a^2}{LC} \frac{1}{p^2 + p\left(\frac{a^2R}{L} + \frac{1}{RC}\right) + \frac{1+a^2}{LC}} \quad (6)$$

Equating the coefficients of like powers of p in equations 5 and 6 yields

$$\omega_{\Delta} \sqrt{2} \sinh \eta = \frac{a^2R}{L} + \frac{1}{RC} \quad (7)$$

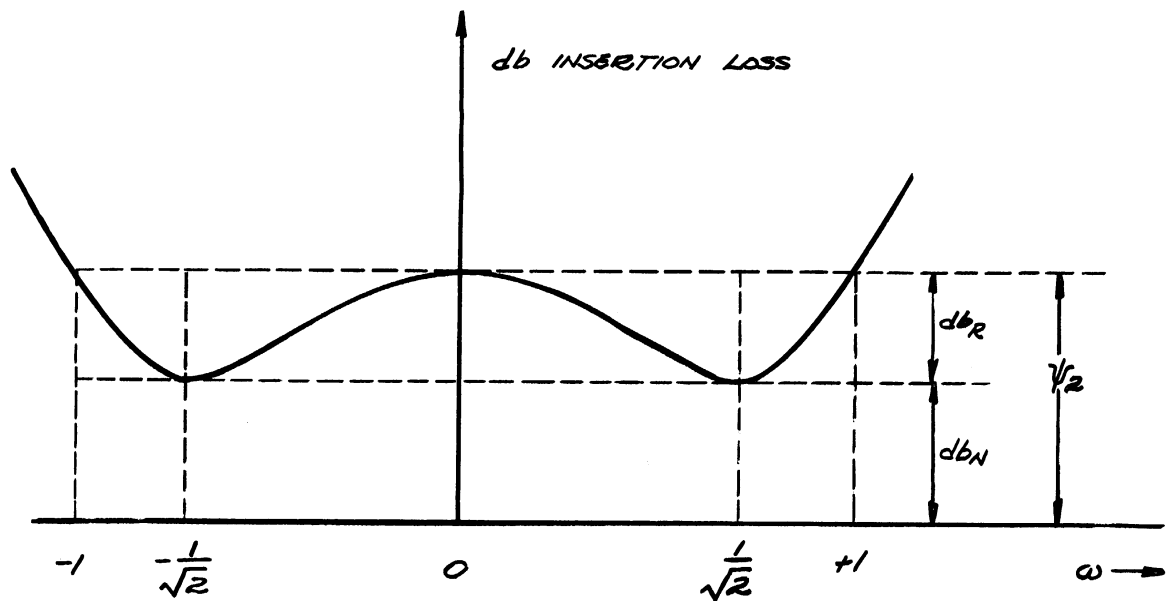
and

$$\frac{\omega_{\Delta}^2}{2} (2 \sinh^2 \eta + 1) = \frac{1+a^2}{LC} \quad (8)$$

Solving these equations results in an expression for a^2

$$a^2 = \frac{\sqrt{2} \omega_{\Delta} RC \sinh \eta - 1}{\frac{(\omega_{\Delta} RC)^2}{2} (2 \sinh^2 \eta + 1) - \sqrt{2} \omega_{\Delta} RC \sinh \eta + 1} \quad (9)$$

The response of the network shown in Figure 3 with the element values adjusted according to equations 7 and 8, and with the bandwidth normalized, is shown in Figure 4.



db_R = RIPPLE IN DECIBELS
 db_N = NOMINAL MISMATCH IN DECIBELS
 ψ_2 = TRANSMISSION LOSS IN DECIBELS

FIG. 4
 FREQUENCY RESPONSE
 OF LOW PASS TWO POLE MATCHING NETWORK.

From equation 2, the points of maximum transmission loss occurs when

$$T_n^2(\omega) = 1,$$

or

$$\omega = \pm 1, 0 \text{ for } n = 2.$$

However, from the circuit of Figure 3 at zero frequency

$$\left. \frac{P_a}{P_2} \right|_{\omega = 0} = \frac{(1 + a^2)^2}{4a^2} \quad (10)$$

$$\text{or } \psi_2 = 10 \log_{10} \left[\frac{(1 + a^2)^2}{4a^2} \right]$$

where P_a is the available power from the generator and P_2 is the power delivered to the load. ψ_2 has its minimum value for $a^2 = 1$. a^2 is

restricted to values less than or equal to one; hence for a small ψ_2 , a^2 should be as large as possible. a^2 is a function of $\omega_{\Delta}RC$ and $\sinh \eta$ as is seen from equation 9. The maximum value of a^2 for a given $\omega_{\Delta}RC$ subject to the restraints imposed on a^2 by equation 9 may be found by evaluating

$$\frac{\partial a^2}{\partial \sinh \eta} = 0 \quad (11)$$

This results in the following two equations

$$a^2 = \frac{1}{\sqrt{1 + (\omega_{\Delta}RC)^2}} \quad (12)$$

and

$$\sinh^2 \eta = \frac{1}{2} \left[\frac{1 + a^2}{1 - a^2} \right] \quad (13)$$

(See Appendix C for this solution to equation 11) Solving equation 8 for L one has

$$L = \frac{1 + a^2}{\frac{\omega_{\Delta}^2 C}{2} (2 \sinh^2 \eta + 1)} \quad (14)$$

Substituting the value of $\sinh^2 \eta$ from equation 13 into equation 14 results in

$$L = \frac{1 - a^4}{\omega_{\Delta}^2 C} \quad (15)$$

We have now determined the parameters of the circuit so that its response has a minimum transmission loss for a given value of $\omega_{\Delta}RC$.

A measure of the efficiency of this network is obtained from Figure 5 where the ratio of the theoretical maximum load power to the load power for this network has been plotted against $\omega_{\Delta}RC$.

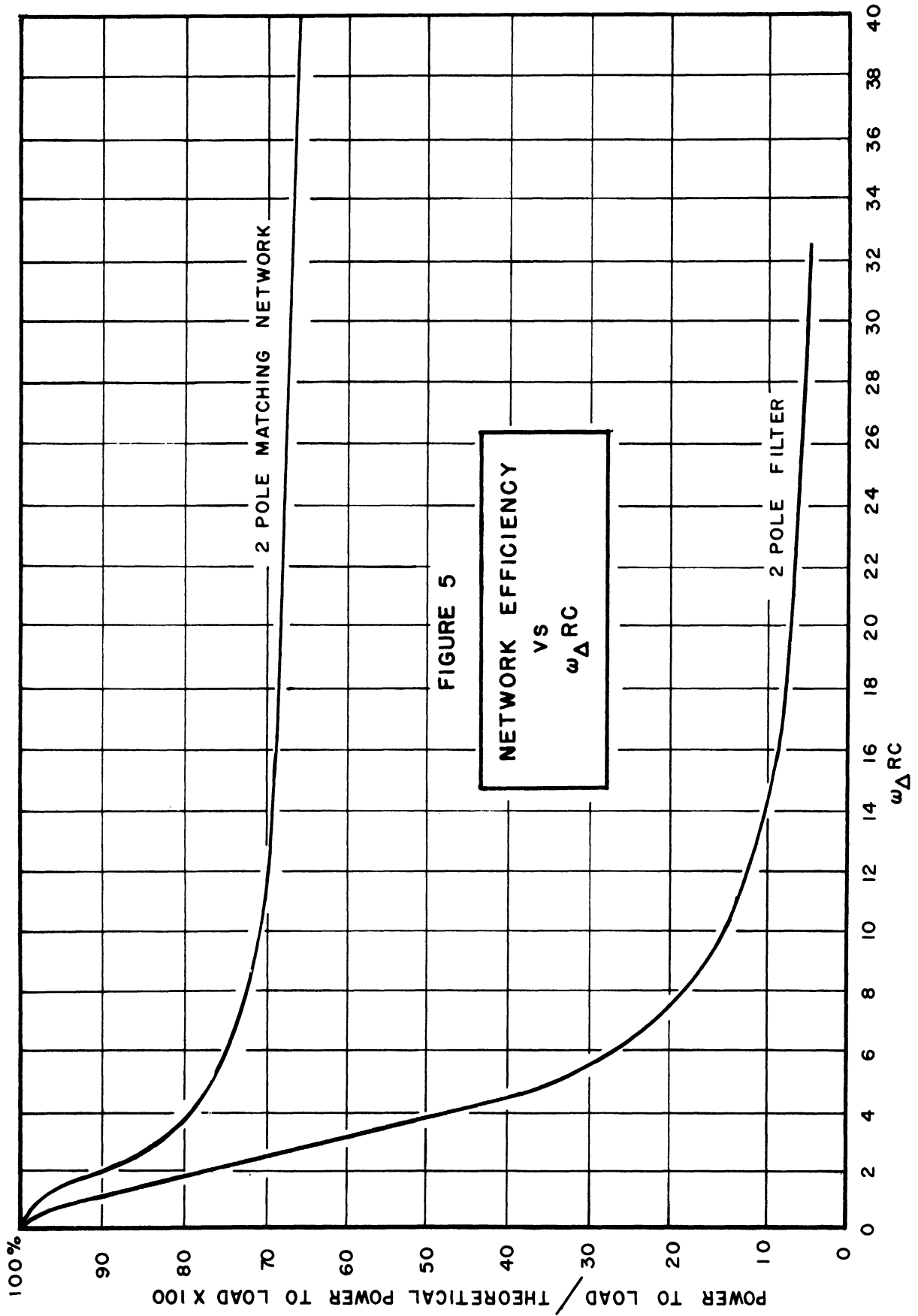


FIGURE 5

NETWORK EFFICIENCY
VS
 $\omega\Delta RC$

The efficiency of the normal two pole filter network is also plotted for comparison.

2. Design Procedure for Two Pole Bandpass Matching Networks with a Tschebyscheff Response

Figures 10 and 11 have been included in this section for the purpose of evaluating the design before it is made. For a required $\omega_{\Delta}RC$, the transmission loss for the network may be determined from Figure 10 and the ripple in its response from Figure 11.

In the following design procedure it is assumed that the bandwidth, resistance levels, and capacitance are given.

Design Procedure

Given: ω_{Δ} , R , C , and ϕ^2 which are defined by Figure 6.

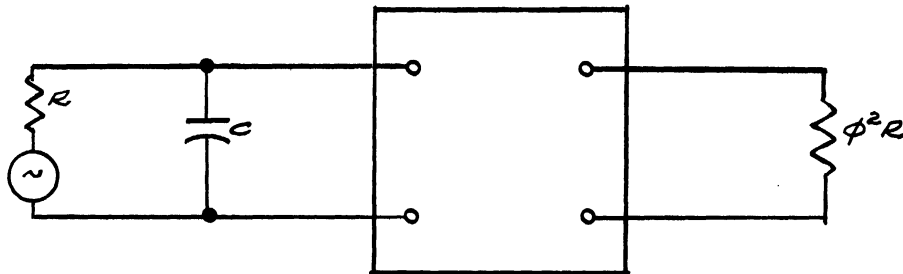


FIG. 6

STEP 1: Determine the value of a^2 for the given value of $\omega_{\Delta}RC$ from Figure 12.

STEP 2: Calculate L from Equation 15

$$L = \frac{1 - a^4}{\omega_{\Delta}^2 C}$$

This determines the values in the network shown in Figure 7.

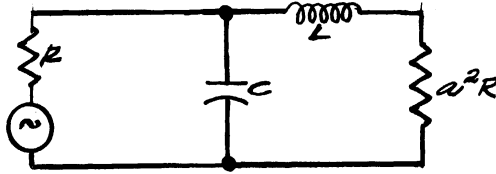


FIG. 7

STEP 3: The low pass network is changed to a bandpass network by resonating the reactive elements at the geometric mean of the upper and lower cutoff frequencies (See Appendix D).

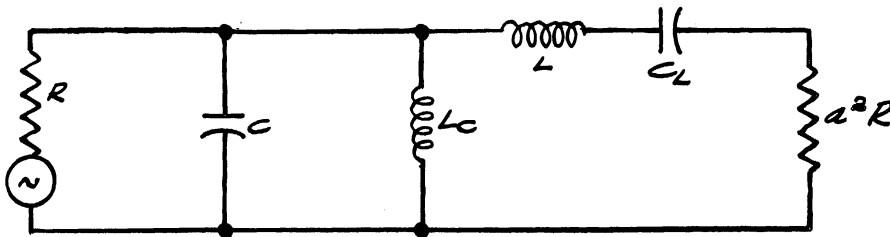


FIG. 8

STEP 4: The output impedance (a^2R) is changed to the desired output impedance level (See Appendix E).

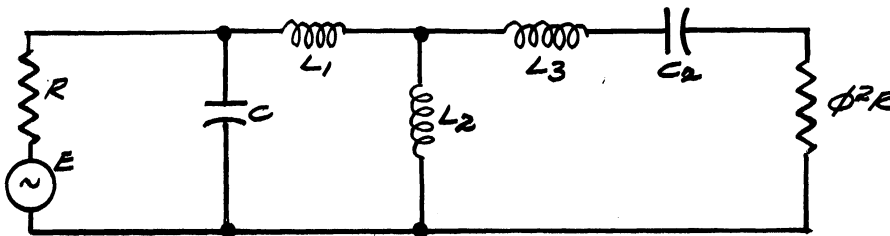


FIG. 9

where

$$L_1 = \frac{L_C(b-1)}{b}$$

$$b^2 = \frac{a^2}{\phi^2}$$

$$L_2 = \frac{L_C}{b}$$

$$C_2 = b^2 C_L$$

$$L_3 = \frac{L + (1-b)L_C}{b^2}$$

Example of Design Procedure

$R = 5000 \text{ ohm}$

$f_{\Delta} = 20 \text{ mc}$

$C = 7 \mu\mu\text{f}$

$f_0 = 200 \text{ mc}$

$\phi^2 R = 50 \text{ ohm}$

$$\omega_{\Delta} RC = 2\pi \times 20 \times 10^6 \times 5 \times 10^3 \times 7 \times 10^{-12} = 4.4$$

From Fig. 10 $\psi_2 = 2.3$ db

From Fig. 11 $db_r = 0.655$

STEP 1: From Fig. 12 for $\omega_{\Delta} RC = 4.4$, $a^2 = 0.225$

STEP 2:

$$L = \frac{1 - a^4}{\omega_{\Delta}^2 C} = 8.6 \mu h.$$

STEP 3:

$$L_C = \frac{1}{\omega_o^2 C} = 0.0907 \mu h$$

$$C_L = \frac{1}{\omega_o^2 L} = 0.0738 \mu \mu f$$

STEP 4: $b^2 = \frac{a^2}{\phi^2} = \frac{0.225}{\frac{50}{5000}} = 22.5$

$$b = 4.75$$

$$L_1 = \frac{L_C(b-1)}{b} = \frac{0.0907(3.75)}{4.75} = 0.0716 \mu h$$

$$L_2 = \frac{L_C}{b} = \frac{0.0907}{4.75} = 0.019 \mu h$$

$$L_3 = \frac{L + (1-b)L_C}{b^2} = \frac{8.6 - 3.75 \times 0.0907}{22.5} = 0.367 \mu h$$

$$C_2 = b^2 C_L = 1.66 \mu \mu f$$

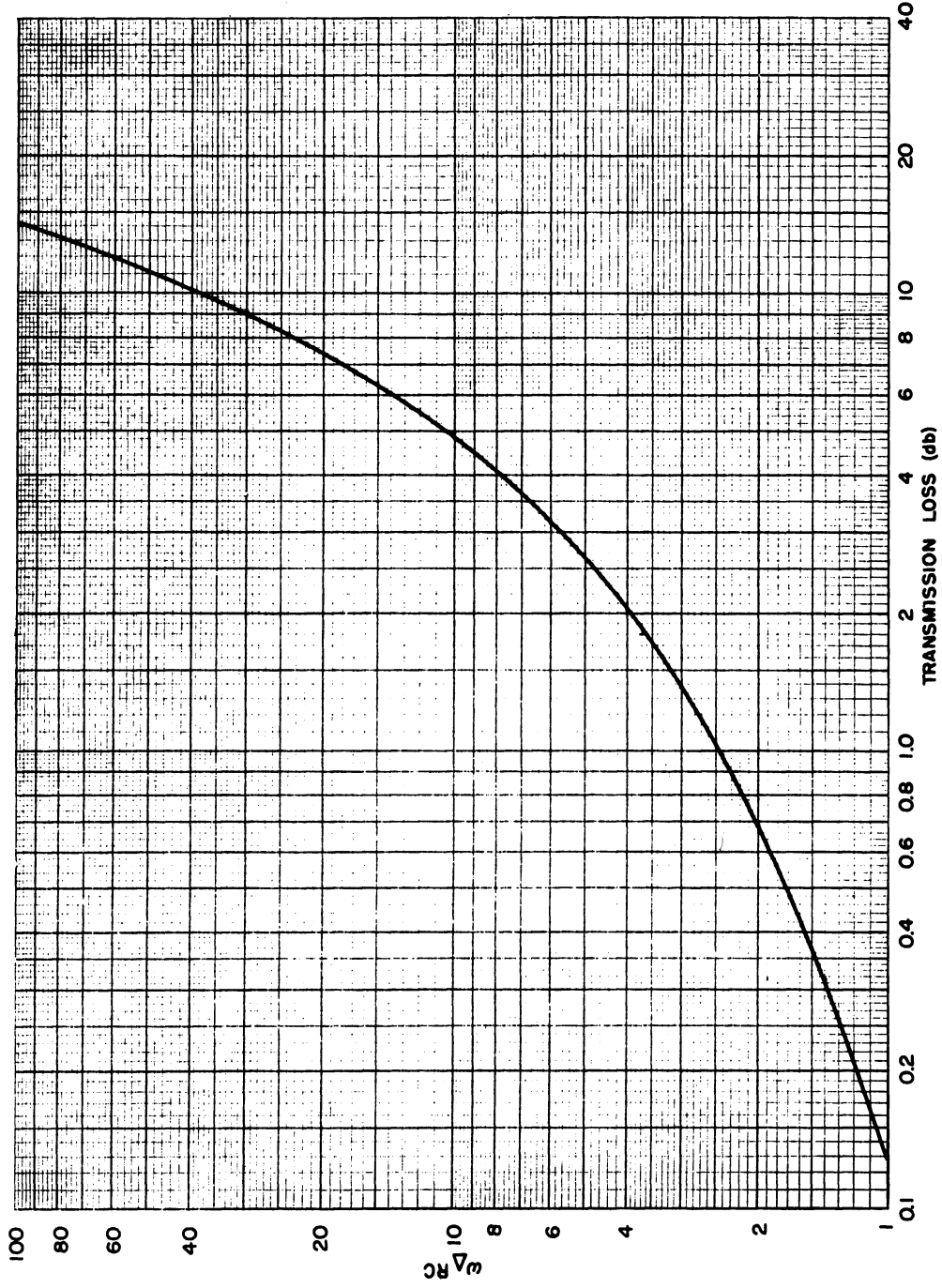


FIG. 10
 $\omega\Delta RC$ VS. TRANSMISSION LOSS

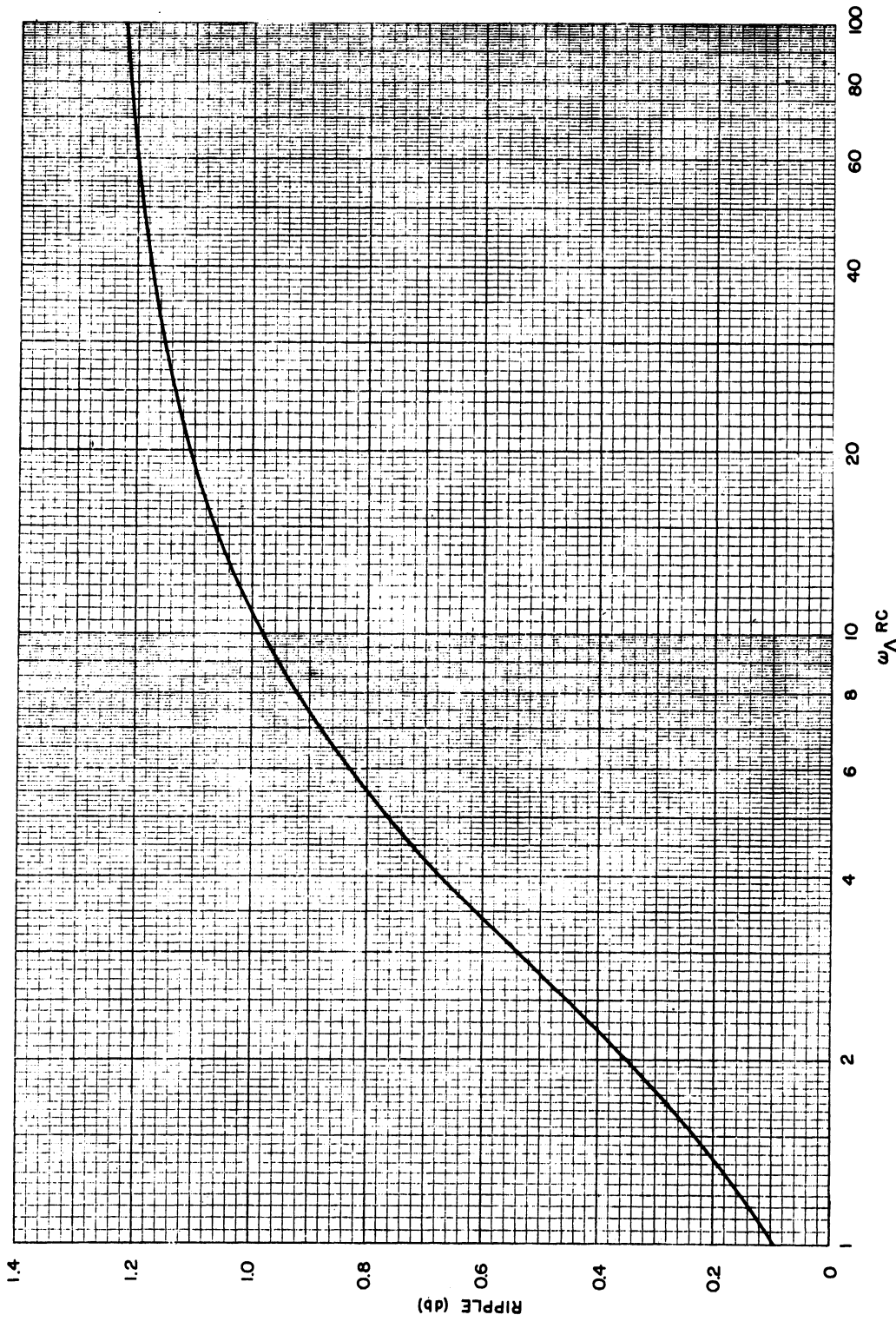


FIG. 11

$\omega_{\Delta} RC$ VS. RIPPLE.

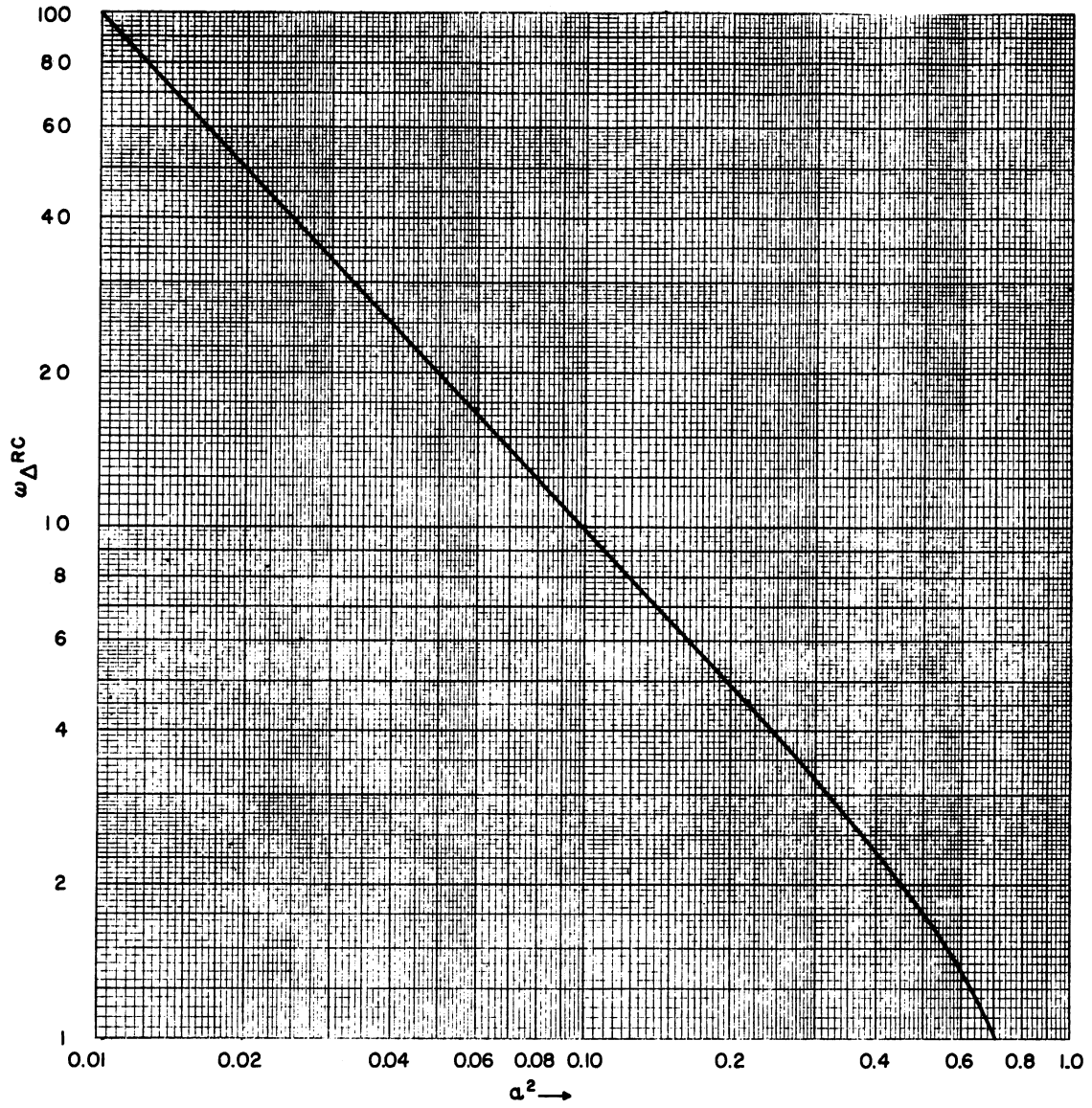


FIG. 12

$\omega_{\Delta} RC$ VS. a^2

APPENDIX A

DERIVATION OF MINIMUM THEORETICAL TRANSMISSION LOSS

The equation due to Bode,¹ representing the minimum realizable reflection coefficient for the network of Figure 1, is

$$\begin{aligned} |\rho| &= \text{Exp} \left[-\frac{\pi}{(\omega_2 - \omega_1) RC} \right] \\ \left| \frac{1}{\rho} \right|^2 &= \text{Exp} \left[\frac{1}{(f_2 - f_1) RC} \right] \end{aligned} \quad (\text{A.1})$$

The absolute value of the reflection coefficient is defined as:

$$\left| \rho \right|^2 = \frac{P_a - P_2}{P_a} \quad (\text{A.2})$$

Where P_a is the power available from the source and P_2 is the power which gets to the load. Hence, from equation 2

$$\frac{P_a}{P_2} = \frac{\left| \frac{1}{\rho} \right|^2}{\left| \frac{1}{\rho} \right|^2 - 1} \quad (\text{A.3})$$

So

$$\psi_o = 10 \log_{10} \left[\frac{\left| \frac{1}{\rho} \right|^2}{\left| \frac{1}{\rho} \right|^2 - 1} \right] \quad (\text{A.4})$$

But, from equations A.1 and A.4

$$\psi_o = 10 \log_{10} \frac{\text{Exp} \left[\frac{1}{(f_2 - f_1) RC} \right]}{\left\{ \text{Exp} \left[\frac{1}{(f_2 - f_1) RC} \right] \right\} - 1}$$

¹
Op. Cit., p. 1.

APPENDIX B

LOCATION OF POLES FOR A TSCHEBYCHEFF RESPONSE

$$\left| \frac{E_2}{E_1}(j\omega) \right|^2 = \frac{A^2}{1 + \epsilon T_n^2(\omega)}$$

$$T_n(\omega) = \cos(n \cos^{-1} \omega)$$

Set: $\cos^{-1} \omega = \varphi$

$$T_n(\omega) = \cos n\varphi$$

Let φ be complex:

$$\varphi = \xi + j\eta$$

$$T_n(\omega) = \cos n(\xi + j\eta)$$

$$T_n(\omega) = \cos n\xi \cosh n\eta + j \sin n\xi \sinh n\eta$$

But:

$$T_n^2(\omega) = -\frac{1}{\epsilon}$$

$$T_n(\omega) = \pm j \sqrt{\frac{1}{\epsilon}}$$

Hence:

$$\pm \sqrt{\frac{1}{\epsilon}} = \sin n\xi \sinh n\eta$$

$$\cos n\xi \cosh n\eta = 0$$

Since:

$$\cosh n\eta \geq 1$$

Then:

$$\cos n\xi = 0$$

Hence:

$$n\xi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$$

$$\xi = \frac{(2k-1)\pi}{2n} \quad \text{for } k = 1, 2, 3, \dots, 2n$$

Hence:

$$\begin{aligned}\sin n\xi &= \pm 1 \\ \sinh n\eta &= \sqrt{\frac{1}{\epsilon}}\end{aligned}$$

and

$$\varphi = \left(\frac{2k-1}{2n}\right)\pi + j\frac{1}{n} \sinh^{-1} \sqrt{\frac{1}{\epsilon}}$$

Remember that:

$$\omega = \cos \varphi = p/j$$

$$\varphi = \xi + j\eta$$

$$\omega = \cos(\xi + j\eta)$$

$$\omega = \cos \xi \cosh \eta + j \sin \xi \sinh \eta$$

$$\omega = \cos\left(\frac{2k-1}{2n}\right)\pi \cosh \eta + j \sin\left(\frac{2k-1}{2n}\right)\pi \sinh \eta \quad (\text{B.1})$$

Define

$$p = \sigma + j\omega$$

The poles of $\frac{E_2(j\omega)^2}{E_1}$ are found by substituting p/j for ω in

equation B.1 and identifying the real and imaginary parts.

Hence:

$$\sigma_k = \sin \frac{(2k-1)\pi}{2n} \sinh \eta$$

$$\omega_k = \cos \frac{(2k-1)\pi}{2n} \cosh \eta \quad (\text{B.2})$$

for $k = 1, 2, \dots, 2n$

The poles specified by equation B.2 lie on an ellipse with foci at $\pm j$. This follows from the fact that:

$$\frac{\sigma_k^2}{\sinh^2 \eta} + \frac{\omega_k^2}{\cosh^2 \eta} = 1$$

The poles in the left half plane are given by p_k , where $p_k = \sigma_k + j\omega_k$

and

$$\sigma_k = -\sin \frac{(2k-1)\pi}{2n} \sinh \eta$$

$$\omega_k = \cos \frac{2k-1}{2n} \cosh \nearrow$$

for $k = 1, 2, \dots, n$

Thus, $\frac{E_2}{E_1}(p)$ may be expressed by:

$$\frac{E_2}{E_1}(p) = \frac{A}{\prod_{k=1}^n (p - p_k)} \quad (\text{B.3})$$

APPENDIX C

DETERMINATION OF PARAMETERS FOR MINIMUM TRANSMISSION

LOSS FOR A GIVEN $\omega_{\Delta}RC$

Equation (9) of the Introduction was

$$a^2 = \frac{\sqrt{2} \omega_{\Delta}RC \sinh \eta - 1}{\frac{(\omega_{\Delta}RC)^2}{2}(2 \sinh^2 \eta + 1) - \sqrt{2}\omega_{\Delta}RC \sinh \eta + 1}$$

It was shown in the Introduction that it was desirable to have a^2 as large as possible in order to minimize the transmission loss. We can do this mathematically by

$$\frac{\partial}{\partial \eta} \frac{a^2}{\sinh \eta} = 0$$

$$\begin{aligned} \sqrt{2} \omega_{\Delta}RC \left[\frac{(\omega_{\Delta}RC)^2}{2} (2 \sinh^2 \eta + 1) - \sqrt{2} \omega_{\Delta}RC \sinh \eta + 1 \right] &= \quad (C.1) \\ -(\sqrt{2}\omega_{\Delta}RC \sinh \eta - 1) \left[\frac{(\omega_{\Delta}RC)^2}{2} 4 \sinh \eta - 2\omega_{\Delta}RC \right] & \\ \hline \left[\frac{(\omega_{\Delta}RC)^2}{2} (2 \sinh^2 \eta + 1) - \sqrt{2}\omega_{\Delta}RC \sinh \eta + 1 \right]^2 & \end{aligned}$$

Since this equation is zero we can equate the numerator to zero and solve for $\sinh \eta$.

$$\sinh \eta = \frac{1}{\sqrt{2}\omega_{\Delta}RC} \left[1 + \sqrt{1 + (\omega_{\Delta}RC)^2} \right] \quad (C.2)$$

Substituting this value for $\sinh \eta$ in the original equation for a^2 yields

$$a^2 = \frac{1}{\sqrt{1 + (\omega_{\Delta}RC)^2}} \quad (C.3)$$

Solving equation C.3 for $\omega_{\Delta} RC$ and substituting in equation C.2 gives

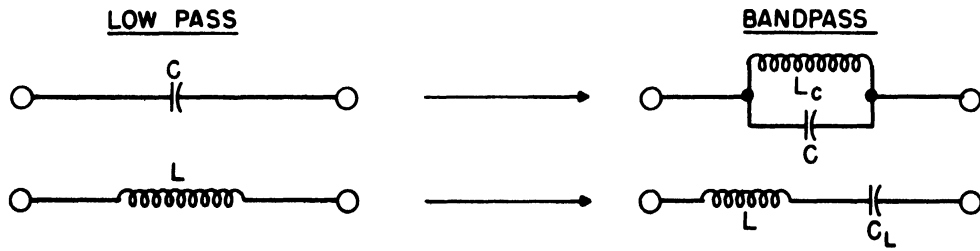
$$\sinh \varphi = \frac{1}{\sqrt{2}} \frac{\sqrt{1+a^2}}{\sqrt{1-a^2}} \quad (\text{C.4})$$

APPENDIX D

LOW PASS TO BAND PASS TRANSFORMATION

A scheme for transforming from low pass to bandpass is given in Fig.

D.1. The resulting two pole bandpass circuit and typical response curve appears in Figure D.2.



WHERE:
$$\left\{ \begin{array}{l} \omega_0^2 = \frac{1}{L_c C} = \frac{1}{L C_L} = \omega_1 \omega_2 \\ \omega_2 - \omega_1 = \omega_\Delta \end{array} \right.$$

FIG. D.1
LOW PASS TO BANDPASS TRANSFORMATION.

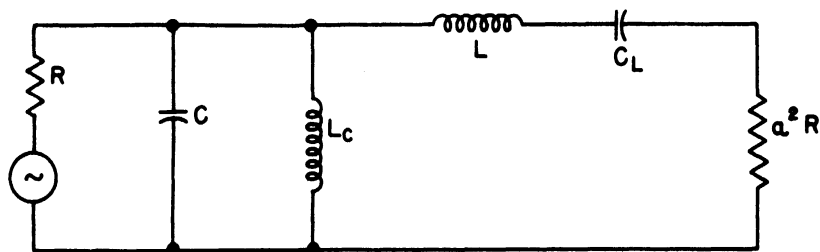
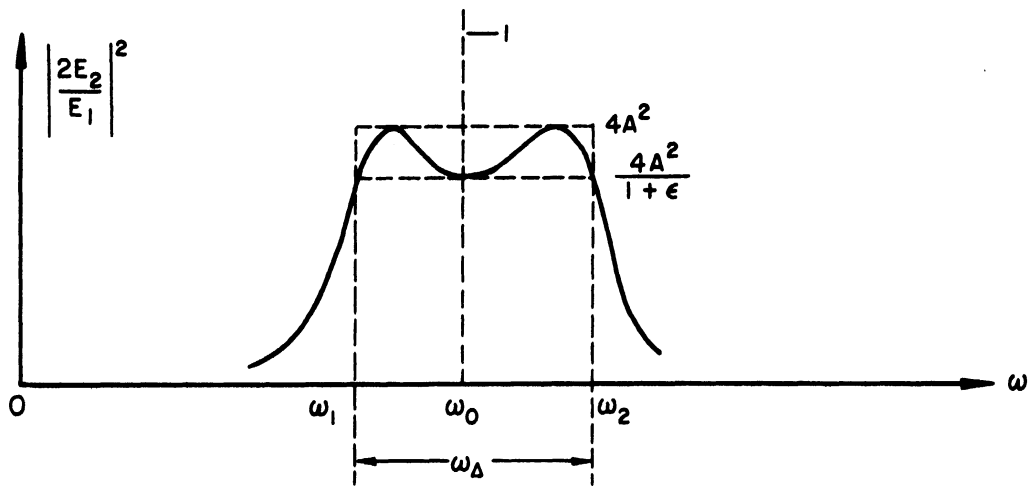


FIG. D.2
BANDPASS TWO POLE MATCHING NETWORK.

APPENDIX E

IMPEDANCE LEVEL TRANSFORMATION

To change the impedance level of the two pole bandpass Tchebycheff network, follow the steps in Figure E.1.

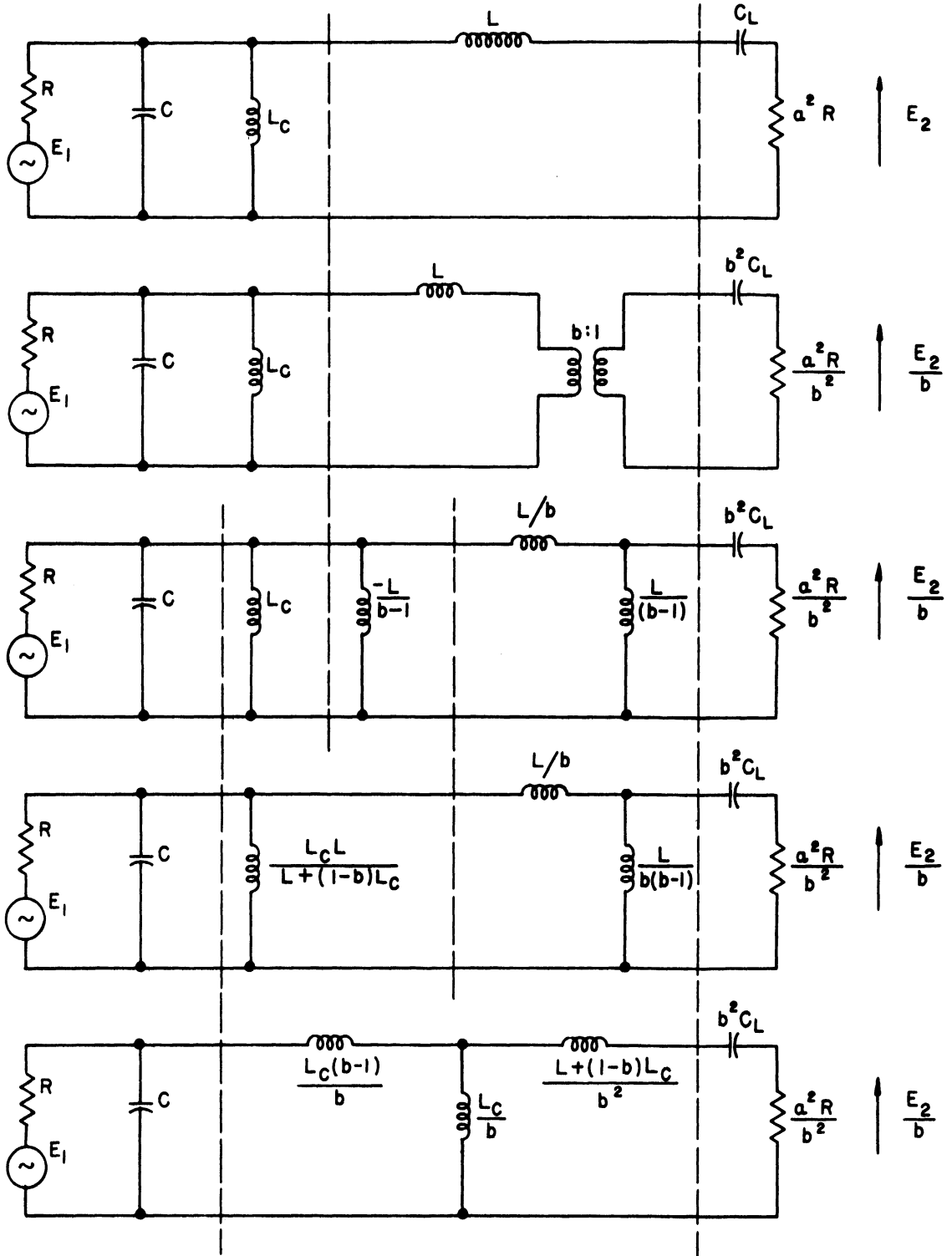


FIG. E.1

FINAL CIRCUIT FOR TWO POLE BANDPASS
MATCHING NETWORK.

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