Multicriterion Structure/Control Design for Optimal Maneuverability and Fault Tolerance of Flexible Spacecraft

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Abstract. A multicriterion design problem for optimal maneuverability and fault tolerance of flexible spacecraft is considered. The maneuverability index reflects the time required to perform rest-to-rest attitude maneuvers for a given set of angles, with the postmaneuver spillover within a specified bound. The performance degradation is defined to reflect the maximum possible attitude error after maneuver due to the effect of faults. The fault-tolerant design is to minimize the worst performance degradation from all admissible faults by adjusting the design of the spacecraft. It is assumed that admissible faults can be specified by a vector of real parameters. The multicriterion design for optimal maneuverability and fault tolerance is shown to be well defined, leading to a minimax problem. Analysis for this nonsmooth problem leads to closed-form expressions of the generalized gradient of the performance degradation function with respect to the fault parameters and structural design variables. Necessary and sufficient conditions for the optimum are derived, and the closed-form expressions of the generalized gradients are applied for their interpretation. The bundle method is applicable to this minimax problem. Approximate methods which efficiently solve this minimax problem with relatively little computational difficulties are presented. Numerical examples suggest that it is possible to improve the fault tolerance substantially with relatively little loss in maneuverability.

Key Words. Optimization, control, structures, spacecraft, fail-safe design.

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1. Introduction

The problem of the combined design of structures and controls for optimal maneuverability has recently received attention in Ref. 1. In that work, a maneuverability index is introduced to directly reflect the time required to perform a rest-to-rest attitude maneuver for a set of given angles. The spacecraft is modeled as a linear, elastic, undamped, nongyroscopic system. The open-loop, bang-bang, time-optimal control history is obtained as a function of the spacecraft design parameters. By designing the flexible appendages of the spacecraft, its maneuverability index is optimized under the constraints of structural properties, and the postmaneuver spillover within a specified bound. The spillover constraint is achieved by retaining an appropriate number of flexible modes in the control design model. The resulting combined design shows that, for large flexible structures, the maneuverability can be much improved while the spillover is kept within specified requirements.

Reliability is an important feature for any system. Although operation without failure is essential, it cannot be guaranteed that a system will be free from faults and their effects during its operational lifetime. In the case of a spacecraft, such faults generally include degradation of the system, damage from the environment, and change of application condition such as change of payload. Furthermore, the spacecraft parameters cannot be known precisely, due to modeling errors and human errors in manufacturing. All such instances will be considered as faults and will generally degrade the performance of the system. The open-loop nature of the time-optimal bang-bang control makes it difficult to compensate for faults by feedback. For this reason, the use of time-optimal control of bang-bang type for flexible spacecraft maneuvers has been criticized (Ref. 2): "near bang-bang controls are usually very sensitive to model errors; therefore, control shaping is an important issue in obtaining robust controls." In order to account for the effects of faults, fault tolerance should be considered as a part of the design problem. Certainly, one way of doing so is to make faults in the system as unlikely as possible. However, this is usually beyond the practical capability of designers. For instance, damages to a system are usually unavoidable and unpredictable. This paper is concerned with improving the fault tolerance of the open-loop system by modifying the design of the structure while considering maneuverability as the primary objective of design.

Similar attempts to overcome the effects of faults have been made in structural design and control design respectively. Taylor summarizes fail-safe design of structures in Ref. 3. In fail-safe design, a system is required to meet a set of performance requirements beyond those dictated by its primary purpose. The alternative performance requirements account for damage or degradation of the primary structure. Studies of such problems are reported in Refs. 4–6, for example. In the area of control design, a fundamental challenge is to account for and accommodate the inaccuracies in the mathematical models of the physical systems used for design. Such requirements lead to the concept of robust control (Ref. 7). Two types of robustness are generally considered in the literature, namely, stability robustness and performance robustness. Stability robustness is defined as the ability to maintain closed-loop system stability, and performance robustness as that of maintaining a satisfactory level of performance, in the presence of modeling errors, including parameter variations. A direct, more heuristic class of method for dealing with the robustness problem is sensitivity minimization (Refs. 8–10). Newsom and Mukhopadhyay studied the multiloop robust controller design (Ref. 11). Kosut, Salzwedel, and Naeini (Ref. 11) used singular-value robustness measures to compare the performance and stability robustness properties of different control design techniques in the presence of residual modal interaction for a flexible spacecraft system. Keel, Lim, and Juang (Ref. 12) developed an algorithm to obtain a state feedback controller that, given an allowable tolerance for the closed-loop eigenvalue perturbation, maximizes the uncertainty tolerance of the open-loop system matrix. Research on integrated structure/control design dealing with robustness has been scarce. Lim and Junkins (Ref. 13) considered the design problems of optimizing structural mass, stability robustness bound of Patel and Toda, and eigenvalue sensitivity with respect to a set of design parameters that included structural and control parameters and actuator locations. Rao, Pan, and Venkayya (Ref. 14) considered the multicriterion, integrated structural/control design problem in which structural weight, stability robustness index, and performance robustness index are objectives.

The objectives of fault-tolerant design and performance robustness design are similar, namely, minimizing the performance degradation of the faulty system. All the work on performance robustness in the literature is done by adjusting the design parameters of the controller to achieve the desired goal, with the plant unmodified. In our present study of fault-tolerant design, however, we minimize the effect of fault by adjusting the structural design, without modifying the control design. In the literature, faults are usually modeled as parameter variations in the system equations, and it is assumed that the worst performance degradation happens with the largest parameter variation. It will be more meaningful in application to directly minimize the worst performance degradation from among those associated with each admissible fault by adjusting the design.

In the present work, we investigate the multicriterion design problem for optimal maneuverability and fault tolerance of flexible spacecraft.

Consider faults which may happen to the system in the process of modeling, manufacturing, or during its operational lifetime. The effect of these admissible faults is to induce a performance degradation, which is defined to reflect the maximum possible attitude error after maneuver. The fault index formulated to reflect the worst performance degradation from all admissible faults is the secondary objective function, while the maneuverability index is of primary concern. The design problem is a nonsmooth optimization problem, because the performance degradation and the fault index may not be differentiable. The following fundamental assumptions are made to model the faults as covered in this study:

(A1) the structural properties remain constant during the maneuver; as a result the induced system dynamics are time-invariant;
(A2) the properties of a fault, i.e., the specification of structural degradation or defect, can be expressed via a vector of real parameters;
(A3) the elements of the vector in (A2) lie within specified bounds, and this set of admissible faults is compact;
(A4) the compact set in (A3) is independent of the spacecraft design;
(A5) the faulty structure is undamped;
(A6) the control input is not changed in the presence of any fault;
(A7) the natural frequencies of the spacecraft are all distinct both in the nominal and any faulty configuration;
(A8) the mass distribution and stiffness distribution of the spacecraft are jointly continuous functions of the structural design variables and the fault parameters;
(A9) the switching times and maneuver time of the time-optimal control (see Appendix B) are continuously differentiable functions of the structural design variables.

To briefly outline the remainder of this paper, we formulate the attitude error, the performance degradation, the fault index, and the multicriterion design problem in Section 2. The properties of the performance degradation and the fault index are also presented. Since the performance degradation function and the fault index may not be differentiable, we develop the nonsmooth mathematical programming preliminaries in Section 3. The closed-form expressions of the generalized gradients of the performance degradation function and the fault index are derived. A necessary condition for optimality in this nonsmooth setting is also presented. In Section 4, we outline the problem-solving procedure which is based on the development from Section 3. Two approximate methods to solve the minimax design problems are introduced. In Section 5, we examine in detail some numerical examples. We conclude by summarizing the original contributions and limitations of this work in Section 6.

2. Dynamics Preliminaries and Problem Formulation

Consider the generic flexible spacecraft of Fig. 1 (Section 5), which has been modeled as a linear, elastic, undamped, nongyroscopic system; the same model was used in an earlier study (see Ref. 1) for the fault-free design. The spacecraft consists of a cylindrical symmetric rigid central body, to which N, $N \geq 2$, identical flexible appendages are attached with uniform spacing between them. For simplicity, we consider the special scalar control case where the spacecraft is controlled by only one torquer located on the rigid central body. It is to be controlled for attitude maneuver, and the amplitude of the torque is limited. The extension to the multi-input case is not difficult given the general results on time-optimal control discussed in Ref. 26. Let θ be the attitude variable of the rigid central body. The primary objective of the design is to minimize the maneuver time of the spacecraft for a specified maneuver angle θ^*, where attitude spillover is required to lie within a specified bound. As indicated earlier, we elaborate on this problem in the present study by extending the model in order to account for structural faults.

2.1. Attitude Error. Due to a fault, the control will in general not drive the system to the specified final state, an underformed rest state where the attitude angle of the central rigid body is the specified maneuver angle θ^*. Let t_f^* be the optimal maneuver time in the absence of structural fault. Let $\theta_e(t)$ be the attitude error after maneuver, defined as

$$\theta_e(t) := |\theta(t) - \theta^*|, \qquad t \geq t_f^*. \tag{1}$$

We use the finite-element method for structural analysis, whereby appendages of the spacecraft are discretized into a finite number of beam elements. As discussed in Ref. 1, we have two mathematical models of the system: the control-evaluation model, which is assumed to represent the dynamics of the system (the finite-element analysis is used in this model); and the control-design model, which is the reduced-order model for the control design. Herein, the performance of the control will be evaluated based on the control-evaluation model. Let there be n flexible modes in the control-evaluation model (n is equal to the number of degrees of freedom in the finite-element analysis), and n flexible modes in the control design model. In this section, we will consider the formulation of the faulty model, attitude error, performance degradation function, and fault index.

Under the assumption that the spacecraft is controlled by only one torquer located on the rigid central body, let $u_0(t)$ be the torque input. From Ref. 1, we obtain the equations of motion for the spacecraft as the following coupled linear differential equations:

$$J^*\ddot{\theta} + m^T\ddot{q} = u_0(t), \tag{2a}$$

$$M\ddot{q} + Kq + m\ddot{\theta} = 0, \tag{2b}$$

where the elements of the $n \times n$ matrices M, K, and the $n \times 1$ vector m are

$$m_i \equiv N \int_0^L \rho(x)(R+x)\phi_i \, dx,$$

$$M_{ij} \equiv N \int_0^L \rho(x)\phi_i\phi_j \, dx,$$

$$K_{ij} \equiv N \int_0^L \{\phi_i\}[EI(x)\{\phi_j\}_{,xx}]_{,xx} \, dx = N \int_0^L EI(x)\{\phi_i\}_{,xx}\{\phi_j\}_{,xx} \, dx,$$

with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$, and where R is radius of the rigid central body, L is the length of the appendage, $EI(x)$ is the elastic rigidity distribution, and $\rho(x)$ is the mass per unit length. The vector

$$q = (q_1(t), q_2(t), q_3(t), \ldots, q_n(t))^T$$

reflects the nodal degrees of freedom, and

$$\phi = (\phi_1(x), \phi_2(x), \phi_3(x), \ldots, \phi_n(x))^T$$

is the elemental Hermite cubic.

The natural frequencies and eigenmodes satisfy

$$V^T[M - (1/J^*)mm^T]V = I, \tag{3a}$$

$$V^TKV = \Omega^2, \tag{3b}$$

where I is a unit matrix, $\Omega^2 = \mathrm{diag}\{\omega_i^2; i = 1, 2, 3, \ldots, n\}$, ω_i is the ith natural frequency, and $\{v\}_i$ is the eigenvector corresponding to ω_i. Without loss of generality, the first nonzero component of $\{v\}_i$ is defined to be positive.

The modal control influence parameters are defined as

$$\beta_0^0 = 1/J^*, \qquad \beta_0^i = -(1/J^*)/\omega_i\{v\}_i^T m. \tag{4}$$

The state vector is defined as

$$x = (x_1, x_2, x_3^1, x_4^1, \ldots, x_3^n, x_4^n)^T,$$

where

$$x_1 = \theta + (1/J^*)m^T q,$$

$$x_2 = \dot{x}_1, \qquad x_3^i = \eta_i, \qquad x_4^i = \dot{\eta}_i/\omega_i,$$

with $i = 1, 2, 3, \ldots, n$. The state space equations are

$$\dot{x}(t) = Ax(t) + Bu_0(t), \tag{5a}$$

$$A = \text{block diag}[A_i], \qquad B = \text{block col}[B_i], \tag{5b}$$

where

$$A_i = \begin{cases} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & i = 0, \\ \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}, & i = 1, 2, \ldots, n, \end{cases} \tag{5c}$$

$$B_i = \begin{bmatrix} 0 \\ \beta_0^i \end{bmatrix}, \tag{5d}$$

with $i = 0, 1, 2, \ldots, n$, with β_0^i defined in (4). Note that the states x_1 and x_2 do not represent the attitude position and velocity of the rigid central body respectively. Indeed by definition these are given by

$$\theta = x_1 + \sum_{i=1}^{n} \{\beta_0^i \omega_i x_3^i\}, \tag{6a}$$

$$\dot{\theta} = x_2 + \sum_{i=1}^{n} \{\beta^0 \omega_i^2 x_4^i\}. \tag{6b}$$

Actually, x_1 and x_2 represent the rigid body mode.

With Assumption (A6), the control input is not changed in the presence of any fault. The computation of the maneuver time and switching times of the time-optimal control problem is discussed in Appendix B of Ref. 15. Since the control input is antisymmetric about the mid maneuver time, we shift the origin of time to the mid maneuver, $t_f^*/2$. Let the switching times be $\{-t_k, -t_{k-1}, \ldots, -t_1, 0, t_1, t_2, \ldots, t_k\}$, where k is the number of switching times in half of the maneuver interval.

The originally specified initial and final states are

$$x(-t_f^*/2) = (-\theta^*/2, 0, \ldots, 0)^T, \qquad x(t_f^*/2) = (\theta^*/2, 0, \ldots, 0)^T.$$

Integrating (2.5) with $x(-t_f^*/2) = \{-\theta^*/2, 0, 0, \ldots, 0\}^T$ and the control mentioned above, we obtain the state variables at the end of the maneuver:

$$x_1(t_f^*/2) = U_0/J^*[(t_f^*/2)^2 - 2t_k^2 + \cdots + 2(-1)^k t_1^2] - \theta_f, \tag{7}$$

$$x_2(t_f^*/2) = 0, \tag{8}$$

$$x_3^i(t_f^*/2) = -2U_0\beta_0^i\bar{C}_i\cos(\omega_i t_f^*/2)/\omega_i, \qquad i=1,2,\ldots,n, \tag{9}$$

$$x_4^i(t_f^*/2) = 2U_0\beta_0^i\bar{C}_i\sin(\omega_i t_f^*/2)/\omega_i, \qquad i=1,2,\ldots,n, \tag{10}$$

where

$$\bar{C}_i = \cos(\omega_i t_f^*/2) - 2\cos(\omega_i t_k) + \cdots + 2(-1)^k\cos(\omega_i t_1) + (-1)^{k+1}, \qquad i=1,2,\ldots,n. \tag{11}$$

We have the following proposition.

Proposition 2.1. With Assumption (A6), the velocity of the rigid body mode at the end of the maneuver defined in Eq. (8) will be equal to zero.

Proof. From (5), we have $\dot{x}_2 = u_0(t)/J^*$, where $u_0(t)$ is the control input and J^* is the rotational moment of the spacecraft. Since the control input $u_0(t)$ is antisymmetric about the mid maneuver time $t_f^*/2$, x_2 is equal to zero at the end of the maneuver. □

Since the value of $x_1(t)$ remains constant after the maneuver, the attitude error (1) has the form

$$\theta_e(t) = |\theta(t) - \theta^*/2| = \left|[x_1(t_f^*/2) - \theta^*/2] - \sum_{i=1}^{n}\{\beta_0^i\omega_i x_3^i(t)]\}\right|, \tag{12}$$

where $t \geq t_f^*/2$.

2.2. Performance Degradation Function. From Assumption (A4), the faults are specified by the fault parameter vector δ, and the set of faults is characterized by specifying bounds on the elements of δ. Let Δ represent the compact set of all possible δ.

After the maneuver, the flexible modes will undergo free vibrations because of nonrest conditions at the end of the maneuver,

$$x_3^i(t) = x_3^i(t_f^*/2)\cos(\omega_i(t - t_f^*/2)) + x_4^i(t_f^*/2)\sin(\omega_i(t - t_f^*/2)), \tag{13a}$$

$$x_4^i(t) = -\omega_i x_3^i(t_f^*/2)\sin(\omega_i(t - t_f^*/2)) + \omega_i x_4^i(t_f^*/2)\cos(\omega_i(t - t_f^*/2)), \tag{13b}$$

Therefore, we have

$$|x_3^i(t)| \leq \sqrt{[x_3^i(t_f^*/2)]^2 + [x_4^i(t_f^*/2)]^2}, \qquad t \geq t_f^*. \tag{14}$$

With (13) and (14), we can derive an upper bound of the attitude error after the maneuver as

$$\begin{aligned}&\left|[x_1(t_f^*/2)-\theta^*/2]-\sum_{i=1}^{n}\{\beta_0^i\omega_i x_3^i(t)\}\right|\\&\le|x_1(t_f^*/2)-\theta^*/2|+\left|\sum_{i=1}^{n}\{\beta_0^i\omega_i x_3^i(t)\}\right|\\&\le|x_1(t_f^*/2)-\theta^*/2|+\sum_{i=1}^{n}|\beta_0^i\omega_i|\sqrt{[x_3^i(t_f^*/2)]^2+[x_4^i(t_f^*/2)]^2},\qquad t\ge t_f^*/2.\end{aligned} \tag{15}$$

The performance degradation, symbolized by $\bar{\theta}_e$, is defined to reflect the maximum possible attitude error after maneuver according to the bound (15) as

$$\bar{\theta}_e := |x_1(t_f^*/2)-\theta^*/2|+\sum_{i=1}^{n}|\beta_0^i\omega_i|\sqrt{[x_3^i(t_f^*/2)]^2+[x_4^i(t_f^*/2)]^2}. \tag{16}$$

with (9)–(11), we have

$$\bar{\theta}_e = |x_1(t_f^*/2)-\theta^*/2|+\sum_{i=1}^{n}\{2U_0(\beta_0^i)^2|\bar{C}_i|\}. \tag{17}$$

This expression (17) will be used to evaluate the fault index.

Remark 2.1. The performance degradation defined in (17) has value zero for the faultless spacecraft.

Note that the performance degradation function is a function of the design variables and the fault parameters, i.e., $\bar{\theta}_e = \bar{\theta}_e(\xi, \delta)$, where ξ is the vector of structural design parameters. Since there are absolute value expressions involved in (17), this expression may not be differentiable.

Proposition 2.2. Under Assumptions (A6)–(A9), the expression inside the absolute value of (17) is jointly continuously differentiable with respect to the structural design variables and the fault parameters.

Corollary 2.1. The performance degradation function defined as (17) is a jointly continuous function with respect to the structural design variables and the fault parameters.

2.3. Fault Index. For our purpose, the fault index is defined to reflect the worst performance degradation from all admissible faults. Given the properties of the set of admissible faults, it is possible to apply optimization analysis to find the specific faulty mode which induces the worst performance degradation of the system. The worst degradation itself, identified here as the fault index FI, is defined via

$$\mathrm{FI}(\xi) := \max_{\delta \in \Delta} [\bar{\theta}_e(\xi, \delta)]. \tag{18}$$

Note that

$$\mathrm{FI}(\xi) = \max_{\delta}[\bar{\theta}_e(\xi, \delta)] = \min_{\delta}[-\bar{\theta}_e(\xi, \delta)]. \tag{19}$$

From Proposition 2.2, we have that the performance degradation function (17) is a jointly continuous function with respect to the fault parameters and the structural design variables. From Corollary 3.2, which will be derived in the next section, the fault index is a continuous function of the structural design variables ξ. Assume that the feasible design space of ξ is compact. Therefore, there exists a local minimum of the fault index with respect to the structural design variables ξ, which implies the existence of an optimal fault-tolerant design for the spacecraft. However, it is apparent as noted that the fault index may not be differentiable for some value of the structural design variables ξ.

2.4. Problem Statement. The objective in this study is to provide a means for the synthesis of designs that are optimal with respect to maneuver time and robust with respect to the consequences of structural faults. Accordingly, both the maneuver time and the fault index are to be minimized with respect to design. Thus, the multicriterion design problem is stated as follows:

$$\min_{\xi \in \Xi} \{t_f^*, \mathrm{FL}(\xi)\}, \tag{20}$$

subject to structural design constraints and control spillover constraint for the primary objective (i.e., maneuverability), where Ξ is the space of structural design variables.

3. Nonsmooth Programming Preliminaries

From Section 2, we know that the performance degradation function (17) and the fault index (19) are nondifferentiable. In what follows, we use generalized gradients in order to treat nonsmooth mathematical programming problems. We will need the following definition.

Definition 3.1. Lipschitz Condition.

(i) Let $X \subset R^n$. A function f: $X \to R$ is locally Lipschitz on X if, for any $x_0 \in X$, there exists a nonempty neighborhood $N(x_0)$ and a nonnegative constant $K(x_0)$ such that

$$|f(x_1) - f(x_2)| \leq K(x_0)\|x_1 - x_2\|, \qquad \forall x_1, x_2 \in N(x_0).$$

(ii) Let $X \subset R^n$. A function $f: X \to R$ is globally Lipschitz on X if there exist a nonnegative constant K independent of x such that

$$|f(x_1) - f(x_2)| \leq K\|x_1 - x_2\|, \qquad \forall x_1, x_2 \in X.$$

Proposition 3.1. Under Assumptions (A6)–(A9), the performance degradation function (17) is a jointly locally Lipschitz function of the structural design variables and fault parameters.

Theorem 3.1. Rademacher's Theorem. Every locally Lipschitz function is differentiable almost everywhere in the sense of the Lebesgue measure.

Definition 3.2. Generalized Gradient. Let $f: R^n \to R$. We define the generalized directional derivative $f^0(x; v)$ at $x \in R^n$ in the direction $v \in R^n$ as

$$f^0(x; v) := \lim_{y \to x,\, \lambda \downarrow 0} \sup(f(y + \lambda v) - f(y))/\lambda. \tag{21}$$

Then, the generalized gradient of f at x, denoted by $\partial_x f(x)$, is defined as (Ref. 19)

$$\partial_x f(x) := \{\xi \in R^n: f^0(x; v) \geq \xi^T v, \text{ for all } v \text{ in } R^n\}.$$

The computation of the generalized gradient from this definition is a formidable task. Fortunately, if f is a locally Lipschitz function, f is differentiable almost everywhere and we can compute $\partial_x f(x)$ as follows. Suppose that f fails to be differentiable at x. We have the following characterization of the generalized gradient (Ref. 19). Let B be a set in the neighborhood of x, with measure zero, at which f fails to be differentiable. The generalized gradient of f at x, $\partial_x f(x)$, is equal to

$$\text{co}\left\{\lim_{i \to \infty} \nabla f(x_i): x_i \to x, x_i \notin B\right\}, \tag{22}$$

where co stands for convex hull, i.e., the set of convex linear combinations of points in a set.

The following basic properties of generalized gradients are cited for future reference:

(P1) if f is continuously differentiable at x, $\partial_x f(x)$ is the singleton $\{\nabla f(x)\}$;

(P2) for any scalar s, one has $\partial_x(sf)(x) = s\partial_x f(x)$;

(P3) let $f_i(x)$, $i = 1, 2, \ldots, n$, be a family of functions each of which is locally Lipschitz; we have $\partial_x(\Sigma f_i)(x) \subset \Sigma\, \partial_x f_i(x)$, where a sum of sets is defined as the set of sums of elements of the sets.

Definition 3.3. Regularity. A function $f: R^n \to R$ is said to be regular at x provided:

(i) For all v, the usual one-sided directional derivative $f'(x; v)$ exists, where

$$f'(x; v) := \lim_{t \downarrow 0} (f(x + tv) - f(x))/t.$$

(ii) For all v, $f^0(x; v) = f'(x; v)$, where $f^0(x; v)$ is the generalized directional derivative defined in (21).

Remark 3.1. See Ref. 19. For (P3), if at a specific point x all f_i are regular, the inclusion can be replaced by an equality.

Lemma 3.1. Let $g: R^n \to R: x \to g(x)$ be continuously differentiable. Let $f: R^n \to R: x \to f(x) = |g(x)|$. Then, $f(x)$ is regular.

In order to facilitate the computational treatment of the design problem, the closed-form expression for the generalized gradient of (17) needs to be derived. When all expressions inside the absolute value are not zero, (17) is continuously differentiable and can be represented as

$$\bar{\theta}_e := [x_1(t_f^*/2) - \theta^*/2] \operatorname{sign}(x_1(t_f^*/2) - \theta^*/2) + \sum_{i=1}^{n} \{2U_0(\beta_0^i)^2 \bar{C}_i \operatorname{sign}(\bar{C}_i)\}.$$

The generalized gradient is the singleton containing the conventional gradient, which can be obtained by application of the chain rule as follows. Suppose that the gradients of the rotational moment, natural frequencies, and control influence coefficients with respect to the structural designs and fault parameters are available. The gradients of (17) with respect to those coefficients have the following expressions:

$$\partial\bar{\theta}_e/\partial J^* = -(U_0/J^{*2})[(t_f^*/2)^2 - 2t_k^2 + \cdots + 2(-1)^k t_1^2] \times \operatorname{sign}(x_1(t_f^*/2) - \theta^*/2), \tag{23a}$$

$$\partial\bar{\theta}_e/\partial\omega_i = -2U_0\{2\beta_0^i(\partial\beta_0^i/\partial\omega_i)\bar{C}_i + (\beta_0^i)^2(\partial\bar{C}_i/\partial\omega_i)\} \operatorname{sign}(\bar{C}_i), \tag{23b}$$

where

$$\partial\bar{C}_i/\partial\omega_i = -t_f^*/2 \sin(\omega_i t_f^*/2) + 2t_k \sin(\omega_i t_k) + \cdots - 2(-1)^k t_1 \sin(\omega_i t_1), \quad i = 1, 2, \ldots, n, \tag{24}$$

$$\partial\bar{\theta}/\partial\beta_0^i = 4U_0\beta_0^i\bar{C}_i \operatorname{sign}(\bar{C}_i), \qquad i = 1, 2, \ldots, n. \tag{25}$$

The quantities β_0^i, ω_i, and J^* are functions of the structural design variables for the fault parameters.

When any expression inside the absolute value function of (17) is zero, the evaluation of the generalized gradient is not so simple. Fortunately, all functions inside the absolute value expression are continuously differentiable. We first introduce the general chain rule.

General Chain Rule. See Ref. 19. Let h: $R^m \to R^n$ (the components of h are denoted by h_i), and let g: $R^n \to R$. Assume that each h_i is Lipschitz near x and g is Lipschitz near $h(x)$. Let $f := g(h(x))$. One has

$$\partial_x f(x) \subset \mathrm{co}\left\{ \sum_{i=1}^{n} \alpha_i v_i : v_i \in \partial_x h_i(x),\ \alpha \in \partial_h g(h(x)), \right.$$
$$\left. \text{where } \alpha_i \text{ are the components of } \alpha \right\}.$$

Remark 3.2. Ref. 19. If g is convex and h is continuously differentiable, the inclusion property of the general chain rule can be replaced by an equality.

Corollary 3.1. Let g: $R^n \to R$: $x \to g(x)$ be continuously differentiable. Let f: $R^n \to R$: $x \to f(x) = |g(x)|$. Suppose that $g(x) = 0$. Then,

$$\partial_x f(x) = \partial_x |g(x)| = \{\alpha \nabla g(x) : \alpha \in [-1, 1]\}.$$

As in Corollary 2.1, (17) can be considered as a sum of absolute values of continuously differentiable functions; with (P3), Remark 3.1, and Lemma 3.1, the generalized gradient of (17) is the sum of the generalized gradients of each term. Moreover, with Corollary 3.1, we have the generalized gradient of each term. Therefore, we have the generalized gradient of the performance degradation function (17) with respect to the structural design variables and the fault parameters.

With the expression of the generalized gradients, we can compute the fault index by using nonsmooth mathematical programming in the next section. The method is formulated to accommodate the following necessary conditions.

3.1. Necessary Condition for Nonsmooth Mathematical Programming. Let f: $R^n \to R$, g_i: $R^n \to R$, $1 \le i \le n_i$, and h_j: $R^n \to R$, $1 \le j \le n_e$, be locally Lipschitz. Consider an optimization problem as follows:

$$\min_{x \in R^n} f(x),$$

subject to the equality and inequality constraints

$$g_i(x) \le 0, \qquad i = 1, 2, \ldots, n_i,$$

$$h_i(x) = 0, \qquad i = 1, 2, \ldots, n_e.$$

Let the Lagrangian $L(\tilde{\lambda}, x, \lambda, \mu): R \times R^n \times R^{n_i} \times R^{n_e} \to R$ be defined by

$$L(\tilde{\lambda}, x, \lambda, \mu) := \tilde{\lambda} f(x) + \sum_{i=1}^{n_{ie}} \lambda_i g_i(x) + \sum_{i=1}^{n_e} \mu_j h_j(x). \tag{26}$$

Let $\bar{x}$ be a local minimum. Then, from Ref. 19, there exist λ_i, $i = 1, 2, \ldots, n_i$, and $\mu_j, j = 1, 2, \ldots, n_e$, such that:

(i) $\tilde{\lambda} \ge 0, \lambda_i \ge 0, i = 1, 2, \ldots, n_i$;

(ii) $\tilde{\lambda}, \lambda_i \ge 0, i = 1, 2, \ldots, n_i$, and $\mu_j, j = 1, 2, \ldots, n_e$, are not all zero;

(iii) $\lambda_i g_i(\bar{x}) = 0, i = 1, 2, \ldots, n_i$;

(iv) $0 \in \partial_x L(\tilde{\lambda}, x, \lambda, \mu)$.

Remark 3.3. If f, h_i, and g_i, for which $g_i = 0$, are all differentiable at $\bar{x}$, condition (iv) becomes $\nabla_x L(\tilde{\lambda}, x, \lambda, \mu) = 0$, yielding the usual necessary conditions for smooth mathematical programming.

To further check whether our candidate solution is indeed a minimizer, we need to apply sufficient conditions for optimization. If the performance degradation function is differentiable at the candidate solution, this task is not difficult and the sufficient conditions are as follows.

3.2. Sufficient Condition for Smooth Mathematical Programming. See Ref. 28. Suppose that $f: R^n \to R$, $g_i: R^n \to R$, $1 \le i \le n_i$, and $h_j: R^n \to R$, $1 \le j \le n_e \in C^2$. Consider an optimization problem as follows:

$$\min_{x \in R^n} f(x),$$

subject to the equality and inequality constraints

$$g_i(x) \le 0, \qquad i = 1, 2, \ldots, n_i,$$

$$h_i(x) = 0, \qquad i = 1, 2, \ldots, n_e.$$

Let the Lagrangian $L(x, \lambda, \mu): R^n \times R^{n_i} \times R^{n_e} \to R$ be defined by

$$L(x, \lambda, \mu) := f(x) + \sum_{i=1}^{n_{ie}} \lambda_i g_i(x) + \sum_{i=1}^{n_e} \mu_j h_j(x).$$

Suppose that at $\bar{x} \in R^n$ all Karush–Kuhn–Tucker necessary conditions hold, i.e.:

(i) $\{\nabla_x h_j(\bar{x}), 1 \le j \le n_e$, and $\nabla g_i(\bar{x}), \forall i$ such that $g_i(\bar{x}) = 0\}$, are linearly independent;

(ii) there exist $\lambda_i \geq 0$, $1 \leq i \leq n_i$, and μ_j, $1 \leq j \leq n_e$, such that $\lambda_i g_i(\bar{x}) = 0$, $1 \leq i \leq n_i$, and $\nabla_x L(\bar{x}, \lambda, \mu) = 0$;

(iii) $g_i(\bar{x}) \leq 0$, $1 \leq i \leq n_i$.

Let $\mathrm{D}(\bar{x})$ be defined as the set

$$\begin{aligned} \mathrm{D}(\bar{x}) = \{d \in R^n\text{: } & \text{(a) } \nabla h_j(\bar{x})^T d = 0, j = 1, 2, \ldots, n_e; \\ & \text{(b) } \nabla g_i(x)^T d = 0, \ \forall i \text{ such that } g_i(\bar{x}) = 0 \text{ and } \lambda_i > 0; \\ & \text{(c) } \nabla g_i(\bar{x})^T d \leq 0, \ \forall i \text{ such that } g_i(\bar{x}) = 0\}. \end{aligned}$$

If for such $\bar{x}$, we also have that $d^T \nabla_x^2 L(\bar{x}, \lambda, \mu) d > 0$, $\forall d \in \mathrm{D}(\bar{x})$ and $d \neq 0$, then $\bar{x}$ is a local minimizer.

Remark 3.4. If at $\bar{x}$, $\forall i$ such that $g_i(\bar{x}) = 0$ we have $\lambda_i > 0$, then $\mathrm{D}(\bar{x})$ is the tangent space of $[\nabla_x h_1(\bar{x}), \nabla_x h_2(\bar{x}), \ldots, \nabla_x h_{n_e}(\bar{x}), \nabla_x g_i(\bar{x}), \ldots]$, where $g_i(\bar{x}) = 0$. Note that, letting

$$P = [\nabla_x h_1(\bar{x}), \nabla_x h_2(\bar{x}), \ldots, \nabla_x h_{n_e}(\bar{x}), \nabla_x g_i(\bar{x}), \ldots]$$

the tangent space is spanned by the columns of $I - P(P^T P)^{-1} P^T$.

At a candidate solution, say δ^*, such that there is an expression inside the absolute value equal to zero, for example $\bar{C}_k = 0$, we have that, in the neighborhood of δ^*,

$$\mathrm{FI}(\xi) = \min_{\delta}[-\bar{\theta}_e(\xi, \delta)] = \min_{\delta}\left[-|x_1(t_f^*/2) - \theta^*/2| + \sum_{i=1}^{n} \{2U_0(\beta_0^i)^2 |\bar{C}_i|\}\right],$$

which is equivalent to

$$\begin{aligned} \min_{\delta} \min \Bigg\{ & -[x_1(t_f^*/2) - \theta^*/2] \operatorname{sign}(x_1(t_f^*/2) - \theta^*/2) \\ & - \sum_{i=1, i\neq k}^{n} \{2U_0(\beta_0^i)^2 \bar{C}_i \operatorname{sign}(\bar{C}_i)\} - 2U_0(\beta_0^k)^2 \bar{C}_k, \\ & -[x_1(t_f^*/2) - \theta^*/2] \operatorname{sign}(x_1(t_f^*/2) - \theta^*/2) \\ & - \sum_{i=1, i\neq k}^{n} \{2U_0(\beta_0^i)^2 \bar{C}_i \operatorname{sign}(\bar{C}_i)\} + 2U_0(\beta_0^k)^2 \bar{C}_k \Bigg\}. \end{aligned} \tag{27}$$

Therefore the sufficient conditions for δ^* to be the optimum for (27) will be the sufficient conditions for δ^* to be the optimum for the fault index computation (19). Note that both functions inside the bracket are continuously differentiable at δ^* and have the same value for $\delta = \delta^*$. Therefore, we will derive sufficient conditions for (27) at the candidate solution δ^*.

Proposition 3.2. Assume that $f_i: R^n \to R$, $1 \le i \le n_r$, $g_i: R^n \to R$, $1 \le i \le n_i$, and $h_j: R^n \to R$, $1 \le j \le n_e$, are differentiable. Consider the function

$$f(x) := \min\{f_1(x), f_2(x), f_3(x), \ldots, f_{n_r}(x)\}.$$

For the optimization problem,

$$\min_{x \in R^n} f(x), \tag{28}$$

subject to the equality and inequality constraints

$$g_i(x) \le 0, \qquad i = 1, 2, \ldots, n_i, \tag{29a}$$

$$h_i(x) = 0, \qquad i = 1, 2, \ldots, n_e. \tag{29b}$$

Suppose that, at the feasible point $\bar{x}$, we have

$$f_1(\bar{x}) = f_2(\bar{x}) = f_3(\bar{x}) = \cdots = f_{n_r}(\bar{x}).$$

Then, $\bar{x}$ is a local minimizer of the problem (28) subject to (29), if and only if $\bar{x}$ is a local minimizer of each optimization problem with $f_i(x)$, $i = 1, 2, \ldots, n_r$, as objective function subject to the constraints (29). □

In summary, Proposition 3.2 is the sufficient conditions for our nonsmooth problem (19) when the objective function (17) is not differentiable at the candidate solution, i.e., there exists a term with value zero inside the absolute value of (17).

Recall the fault index defined in (19) as

$$\mathrm{FI}(\xi) = \min_{\delta} [\bar{\theta}_e(\xi, \delta)],$$

where $\bar{\theta}_e$ is the performance degradation function. It is necessary to examine the minimization problem (20) with the fault index as the objective, including the computation of its generalized gradient.

First, we show that the fault index is locally Lipschitz. The following notation is adopted: let the performance degradation function be represented by f, the fault index by f, the fault parameters by y, and the structural design variables by x. Thus,

$$\mathrm{f}(x) = \max_{y} f(x, y).$$

From Proposition 3.1, the performance degradation function $f(x, y)$ is a jointly locally Lipschitz function of the fault parameters y and the structural design variables x. We have the following proposition.

Proposition 3.3. Let $X \subset R^{n_x}$, $Y \subset R^{n_y}$, with X and Y nonempty compact. Let the function $f: X \times Y \to R: (x, y) \to f(x, y)$ be jointly locally Lipschitz. Let

$$\mathrm{f}(x) := \max_{y \in Y} f(x, y).$$

Then, f(x) is globally Lipschitz on X.

Proposition 3.4. Under Assumptions (A1)–(A9), the fault index defined as

$$\mathrm{FI}(\xi) = \max_{\delta \in \Delta} [\bar{\theta}_e(\xi, \delta)],$$

where $\bar{\theta}_e$ is the performance degradation function (17), is a globally Lipschitz function on the compact set Δ.

Corollary 3.2. The fault index defined above is a continuous function of ξ.

Proof. This follows from Proposition 3.4 and the fact that, for $D \subset R^n$, if $D \to R$ is locally Lipschitz on D, f is continuous on D. □

As a consequence of Proposition 3.4, the fault index has a gradient almost everywhere, and we can compute the generalized gradient by applying Definition 3.2. Moreover, for designs and admissible faults such that all expressions inside absolute values in the performance degradation function (17) are nonzero, the performance degradation function is analytic. We have Proposition 3.4 and Corollary 3.2 to simplify the task of computing the generalized gradients.

Proposition 3.5. Assume that $f: R^{n+m} \to R$, $g_i: R^m \to R$, $1 \le i \le n_i$. Consider the problem

$$\mathrm{f}(x) := \max_{y} f(x, y), \tag{30a}$$

$$g_i(x) \le 0, \qquad i = 1, 2, \ldots, n_i. \tag{30b}$$

Note that, without loss of generality, we consider only inequality constraints. Define the Lagrangian function $L(x, y, \lambda): R^m \times R^n \times R^{n_i} \to R$ as

$$L(x, y, \lambda) := f(x, y) - \sum_{i=1}^{n_{ie}} \lambda_i g_i(y).$$

Suppose that:

(i) For $x = \hat{x}$ there exists a unique solution y^* to the problem (30);

(ii) $f \in C^1$ with respect to x and $f \in C^2$ with respect to y at $(\hat{x}, y^*)$;

(iii) $g_i(y) \in C^2$ at $y = y^*$, $i = 1, 2, \ldots, n_i$;

(iv) the constraint qualification holds for y^*.

With assumptions (i)–(iv), the Kuhn–Tucker conditions assert that there exists $\lambda^* \in R^{n_i}$ such that $\lambda_i \geq 0$, $\lambda_i g_i(y) = 0$, $i = 1, 2, \ldots, n_i$, and $\nabla_y L(\hat{x}, y^*, \lambda^*) = 0$. Without loss of generality, assume that $g_i(y)$, $i = 1, 2, \ldots, r$, are such that $g_i(y) = 0$ and $\lambda_i > 0$. Let $z := [y^t, \lambda_1, \lambda_2, \ldots, \lambda_r]^t$ and $z^* = [y^{*t}, \lambda_1^*, \lambda_2^*, \ldots, \lambda_r^*]^t$. Let $F: R^m \times R^n \times R^r \to R^{n+r}: (x, z) \to F(x, z)$ be defined as $[\nabla_y L(x, z)^t, \lambda_1 g_1(y), \lambda_2 g_2(y), \ldots, \lambda_r g_r(y)]^t$. Further,

(v) assume that the Jacobian matrix $[\nabla_z F(\hat{x}, z^*)]$ is not singular.

Then, we have

$$\nabla_x \mathrm{f}(x) = \partial f(\hat{x}, y^*)/\partial x + [\partial y/\partial x]_{(\hat{x}, z^*)} \{\partial f(\hat{x}, y^*)/\partial y\},$$

where $[\partial y/\partial x]_{(\hat{x}, z^*)}$ is the first $m \times n$ submatrix of $-[\nabla_x F(\hat{x}, z^*)]^T [\nabla_z F(\hat{x}, z^*)]^{-T}$.

Corollary 3.3. Suppose that:

(i) At $x = \hat{x}$, there exist more than one solution y to the problem (30), say $y^{*1}, y^{*2}, \ldots, y^{*k}$. Assume that $g_j(y) = 0$ and $\lambda_j > 0$, $1 \leq j \leq r^i$, for each y^{*i}. Let $z^{*i} := [y^{*it}, \lambda_1^*, \lambda_2^*, \ldots, \lambda_r^{i*}]^T$, $1 \leq i \leq k$.

(ii) The hypotheses (ii)–(v) of Proposition 3.4 hold for each $(\hat{x}, x^{*i})$. Then, the generalized gradient of f at $x = \hat{x}$ is

$$\mathrm{co}\{\nabla_x \mathrm{f}(\hat{x})|_{y = y^{*1}}, \nabla_x \mathrm{f}(\hat{x})|_{y = y^{*2}} \cdots \nabla_x \mathrm{f}(\hat{x})|_{y = y^{*k}}\},$$

where

$$\nabla_x \mathrm{f}(\hat{x})|_{y = y^{*i}} = \partial f(\hat{x}, y^{*i})/\partial x + [\partial y/\partial x]_{(\hat{x}, z^{*i})} \{\partial f(\hat{x}, y^{*i})/\partial y\},$$

and $[\partial y/\partial x]_{(\hat{x}, z^{*i})}$ is the first $m \times n$ submatrix of $-[\nabla_x F(\hat{x}, z_k^*)]^T [\nabla_z F(\hat{x}, z_k^*)]^{-T}$.

Of course it becomes difficult to solve the worst-case design problem (19) when the performance degradation function (17) is not differentiable. The problem-solving procedure is discussed in the next section. The proofs of Propositons 2.2, 3.1, 3.2, 3.3, 3.5, Lemma 3.1, and Corollaries 3.1, 3.3 or elements thereof are given in the Appendices to Ref. 15.

4. Problem-Solving Procedure

In this section we present algorithms to solve the multicriterion, worst-case design problem (19). We also introduce some approximate methods which efficiently solve the design problem with relatively little computational difficulty.

In Ref. 23, Osyczka surveys several approaches to solve multicriterion optimization problems. The advantages and disadvantages of each approach are also discussed. In the present paper, we use the method of weighted objectives. Recall that our problem is

$$\min_{\xi \in \Xi} \{t_f^*, \mathrm{FI}(\xi)\},$$

where ξ are the structural design variables and Ξ is the feasible space of ξ.

Define a new scalar objective,

$$\Gamma(\xi) := w_1 t_f^*(\xi) + w_2 s\, \mathrm{FI}(\xi), \tag{31}$$

where $w_i \geq 0$, $i = 1, 2$, are the weighting factors with $w_1 + w_2 = 1.0$, and s is the scaling factor such that the two original objectives are of the same order. Thus, our problem is transformed into minimizing $\Gamma(\xi)$ for $\xi \in \Xi$.

Since $t_f^*(\xi)$ is independent of δ, $\Gamma(\xi)$, which is

$$w_1 t_f^*(\xi) + w_2 s \max_{\delta}[\bar{\theta}_e(\xi, \delta)],$$

is equal to

$$\max_{\delta}[w_1 t_f^*(\xi) + w_2 s \bar{\theta}_e(\xi, \delta)], \tag{32}$$

where $\bar{\theta}_e$ is the performance degradation function. Therefore, our multicriterion design problem is a minimax problem, i.e.,

$$\min_{\xi} \Gamma(\xi) = \min_{\xi} \max_{\delta}[w_1 t_f^*(\xi) + w_2 s \bar{\theta}_e(\xi, \delta)]. \tag{33}$$

There are two levels of optimization in this problem: maximizing $[w_1 t_f^*(\xi) + w_2 s \bar{\theta}_e(\xi, \delta)]$ with respect to δ [see (32)], and then minimizing $\Gamma(\xi)$ with respect to ξ [see (33)]. We note again that the objective functions of the two levels are not differentiable everywhere, leading to nonsmooth optimizations. The algorithm to perform nonsmooth optimization in this work is based on the so-called bundle method (Ref. 21). A detailed discussion the bundle method can also be found in Ref. 20, Chapter 3. Together with the bundle method, we use the following proposition.

Proposition 4.1. Let f: $X \subset R^n \to R$. Let v be a vector in $\partial_x f(\hat{x})$, where $\hat{x} \in X$. Suppose that v has minimum norm and $v \neq 0$. Then $-v$ is a descent direction for f at $\hat{x}$.

A proof is provided in Ref. 15.

As a consequence of Proposition 4.1, even though the objective function is not convex, a descent direction can be found everywhere except at a minimum.

In Section 3, we have derived the closed-form expression of the generalized gradient of the performance degradation function $\bar{\theta}_e(\xi, \delta)$ [see (17)], i.e., the objective function for the maximization (32) with respect to ξ and δ. With this closed-form expression, we can easily obtain a descent direction through Proposition 4.1. The computation is more accurate than obtaining the generalized gradient through the bundle technique. It is easy to check the solution with the necessary conditions developed in Section 3.

However, we cannot find a closed-form expression of the generalized gradient for $\Gamma(\xi)$, the objective function of the second level (33). We can only compute its generalized gradient by Corollary 3.3. In application, we have encountered the following computational difficulties with this approach.

(i) For each fixed ξ, the maximization (32) requires substantial effort, because we need to perform a finite-element analysis to compute $\bar{\theta}_e(\xi, \delta)$.

(ii) The evaluation of generalized gradients of $\Gamma(\xi)$ by Corollary 3.2 is difficult. Also, the possibility of multiple local solutions δ for the maximization problem (32) complicates matters. This issue should be treated in future work. Furthermore, when the performance degradation function $\bar{\theta}_e(\xi, \delta)$ is not differentiable, we can only solve for the generalized gradient through the bundle technique with all the gradients in the bundle obtained by the finite-difference technique. In this case, the generalized gradient becomes corrupted by numerical errors.

Since we have closed-form expressions of the generalized gradient of $\bar{\theta}_e(\xi, \delta)$ with respect to ξ and δ, we can overcome these computational difficulties by avoiding the two-level optimization (33). We replace the minimax problem by iterations of minimization and maximization of $w_1 t_f^*(\xi) + w_2 s\bar{\theta}_e(\xi, \delta)$ with respect to ξ and δ, respectively. Each of these optimization problems is relatively easy to solve. In what follows, we will discuss possible ways to solve the minimax problem without solving a two-level optimization problem.

Finding a sandle-point type solution is possibly the simplest way to solve a minimax problem. Specifically, we locate solutions satisfying

$$\min_{\xi} \max_{\delta}[w_1 t_f^*(\xi) + w_2 s\bar{\theta}_e(\xi, \delta)] = \max_{\delta} \min_{\xi}[w_1 t_f^*(\xi) + w_2 s\bar{\theta}_e(\xi, \delta)].$$

Unfortunately, in numerical studies we have found that the optimal design is usually not a saddle point. We develop another method for solving minimax problems whose exact solution may not be a saddle point based on Ref. 22. This method is called the sequential iterating search method.

4.1. Sequential Iterating Search Method. The basic idea of this method is to approximate the whole admissible space of the fault parameter δ by a finite number of different admissible values of δ. Therefore, the optimization in the space of δ is transformed into a finite search for the worst performance degradation associated with admissible values of δ. Let $\delta^k, k = 1, 2, \ldots$, represent admissible values of δ and let $\tilde{\Delta}^k$ represent a set containing k different values of δ. The iteration starts with an arbitrary value of δ as δ^1, i.e., $\tilde{\Delta}^k = \{\delta^1\}$ for $k = 1$. In each iteration, we include in $\tilde{\Delta}^k$ a new value $\delta^{k+1}, k = 1, 2, \ldots$, of concern. The computation of δ^{k+1} is discussed below. In the kth iteration, let the set of different values of the fault parameters δ of concern be $\tilde{\Delta}^k = \{\delta^1, \delta^2, \ldots, \delta^k\}$. Let

$$\mathrm{E}_k^m := \min_{\xi} \max_{\delta \in \tilde{\Delta}^{k-1}} [w_1 t_f^*(\xi) + w_2 s\hat{\theta}_e(\xi, \delta)],$$

and let the solution for ξ be ξ^k. Let

$$\mathrm{E}_k^M := \max_{\delta \in \Delta} [w_1 t_f^*(\xi) + w_2 s\bar{\theta}_e(\xi, \delta)],$$

and let the solution for δ be δ^k. Let the exact solution of

$$\min_{\xi} \max_{\delta \in \Delta} [w_1 t_f^*(\xi) + w_2 s\bar{\theta}_e(\xi, \delta)]$$

be represented by E. From Ref. 22, we have the following facts as the basis of this method.

Fact 4.1. For any k, we have $\mathrm{E}_k^m \leq \mathrm{E} \leq \mathrm{E}_k^M$.

Fact 4.2. If for some k we have $\mathrm{E}_k^m = \mathrm{E}_k^M$, the optimal design is ξ^k and the worst faulty mode is δ^k.

Fact 4.3. E_k^m is a monotonic increasing sequence.

Fact 4.4. The sequences $\mathrm{E}_k^m, \mathrm{E}_k^M$ both converge to E.

Based on these facts, the iteration procedure is convergent. As a termination criterion, the relative accuracy E_k^d, which is defined as $(\mathrm{E}_k^M - \mathrm{E}_k^m)/\mathrm{E}_k^M$, must be less than a specified value. When the termination criterion is satisfied at iteration k, the optimal design corresponds to the design variable ξ^k and the fault index is E_k^M, with the worst faulty mode δ^k. Note that

$$\min_{\xi} \max_{\xi \in \tilde{\Delta}^{k-1}} [w_1 t_f^*(\xi) + w_2 s\bar{\theta}_e(\xi, \delta)]$$
$$= \min_{\xi} \max\{[w_1 t_f^*(\xi) + w_2 s\theta_e(\xi, \delta^1)], \ldots, [w_1 t_f^*(\xi) + w_2 s\bar{\theta}_e(\xi, \delta^{k-1})]\}. \tag{34}$$

By introducing a variable λ, we can transform (34) into an equivalent scalar minimization problem as follows:

$$\min_{\xi} \lambda,$$

subject to

$$[w_1 t_f^*(\xi) + w_2 s\bar{\theta}_e(\xi, \delta^i)] \leq \lambda, \qquad i = 1, 2, \ldots, k-1.$$

The overall algorithm is as follows.

Step 1. Begin with a reasonable baseline design value for the structural design vector ξ^1, set $k = 1$, find E_k^M and δ^1, and set $\tilde{\Delta}^k = \{\delta^1\}$.

Step 2. $k = k + 1$. Solve the optimization problem to find E_k^m and ξ^k.

Step 3. For ξ^k, find E_k^M and δ^{k+1}.

Step 4. If $E_k^d = (E_k^M - E_k^m)/E_k^M$ is less than the required accuracy, stop. Otherwise, set $\tilde{\Delta}^{k+1} = \tilde{\Delta}^k \cup \{\delta^{k+1}\}$, and go to Step 2.

From our numerical studies, we have observed that this procedure generally converges in fewer than eight iterations.

To summarize this section, the minimax problem (33) is usually not easy to solve, especially when the objective function is not smooth. However, with the sequential iterating search method, we can solve the problem efficiently. Furthermore, with the closed-form expressions of the generalized gradient of the objective functions, it is relatively easy to apply necessary and sufficient conditions.

5. Numerical Example

In this section, we illustrate the multicriterion design problem by designing the flexible appendages of the spacecraft described schematically in Fig. 1. This is accomplished by adjusting their cross section. The appendages are I-beams as shown in Fig. 2. Our goal is to obtain the optimal flange depth distribution of the appendages, assuming that the width of the web flange and the thickness of the web and flange are constant. The flange depth is symmetric about a central line passing through the cross section. We use two spline polynomials as the assumed shape functions to describe the half flange depth,

$$h(x) = \begin{cases} \xi_1 + \xi_2(x/L) + \xi_3(x/L)^2 + \xi_4(x/L)^3, & 0 \leq x \leq L/2, \\ h(L/2) + h'(L/2)(x - L/2) + \xi_5(x - L/2)^2/L^2 + \xi_6(x - L/2)^3/L^3, & L/2 < x \leq L, \end{cases} \quad (35)$$

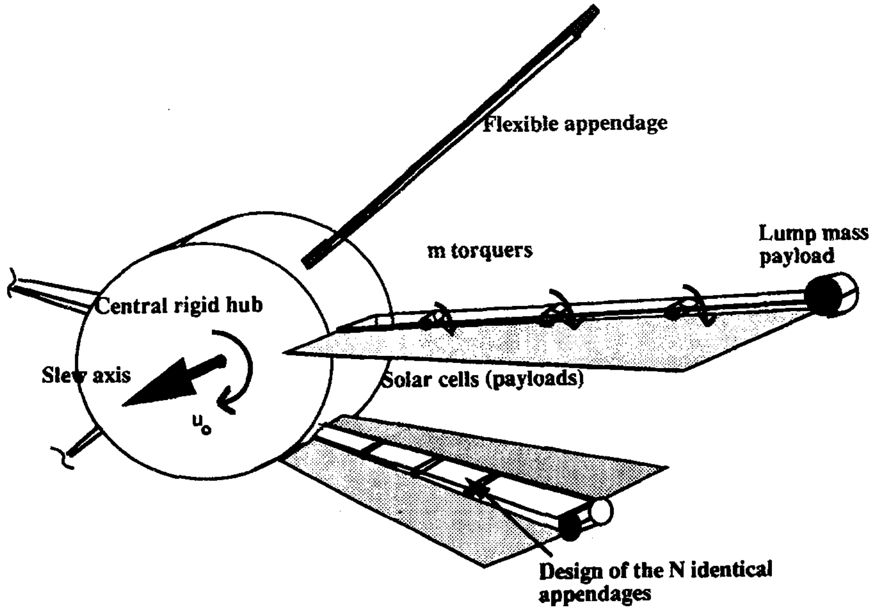

Fig. 1. Generic spacecraft model.

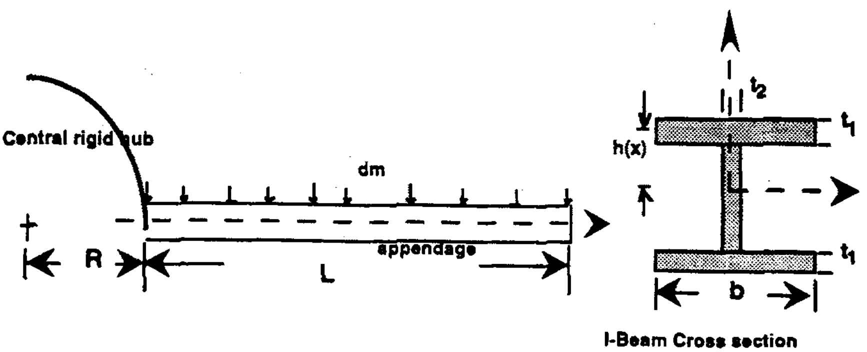

Fig. 2. Cross-section design of spacecraft appendages.

where ξ_i, $i = 1, 2, \ldots, 6$, are the design variables. For practical reasons, $h(x)$ and $dh(x)/dx$ must be continuous at $x = L/2$. Each of the two subdomains for the polynomial is discretized into 15 elements in the finite-element analysis.

Suppose that there is dimension error $\delta(x)$, $0 \leq x \leq L$, in the flange depth of the appendage. For simplicity, we assume that the dimension error

$\delta(x)$ is the same in all appendages. Let $\delta(x)$ be locally bounded between the lower bound $\underline{\delta}(x)$ and the upper bound $\bar{\delta}(x)$. Then, the fault index is

$$\mathrm{FI}(\xi) = \max \bar{\theta}_e(\xi, \delta(x)), \tag{36a}$$

subject to

$$\underline{\delta}(x) \leq \delta(x) \leq \bar{\delta}(x), \qquad 0 \leq x \leq L, \tag{36b}$$

$$\int_0^L |\delta(x)|\, dx \leq \Lambda, \tag{36c}$$

where Λ is a given datum. We represent the distributed function $\delta(x)$ using the same type of assumed functions as in the design of the appendages. Consequently the error distribution is specified by

$$\delta(x) = \begin{cases} \delta_1 + \delta_2(x/L) + \delta_3(x/L)^2 + \delta_4(x/L)^3, & 0 \leq x \leq L/2, \\ \delta(L/2) + \delta'(L/2)(x - L/2) + \delta_5(x - L/2)^2/L^2 + \delta_6(x - L/2)^3/L^3, & \\ & L/2 < x \leq L, \end{cases} \tag{37}$$

where δ_i, $i = 1, 2, \ldots, 6$, are the fault parameters. For a value of structural design variables and the fault parameters, the performance degradation function $\bar{\theta}_e(\xi, \delta)$ is obtained by (17).

Table 1. Spacecraft data and design constraints.

Appendage material density	$\rho = 1880.00$ kg/m^3
Appendage material elasticity	$E = 2.76e11$ N/m^2
Radius of the rigid central body	$R = 12.00$ m
Mass of the rigid central body	4500.00 kg
Length of one appendage	$L = 30.00$ m
Maximum torque available	$V_0^* = 6.0e04$ Nm
Width of the web	$b = 5.00$ cm
Thickness of the web	$t_1 = 1.75$ cm
Thickness of the flange	$t_2 = 0.75$ cm
Distributed payload mass	$dm = 9.00$ kg/m
Concentrated payload mass M at $x = L$	None
Design constraints:	
resource constraint of two appendages	450.0 kg
minimal flange depth	2.00 cm
maximal flange depth	12.00 cm
Constraints on the fault parameters:	
$\underline{\delta}(x) = -0.2$ cm, $\bar{\delta}(x) = 0.2$ cm, $\Lambda = 0.75(L \cdot 0.2$ cm$)$	

Consider the single-maneuver case with specified maneuver angle 90 deg for a spacecraft with two identical flexible appendages. The appendages are made of a single uniform material. The spacecraft data are listed in Table 1. In the design for optimal maneuverability, the postmaneuver spillover from uncontrolled flexible modes should be within a specified bound. This constraint is achieved by retaining an appropriate number of flexible modes in the control design model. The formulation of the control design model is briefly outlined in Ref. 15. Assume that the error from the maximum postmaneuver spillover is limited to 0.1 deg error in attitude of the central body. Accordingly, three flexible modes should be retained in the control design model (see Ref. 15, Appendix B). Other constraints are also listed in Table 1.

To understand qualitatively the behavior of the performance degradation as a function of the design variables, we examine the performance degradation function with a fixed fault for some designs of spacecraft. Consider the designs of the spacecraft with constant flange depth between 2.5 cm and 9.0 cm. Suppose that the dimension error is a constant undersizing of flange depth of 0.4 cm, i.e., $\delta(x) = -0.4$ cm. The performance degradation and the maneuver time for these designs of spacecraft are shown simultaneously in Fig. 3.

The performance degradation is typically much more sensitive to design changes than the maneuver time. As shown in Fig. 3, the worst performance degradation (0.18 rad) is more than 6 times the best one (0.027 rad). However, the difference of maneuver time between the smallest and the largest values is only about 10%. Consequently, it is possible to

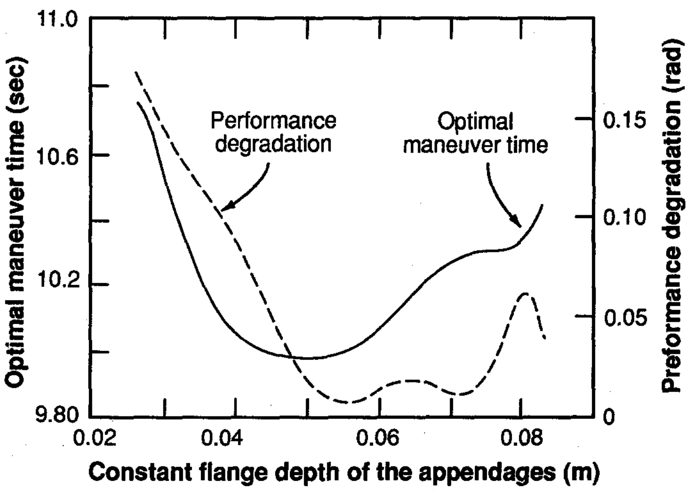

Fig. 3. Performance degradation and optimal maneuver time for a class of spacecraft designs.

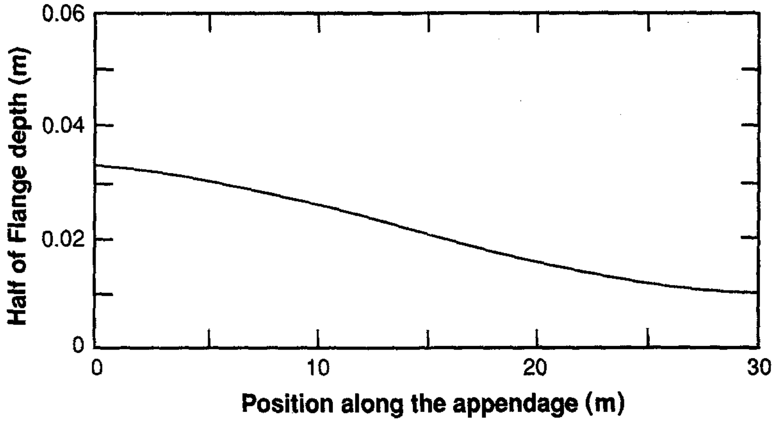

Fig. 4. Design for optimal maneuverability (Case 1).

improve fault tolerance of the system substantially with relative little sacrifice of maneuverability, and a fault-tolerant design need not necessarily be one with low maneuverability.

We examine three cases of design with different weighting for the two objectives. The results are as follows.

Case 1. Design of optimal maneuverability, i.e., $w_1 = 1.0, w_2 = 0$. Here, the only objective is to minimize the maneuver time. The flange depth distribution of the optimal design is shown in Fig. 4. The natural frequencies of the first three flexible modes are 1.35786, 5.00003, and 12.5944 rad/sec. The optimal maneuver time is 9.91915 sec. The fault index of this design is obtained as $2.60572e-02$ rad. The comparison of the maneuver trajectories between the faultless spacecraft and that in the worst faulty mode is shown in Fig. 5.

Case 2. Design for optimal fault tolerance, i.e., $w_1 = 0, w_2 = 1.0$. Here, we only take into account the fault tolerance without considering the primary objective, the maneuverability. This extreme case is not realistic in application. However, the result is useful for comparison.

We use the sequential iterating search method outlined in Section 4 to solve the problem. We begin with the baseline design for optimal maneuverability obtained from Case 1. The fault parameter for the worst performance degradation associated with this design is δ^1, and $\tilde{\Delta}^k = \{\delta^1\}$, $k = 1$. From Case 1, we have that the fault index associated with this design is $2.60572e-02$. Thus, $\mathrm{E}_1^M = 2.60572e-02$ for the first iteration. After four iterations, we obtain a design with $\mathrm{E}_k^m = 7.90757e-03$ and $\mathrm{E}_k^M = 7.908236e-03$, $k = 4$; thus $\mathrm{E}^d = 5.075\mathrm{e}-3\%$. We accept this design

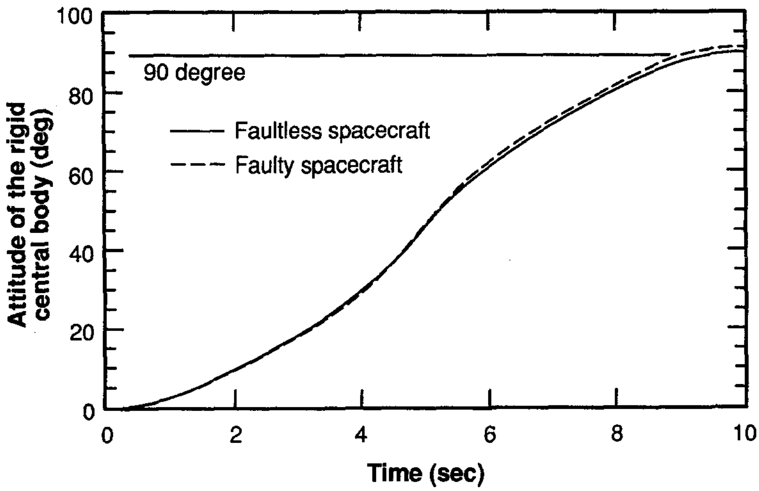

Fig. 5. Comparison of maneuver trajectory between the faultless spacecraft and one with the worst faulty mode (Case 1).

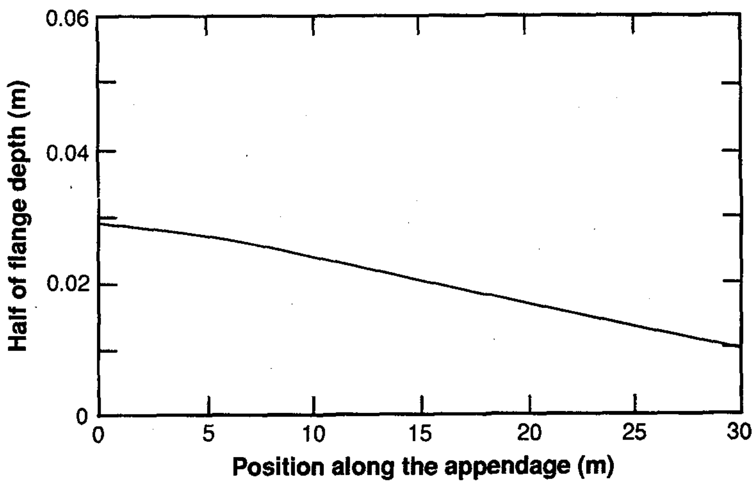

Fig. 6. Designator optimal fault tolerance (Case 2).

as the solution for optimal fault tolerance. Therefore, the fault index is $E_k^M = 7.90757e-03$ rad. The optimal maneuver time of this design is 9.92519 sec. The flange depth distribution of this design is shown in Fig. 6, and the worst dimension error distribution is shown in Fig. 7. The comparison of the maneuver trajectories between the faultless spacecraft and that in the worst faulty mode is shown in Fig. 8.

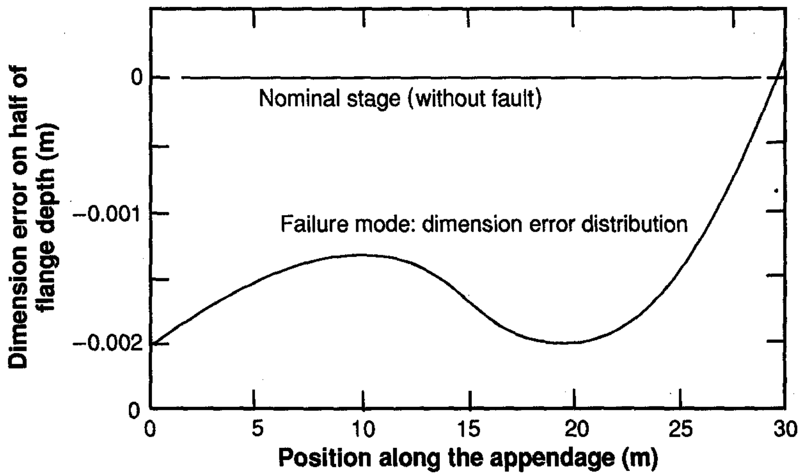

Fig. 7. Distribution of dimension error of the worst failure mode for the optimal design of Case 2.

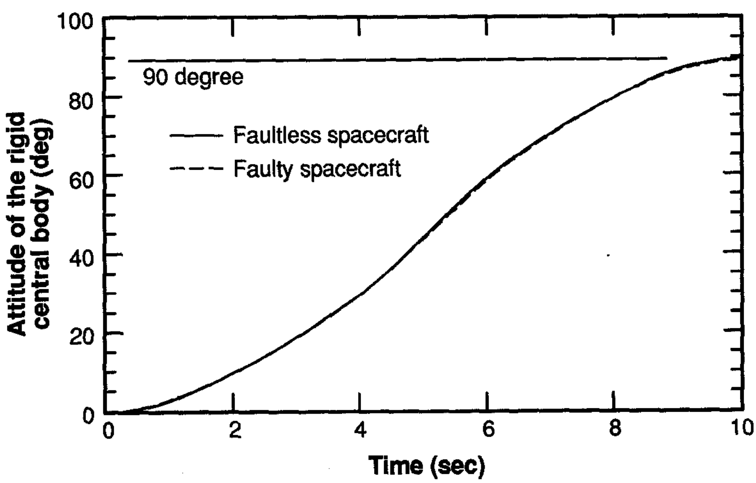

Fig. 8. Comparison of maneuver trajectory between the faultless spacecraft and one with the worst faulty mode (Case 2).

Comparing Case 2 with Case 1, it appears that a reduction of about 60% in fault sensitivity costs only 1% in maneuver time.

Case 3. Multicriterion design for optimal maneuverability and fault tolerance, i.e., $w_1 = 0.5$, $s = 1.0e3$, $w_2 = 0.5$. The weighting factors

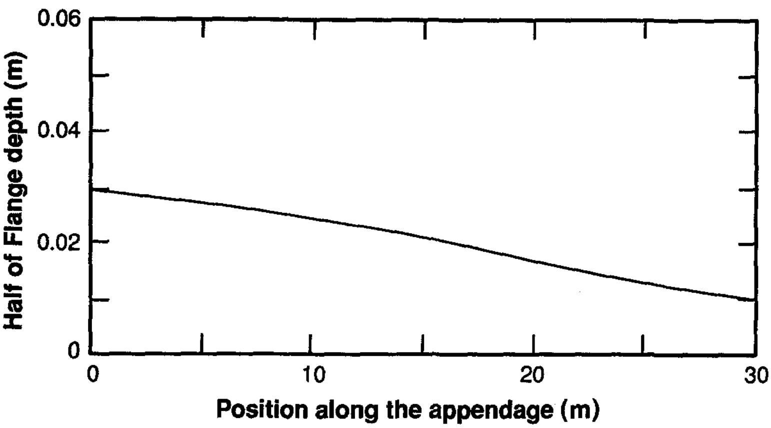

Fig. 9. Multicriterion design results for $w_1 = 0.5$, $w_2 = 0.5$ (Case 3).

w_1 and w_2 have equal value for this example. The design of optimal fault tolerance from Case 2 is the baseline design. The fault parameter for the worst performance degradation associated with this design is δ^1; and $\bar{\Delta}^k = \{\delta^1\}, k = 1$. After six iterations, we obtain a design with $E_k^m = 7.88475e-03$. For this design, we obtain $E_k^M = 7.92497e-03$, thus $E^d = 5.075e-01\%$. We accept this design as the solution for optimal fault tolerance. Therefore, the fault index is $E_k^M = 7.90757e-03$ rad. The optimal maneuver time of this design is 9.924508 sec. The flange depth distribution of this design is shown in Fig. 9, and the worst dimension error distribution is shown in Fig. 10. Note that the design in Case 3 is very similar to that in Case 2, both in optimal maneuver time and fault index. The comparison of the maneuver trajectories between the faultless spacecraft and that in the worst faulty mode is shown in Fig. 11.

The results of these cases are summarized in Table 2.

It is observed from Table 2 that we can actually improve the fault tolerance substantially with relatively little sacrifice of the primary objective, the maneuverability. The fault index of Case 1 is about 3 times its value for Case 3, while the difference of maneuver time between them is only 1%.

6. Conclusions and Future Work

The multicriterion design problem for optimal maneuverability and fault tolerance of flexible spacecraft has been considered. We have devel-

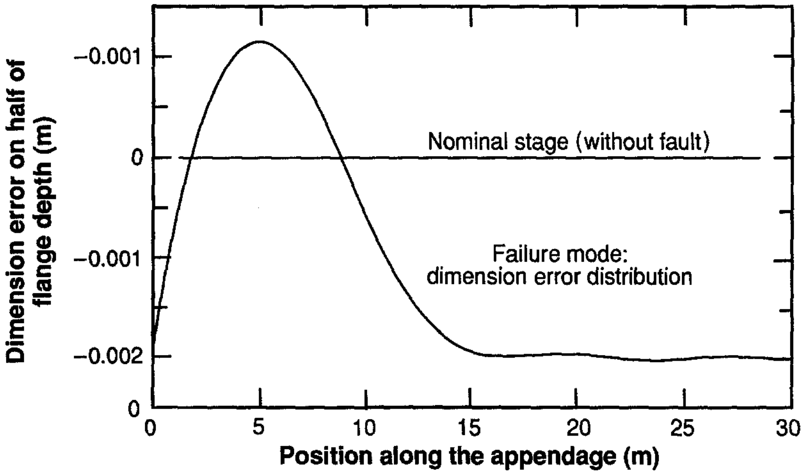

Fig. 10. Distribution of dimension error of the worst failure mode for the optimal design (Case 3).

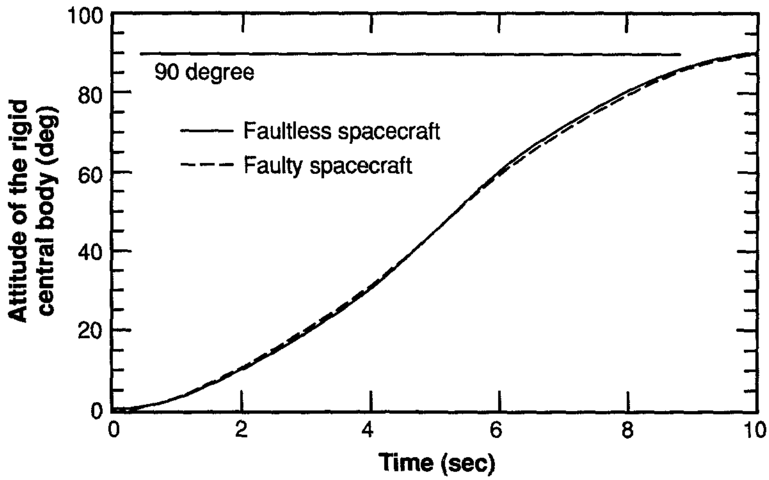

Fig. 11. Comparison of maneuver trajectory between the faultless spacecraft and one with the worst faulty mode (Case 3).

oped a theoretical and practical framework for solving this problem. The main results of the present work are:

(i) The problem formulation for fault tolerance is a minimax problem.

(ii) The multicriterion design problem is shown to possess a solution.
(iii) The performance degradation and the fault index are shown to be locally Lipschitz functions of the fault parameters and the structural design variables respectively.
(iv) A closed-form expression of the generalized gradient of the performance degradation function with respect to the fault parameters or the structural design variables is obtained.
(v) The generalized gradient of a marginal maximization function is obtained by the implicit function theorem.
(vi) Necessary conditions and sufficient conditions to find the worst performance degradation are obtained.
(vii) Approximate methods which solve the minimax problem with relatively little computational difficulties have been introduced.

The problem-solving procedure developed in the present paper can be applied to general designs of flexible spacecraft. Numerical examples suggest that it is possible to improve the fault tolerance substantially with relatively little loss in the primary objective, that is, the maneuverability.

It is well known that feedback control strategy can provide robustness (Ref. 8); traditionally, the robustness of a system has been achieved by designing the feedback control system while leaving its plant unmodified. However, it is clear that, if we can modify the plant simultaneously, the control design can be improved and the cost of implementation can be reduced. This is justified simply because robustness is actually a coupled function of the controller and the plant. For example, we can improve the fault tolerance substantially with relatively little loss in the maneuverability. This study should therefore be viewed as a preliminary work in the direction of combined design of control and plant for robustness.

Table 2. Summary of results.

Case	Performance index	Optimal maneuver time	Fault index
1	Design of optimal maneuverability, $w_1 = 1.0, w_2 = 0.$	9.91915 sec	$2.60572e - 02$ rad (1.493 deg)
2	Design of optimal fault tolerance, $w_1 = 0, w_2 = 1.0.$	9.92519 sec	$7.90757e - 03$ rad (0.453 deg)
3	Multicriterion design of optimal maneuverability and fault tolerance, $w_1 = 0.5, w_2 = 0.5.$	9.924508 sec	$7.92497e - 03$ rad (0.454 deg)

It would be of interest to model the effects of other sources of error, such as unmodeled nonlinear structural factors (e.g., joint stiffness, variations in support stiffness), and possibly discrete forms of damage such as cracks. Also, for a fully integrated treatment, the control design variables should be taken into account in the design problem. In this work, the set of admissible fault parameters is independent of the structural design. It is worth investigating problems without this constraint. The extension of the present work to discrete faults which induce discontinuous responses of the system is also indicated for future study.

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