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Infinite Prandtl Number Convection

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We prove an inequality of the type $N \leqslant CR^{1/3}(1+\log_+ R)^{2/3}$ for the Nusselt number N in terms of the Rayleigh number R for the equations describing three-dimensional Rayleigh–Bénard convection in the limit of infinite Prandtl number.

KEY WORDS: Nusselt number; convection; heat transport; turbulence.

1. INTRODUCTION

The bulk average of heat transport in Rayleigh–Bénard convection, the Nusselt number N, is measured in experiments and numerical simulations.[(1)]

Under a variety of conditions it is found that

$$N \sim R^q$$

where the Rayleigh number R is proportional to the amount of heat supplied externally. Many experiments report values for q that belong approximately to the interval $[\frac{2}{7}, \frac{1}{3}]$ for large R. The mathematical description is based on the three dimensional Bourssinesq equations for Rayleigh–Bénard convection.[(2)] These are a system of equations coupling the three dimensional Navier–Stokes equations to a heat advection-diffusion equation. The only known rigorous upper bound for N[(3)] at large R is of the order $R^{1/2}$; the bound is valid for all weak solutions (the global existence of smooth solutions is not known). Although one can describe sufficient conditions that ensure bounds with lower exponents,[(4)] no rigorous derivation of the exponents $\frac{2}{7}$ or $\frac{1}{3}$ exists, to our knowledge. These exponents have been

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discussed by several authors using physical reasoning and dimensional analysis.[5]

In this paper five derive rigorously an upper bound of the form

$$N \leqslant 1 + C_1 R^{1/3}(1 + \log_+(R))^{2/3}$$

for the three dimensional equations for Rayleigh–Bénard convection obtained in the limit of infinite Prandtl number. The Prandtl number is the ratio of the fluid's viscosity to the fluid's heat conduction coefficient. These equations are easier to analyze than the Boussinesq equations; in particular one can prove global existence and uniqueness of smooth solutions. We obtain bounds for higher derivatives (average Laplacian squared) of the temperature and a pointwise logarithmic bound for the second derivative of vertical velocity. The latter is obtained via a logarithmic L^∞ bound for the bi-Laplacian with homogeneous Dirichlet and Neumann boundary conditions. The logarithmic $\frac{1}{3}$ bound for the Nusselt number follows.

In previous work we have developed and applied a general variational method[6] to estimate bulk dissipation quantities in systems in which energy is supplied by boundary conditions. The method starts by translating the equation in function space by a background—a time independent function that obeys the driving boundary conditions. A quadratic form is associated naturally to each background, and the method consists in selecting those backgrounds for which this quadratic form is positive semi-definite and then minimizing a certain integral of the background. In order to obtain our present result we need to use more PDE information. The additional information concerns higher derivatives and cannot be deduced from stability considerations. Consequently we find that the method is substantially modified: the quadratic form is no longer required to be semidefinite. Instead, the additional information coming from the evolution equation is incorporated in the constraints of a new mini-max procedure. This additional information is however the essential new element and the mini-max procedure is its natural by-product.

2. EQUATIONS

We consider the infinite Prandtl number equations for Rayleigh–Bénard convection in the Boussinesq approximation. They are a system of five equations for velocities (u, v, w), pressure p and temperature T in three spatial dimensions. The velocity and pressure are determined from the temperature by solving time independent equations:

$$-\Delta u + p_x = 0 \tag{1}$$

together with

$$-\Delta v + p_y = 0 \tag{2}$$

and

$$-\Delta w + p_z = RT \tag{3}$$

The velocity is divergence-free

$$u_x + v_y + w_z = 0 \tag{4}$$

The temperature is advected and diffuses according to the active scalar equation

$$(\partial_t + \mathbf{u} \cdot \nabla)\, T = \Delta T \tag{5}$$

where $\mathbf{u} = (u, v, w)$. R represents the Rayleigh number. The horizontal independent variables (x, y) belong to a basic square $Q \subset \mathbf{R}^2$ of side L. Sometimes we will drop the distinction between x and y and denote both horizontal variables x. The vertical variable z belongs to the interval $[0, 1]$. The non-negative variable t represents time. The boundary conditions are as follows: all functions $((u, v, w), p, T)$ are periodic in x and y with period L; u, v, and w vanish for $z = 0, 1$, and the temperature obeys $T = 0$ at $z = 1$, $T = 1$ at $z = 0$.

We take a function $\tau(z)$ that satisfies $\tau(0) = 1$, $\tau(1) = 0$, and write $T = \tau + \theta(x, y, z, t)$. The role of τ is that of a convenient background; there is no implied smallness of θ, but of course θ obeys the same homogeneous boundary conditions as the velocity.

The equation obeyed by θ is

$$(\partial_t + u \cdot \nabla - \Delta)\, \theta = \tau'' - w\tau' \tag{6}$$

where we used $\tau' = d\tau/dz$.

We will write

$$\|f\|^2 = \frac{1}{L^2} \int_0^1 \iint |f(x, y, z)|^2 \, dz\, dx\, dy$$

for the (normalized) L^2 norm on the whole domain. We denote by Δ_D the Laplacian with periodic-Dirichlet boundary conditions. We will denote by

Δ_h the Laplacian in the horizontal directions x and y. We will use $\langle \cdots \rangle$ for long time average:

$$\langle f \rangle = \lim_{t \to \infty} \sup \frac{1}{t} \int_0^t f(s)\, ds$$

The Nusselt number is

$$N = \langle \|\nabla T\|^2 \rangle$$

The system has global smooth solutions for arbitrary smooth initial data. Let us record here, without proof, a few elementary bounds:

$$\|\Delta w\| \leqslant CR$$

at each instance of time. Because the temperature just advects and diffuses we have also

$$0 \leqslant T \leqslant 1$$

pointwise in space and time. Also

$$\|\nabla \mathbf{u}\|^2 \leqslant CR^2$$

holds at each instance of time. Actually

$$\|\nabla \mathbf{u}\|^2 = \frac{R}{L^2} \int Tw\, dx\, dy\, dz = \frac{R}{L^2} \int \theta w\, dx\, dy\, dz$$

so the time average is

$$\langle \|\nabla \mathbf{u}\|^2 \rangle = R(N-1)$$

The inequality

$$\|\nabla \theta\|_{L^4}^2 \leqslant C\, \|\theta\|_{L^\infty}\, \|\Delta \theta\|_{L^2}$$

is true in any dimension. From the bounds above and (6) it is not difficult to see that there exists a positive constant C_Δ such that

$$\langle \|\Delta \theta\|^2 \rangle \leqslant C_\Delta \left\{ RN + \int_0^1 \left[(\tau''(z))^2 + Rz(\tau'(z))^2 \right] dz \right\} \tag{7}$$

holds. By Sobolev embedding it follows that averages of squares of spatial $C^{0,\alpha}$ norms of θ are controlled also.

3. THE VERTICAL VELOCITY

Because of horizontal periodicity the unknown functions can be written as Fourier series with z-dependent coefficients, for instance

$$\mathbf{u}(x, z) = \sum_{k \in Z^2} \mathbf{u}_k(z)\, e^{(2\pi i/L)(x \cdot k)}$$

The equation $\Delta_D u = f$ decomposes in the ordinary differential equations

$$\left(\frac{d^2}{dz^2} - m_k^2\right) u_k = f_k$$

where

$$m_k = \frac{2\pi}{L}\, |k| \tag{8}$$

For $m > 0$, the solution to

$$\left(\frac{d^2}{dz^2} - m^2\right) u = f$$

with zero Dirichlet boundary conditions,

$$u(z) = 0 \qquad \text{at} \quad z = 0, 1$$

is given using the Green's function $G^{(m)}(z, \zeta)$ by

$$u(z) = G^{(m)}(f)(z) = \int_0^1 G^{(m)}(z, \zeta)\, f(\zeta)\, d\zeta$$

The Green's function is calculated from two independent solutions of the homogeneous ODE, with one point boundary conditions.

$$G^{(m)}(x, \zeta) = \frac{1}{W} \begin{cases} y_1(z)\, y_2(\zeta) & \text{if} \quad z < \zeta \\ y_1(\zeta)\, y_2(z) & \text{if} \quad z \geqslant \zeta \end{cases}$$

We will take

$$y_1(z) = \frac{e^{mz} - e^{-mz}}{2m}, \qquad y_2(z) = y_1(z-1) = -y_1(1-z)$$

and therefore the Wronskian $W = y_1 y_2' - y_1' y_2$ equals

$$W = y_1(1) = \frac{e^m - e^{-m}}{2m}$$

(we use the notation $y' = dy/dz$). Clearly $y_1 = y_1^{(m)}$ depends on m but we will keep the notation light. Recall that $G^{(m)}$ is continuous, non-positive $G^{(m)}(z, \zeta) \leqslant 0$, symmetric $G^{(m)}(z, \zeta) = G^{(m)}(\zeta, z)$, solves the homogeneous ODE in z and ζ separately for $z \neq \zeta$, and the left-to-right jump across the diagonal of the z partial derivatives equals one:

$$G_z^{(m)}(z, z-0) - G_z^{(m)}(z, z+0) = 1$$

Also note that the Green's function is explicitly

$$G^{(m)}(z, \zeta) = \frac{1}{2m^2 W} (\cosh(m(z+\zeta-1)) - \cosh(m(1-|z-\zeta|))) \tag{9}$$

The Green's function for the Laplacian with Dirichlet-periodic boundary conditions is

$$G(x-y, z, \zeta) = \sum_{k \in \mathbf{Z}^2} e^{(2\pi/L)\, ik \cdot (x-y)} G^{(m_k)}(z, \zeta) \tag{10}$$

Because of the divergence-free condition, the pressure obeys

$$\Delta p = RT_z$$

Differentiating this with respect to z and substituting, the equation for w becomes

$$\Delta^2 w = -R \Delta_h T \tag{11}$$

and consequently

$$\left(\frac{d^2}{dz^2} - m_k^2 \right)^2 w_k = R m_k^2 T_k \tag{12}$$

The boundary conditions are

$$w_k(0) = w_k'(0) = w_k(1) = w_k'(1) = 0 \tag{13}$$

Now we will seek the solution of

$$\left(\frac{d^2}{dz^2} - m_k^2 \right)^2 w = f$$

by solving

$$w = g + h$$

where

$$g = G^{(m)}G^{(m)}\, f$$

and

$$h = \frac{1}{W^2 - 1}\,(g'(0) + Wg'(1))(1 - z)\; y_1(z) + \frac{1}{W^2 - 1}\,(Wg'(0) + g'(1))\; zy_2(z) \tag{14}$$

Note that h solves the homogeneous equation

$$\left(\frac{d^2}{dz^2} - m_k^2\right)^2 h = 0$$

and that is satisfies $h = 0$ and $h' = -g'$ boundary conditions. Note that

$$g'(0) = \int_0^1 \int_0^1 \frac{y_2(\zeta')}{W}\, G^{(m)}(\zeta', \zeta)\, f(\zeta)\, d\zeta'\, d\zeta \tag{15}$$

and

$$g'(1) = \int_0^1 \int_0^1 \frac{y_1(\zeta')}{W}\, G^{(m)}(\zeta', \zeta)\, f(\zeta)\, d\zeta'\, d\zeta \tag{16}$$

Thus the solution w can be expressed as

$$w(z) = \int_0^1 (G^{(m)}(z, \zeta) + H^{(m)}(z, \zeta))\; G^{(m)}(f)(\zeta)\, d\zeta \tag{17}$$

with

$$\begin{aligned} H^{(m)}(z, \zeta) = {} & \frac{1}{W^2 - 1}\,[(1 - z)\; y_1(z) + Wzy_2(z)]\,\frac{y_2(\zeta)}{W} \\ & + \frac{1}{W^2 - 1}\,[\,W(1 - z)\; y_1(z) + zy_2(z)]\,\frac{y_1(\zeta)}{W} \end{aligned} \tag{18}$$

We will consider now the operator

$$B=\frac{\partial^2}{\partial z^2}(\Delta_{DN}^2)^{-1}\,\Delta_h$$

where $w=(\Delta_{DN}^2)^{-1}f$ is the solution of

$$\Delta^2 w=f$$

with horizontally periodic and vertically Dirichlet and Neumann boundary conditions $w=w'=0$.

We will prove that B obeys logarithmic L^∞ estimates. More precisely

Theorem 1. For any $\alpha\in(0,1)$ there exists a positive constant C_α such that every Hölder continuous function θ that is horizontally periodic and vanishes at the vertical boundaries satisfies

$$\|B\theta\|_{L^\infty}\leqslant C_\alpha\,\|\theta\|_{L^\infty}\,(1+\log_+\|\theta\|_{C^{0,\alpha}})^2 \tag{19}$$

The spatial $C^{0,\alpha}$ norm is defined as

$$\|\theta\|_{C^{0,\alpha}}=\sup_{X=(x,y,z)\in Q\times[0,1]}|\theta(X,t)|+\sup_{X\neq Y}\frac{|\theta(X,t)-\theta(Y,t)|}{|X-Y|^\alpha}$$

We will derive this result using the calculation above; we express $B\theta$ as the sum

$$B\theta=(I-B_1+B_2)\,B_1\theta$$

where

$$B_1(\theta)=\Delta_h(\Delta_D)^{-1}\,\theta$$

and

$$B_2(\theta)(x,z)=L^{-2}\iint_0^1\sum_k e^{i(2\pi/L)\,k\cdot(x-y)}\,\frac{\partial^2}{\partial z^2}H^{(m_k)}(z,\zeta)\,\theta(y,\zeta)\,dy\,d\zeta$$

and prove for both B_j, $j=1,2$ the estimates

$$\|B_j\theta\|_{L^\infty}\leqslant C_\alpha\,\|\theta\|_{L^\infty}\,(1+\log_+\|\theta\|_{C^{0,\alpha}}) \tag{20}$$

These estimates are well-known for singular integral operators of the classical Calderon–Zygmund type. In our case the operators are not translationally invariant and their kernels are not explicit. We will consider first the operator B_1. B_1 is an integral operator with kernel $K = \Delta_h G$

$$K(x-y, z, \zeta) = -\sum_{k \in \mathbf{Z}^2} e^{(2\pi/L)\, ik \cdot (x-y)} m_k^2 G^{(m_k)}(z, \zeta) \tag{21}$$

The operator is given by

$$B_1(\theta)(x, z) = L^{-2} \int_Q \int_0^1 K(x-y, z, \zeta)(\theta(y, \zeta) - \theta(x, z))\, dy\, d\zeta \tag{22}$$

Now we claim that there exists a constant such that

$$|K(x-y, z, \zeta)| \leqslant C(|x-y|^2 + |z-\zeta|^2)^{-3/2} \tag{23}$$

Notice that the estimate (20) follows immediately from (23). The proof of (23) is done using the explicit representation of the Green's function $G^{(m_k)}$, the Poisson summation formula and the Poisson kernel. Note first that from the explicit form (9) it follows that

$$0 \leqslant -m_k^2 G^{(m_k)}(z, \zeta) \leqslant C m_k e^{-m_k |z-\zeta|}$$

If we ignore the horizontal oscillatory sum we obtain a pointwise bound that diverges like $|z-\zeta|^{-3}$; this is not sufficient for our purposes. In order to use the cancellations in the oscillatory sum we need to sum for instance

$$I = \sum_k e^{(2\pi/L)\, ik \cdot (x-y)} m_k \frac{\cosh(m_k(1-|z-\zeta|))}{\sinh(m_k)}$$

We express

$$\frac{\cosh(m_k(1-|z-\zeta|))}{\sinh(m_k)} = e^{-m_k |z-\zeta|} + e_k$$

with

$$0 \leqslant e_k \leqslant 2e^{-m_k}$$

The contribution coming from the sum of e_k is not singular and is incorporated in the prefactor. Below we will use the Poisson summation formula to obtain pointwise inequalities for exponential-oscillatory sums of the type

$$\sum_{k \in \mathbf{Z}^2} e^{(2\pi/L)\, ik \cdot (x-y)} c_k\, e^{-\varepsilon m_k}$$

where $\varepsilon = \varepsilon(z, \zeta)$ is nonnegative and c_k grows at most algebraically in $|k|$. The sums are singular where ε vanishes, typically at $z = \zeta$ or $z, \zeta = 0, 1$. These are co-dimension one surfaces and the Poisson summation formula allows one to replace these singular surfaces with single points (e.g., $x = y$, $z = \zeta$). The Poisson summation formula is[7]

$$\sum_{j \in \mathbf{Z}^2} P(x + Lj) = L^{-2} \sum_{k \in \mathbf{Z}^2} \hat{P}\left(\frac{2\pi}{L} k\right) e^{(2\pi/L)\, ix \cdot k}$$

where the Fourier transform is

$$\hat{P}(\xi) = \int_{\mathbf{R}^2} e^{-i\xi \cdot x} P(x)\, dx$$

it follows that

$$\sum_k e^{(2\pi/L)\, ik \cdot (x-y)} m_k\, e^{-m_k |z-\zeta|} = -L^2 \sum_j \{P'(x - y + Lj, |z-\zeta|)\}$$

where $P(x-y, \varepsilon)$ is the Poisson kernel in 2 dimensions

$$P(x-y, \varepsilon) = c_2 \varepsilon (|x-y|^2 + \varepsilon^2)^{-3/2}$$

which is the inverse of Fournier transform of $e^{-\varepsilon |\xi|}$,

$$e^{-\varepsilon |\xi|} = \int_{\mathbf{R}^2} e^{-i\xi \cdot x} P(x, \varepsilon)\, dx$$

and

$$P'(x, \varepsilon) = \frac{\partial P(x, \varepsilon)}{\partial \varepsilon}$$

The estimate (23) for I follows directly by evaluating the derivative P' and noting that only a few terms ($|j| \leqslant 2$ for $x, y \in Q$) in the sum in the right hand side of the Poisson summation formula are significant. There is a second term in the kernel; its estimate is similar.

The operator B_2 has also a kernel

$$H''(x-y, z, \zeta) = \sum_k e^{i(2\pi/L)\, k \cdot (x-y)} \frac{\partial^2}{\partial z^2} H^{(m_k)}(z, \zeta)$$

We will decompose the kernel in two parts, corresponding to the upper and lower boundaries:

$$\frac{\partial^2}{\partial z^2} H^{(m_k)}(z, \zeta) = S^{(m_k)}(z, \zeta) + J^{(m_k)}(z, \zeta)$$

where

$$J^{(m_k)}(z, \zeta) = \frac{1}{W^2 - 1} [\, W(1-z)\; y_1(z) + z y_2(z)]'' \,\frac{y_1(\zeta)}{W}$$

and

$$S^{(m_k)}(z, \zeta) = \frac{1}{W^2 - 1} [(1-z)\; y_1(z) + W z y_2(z)]'' \,\frac{y_2(\zeta)}{W}$$

We form the corresponding kernels

$$J(x-y, z, \zeta) = \sum_k e^{i(2\pi/L)\, k\cdot(x-y)} J^{(m_k)}(z, \zeta)$$

and

$$S(x-y, z, \zeta) = \sum_k e^{i(2\pi/L)\, k\cdot(x-y)} S^{(m_k)}(z, \zeta)$$

The operator B_2 can be written as $B_2 = S + J$ where

$$Jf(x, z) = L^{-2} \int_Q \int_0^1 J(x-y, z, \zeta)(f(y, \zeta) - f(y, 1))\, dy\, d\zeta$$

and

$$Sf(x, z) = L^{-2} \int_Q \int_0^1 S(x-y, z, \zeta)(f(y, \zeta) - f(y, 0))\, dy\, d\zeta$$

for any continuous function f that obeys the homogeneous boundary conditions (so that $f(y, 0) = f(y, 1) = 0$). Now the calculation of the individual kernels is again done using the Poisson summation formula. A typical term, arising in J is

$$\sum_k e^{i(2\pi/L)\, k\cdot(x-y)} m_k^2 (1-z)\, \frac{(\cosh(m_k(z+\zeta)) - \cosh(m_k(z-\zeta)))}{(\sinh m_k)^2}$$

The main contribution comes from the first term:

$$I = \sum_k e^{i(2\pi/L)\, k \cdot (x-y)} m_k^2 (1-z) \cosh(m_k(z+\zeta-2))$$

After Poisson summation we obtain

$$I = C(1-z)\, L^2 \sum_j (\Delta_h P(x-y+jL, 2-z-\zeta))$$

and the fact that $0 \leqslant 1-z \leqslant 2-z-\zeta$ is used to deduce

$$I \leqslant C(|x-y|^2 + |2-z-\zeta|^2)^{-3/2} \leqslant C(|x-y|^2 + |1-\zeta|^2)^{-3/2}$$

This term is the leading order term in the evaluation of J; we obtain

$$|J(x-y, z, \zeta)| \leqslant C(|x-y|^2 + |1-\zeta|^2)^{-3/2} \tag{24}$$

Similarly,

$$|S(x-y, z, \zeta)| \leqslant C(|x-y|^2 + |\zeta|^2)^{-3/2} \tag{25}$$

The estimate (20) for $B_2 = J + S$ follows from (24) and (25).

4. HEAT FLUX

The object of interest here is the function $b(z, t)$ defined by

$$b(z, t) = \frac{1}{L^2} \int_Q w(\cdot, z)\, T(\cdot, z)\, dx$$

Its average is related to the Nusselt number:

$$N - 1 = \left\langle \int_0^1 b(z)\, dz \right\rangle$$

From the equation (6) it follows that

$$N + \langle \|\nabla\theta\|^2 \rangle = 2 \left\langle \int_0^1 -\tau'(z)\, b(z)\, dz \right\rangle + \int_0^1 (\tau'(z))^2 \tag{26}$$

Let us write now

$$b(z, t) = \frac{1}{L^2} \int_Q \int_0^z \int_0^{z_1} w_{zz}(x, z_2, t)\, \theta(x, z)\, dx\, dz\, dz_2\, dz_1$$

It follows that

$$|b(z, t)| \leqslant z^2(1 + \|\tau\|_{L^\infty}) \|w_{zz}\|_{L^\infty(dz;\, L^1(dx))} \tag{27}$$

Applying the logarithmic bound

$$\|w_{zz}\|_{L^\infty} \leqslant CR(1 + \|\tau\|_{L^\infty})[1 + \log_+(R\|\Delta\theta\|)]^2$$

and using it together with (7) in (27) we obtain from (26)

$$\begin{aligned} N \leqslant \int_0^1 (\tau'(z))^2\, dz + CR(1 + \|\tau\|_{L^\infty})^2 \left[\int_0^1 z^2\, |\tau'|\, dz\right] \\ \times \left[1 + \log_+ \left\{RN + \int_0^1 [(\tau''(z))^2 + Rz(\tau'(z))^2]\, dz\right\}\right] \end{aligned} \tag{28}$$

Choosing τ to be a smooth approximation of $\tau(z) = (1-z)/\delta$ for $0 \leqslant z \leqslant \delta$ and $\tau = 0$ for $z \geqslant \delta$ and optimizing in δ we obtain

Theorem 2. There exists a constant C_0 such that the Nusselt number for the infinite Prandtl number equation is bounded by

$$N \leqslant N_0(R)$$

where

$$N_0(R) = 1 + C_0\, R^{1/3}(1 + \log_+ R)^{2/3}$$

In fact we proved also

Theorem 3. The Nusselt number for the infinite Prandtl number equation is bounded by the constrained mini-max procedure

$$N \leqslant \inf_{\tau} \sup_{\theta \in C_\tau} \left\{ -\langle \|\nabla\theta\|^2 \rangle + 2 \left\langle \int_0^1 -\tau'(z)\, b(z)\, dz \right\rangle + \int_0^1 (\tau'(z))^2 \right\}$$

where C_τ is the set of smooth, time dependent functions θ that obey periodic-homogeneous Dirichlet boundary conditions and the inequality

$$\langle \|\Delta\theta\|^2 \rangle \leqslant C_\Delta \left\{ RN_0(R) + \int_0^1 [\tau''(z))^2 + Rz(\tau'(z))^2]\, dz \right\}$$

The functions $b(z, t)$ are computed via

$$b(z, t) = \frac{1}{L^2} \iint_Q w(x, y, z, t)\, \theta(x, y, z, t)\, dx\, dy$$

and the functions $w(x, y, z, t)$ are computed by solving

$$\Delta^2 w = -R\Delta_h \theta$$

with periodic-homogeneous Dirichlet and Neumann boundary conditions.

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