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Sufficient Conditions for Bang-Bang Control in Hilbert Space[1]

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Abstract. Sufficient conditions for bang-bang and singular optimal control are established in the case of linear operator equations with cost functionals which are the sum of linear and quadratic terms, that is, $Ax = u$, $J(u) = (r, x) + \beta(x, x)$, $\beta > 0$. For example, if A is a bounded operator with a bounded inverse from a Hilbert space H into itself and the control set U is the unit ball in H, then an optimal control is bang-bang (has norm 1) if $0 \leqslant \beta < \frac{1}{2} \| A^{-1*} r \| \cdot \| A^{-1} \|^{-2}$, but is singular (an interior point of U) if $\beta > \frac{1}{2} \| A^{-1*} r \| \cdot \| A \|^{2}$.

1. Introduction

Control processes described by differential equations linear in the control but with quadratic cost functionals are often not bang-bang (Refs. 1–3). That is, for such processes, the optimal control does not lie on the boundary of the control region. A simple example of such a process is described in Section 2, namely, to minimize

$$J(u) = \beta \int_0^T x^2 \, dt, \qquad \beta > 0$$

for u piecewise constant and $| u(t)| \leqslant 1$ on $[0, T]$ if

$$dx/dt = u, \qquad x(0) = x_0, \qquad x(T) = x_1$$

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Here, x_0, x_1, T are fixed. Whether or not a given finite-dimensional process is bang-bang can be determined from Pontryagin's maximum principle, provided the adjoint problem can be solved and has a nonzero solution. However, a maximum principle for control problems involving operator equations in a Hilbert space is not presently available except in special cases (Refs. 4–5).

Here, we establish, without a maximum principle, sufficient conditions for bang-bang and singular optimal control (in the sense defined in Section 3) in the case of linear operator equations with cost functionals which are the sum of linear and quadratic terms, that is,

$$Ax = u, \qquad J(u) = (r, x) + \beta(x, x), \qquad \beta > 0$$

These are stated as Theorems 3.1, 4.1, 4.2 in Sections 3 and 4. We show, for example, that, if A is a bounded operator with a bounded inverse from a Hilbert space H into itself and the control set U is the unit ball in H, then an optimal control is bang-bang (has norm 1) if

$$0 \leqslant \beta < \tfrac{1}{2} \| A^{-1*} r \| \cdot \| A^{-1} \|^{-2}$$

but is singular (an interior point of U) if

$$\beta > \tfrac{1}{2} \| A^{-1*} r \| \cdot \| A \|^2$$

We were led to our results through consideration of the example mentioned above. Our approach to guarantee bang-bang control is the naive one: we assume that the optimal control $\tilde{u}$ is singular and, with simple estimates, show that there exists a $\lambda > 0$ such that $J(\lambda\tilde{u}) < J(\tilde{u})$. In essence, our results are: if J is close to linear, an optimal control is bang-bang; but, if J is close to quadratic, the optimal control is singular. They appear to be reasonably close to best possible.

2. Finite-Dimensional Example

Consider the problem represented by

$$dx/dt = u, \qquad x(0) = x_0, \qquad x(T) = x_1 \tag{1}$$

with

$$J(u) = \int_0^T (\beta x^2 + rx)\, dt, \qquad \beta > 0 \tag{2}$$

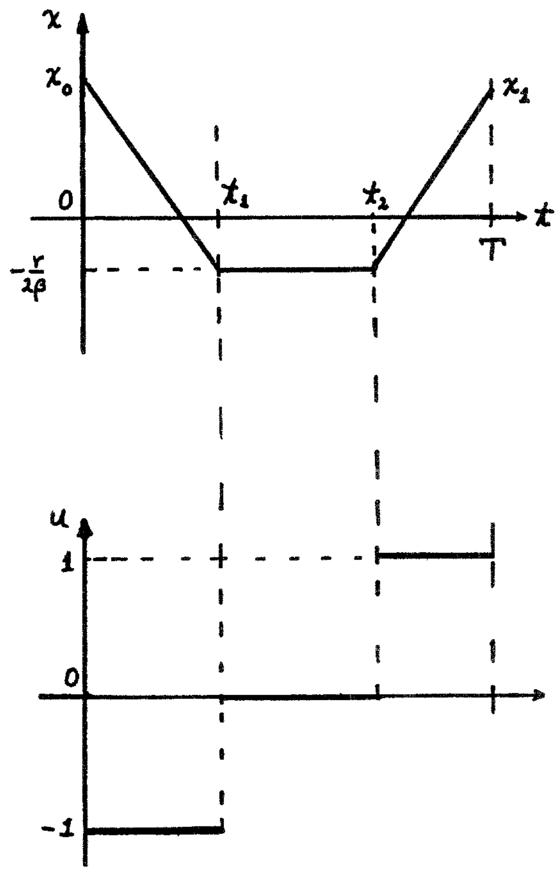

Fig. 1

Here, x_0, x_1, T are fixed; r, β are real numbers; and $u \in U = \{u \mid u \text{ is piecewise constant and } |u(t)| \leqslant 1 \text{ on } [0, T]\}$. The problem is to minimize $J(u)$ for $x(t, u)$ satisfying (1) and $u \in U$. We assume that x_1 is accessible from x_0 in the given time T.

The minimum of $\beta x^2 + rx$ occurs at $x = -r/2\beta$. Optimum passage from x_0 to x_1 in time T is obtained by moving as fast as possible to $x = -r/2\beta$ (with $u = -1$ if $x_0 > -r/2\beta$, but with $u = +1$ if $x_0 < -r/2\beta$). The process continues at $x = -r/2\beta$ ($u \equiv 0$) until we can apply a control $u = \pm 1$, the sign depending upon whether x_1 is larger or smaller than $-r/2\beta$, to reach x_1 at time T. The switching times are

$$t_1 = x_0 + (r/2\beta), \qquad t_2 = T - [x_1 + (r/2\beta)]$$

The phase trajectory and control are shown for a typical case (x_0 and x_1 exceed $-r/2\beta$) in Fig. 1. Thus, if both x_0 and x_1 exceed $-r/2\beta$, a necessary and sufficient condition for bang-bang control is $t_2 \leqslant t_1$, that is,

$$r/\beta \geqslant T - (x_0 + x_1)$$

Analogous results hold for other configurations of x_0 and x_1.

The reader should be aware that, while the example just described motivated the discovery of the theorems to follow, we have been unable to fit the example in the abstract setting of the theorems. Our difficulty is that some nonextremal points of the control set in the example are identified as extremal points when we try to describe the example in terms of the abstract setting of the theorems.

3. Bang-Bang Control

Let H_i, $i = 1, 2$, be two real Hilbert spaces with inner products and norms $(.,.)_i$ and $\|\cdot\|_i$, respectively. We assume that A is a linear operator from H_1 into H_2, that is,

$$A\colon\ \mathscr{D}(A) \overset{1-1}{\leftrightarrow} R(A)$$

with $R(A)$ dense in H_2. For the control region U, we choose a subset of $R(A)$. In all that follows, it is sufficient to assume that A has a right inverse and that $R(A)$ is dense in a subspace of H_2. We have made the stronger and simpler assumptions that A has an inverse and $R(A)$ is dense in H_2.

Let $r \in \mathscr{D}(A^{-1*})$ be a fixed vector in H_1. We study controls u that minimize the functional

$$J(u) = (r, x)_1 + \beta(x, x)_1, \qquad \beta \geqslant 0 \tag{3}$$

subject to the constraint

$$Ax = u \tag{4}$$

We assume that an optimal control exists; namely, for all $\beta \geqslant 0$, there exists at least one $\tilde{u} \in U$ and $\tilde{x} \in \mathscr{D}(A)$ such that

$$A\tilde{x} = \tilde{u} \qquad \text{and} \qquad J(\tilde{u}) = \inf_{u \in U} J(u)$$

Note that $\tilde{u}$ and $\tilde{x}$ in general depend on β. Also, we need not assume uniqueness of the optimal control. However, it turns out that, if U is convex, then the optimal control must be unique. In fact, let u_0 and u_1 be two optimal controls corresponding to x_0 and x_1, with $J(u_0) = J(u_1)$. Then, $(1 - \lambda) u_0 + \lambda u_1$ is an admissible control corresponding to $(1 - \lambda) x_0 + \lambda x_1$. A simple computation shows that

$$J(x_\lambda) = (1 - \lambda) J(u_0) + \lambda J(u_1) - \lambda(1 - \lambda) \beta(x_0 - x_1, x_0 - x_1)$$
$$< J(x_0) \quad \text{if} \quad u_0 \neq u_1$$

Further, it is natural to impose the condition that, even when $\beta = 0$, the control action affects J; namely, if $\beta = 0$, there exists a $v \in U$ such that

$$J(v) = (r, A^{-1}v)_1 + 0(A^{-1}v, A^{-1}v)_1 = (A^{-1*}r, v)_2 = \theta \neq 0 \tag{5}$$

In what follows, v is an optimal control for $\beta = 0$.

Note that, if U is a subset of the unit ball and

$$-r^*/\| r^* \| \in U \qquad (r^* = A^{-1*}r)$$

then it is an optimal control for $\beta = 0$, and the minimum value of J is $-\| r^* \|_2$. In the case $\beta = 0$, whether or not $-r^*/\| r^* \| \in U$, the optimal control process is bang-bang. On the other hand, if $r = 0$, then $J(u) = \beta(x, x)$ is least for $u = x = 0$. In this case, therefore, the optimal control is singular. The theorems to follow give various sufficient conditions to guarantee bang-bang control or singular control in the intermediate cases where neither r nor β is zero.

We next describe the kind of control sets we shall admit, and define precisely what we mean by bang-bang control.

Definition 3.1. A control set U is *star-shaped* if, and only if, for each $u \in U$, there exists an $\epsilon > 0$ such that $\lambda u \in U$ for $-\epsilon < \lambda < 1$.

Definition 3.2. A point u in a control set U is an *extremal* point of U if, and only if, $\lambda u \notin U$ for each $\lambda > 1$.

A star-shaped control set, according to Definition 3.1, may be a thin spiny set or one with "faces" containing "radial line segments". Hence, not all the boundary points of a star-shaped control set are necessarily extremal points. By a bang-bang control, we mean an extremal point of U. By a singular control, we mean one that is not bang-bang. A singular control may be a boundary point of U.

It is easy to see that extremal points are always boundary points and, if U is a convex body with the origin in its interior, then it is also star-shaped.

Theorem 3.1. If U is a star-shaped subset of the unit ball of H_2 and if

$$A\colon \ \mathscr{D}(A) \overset{1-1}{\leftrightarrow} R(A) \subset H_2$$

has a bounded inverse with $\| A^{-1} \|^2 = K$, then an optimal control for the problem (3)–(4) is always an extremal point of U for all nonnegative

$$\beta < | \theta |/2K \tag{6}$$

where $\theta = \min_{u \in U} J(u)$ for $\beta = 0$.

Note that A need not be bounded, nor must $\tilde{u}$ be unique.

Proof. Suppose that $\tilde{u}$, an optimal control, is not an extremal point of U. Then, since U is star-shaped, there exists an $\epsilon_0 > 0$ such that, if $1 \leqslant \lambda < 1 + \epsilon_0$, $\lambda\tilde{u} \in U$. Thus, we may consider

$$J(\lambda\tilde{u}) - J(\tilde{u}) = (\lambda - 1)[(r^*, \tilde{u})_2 + \beta(L_1\tilde{u}, \tilde{u})_2 + \lambda\beta(L_1\tilde{u}, \tilde{u})_2] \tag{7}$$

where

$$r^* = A^{-1*}r \qquad \text{and} \qquad L_1 = A^{-1*}A^{-1}$$

By (6), there exists an $\epsilon > 0$ such that

$$\beta < | \theta |/(2 + \epsilon) K < | \theta |/2K$$

Therefore, recalling the definition of v just after (5), we obtain

$$\begin{aligned} (r^*, \tilde{u})_2 + \beta(L_1\tilde{u}, \tilde{u})_2 \equiv J(\tilde{u}) &\leqslant J(v) \equiv (r^*, v)_2 + \beta(L_1 v, v)_2 \\ &\leqslant \theta + \beta \,|(L_1 v, v)_2 \,| \leqslant \theta + \beta K < \theta + | \theta |/(2 + \epsilon) \\ &= (1 + \epsilon)\, \theta/(2 + \epsilon) < 0 \end{aligned}$$

The last equality holds, since the hypotheses that U is star-shaped and the control action affects J if $\beta = 0$ guarantee that $\theta < 0$.

We next estimate the remaining term in (7). Suppose that $\epsilon_0 < \epsilon/2$. Then,

$$| \lambda\beta(L_1\tilde{u}, \tilde{u})_2 \,| < \lambda \,| \theta |/(2 + \epsilon) < (1 + \epsilon/2) | \theta |/(2 + \epsilon) = -\theta/2$$

Thus,

$$\begin{aligned} J(\lambda\tilde{u}) - J(\tilde{u}) &< (\lambda - 1)\{[(1 + \epsilon)\, \theta/(2 + \epsilon)] - (\theta/2)\} \\ &\leqslant (\epsilon/2)\, \epsilon\theta/2(2 + \epsilon) < 0 \end{aligned}$$

This contradicts the optimality of $\tilde{u}$.

Corollary 3.1. If, in addition to the hypotheses of the theorem,

$$r^*/\| r^* \| \in U$$

then $\tilde{u}$ is always an extremal point of U for all nonnegative

$$\beta < \| r^* \|_2/2 \| A^{-1} \|^2$$

Recall that a *regular* convex body is one in which every hyperplane of support intersects the body in exactly one point. An *extreme* point of a convex set U is one that is not a nontrivial convex combination of other points in U.

Corollary 3.2. Let the properties of A be as in Theorem 3.1. Let U be a regular convex body contained in the unit ball and containing the origin. If

$$\beta < | \theta |/2K$$

then the unique optimal control $\tilde{u}$ is an extreme point of U.

Proof. The uniqueness follows from the convexity of U and the remarks preceding (5). Now, suppose that there existed y, $w \in U$, and $0 < t < 1$ such that $\tilde{u} = ty + (1 - t) w$. By Theorem 3.1, $\tilde{u}$ is extremal and, hence, for all $\lambda > 1$, $\lambda \tilde{u} \notin U$. We now show that the segment $[y, w]$ is composed only of extremal points. For $\alpha > 1$, if $\alpha y \in U$, then the segment $[w, \alpha y] \subset U$. But this segment cuts the segment $[0, \lambda \tilde{u}]$ at some point $\bar{\lambda} \tilde{u}$, $\bar{\lambda} > 1$, contradicting the extremality of $\tilde{u}$. Thus, y and, similarly, w are extremal points of U. The same argument shows that all points in the segment $[y, w]$ must be extremal and, hence, boundary points of U. By a well-known theorem (Ref. 6, p. 64), there exists a closed hyperplane separating $[y, w]$ and int U, the interior of U. It is easy to see that this must be a hyperplane of support of U, containing $[y, w]$, and thus contradicting the regular convexity of U.

4. Singular Control

For A unbounded and A^{-1} bounded, we have not been able to prove, as we would have liked to have done, that an optimal control for the problem (4)–(5) is never an extremal point of U if

$$\beta > | \theta |/2K$$

In the way of an *only if* part of Theorem 3.1 we have only been able to prove that optimal controls are singular if β is sufficiently large. How large we do not know.

Under alternate hypotheses to those of Theorem 3.1, however, we can do better.

Theorem 4.1. If (*a*) U is a star-shaped subset of the unit ball in H_2, (*b*) $A: \mathscr{D}(A) \overset{1-1}{\leftrightarrow} R(A) \subset H_2$ is a bounded linear operator from H_1 to H_2, with A^{-1} bounded or unbounded, and (*c*)

$$\inf\{\| u \| \mid u \in U \quad \text{and} \quad u \text{ is extremal}\} = m > 0 \tag{8}$$

then an optimal control $\tilde{u}$ for the problem (4)–(5) is never an extremal point (is always singular) if

$$\beta > \| r^* \|_2 \| A \|^2 / 2m^2 \tag{9}$$

Proof. Suppose that β satisfies (9) and $\tilde{u}$, an optimal control, is an extremal point of U. Since U is star-shaped, $\lambda\tilde{u} \in U$ for all $\lambda \in [0, 1]$. For such λ, we may consider

$$J(\lambda\tilde{u}) - J(\tilde{u}) = (\lambda - 1)[(r^*, \tilde{u})_2 + (\lambda + 1)\beta(L_1\tilde{u}, \tilde{u})_2]$$

where

$$r^* = A^{-1*} r \qquad \text{and} \qquad L_1 = A^{-1*} A^{-1}$$

Now,

$$|(r^*, \tilde{u})_2| \leqslant \| r^* \|_2 \tag{10}$$

since $\| \tilde{u} \| \leqslant 1$. Also, by the hypothesis (8),

$$m \leqslant \| \tilde{u} \|_2 = \| A\tilde{x} \|_2 \leqslant \| A \| \| \tilde{x} \|_1 \qquad (A\tilde{x} = \tilde{u})$$

or

$$\| \tilde{x} \|_1^2 \geqslant m^2 / \| A \|^2$$

Therefore,

$$(L_1\tilde{u}, \tilde{u})_2 = \| \tilde{x} \|_1^2 \geqslant m^2 / \| A \|^2 \tag{11}$$

It now follows from (10)–(11) that

$$\begin{aligned}(r^*, \tilde{u})_2 + (\lambda + 1)\beta(L_1\tilde{u}, \tilde{u})_2 &\geqslant -|(r^*, \tilde{u})_2| + \beta(\lambda + 1)(L_1\tilde{u}, \tilde{u})_2 \\ &\geqslant -\| r^* \|_2 + \beta(\lambda + 1)[m^2 / \| A \|^2] \end{aligned} \tag{12}$$

But, for β satisfying (9), there exists an $\epsilon \in (0,1)$ such that

$$\beta > \| r^* \| \cdot \| A \|^2/(\epsilon + 1)\, m^2 \tag{13}$$

Having chosen $\epsilon \in (0,1)$, we finally choose $\lambda \in (\epsilon, 1)$. Then, by (12)–(13),

$$\frac{J(\lambda\tilde{u}) - J(\tilde{u})}{\lambda - 1} > -\| r^* \|_2 + \frac{\| r^* \|_2 \| A \|^2}{(\epsilon + 1)\, m^2} (\lambda + 1) \frac{m^2}{\| A \|^2}$$

$$= \| r^* \|_2 \left[-1 + \frac{1 + \lambda}{1 + \epsilon} \right] > 0$$

Since $\lambda - 1 < 0$, we now are forced to conclude that

$$J(\lambda\tilde{u}) - J(\tilde{u}) < 0$$

This contradicts the optimality of $\tilde{u}$.

Corollary 4.1. If (*a*) U is the unit ball in H_2, (b) A is a bounded linear operator from $H_1 \to H_2$ with a bounded inverse, and (c)

$$\| A \|^2 = \| A^{-1} \|^{-2} = K$$

then an optimal control $\tilde{u}$ for the problem (4)–(5) is bang-bang if

$$0 \leqslant \beta < \tfrac{1}{2} \| r^* \|_2 K$$

and is singular if

$$\beta > \tfrac{1}{2} \| r^* \|_2 K$$

Example 4.1. Suppose that $H_1 = H_2 = R^2$, the plane, and suppose that U is the closed unit disc. Consider

$$r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and assume A to be a rotation of R^2 about the origin counterclockwise through an angle θ. Then, $\| A \| = \| A^{-1} \|^{-1} = 1$, and

$$J(u) = (\cos\theta)\, u_1 + (\sin\theta)\, u_2 + \beta(u_1{}^2 + u_2{}^2)$$

For $\beta > 0$, the minimum value of $J(u)$ over all of R^2 is $-1/4\beta$ and is taken for

$$u = \begin{bmatrix} u_1 \\ u \end{bmatrix} = \begin{bmatrix} -\cos\theta/2\beta \\ -\sin\theta/2\beta \end{bmatrix} \tag{14}$$

It is easy to see geometrically that, if $0 < \beta \leqslant \frac{1}{2}$, the minimum of $J(u)$ for u in the closed unit disc U is $-1 + \beta$ and is taken on for

$$u = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\cos\theta \\ -\sin\theta \end{bmatrix}$$

(If $0 < \beta < \frac{1}{2}$, min $J(u)$ over R^2 is attained outside U.) This control is bang-bang.

For $\beta \geqslant \frac{1}{2}$, as $\beta \nearrow +\infty$, the minimum of $J(u)$ on U is its minimum over all of R^2, namely, $-1/4\beta$, which is taken at the point given by (14). Thus, as $\beta \nearrow \infty$, $\tilde{u}$ approaches 0 along the radius with endpoint ($-\cos\theta$, $-\sin\theta$). This control is singular. For this example, Theorems 3.1 and 4.1 are the best possible.

Finally, we give the *only if* part of Theorem 3.1.

Theorem 4.2. If (a) U is a subset of the unit ball in H_2, (b) the set

$$\mathscr{E} = \{u \mid u \in U \quad \text{and} \quad u \text{ is extremal}\}$$

is compact, (c) there exists an $\epsilon > 0$ such that, if $u \in \mathscr{E}$, $\| u \| \geqslant \epsilon$, and (d) A is a (not necessarily bounded) linear operator from H_1 into H_2 with a bounded inverse, then an optimal control $\tilde{u}$ for the problem (4)–(5) is always singular for β sufficiently large.

Proof. Suppose that the theorem is false. Then, there exist $\beta_n \nearrow \infty$ such that, to each β_n, there corresponds an optimal control $\tilde{u}_n$ with $\tilde{u}_n \in \mathscr{E}$. Since $J(0) = 0$ for all β_n,

$$J(\tilde{u}_n) \equiv (r^*, \tilde{u}_n)_2 + \beta_n(L_1\tilde{u}_n, \tilde{u}_n)_2 \leqslant 0 \tag{15}$$

But

$$(L_1\tilde{u}_n, \tilde{u}_n) = \| A^{-1}\tilde{u}_n \|^2 \leqslant \| A^{-1} \|^2 \| u_n \|^2 < \infty$$

Therefore, it follows from (15) that, since $\beta_n \nearrow \infty$, either

$$(r^*, \tilde{u}_n) \to -\infty$$

(which is impossible) or

$$\| A^{-1}\tilde{u}_n \| \to 0 \tag{16}$$

In the latter case, since $\mathscr{E}$ is compact, there exists a convergent subsequence of $\{\tilde{u}_n\}$, which we also call $\{\tilde{u}_n\}$, such that

$$\tilde{u}_n \xrightarrow[n\to\infty]{} \tilde{u} \in \mathscr{E} \quad \text{in} \quad H_2$$

Since A^{-1} is bounded,

$$A^{-1}\tilde{u}_n \xrightarrow[n\to\infty]{} A^{-1}\tilde{u} \quad \text{in} \quad H_1$$

The alternative (16) thus implies that $\tilde{x} = A^{-1}\tilde{u} = 0$ and, in turn, $\tilde{u} = 0$. This contradicts hypothesis (C) of the theorem.

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