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Remarks on Bang–Bang Control in Hilbert Space[1]

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Abstract. In this note, a natural definition of bang–bang control in Hilbert space is given, and some of the theory of the authors' paper (Ref. 1) is rebuilt upon it. An elliptic boundary-value problem illustrating the theory is given. In the last part of this note, the results of Ref. 1 are extended to nonlinear perturbations of linear operators and to homogeneous nonlinear operators.

1. Introduction

Let H_1 and H_2 be two, real Hilbert spaces with inner product and norm $(\ ,\ )_i$ and $\|\ \|_i$, $i = 1, 2$. Let A be a linear operator with dense domain in H_1 mapping 1-1 into H_2. We admit as control sets U bounded closed convex subsets of $R(A)$, the range of A.

In Ref. 1, we defined an *extremal* point of U to be any point $u \in U$ such that $\lambda u \notin U$ for each $\lambda > 1$. We considered there the following problem.

Problem 1.1. Given $r \in D(A^{-1*}) \subset H_1$ and $\beta > 0$, to find $u \in U$ and $x(u) \in D(A)$ such that

(i) $\quad Ax = u$

and

(ii) $\quad J(u) = (r, x)_1 + \beta(x, x)_1$ is minimized.

Our main result was the following.

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Theorem 1.1. If U is a closed convex subset of the unit ball in H_2 containing 0 and if A has a bounded inverse with $\| A^{-1} \|^2 = K$, then an optimal control for Problem 1.1 is bang–bang (extremal) provided

$$0 \leqslant \beta \leqslant | \theta |/2K,$$

where

$$\theta = \min_{\substack{u \in U \\ \beta = 0}} J(u).$$

The definition of an extremal or bang–bang control used above, however, is not always a natural extension of the usual non-Hilbert space definition. For example, let $U = \{u \in L^2(\Omega) \mid | u(t)| \leqslant 1 \text{ a.e.}\}$, $\Omega = [0, 1]$, $H_2 = L^2(\Omega)$, and let

$$u(t) = 0 \quad \text{for} \quad t \in (\delta, 1], \quad \delta > 0,$$
$$u(t) = 1 \quad \text{for} \quad t \in (0, \delta];$$

then, $u \in U$ and u is an extremal point of U, but is far from being a bang–bang control in the engineering sense. In a less abstract setting than in Ref. 1, we can largely overcome this objection.

Let Ω be a bounded domain in R^n. Let $H_1 = H_2 = L^2(\Omega)$. We assume that the admissible control set U includes all functions in $L^2(\Omega)$ such that $| u(t)| < 1$ a.e. in Ω. A more natural extension, than the above definition from Ref. 1, of the usual engineering concept of a bang–bang control is the following.

Definition 1.1. A function $u \in U$ is a *singular* (not bang–bang) control on Ω iff there exists a set E of positive measure, $E \subset \Omega$, such that ess sup $| u(t)| < 1$ on E. A function $u \in U$ is a *bang–bang* (extremal) control on Ω iff it is nonsingular on Ω.

With the restriction of our attention to $L^2(\Omega)$, we can prove a result much like Theorem 1.1 using Definition 1.1. Apart from the definition of bang–bang control being used, the essential difference between Theorem 1.1 and our main result, Theorem 1.2 below, is that we are able to prove that the optimal control is bang–bang for β positive only on subdomains of Ω over which ess inf $| A^{-1*}r |$ is positive. It will be seen that this restriction is inherent in our method of proof. We do not know if it or a similar hypothesis is a necessary condition for bang–bang control for β positive.

Theorem 1.2. If

(i) A is a linear, densely defined operator on $L^2(\Omega)$ with A^{-1} bounded,

(ii) there exists a *Green's* function G on $\Omega \times \Omega$ and a positive constant M such that

$$[A^{-1}u](t) = \int_\Omega G(t, \tau)\, u(\tau)\, d\tau, \qquad \sup_t \int_\Omega |\, G(t, \tau)\, d\tau \,| < M,$$

$$\sup_\tau \int_\Omega |\, G(t, \tau)\, dt \,| < M,$$

(iii) $U = \{u \in L^2(\Omega) : |\, u(t)| \leqslant 1 \text{ a.e. in } \Omega\}$,

(iv) Ω_1 is a subdomain of Ω on which $\operatorname{ess\,inf}_{\Omega_1} |\, A^{-1*}r \,| = \delta > 0$,

then, for $\beta < \delta/6M^2$, the optimal control $\tilde{u}$ is bang–bang on Ω_1.

Remark 1.1. If $\operatorname{ess\,inf} |\, A^{-1*}r \,| = \delta > 0$ on all of Ω, then of course the optimal control for such β will be bang–bang on Ω. In the case of a self-adjoint differential operator with homogeneous boundary conditions, $\operatorname{ess\,inf}_\Omega |\, A^{-1*}r \,|$ may be zero; see the example below.

Remark 1.2. We have tacitly assumed that, for each β, an optimum control for Problem 1.1 exists. This follows from the fact that U is a closed bounded convex set and $J(u)$ is weakly lower semicontinuous (see Ref. 2, page 5). Convexity is required only for the proof of existence of an optimal control.

Remark 1.3. From the proof of Theorem 1.2, it will be apparent to the reader that any closed bounded convex set containing all functions cut to ± 1 on measurable subsets of Ω would serve as the admissible control region U. The proof can also be modified to admit as controls square-integrable functions for which $-a \leqslant u \leqslant b$ a.e. in Ω, where a and b are positive. Of course, the bound on β in the conclusion must be appropriately modified.

2. Proof of Theorem 1.2

We shall need the following lemma.

Lemma 2.1. If v is an optimal control Problem 1.1 with $\beta = 0$ and $r^* = A^{-1*}r$ and Ω_1 satisfying condition (iv) of Theorem 1.2, then for each $E \subset \Omega_1$ with $\mu(E) > 0$, $(r^*, v)_E < 0$.

Indeed, $v = -\operatorname{sign} r^*$ in E.

Proof. Suppose that the lemma is false and that there exists an $E \subset \Omega_1$ with $\mu(E) > 0$ and such that $(r^*, v)_E \geqslant 0$. Consider

$$w(t) = \begin{cases} v(t) & \text{if} \quad t \notin E, \\ -\operatorname{sign} r^*(t) & \text{if} \quad t \in E. \end{cases}$$

Then, $w \in U$. Now,

$$(r^*, v)_\Omega \leqslant (r^*, w)_\Omega, \tag{1}$$

since v minimizes $(r^*, u)_\Omega = J(u)$ when $\beta = 0$. But

$$(r^*, v)_\Omega - (r^*, w)_\Omega = (r^*, v)_E - (r^*, w)_E > 0,$$

since

$$(r^*, w)_E \geqslant 0 \qquad \text{and} \qquad -(r^*, w)_E = (r^*, \operatorname{sign} r^*)_E > 0.$$

This contradicts (1).

Proof of Theorem 1.2. Suppose that $\tilde{u}$ is an optimal control for Problem 1.1, and suppose that $\tilde{u}$ is not bang–bang on all of Ω_1. Then, there exist a measurable set $E \subset \Omega_1$ with $\mu(E) > 0$ and a $\lambda > 1$ such that $| \lambda u(t)| < 1$ for $t \in E$. Define

$$u(t) = \begin{cases} \tilde{u}(t), & t \in \Omega - E, \\ \lambda \tilde{u}(t), & t \in E. \end{cases}$$

Then, $u \in U$. We shall reach a contradiction by showing that $\Delta J = J(u) - J(\tilde{u}) < 0$, which implies that $\tilde{u}$ is not optimal. We first observe that an easy computation shows that

$$\Delta J = (\lambda - 1)(r^*, \tilde{u})_E + \beta[\| A^{-1}u \|_\Omega^2 - \| A^{-1}\tilde{u} \|_\Omega^2]. \tag{2}$$

Now, let

$$v_1(t) = \begin{cases} \tilde{u}(t), & t \in \Omega - E, \\ v(t), & t \in E, \end{cases}$$

where v is optimal for $\beta = 0$. Since $J(\tilde{u}) \leqslant J(v_1)$,

$$(r^*, \tilde{u})_\Omega + \beta \| A^{-1}\tilde{u} \|_\Omega^2 \leqslant (r^*, v_1)_\Omega + \beta \| A\ ^{1}v_1 \|_\Omega^2 .$$

Using the properties of the scalar product and the definitions of v_1, we obtain

$$(r^*, \tilde{u})_E \leqslant (r^*, v)_E + \beta[\| A^{-1}v_1 \|_\Omega^2 - \| A^{-1}\tilde{u} \|_\Omega^2].$$

Substituting in (2), we get

$$\Delta J \leqslant (\lambda - 1)\{(r^*, v)_E + \beta[\| A^{-1}v_1 \|_\Omega^2 - \| A^{-1}\tilde{u} \|_\Omega^2]\} + \beta[\| A^{-1}u \|_\Omega^2 - \| A^{-1}\tilde{u} \|_\Omega^2].$$

Hence,

$$\begin{aligned} \Delta J &\leqslant (\lambda - 1) \left\{(r^*, v)_E + \beta \int_\Omega (A^{-1}v_1)^2 - (A^{-1}\tilde{u})^2 \, dt\right\} \\ &\quad + \beta \int_\Omega (A^{-1}u)^2 - (A^{-1}\tilde{u})^2 \, dt \\ &= (\lambda - 1) \left\{(r^*, v)_E + \beta \int_\Omega (A^{-1}v_1 - A^{-1}\tilde{u})(t)(A^{-1}v_1 + A^{-1}\tilde{u})(t) \, dt\right\} \\ &\quad + \beta \int_\Omega (A^{-1}u - A^{-1}\tilde{u})(t) \cdot (A^{-1}\tilde{u} + A^{-1}u)(t) \, dt. \end{aligned} \tag{3}$$

By hypothesis (ii) and the fact that u and v are in U,

$$| A^{-1}(v_1 + \tilde{u})(t)| = \left| \int_\Omega G(t, \tau)(v_1 + \tilde{u})(\tau) \, d\tau \right| \leqslant 2 \int_\Omega | G(t, \tau)| \, d\tau \leqslant 2M. \tag{4}$$

Also,

$$\int_\Omega | A^{-1}(v_1 - \tilde{u})| \, dt \leqslant \int_\Omega \int_\Omega | G(t, \tau)| \, |(v_1 - \tilde{u})(\tau)| \, d\tau \, dt. \tag{5}$$

Interchanging the order of integration, which is permissible since the integrand is measurable and nonnegative, we get

$$\begin{aligned} \int_\Omega | A^{-1}(v_1 - \tilde{u})| \, dt &\leqslant \int_\Omega \left[|(v_1 - \tilde{u})(\tau)| \int_\Omega | G(t, \tau)| \, dt\right] d\tau \\ &\leqslant M \int_E | v_1(\tau) - \tilde{u}(\tau)| \, d\tau \leqslant 2M\mu(E). \end{aligned}$$

Combining (4)–(5), we see that the first integral in (3) is less than $4M^2\mu(E)$. A similar estimate shows the second integral in (3) is less than $2M^2(\lambda - 1)\,\mu(E)$. Therefore,

$$\Delta J \leqslant (\lambda - 1)\{(r^*, v)_E + 6\beta M^2\mu(E)\}. \tag{6}$$

Since from Lemma 2.1,

$$(r^*, v)_E = - \int_E | r^* |,$$

we see that

$$\begin{aligned} \Delta J &\leqslant (\lambda - 1) \, |(r^*, v)_E | \, [-1 + 6\beta M^2\mu(E)/|(r^*, v)_E |] \\ &\leqslant (\lambda - 1) \, |(r^*, v)_E | \, [-1 + 6\beta M^2\mu(E)/\mathrm{ess}_E\mathrm{inf} \, | r^* | \cdot \mu(E)], \end{aligned}$$

where we have used the inequality

$$\int_E |r^*| \geqslant (\mathrm{ess}_E \inf |r^*|) \cdot \mu(E).$$

Since

$$\delta = \mathrm{ess}_{\Omega_1} \inf |r^*| \leqslant \mathrm{ess}_E \inf |r^*|,$$

and since, by hypothesis (iv), $\delta > 0$, we obtain the inequality

$$\Delta J \leqslant (\lambda - 1)\,|(r^*, v)_E|\,[-1 + 6\beta M^2/\delta]. \tag{7}$$

If $\beta < \delta/6M^2$, the sum in the square brackets is negative, while the other factors are strictly positive. Thus, $\Delta J < 0$, contradicting the optimality of $\tilde{u}$.

3. Example

We now consider a simple example from partial differential equations. Let Ω be the unit disk in R^2. We consider the problem of minimizing the functional

$$J(u) = \int_\Omega rw + \beta w^2 \, d\Omega,$$

where r is a constant function, $\beta > 0$, and w is the solution to

$$\Delta w = u, \qquad w|_{\partial\Omega} = 0.$$

We take

$$U = \{u \in L^2(\Omega) \mid |u(\rho, \theta)| \leqslant 1 \text{ a.e.}\}.$$

This problem may be put into the setting of Theorem 1.2 by taking A to be the Friedrichs' self-adjoint extension of the Laplacian (Ref. 3). Note that $D(A) \subseteq \mathring{H}^1(\Omega)$, and functions in $D(A)$ vanish on $\partial\Omega$ in the usual generalized sense. Then, for each $u \in U$, there exists a unique (possibly generalized) solution to the above Dirichlet problem. Moreover, by the remark after Theorem 1.2, for each $r \in L^2(\Omega)$, $\beta > 0$, there exists an optimal control $u \in U$. We will now show that the hypotheses of Theorem 1.2 are satisfied and give sufficient conditions to guarantee that the optimal control is bang–bang on subdomains of Ω.

That condition (i) is satisfied is obvious. If (ρ, θ) are the polar coordinates of a point in the disk, and $r = \alpha = \text{const}$, it is readily seen that

$$r^* = A^{-1*}\alpha = A^{-1}\alpha = \alpha(\rho^2 - 1)/4, \; Ar^* = \alpha, \qquad r^*|_{\partial\Omega} = 0.$$

Since

$$\int_\Omega G = \int_\Omega G \cdot 1$$

is the solution to

$$\Delta w = 1, \qquad w\,|_{\partial\Omega} = 0,$$

and this solution is smooth, it follows that

$$\sup_t \int_\Omega |\, G(t, \tau)|\, d\tau = \sup_\tau \int_\Omega |\, G(t, \tau)|\, dt$$

is bounded, and condition (ii) is satisfied.

It is clear that $\mathrm{ess}_\Omega \inf |\, r^* \,| = 0$ and that hypothesis (iv) is not satisfied on Ω. For any subdomain $\Omega_1 \subset \Omega$, however, where Ω_1 has a positive distance from $\partial\Omega$,

$$\mathrm{ess}_{\Omega_1} \inf |\, r^* \,| = \delta > 0;$$

and, for positive β sufficiently small, the optimal control is bang–bang on Ω_1. For example, if

$$\Omega_1 = \{(\rho, \theta) \mid \rho \leqslant \rho_0 \leqslant 1\},$$

then

$$\mathrm{ess}_{\Omega_1} \inf |\, r^* \,| = \alpha(1 - \rho_0{}^2)/4;$$

and, for

$$\beta < \alpha(1 - \rho_0{}^2)/24M^2,$$

the optimal control $\tilde{u}$ is bang–bang on Ω_1. In other words, $|\, \tilde{u}(\rho, \theta)| = 1$ a.e. for $\rho \leqslant \rho_0$.

For the above example, we see that, if $0 < \beta < \alpha/24M^2$, then, for

$$\rho < \rho_0 = \sqrt{[1 - 24M^2\beta/\alpha]},$$

$\tilde{u}$ is bang–bang.

4. Nonlinear Perturbations

In this section, we revert to the general setting and definitions of Ref. 1 and extend some of the results obtained there to nonlinear operators.

An examination of the proof of Theorem 3.1 of Ref. 1 (Theorem 1.1 in this paper) shows that it is valid if the linear operator A is replaced by a homogeneous but nonlinear operator, provided the inverse is bounded on the unit sphere. By a homogeneous operator, we mean one for which $A(\lambda x) = \lambda^\alpha A(x)$, α real. Many of the other theorems of Ref. 1 also hold for homogeneous nonlinear operators, provided some obvious modifications are made. The verification of whether the inverse of a nonlinear operator (if it exists) is bounded on the unit sphere, however, is in general no simple task.

We now extend Theorem 1.1 to the case of a nonlinear operator that is a perturbation of the original linear operator.

Let $A = A_0$ be a linear (not necessarily bounded) operator mapping $H_1 \to H_2$; A^{-1} is assumed to exist and to be compact[4] as an operator from H_2 to H_1. Let

$$A_\epsilon x = Ax + \epsilon f(x),$$

where f is a nonlinear function mapping H_1 into H_2 and satisfying the Lipschitz condition

$$\| f(x_1) - f(x_2) \|_2 \leqslant N \| x_1 - x_2 \|_1 .$$

We seek to minimize the functional

$$J_{\epsilon\beta}(u) = (r, x)_1 + \beta(x, x)_1 ,$$

subject to

$$A_\epsilon x = (A + \epsilon f)x = u.$$

The control region U is assumed to satisfy the conditions of Theorem 1.1.

Theorem 4.1. *If* $\beta < | \theta |/2K$ $[\theta = \min J(u)$ for $\beta = \epsilon = 0]$ and $K = \| A^{-1} \|^2$, then, for $\epsilon > 0$ sufficiently small, u optimal is an extremal point of U.

We first prove a few lemmas.

Lemma 4.1. For $0 < \epsilon < 1/\| A^{-1} \| N$, A_ϵ^{-1} exists and is compact and continuous.

Proof. Let

$$A_\epsilon x_i = Ax_i + \epsilon f(x_i) = u_i , \qquad i = 1, 2. \tag{8}$$

Then,

$$A(x_1 - x_2) = u_1 - u_2 - \epsilon[f(x_1) - f(x_2)]$$

[4] See remark at end of this section.

and

$$x_1 - x_2 = A^{-1}(u_1 - u_2) - \epsilon A^{-1}[f(x_1) - f(x_2)].$$

Thus,

$$\| x_1 - x_2 \|_1 \leqslant \| A^{-1}(u_1 - u_2)\|_1 + \epsilon \| A^{-1} \| N \| x_1 - x_2 \|_1 ,$$

and hence,

$$\| x_1 - x_2 \|_1 [1 - \epsilon \| A^{-1} \| N] \leqslant \| A^{-1}(u_1 - u_2)\|_1 . \tag{9}$$

For $\epsilon < 1/\| A^{-1} \| N$,

$$\| u_1 - u_2 \|_2 \to 0 \Rightarrow \| x_1 - x_2 \|_1 \to 0.$$

Thus, A_ϵ is 1-1, and A_ϵ^{-1} exists and is continuous.

Remark 4.1. That the range $(A_\epsilon) \supseteq U$ for such ϵ follows readily from the properties of A and f and a simple successive approximation argument. To show *compactness*, let u_1 converge weakly to u_2. It suffices to show that $x_1 \to x_2$ strongly. But $A^{-1} = A_0^{-1}$ is compact from H_2 to H_1. Thus, $A^{-1}(u_1 - u_2) \to 0$ strongly in H_1. It follows from (9) that $x_1 \to x_2$ strongly.

Lemma 4.2. For $\epsilon < 1/\| A^{-1} \| N$, the inverse of the perturbed operator is a compact perturbation of the inverse of the unperturbed operator and has the representation

$$A_\epsilon^{-1}(u) = A^{-1}(u) + \epsilon \mathscr{K}(u),$$

where $\mathscr{K}$ is a compact nonlinear operator.

Proof. Let

$$A_\epsilon x = Ax + \epsilon f(x) = u.$$

Then,

$$\begin{aligned} x = A_\epsilon^{-1}(u) &= A^{-1}(u) - \epsilon A^{-1}(f(x)) \\ &= A^{-1}(u) - \epsilon A^{-1} \circ f \circ A_\epsilon^{-1}(u). \end{aligned}$$

But, for $\epsilon < 1/\| A^{-1} \| N$, A_ϵ^{-1} is compact by Lemma 4.1; and $\mathscr{K} = A^{-1} \circ f \circ A_\epsilon^{-1}$ is the composition of a compact operator followed by a continuous operator, followed by a compact operator, and hence is compact.

Proof of Theorem 4.1. Suppose that u optimal is not extremal. Then, there exists $\epsilon_0 > 0$ such that, for $1 \leqslant \lambda \leqslant 1 + \epsilon_0$, $\lambda u \in U$. Consider

$$\begin{aligned} J_{\epsilon\beta}(\lambda u) - J_{\epsilon\beta}(u) &= (\lambda - 1)[(r, A^{-1}u)_1 + \beta(\lambda + 1)(A^{-1}u, A^{-1}u)_1] \\ &\quad + \epsilon[(r, \mathscr{K}(\lambda u))_1 - (r, \mathscr{K}u)_1] \\ &\quad + 2\beta\epsilon[(\lambda A^{-1}u, \mathscr{K}(\lambda u))_1 - (A^{-1}u, \mathscr{K}u)_1] \\ &\quad + \beta\epsilon^2[(\mathscr{K}(\lambda u), \mathscr{K}(\lambda u))_1 - (\mathscr{K}u, \mathscr{K}u)_1], \end{aligned}$$

which for brevity we write as

$$\begin{aligned} &J_{\epsilon\beta}(\lambda u) - J_{\epsilon\beta}(u) \\ &\quad = (\lambda - 1)[(r, A^{-1}u)_1 + \beta(A^{-1}u, A^{-1}u)_1 + \lambda\beta(A^{-1}u, A^{-1}u)_1] + \epsilon C, \end{aligned}$$

where, since ϵ, u, λ, β may all be considered to run over bounded sets and the operators A^{-1} and $\mathscr{K}$ are compact, C may hence be taken to be bounded uniformly. We estimate

$$(r, A^{-1}u)_1 + \beta(A^{-1}u, A^{-1}u)_1 = J_{\epsilon\beta}(u) - \epsilon C_1,$$

where C_1 is bounded uniformly as C above. If v is the optimal control for $\epsilon = 0$, $\beta = 0$, then $J_{\epsilon\beta}(u) \leqslant J_{\epsilon\beta}(v)$ (u was assumed optimal for $J_{\epsilon\beta}$). Hence,

$$\begin{aligned} T &\equiv (r, A^{-1}u)_1 + \beta(A^{-1}u, A^{-1}u)_1 \leqslant J_{\epsilon\beta}(v) - \epsilon C_1 \\ &= (r, A^{-1}v)_1 + \beta(A^{-1}v, A^{-1}v)_1 + 0(\epsilon) \leqslant \theta + \beta K + 0(\epsilon). \end{aligned}$$

Let $\bar{\epsilon} > 0$ be such that

$$\beta < |\theta|/(2 + \bar{\epsilon})K < |\theta|/2K.$$

Then,

$$\begin{aligned} T &\leqslant \theta + |\theta|/(2 + \bar{\epsilon}) + 0(\epsilon) \\ &= \theta - \theta/(2 + \bar{\epsilon}) + 0(\epsilon) = \theta(1 + \bar{\epsilon})/(2 + \bar{\epsilon}) + 0(\epsilon). \end{aligned}$$

Also,

$$|\beta\lambda(A^{-1}u, A^{-1}u)_1| \leqslant \lambda\beta K < \lambda|\theta|/(2 + \bar{\epsilon}).$$

Now, take ϵ_0 sufficiently small, so that $\epsilon_0 < \bar{\epsilon}/2$, where

$$1 \leqslant \lambda \leqslant 1 + \epsilon_0 < 1 + \bar{\epsilon}/2.$$

Then,

$$\lambda|\theta|/(2 + \bar{\epsilon}) < -\theta/2.$$

(Recall that $\theta < 0$). We thus obtain the inequalities

$$J_{\epsilon\beta}(\lambda u) - J_{\epsilon\beta}(u) < (\lambda - 1)[(1 + \bar{\epsilon})/(2 + \bar{\epsilon})\theta - \theta/2] + 0(\epsilon)$$
$$< (\bar{\epsilon}/2)[\bar{\epsilon}\theta/2(2 + \bar{\epsilon})] + 0(\epsilon).$$

Since $0(\epsilon) \to 0$ as $\epsilon \to 0$, by choosing $\epsilon > 0$ sufficiently small, we can guarantee that the right-hand side is negative, contradicting the optimality of u.

Remark 4.2. The proof of Theorem 4.1 also goes through if $A_\epsilon^{-1} = A_0^{-1} + \epsilon B$ where B is bounded on bounded sets. However, simple conditions to guarantee that such a representation exists for the inverse of the perturbed operator are not known.

Remark 4.3. The assumption that A^{-1} is compact from H_2 to H_1 is rather strong. If we assume that $H_1 \subseteq H_2$ and that A^{-1} is compact as an operator from $H_2 \to H_2$ and only bounded from $H_2 \to H_1$, then the theorem and lemmas of this section go through essentially unchanged, provided the functional being minimized is

$$\bar{J}(u) = (r, x_2)_2 + \beta(x, x)_2$$

and *not*

$$\bar{J}(u) = (r, x)_1 + \beta(x, x)_1 .$$

References

1. Rogak, E. D., and Kazarinoff, N. D., *Sufficient Conditions for Bang–Bang Control in Hilbert Space,* Journal of Optimization Theory and Applications, Vol. 5, No. 1, 1970.
2. Lions, J. L., *Contrôle Optimal de Systèmes Gouvernés par des Équations aux Dérivées Partielles,* Dunod, Paris, 1968.
3. Smirnov, V. I., *A Course in Higher Mathematics,* Vol. 5, Moscow, 1960.