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Random Walks Associated with Non-Divergence Form Elliptic Equations

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This paper is concerned with the study of the diffusion process associated with a nondivergence form elliptic operator in d dimensions, $d \geqslant 2$. The authors introduce a new technique for studying the diffusion, based on the observation that the probability of escape from a $d-1$ dimensional hyperplane can be explicitly calculated. They use the method to estimate the probability of escape from $d-1$ dimensional manifolds which are $C^{1,\alpha}$, and also $d-1$ dimensional Lipschitz manifolds. To implement their method the authors study various random walks induced by the diffusion process, and compare them to the corresponding walks induced by Brownian motion.

KEY WORDS: Diffusion process; elliptic operator; Lipschitz manifolds; random walks.

1. INTRODUCTION

In this paper we are concerned with random walks associated with an elliptic operator L. The operator L acts on functions with domain $\mathbb{R}^d$ and is defined by

$$L = \sum_{i,\, j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \, \partial x_j}, \qquad x = (x_1,..., x_d) \in \mathbb{R}^d \tag{1.1}$$

The symmetric matrix $A(x) = [a_{ij}(x)]$ is assumed to satisfy inequalities

$$\lambda I \leqslant A(x) \leqslant \Lambda I, \qquad x \in \mathbb{R}^d \tag{1.2}$$

for some constants λ, Λ with $0 < \lambda < \Lambda < \infty$.

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It is well known[12] that if $A(x)$, $x \in \mathbb{R}^d$ is a smooth function, then there is a diffusion process $X_L(t)$ associated with the operator L of (1.1). The goal of this paper is to compare the behavior of $X_L(t)$ to that of Brownian motion. The comparison will be given in terms of the constants λ, Λ only, and therefore does not depend on the degree of oscillation of $A(x)$. In probabilistic language, we shall be studying the diffusion $X_L(t)$ using only the knowledge that the local variance of the process is bounded above and below.

Problems of this nature arise in finance. Consider the situation where one wishes to estimate the Black-Scholes price[8] of a stock option, knowing only that the volatility of the stock lies in an interval $[\sigma_{\min}, \sigma_{\max}]$. It has been shown[1] that the Black-Scholes price is larger than the Black-Scholes formula[8] at constant volatility $\sigma_{\min}$, and less than the Black-Scholes formula at constant volatility $\sigma_{\max}$. For an option depending on two or more stocks the problem is not exactly solvable. One has to estimate[11] the solution of the parabolic equation associated with L, assuming only (1.2). This in turn leads to the study of a Bellman equation, which in this case is a fully nonlinear equation.[4]

Here we introduce a new technique for the study of the diffusion $X_L(t)$, assuming only that the local variance matrix satisfies (1.2). We hope that, with further development, this technique can be applied to the study of the difficult problems mentioned in the previous paragraph. The basis of our technique is the observation that the probability $X_L(t)$ escapes a $d-1$ dimensional hyperplane is the same as the corresponding probability for Brownian motion. Thus, let S be a $d-1$ dimensional hyperplane and $x \in \mathbb{R}^d$ a point a distance r from S. Let $p(x)$ be the probability that $X_L(t)$, started at x, escapes to a distance $R > r$ from S without hitting it. Then one can easily see that $p(x) = r/R$.

We are able to generalise this fact to $d-1$ dimensional manifolds which are $C^{1,\alpha}$, and even to Lipschitz manifolds S. In fact if S is $C^{1,\alpha}$ we show in Section 3 that

$$\inf\{p(x) : d(x, S) = r\} \geqslant cr/R \tag{1.3}$$

for some constant $c > 0$ depending only on λ, Λ, d, α. In Section 4 we prove an averaged version of (1.3) when S is a Lipschitz manifold. Thus if Av_R denotes average value over a length scale R, we prove that

$$Av_R\{p(x) : d(x, S) = r\} \geqslant \frac{cr}{R} \exp[-C\,\ell n(R/r)^{1/2}] \tag{1.4}$$

for some constants c, $C > 0$ depending only on λ, Λ, d and the Lipschitz constant.

We can use the inequality (1.3) to estimate the expected time the diffusion $X_L(t)$ spends in a neighborhood of a $d-1$ dimensional $C^{1,\alpha}$ manifold. Let $f: \mathbb{R}^d \to \mathbb{R}$ be a bounded function, $R > 0$, and τ be the first hitting time on the sphere $|y| = R$ for the diffusion $X_L(t)$ started at some point x, $|x| < R$. Let $u(x)$ be the expectation,

$$u(x) = E_x\left[\int_0^\tau f(X_L(t))\, dt\right], \qquad |x| < R$$

where E_x denotes expectation value for the diffusion started at x. It is well known[12] that $u(x)$ solves the Dirichlet problem,

$$\begin{aligned} -Lu(x) &= f(x), \qquad &|x| < R \\ u(x) &= 0, \qquad &|x| = R \end{aligned} \tag{1.5}$$

Suppose now that f is the characteristic function of a neighborhood of a $d-1$ dimensional $C^{1,\alpha}$ manifold with radius $r < R$. We show in Section 3 that (1.3) implies that there is a constant C depending only on d and the uniform ellipticity constants λ, Λ of (1.2) such that

$$\|u\|_\infty \leqslant CRr \tag{1.6}$$

It is interesting to compare (1.6) with the corresponding estimate given by the Alexandroff, Bakelman, Pucci (ABP) inequality.[4, 10] This yields

$$\|u\|_\infty \leqslant CR^{2-1/d} r^{1/d} \tag{1.7}$$

a significantly worse estimate than (1.6). On the other hand (1.7) continues to hold for Lipschitz manifolds while we are unable to prove that (1.6) holds for Lipschitz manifolds.

In the case of estimating the expected time the diffusion spends in a neighbourhood of a zero dimensional manifold with radius r i.e. a ball of radius r, the estimate we obtain is only slightly better than that given by ABP. In fact ABP gives the inequality (1.6), while in Section 2 we obtain

$$\|u\|_\infty \leqslant CR^{1-\varepsilon} r^{1+\varepsilon} \tag{1.8}$$

for some C, $\varepsilon > 0$ depending only on λ, Λ, d. Once can see from the example of Pucci[10] that ε can be arbitrarily small for appropriate choice of λ, Λ.

To prove (1.3), (1.4) we consider various random walks induced by the diffusion process X_L, and show that they behave roughly like the corresponding walks induced by Brownian motion. The final section, Section 5, is devoted to studying the detailed properties of such a walk. To describe it, let S_λ be the $d-1$ dimensional hyperplane $S_\lambda = \{x = (x_1, ..., x_d): x_d = \lambda\}$, $\lambda \in \mathbb{R}$. Then we can define a walk on S_0 as follows: Suppose at integer time n, the walk is at the point $Y(n) \in S_0$. Then $Y(n+1)$ is the first hitting point on S_0 for the diffusion started at $Y(n)$, after it hits S_1. In the case of Brownian motion $Y(n)$ is a Cauchy process. This continues to hold in an approximate sense for the case of the diffusion generated by L.

There is a considerable literature on elliptic operators in non divergence form. Much of this is concerned with the Harnack inequality.[9, 2] The paper[3] makes strong use of the ABP inequality to prove existence of a positive eigenfunction for the operator L, only assuming the conditions (1.2). The paper[7] is concerned with boundary behavior of solutions to the equation $Lu = 0$. See Ref. 9 for a general review.

In the subsequent work we shall take dimension $d = 2$. All arguments can however be adapted to the case $d > 2$ as stated in this introduction.

2. CIRCLES AND LINES

In this section we shall be interested in estimating the amount of time the diffusion process spends in a disc of radius r and in an r neighborhood of a line before going a distance $R > r$. First we consider the amount of time taken to exit a disc of radius R.

Lemma 1. Let X_L be the diffusion process in $\mathbb{R}^2$ with generator L given by (1.1). For $x \in \mathbb{R}^2$ with $|x| < R$, let τ_R be the time taken for X_L to exit the disc $\{y : |y| < R\}$. If $u(x) = E_x[\tau_R]$, then there are constants c, C depending only on λ, Λ such that

$$c[R^2 - |x|^2] \leqslant u(x) \leqslant C[R^2 - |x|^2] \tag{2.1}$$

Proof. Let $v(x) = R^2 - |x|^2$. Evidently, $v(x) = 0$, $|x| = R$ and

$$4\lambda \leqslant -Lv(x) \leqslant 4\Lambda, \qquad |x| < R$$

Since $u(x)$ satisfies the equation

$$-Lu(x) = 1, \qquad |x| < R, \qquad u(x) = 0, \qquad |x| = R$$

it follows by the maximum principle that (2.1) holds with $C = 1/4\lambda$, $c = 1/4\Lambda$. QED

Lemma 2.1 tells us that the amount of time taken for the diffusion process to exit a disc of radius R is comparable to Brownian time. Next let D_r be the disc of radius r, $\{y : |y| < r\}$ and for x with $|x| < R$, let $u_r(x)$ be the time spent in D_r before exiting $\{y : |y| < R\}$. Thus

$$-Lu_r(x) = \chi_{D_r}(x), \qquad |x| < R, \qquad u_r(x) = 0, \qquad |x| = R \tag{2.2}$$

where χ_{D_r} is the characteristic function of D_r. We can obtain an upper bound on u_r by comparison with the solution of a one dimensional problem.

Lemma 2. There exists a constant C depending only on λ such that $u_r(x) \leqslant CrR$, $|x| < R$.

Proof. Let $v_r(x_1)$ be the solution of the one dimensional Dirichlet problem,

$$-v_r''(x_1) = \chi_{(-r,r)}(x_1), \qquad |x_1| < R, \qquad v_r(R) = v_r(-R) = 0 \tag{2.3}$$

where $\chi_{(-r,r)}$ is the characteristic function of the interval $(-r, r)$. Hence $v_r(x_1)$ is the time one dimensional Brownian motion spends in the interval $(-r, r)$ before exiting the interval $(-R, R)$. The function v_r is given explicitly by

$$v_r(x_1) = \int_{-r}^{r} G(x_1, z)\, dz$$

where the Green's function $G(x_1, z)$ is defined by

$$G(x_1, z) = \begin{cases} \dfrac{1}{2R}(R - x_1)(R + z), & -R < z < x_1 < R, \\[2ex] \dfrac{1}{2R}(R + x_1)(R - z) & -R < x_1 < z < R \end{cases}$$

Now let $w_r(x_1, x_2) = \lambda^{-1} v_r(x_1) - u_r(x_1, x_2)$. Evidently one has $w_r(x) \geqslant 0$, $|x| = R$. In view of (2.3) it follows that $-Lw_r(x) \geqslant 0$, $|x| < R$. The result follows from the maximum principle. QED

The upper bound in Lemma 2.2 can be much larger than the corresponding Brownian time which is r^2. We wish to obtain an improvement on the bound in Lemma 2.2. To do this we shall first given an alternative proof of Lemma 2.2. This is based on estimating the time spent in D_r by r^2 times the number of recurrences to the circle $\{|y| = r\}$ before hitting the circle $\{|y| = R\}$.

We can estimate recurrences by estimating exit probabilities from an annulus as in Ref. 5. Thus consider the annulus $A_R = \{x : R/2 < |x| < 2R\}$. Let $u(x)$ be the solution of the Dirichlet problem,

$$Lu(x) = 0, \quad x \in A_R, \quad u(x) = 1, \quad |x| = 2R, \quad u(x) = 0, \quad |x| = R/2 \tag{2.4}$$

Thus $u(x)$ is the probability that the diffusion process started at $x \in A_R$ exits the annulus through the outer boundary. Now for Brownian motion, L is just the Laplacian and one can explicitly compute the solution of (2.4). In particular $u(x) = 1/2$ if $|x| = R$. We can get a lower bound on $u(x)$ by comparison with a one dimensional problem again.

Lemma 3. Let $u(x)$ be the solution of (2.4). Then $\inf_{|x|=R} u(x) \geqslant 1/3$.

Proof. Let $v(x_1)$ be the solution of the one dimensional Dirichlet problem,

$$v''(x_1) = 0, \qquad R/2 < x_1 < 2R, \qquad v(2R) = 1, \qquad v(R/2) = 0$$

The function $v(x_1)$ is explicitly given by the formula $v(x_1) = 2(x_1 - R/2)/3R$. Next let $w(x_1, x_2)$ be defined by $w(x_1, x_2) = u(x_1, x_2) - v(x_1)$. It is easy to see that $w(x) \geqslant 0$, $x \in \partial A_R$, and also that $Lw(x) = 0$, $x \in A_R$. Hence by the maximum principle $w(x) \geqslant 0$, $x \in A_R$. In particular it follows that $u(R, 0) \geqslant 1/3$. The result is a consequence of this last inequality since one can rotate any point x with $|x| = R$ to the point $(R, 0)$. QED

Remark. Observe the inequality in Lemma 2.3 depends only on the ellipticity of L and not on the actual bounds λ, Λ.

We can use Lemmas 2.1 and 2.3 to give a new proof of Lemma 2.2 as follows:

Proof of Lemma 2.2. For $k = 0, 1, 2,...$ we define radii $r_k = 0, 1, 2,...$ by $r_k = 2^k r$ and let M be the smallest integer such that $r_M \geqslant R$. Let S_k, $k = 0, 1, 2,...$ be circles centered at the origin with radii r_k. The diffusion process then induces a random walk on the circles S_k. For $x \in S_k$, $0 \leqslant k < M$, let N_x be the number of recurrences of the random walk to S_0 before it hits S_M. Then, following the proof of Lemma 3.17 of Ref. 6, we have from Lemma 2.3 that

$$E[N_x] \leqslant 1 + \sum_{j=1}^{M-1} 2^j = 2^M - 1$$

For $x \in S_0$ let τ_x be the time taken for the diffusion process to reach the circle S_1. In view of Lemma 2.1 we have the estimate

$$E[\tau_x] \leqslant 4Cr^2 \tag{2.5}$$

If we follow the proof of Lemma 3.11 of Ref. 6 and use these last two estimates we conclude that the function $u_r(x)$ satisfies the inequality

$$u_r(x) \leqslant 4Cr^2[2^M - 1]$$

The lemma follows from the inequality $2R > 2^M r \geqslant R$. QED

The advantage of our second proof of Lemma 2.2 is that we can immediately deduce an improvement on Lemma 2.2 if we can obtain an improvement on Lemma 2.3.

Lemma 4. Let $u(x)$ be the solution of (2.4). Then there is a constant γ, $1/3 < \gamma < 1$, depending only on λ/Λ such that $\inf_{|x|=R} u(x) \geqslant \gamma$.

Proof. Let $v_\varepsilon(x_1, x_2)$, be the function,

$$v_\varepsilon(x_1, x_2) = 2(x_1 - R/2)/3R + \frac{\varepsilon}{R^2}[(2R - x_1)(x_1 - R/2) + \Lambda x_2^2/\lambda]$$

It is clear that for $\varepsilon > 0$, $Lv_\varepsilon(x) \geqslant 0$, $x \in A_R$. Hence if we can show that $v_\varepsilon \leqslant 0$ on the circle $\{|x| = R/2\}$ and $v_\varepsilon \leqslant 1$ on $\{|x| = 2R\}$, the maximum principle implies that $u(x) \geqslant v_\varepsilon(x)$, $x \in A_R$. Taking $x = (R, 0)$ in this inequality, we conclude that $u(R, 0) \geqslant 1/3 + \varepsilon/2 = \gamma > 1/3$.

To show that $v_\varepsilon \leqslant 0$ on the circle $\{|x| = R/2\}$ we need to prove that

$$2(x_1 - R/2)/3R + \frac{\varepsilon}{R^2}\left[(2R - x_1)(x_1 - R/2) + \frac{\Lambda}{\lambda}\left(\frac{R^2}{4} - x_1^2\right)\right]$$

$$\leqslant 0, \qquad \frac{-R}{2} \leqslant x_1 \leqslant \frac{R}{2}$$

This is the same as showing that

$$\frac{2}{3} + \varepsilon\left(2 - \frac{x_1}{R}\right) \geqslant \frac{\varepsilon\Lambda}{\lambda}\left(\frac{1}{2} + \frac{x_1}{R}\right), \qquad \frac{-R}{2} \leqslant x_1 \leqslant \frac{R}{2}$$

Evidently, this last inequality holds if we choose ε to satisfy the inequality $0 < \varepsilon < 2\lambda/3\Lambda$. To show that $v_\varepsilon \leqslant 1$ on the circle $\{|x| = 2R\}$ we need to prove that

$$2(x_1 - 2R)/3R + \frac{\varepsilon}{R^2}\left[(2R - x_1)(x_1 - R/2) + \frac{\Lambda}{\lambda}(4R^2 - x_1^2)\right]$$
$$\leqslant 0, \qquad -2R \leqslant x_1 \leqslant 2R$$

This is the same as showing that

$$\frac{2}{3} \geqslant \varepsilon\left(\frac{x_1}{R} - \frac{1}{2}\right) + \frac{\varepsilon\Lambda}{\lambda}\left(2 + \frac{x_1}{R}\right), \qquad -2R \leqslant x_1 \leqslant 2R$$

This last inequality holds if ε satisfies $0 < \varepsilon < 4\lambda/33\Lambda$. The result follows on taking $\gamma = 1/3 + 2\lambda/33\Lambda$. QED

Corollary 1. Let $u_r(x)$ be the solution of (2.2). Then there is a constant C depending only on λ and a constant $\delta > 0$ depending only on λ/Λ such that

$$u_r(x) \leqslant C r^{1+\delta} R^{1-\delta}, \qquad |x| < R$$

Proof. We use the second proof of Lemma 2.2. In view of Lemma 2.4 we have that

$$E[N_x] \leqslant 1 + \sum_{j=1}^{M-1} \left(\frac{1-\gamma}{\gamma}\right)^j \leqslant \frac{1}{1-2\gamma}\left(\frac{1-\gamma}{\gamma}\right)^M$$

where we assume $\gamma < 1/2$. Taking $\gamma = \min[\frac{2}{5}, \frac{1}{3} + 2\lambda/33\Lambda]$, it is easy to see that there is a universal constant $k > 0$ such that

$$E[N_x] \leqslant 5(R/r)^{1 - k\lambda/\Lambda}$$

We conclude from this last inequality and (2.5) that

$$u_r(x) \leqslant 20 C r^{1+\delta} R^{1-\delta}$$

where $\delta = k\lambda/\Lambda$. QED

Next we consider the problem of finding a lower bound on the function $u_r(x)$ defined by (2.2). We shall proceed in a similar manner to the second proof of Lemma 2.2. Hence we need an upper bound on the function $u(x)$ defined by (2.4). The crucial observation is the following:

Lemma 5. Let $w(x) = 1/|x|^{d-2}$, $d > 2$. Then for d sufficiently large, depending on Λ/λ, one has $Lw(x) \geqslant 0$, $x \neq 0$.

Proof. We have that for $x = (x_1, x_2)$,

$$\frac{\partial^2 w}{\partial x_1^2} = \frac{-(d-2)}{|x|^d} + \frac{d(d-2)\, x_1^2}{|x|^{d+2}}$$

$$\frac{\partial^2 w}{\partial x_1\, \partial x_2} = \frac{d(d-2)\, x_1 x_2}{|x|^{d+2}}$$

$$\frac{\partial^2 w}{\partial x_2^2} = \frac{-(d-2)}{|x|^d} + \frac{d(d-2)\, x_2^2}{|x|^{d+2}}$$

Hence we have

$$Lw(x) = \frac{d(d-2)}{|x|^{d+2}} \langle x, A(x)\, x \rangle - \frac{(d-2)}{|x|^d} [a_{11}(x) + a_{22}(x)]$$

$$\geqslant \frac{(d-2)}{|x|^d} [d\lambda - 2\Lambda] \geqslant 0, \qquad \text{provided} \quad d \geqslant \frac{2\Lambda}{\lambda} \qquad \text{QED}$$

Corollary 2. Let $u(x)$ be the solution of (2.4). Then there exists a constant γ, $0 < \gamma < 1$, depending only on λ, Λ such that $\inf_{|x| = R} [1 - u(x)] \geqslant \gamma$.

Proof. Let $v(x)$ be the function,

$$v(x) = \left[\left(\frac{2R}{|x|} \right)^{d-2} - 1 \right] \Big/ [4^{d-2} - 1]$$

It is clear that $v(x) = 0$, $|x| = 2R$, $v(x) = 1$, $|x| = R/2$. From Lemma 2.5 we have that $Lv(x) \geqslant 0$, $x \neq 0$, if d is sufficiently large. Hence by the maximum principle $1 - u(x) \geqslant v(x)$, $R/2 < |x| < 2R$, if d is large. The result follows on taking $|x| = R$. QED

Corollary 3. Let $u_r(x)$ be the solution of (2.2). Then there is a constant $c > 0$ depending only on Λ and a constant $\alpha > 0$ depending only on Λ/λ such that

$$u_r(x) \geqslant c r^2 [r/(r + |x|)]^\alpha, \qquad |x| \leqslant R/2$$

Proof. For $|x| \leqslant r/2$ the result follows from Lemma 2.1. Suppose $|x| > r/2$ and let p_x be the probability that the diffusion process started at x hits the circle $|y| = r/2$ before it hits the circle $|y| = R$. In view of Corollary 2.2 and Lemma 6.3 of Ref. 6 it follows that there is a constant $\alpha > 0$ depending on Λ/λ such that

$$p_x \geqslant [r/(r + |x|)]^\alpha$$

The result follows now from this last inequality and Lemma 2.1. QED

We can see that the estimates in Corollaries 2.1 and 2.3 are in some sense sharp by considering the example of Pucci.[10] Thus let M be the differential operator,

$$M = \frac{x_1^2}{|x|^2} \frac{\partial^2}{\partial x_1^2} + \frac{2x_1 x_2}{|x|^2} \frac{\partial^2}{\partial x_1 \, \partial x_2} + \frac{x_2^2}{|x|^2} \frac{\partial^2}{\partial x_2^2}$$

If w is just a function of $|x|$, $w(x) = v(|x|)$, then $Mw(x) = v''(|x|)$. For $-\infty < \varepsilon < \infty$, let L_ε be the operator $L_\varepsilon = \Delta + \varepsilon M$. It is clear that L_ε is uniformly elliptic if $\varepsilon > -1$. Further, with $w(x) = v(|x|)$, we have

$$L_\varepsilon w(x) = (1 + \varepsilon) \, v''(|x|) + |x|^{-1} \, v'(|x|) \tag{2.6}$$

In view of this last identity we can explicitly compute the function $u(x)$ of (2.4) when $L = L_\varepsilon$, $\varepsilon > -1$. It is given by the formula

$$u(x) = \left[1 - \left(\frac{2\,|x|}{R} \right)^{\varepsilon/(1+\varepsilon)} \right] \Big/ [1 - 4^{\varepsilon/(1+\varepsilon)}], \qquad R/2 < |x| < 2R$$

Evidently then,

$$u(x) = [1 - 2^{\varepsilon/(1+\varepsilon)}]/[1 - 4^{\varepsilon/(1+\varepsilon)}], \qquad |x| = R$$

It is clear now that by taking ε sufficiently large that we can make $\inf_{|x| = R} u(x)$ come arbitrarily close to the value 1/3. This is consistent with Lemma 2.4. We also have

$$\begin{aligned} 1 - u(x) &= [2^{\varepsilon/(1+\varepsilon)} - 4^{\varepsilon/(1+\varepsilon)}]/[1 - 4^{\varepsilon/(1+\varepsilon)}] \\ &= 2^{\varepsilon/(1+\varepsilon)}/[1 + 2^{\varepsilon/(1+\varepsilon)}] \end{aligned}$$

We see from this last formula that by choosing ε sufficiently close to -1 we can make $\inf_{|x| = R}[1 - u(x)]$ come arbitrarily close to 0. This is consistent with Corollary 2.2.

We can also explicitly compute the function $u_r(x)$ defined by (2.2) when $L = L_\varepsilon$. In fact we have

$$u_r(x) = \frac{(1+\varepsilon)\, r^{(2+\varepsilon)/(1+\varepsilon)}}{\varepsilon(2+\varepsilon)} \left[R^{\varepsilon/(1+\varepsilon)} - |x|^{\varepsilon/(1+\varepsilon)} \right], \qquad r \leqslant |x| \leqslant R$$

If we let ε become large in this last formula, we see that for $|x| \leqslant R/2$, $u_r(x)$ is of order $r^{1+\delta} R^{1-\delta}$, where $\delta > 0$ can be made arbitrarily small. This is consistent with Corollary 2.1. If we let ε get close to -1, we see that, for $|x| \leqslant R/2$, $u_r(x)$ is of order $r^2[r/(r+|x|)]^\alpha$, where $\alpha > 0$ can be made arbitrarily large. This is consistent with Corollary 2.3.

The estimates of Corollaries 2.1 and 2.3 show that the time the diffusion process spends in an r neighborhood of a point before exiting a disc of radius R can be very different from the Brownian time. The next proposition shows that the time the diffusion process spends in an r neighborhood of a line before exiting a disc of radius R is comparable to Brownian time.

For $0 < r < R$ let w_r be the solution of the Dirichlet problem,

$$-Lw_r(x) = \chi_{(-r,r)}(x_1), \qquad |x| < R, \quad w_r(x) = 0, \quad |x| = R$$

Thus $w_r(x)$ is the expected time the diffusion process started at x spends in the strip $\{x : -r < x_1 < r\}$ before exiting the disc $|x| \leqslant R$.

Proposition 1. There exist positive constants C, c depending only on λ, Λ such that

$$crR \leqslant w_r(x) \leqslant CrR, \qquad |x| \leqslant R/2. \tag{2.7}$$

Proof. The upper bound in (2.7) follows from the argument of Lemma 2.2. To get the lower bound we show first that if v_r is the function defined by (2.3) then one has

$$v_r(x_1) - rx_2^2/R \leqslant 0, \qquad |x| = R \tag{2.8}$$

To prove (2.8) we consider first the case when $r < x_1 < R$. Then one has

$$\begin{aligned} v_r(x_1) - rx_2^2/R &= (R - x_1)\, r - r(R^2 - x_1^2)/R \\ &= -(R - x_1)\, rx_1/R < 0 \end{aligned}$$

For $0 \leqslant x_1 \leqslant r$, we have

$$v_r(x_1) - rx_2^2/R = \frac{(R-x_1)}{2R} \int_{-r}^{x_1} (R+z)\,dz + \frac{(R+x_1)}{2R} \int_{x_1}^{r} (R-z)\,dz$$
$$-r(R^2 - x_1^2)/R$$
$$= rx_1^2/R - \frac{1}{2}r^2 - \frac{1}{2}x_1^2 \leqslant x_1^2 - \frac{1}{2}r^2 - \frac{1}{2}x_1^2$$
$$= \frac{1}{2}x_1^2 - \frac{1}{2}r^2 \leqslant 0, \qquad \text{since} \quad 0 \leqslant x_1 \leqslant r$$

Now (2.8) follows for all x, $|x| = R$, by symmetry.

Consider next the function $u_\gamma(x)$ given by

$$u_\gamma(x_1, x_2) = v_r(x_1) - rx_2^2/R - \gamma r[R^2 - x_1^2 - x_2^2]/R, \qquad |x| < R \tag{2.9}$$

where $\gamma > 0$ is a parameter. Evidently from (2.8) we have that $u_\gamma(x) \leqslant 0$, $|x| = R$. We also have

$$-Lu_\gamma(x_1, x_2) = a_{11}(x)\,\chi_{(-r,r)}(x_1) + 2ra_{22}(x)/R$$
$$-2\gamma ra_{11}(x)/R - 2\gamma ra_{22}(x)/R$$

Hence if we choose γ to satisfy $\gamma > (1 + \lambda/\Lambda)^{-1}$, then

$$-Lu_\gamma(x_1, x_2) \leqslant a_{11}(x)\,\chi_{(-r,r)}(x_1)$$

It follows therefore from the maximum principle that

$$w_r(x) \geqslant \Lambda^{-1}u_\gamma(x), \qquad |x| \leqslant R \tag{2.10}$$

Observe next that for $r < x_1 < R$, we have $u_\gamma(x_1, 0) = (R - x_1)\,r - \gamma r[R^2 - x_1^2]/R = (R - x_1)\,r[1 - \gamma - \gamma x_1/R]$. Hence if we choose γ to satisfy $1/2 < \gamma < 1$, then

$$u_\gamma(x_1, 0) \geqslant \tfrac{1}{4}(1-\gamma)\,Rr, \qquad r \leqslant x_1 < (1-\gamma)\,R/2\gamma$$

For $0 \leqslant x_1 \leqslant r$, we have

$$u_\gamma(x_1, 0) = (1-\gamma)\,rR - r^2/2 - (1 - 2\gamma r/R)\,x_1^2/2$$
$$\geqslant (1-\gamma)\,rR - r^2 \geqslant \tfrac{1}{4}(1-\gamma)\,Rr, \quad \text{provided } r \leqslant 3(1-\gamma)\,R/4$$

We conclude from these last two inequalities that

$$u_\gamma(x_1, 0) \geqslant \tfrac{1}{4}(1-\gamma)\, Rr, \qquad |x_1| < (1-\gamma)\, R/2\gamma$$

provided $r \leqslant 3(1-\gamma)\, R/4$, $\tfrac{1}{2} < \gamma < 1$. It easily follows from this last inequality and (2.9) that

$$u_\gamma(x_1, x_2) \geqslant \tfrac{1}{8}(1-\gamma)\, Rr, \qquad |x_1| < (1-\gamma)\, R/2\gamma, \quad |x_2| \leqslant R/2\sqrt{2}$$

Hence from (2.10) we conclude that

$$w_r(x) \geqslant \frac{1}{8\Lambda}(1-\gamma)\, Rr, \qquad |x| \leqslant (1-\gamma)\, R/4\gamma$$

subject to the restriction $r \leqslant 3(1-\gamma)\, R/4$, $\tfrac{1}{2} < \gamma < 1$. Now for x satisfying $|x| \leqslant R/2$, let p_x be the probability that the diffusion process started at x hits the circle $|y| = (1-\gamma)\, R/4\gamma$ before hitting the circle $|y| = R$. From Corollary 2.2 and the proof of Corollary 2.3 we have that $p_x \geqslant \delta > 0$, where δ depends only on λ/Λ. Hence we have

$$w_r(x) \geqslant \frac{\delta}{8\Lambda}(1-\gamma)\, Rr, \qquad |x| \leqslant R/2 \qquad \text{QED}$$

3. $C^{1,\alpha}$ CURVES

In this section we shall prove the analogue of Proposition 2.1 for $C^{1,\alpha}$ curves. Let $g\colon [-R, R] \to \mathbb{R}$ be a differentiable function satisfying $g(0) = g'(0) = 0$ whose derivative g' is Holder continuous with exponent α. Thus there is a constant M such that

$$|g'(x) - g'(y)| \leqslant M\left[\frac{|x-y|}{R}\right]^\alpha, \qquad x, y \in (-R, R) \tag{3.1}$$

We consider the graph $x_2 = g(x_1)$, $-R < x_1 < R$, in the plane and for $r > 0$ define an r neighborhood U_r of this graph by

$$U_r = \{(x_1, x_2)\colon |x_2 - g(x_1)| < r,\ -R < x_1 < R\}$$

Let χ_{U_r} be the characteristic function of U_r and $w_r(x)$ be the solution of the Dirichlet problem,

$$-Lw_r(x) = \chi_{U_r}(x), \qquad |x| < R, \quad w_r(x) = 0, \quad |x| = R$$

Theorem 1. There exists positive constants C, c depending only on λ, Λ, M, α such that

$$crR \leqslant w_r(x) \leqslant CrR, \qquad |x| \leqslant R/2$$

The proof of the theorem is a consequence of the following lemmas:

Lemma 1. Let D be the rectangle

$$D = \{x = (x_1, x_2) : |x_2| < R,\ |x_1| < mR\},$$

where $m \geqslant 1$. Suppose $u(x)$ is the solution of the Dirichlet problem,

$$Lu(x) = 0, \quad x \in D; \qquad u(x) = 0, \quad |x_2| = R; \qquad u(x) = 1, \quad x_1 = mR$$

Then there are constants β, $\gamma > 0$ depending only on λ, Λ such that

$$\exp[-\gamma m] \leqslant u(0, 0) \leqslant \exp[-\beta m] \tag{3.2}$$

Proof. Consider the function $w_\delta(x_1, x_2)$ given by

$$w_\delta(x_1, x_2) = \left[2 - \frac{x_2^2}{R^2}\right] \cosh\left(\frac{\delta x_1}{R}\right) \Big/ \cosh \delta m$$

where $\delta > 0$ is a parameter. It is easy to see that $w_\delta(x) \geqslant 0$, $x \in D$, and $w_\delta(\pm mR, x_2) \geqslant 1$, $|x_2| \leqslant R$. We also have that

$$\begin{aligned} Lw_\delta(x_1, x_2) = a_{11}(x) \frac{\delta^2}{R^2}\left[2 - \frac{x_2^2}{R^2}\right] \cosh\left(\frac{\delta x_1}{R}\right) \Big/ \cosh \delta m \\ - a_{12}(x)\left(\frac{4\delta}{R^2}\right)\left(\frac{x_2}{R}\right) \sinh\left(\frac{\delta x_1}{R}\right) \Big/ \cosh \delta m \\ - a_{22}(x)\left(\frac{2}{R^2}\right) \cosh\left(\frac{\delta x_1}{R}\right) \Big/ \cosh \delta m \leqslant 0, \qquad x \in D \end{aligned}$$

provided δ is sufficiently small. It follows then from the maximum principle that $u(x) \leqslant w_\delta(x)$, $x \in D$, provided δ is small. This proves the upper bound in (3.2).

To get the lower bound we use Corollary 2.2 to construct a set of paths in D starting at $(0, 0)$ which exit D through the boundary $x_1 = mR$. To see this suppose the diffusion process starts at the point (x_1, x_2) where $|x_2| < R/2$. Let S_1 be the circle with center $(x_1 + R/2, 0)$ and radius $R/4$.

Similarly let S_2 and S_3 be circles concentric with S_1 and radii $R/\sqrt{2}$, R respectively. Arguing as in Corollary 2.2, we see there is a number $\gamma > 0$ depending only on λ, Λ such that a particle started on S_2 exits the region between S_1, and S_3 through S_1 with probability at least γ. Hence the particle started at (x_1, x_2) hits S_1 before exiting D with probability at least γ. Since all points on S_1 have $|x_2| < R/2$ we have shown that a particle in the set $|x_2| < R/2$ jumps a distance at least $R/4$ to the right without exiting D and landing again in the set $|x_2| < R/2$ with probability at least γ. We conclude that

$$u(0, 0) \geqslant \gamma^{4(m+1)} \qquad \text{QED}$$

We extend the function $g(x)$ to all of $\mathbb{R}$ by setting $g'(x) = g'(R)$, $x > R$, $g'(x) = g'(-R)$, $x < -R$. For $k = 0, 1, 2,...$, let g_k be the function $g_k(x) = g(x) + r2^k$. We consider the diffusion as a random walk on the graphs $x_2 = g_k(x_1)$, $k = 0, 1, 2,....$ For the diffusion started at a point (x_1, x_2) on the graph of g_1 let $p_N(x_1)$ be the probability that it hits the graph of g_N before hitting the graph of g_0.

Lemma 2. Suppose N is an integer satisfying $N \geqslant 2$ and the inequality $r2^N \leqslant KR$, for some constant K. Then there are positive constants C, c depending only on λ, Λ, M, α, K such that

$$\frac{c}{2^N} \leqslant p_N(x_1) \leqslant \frac{C}{2^N}, \qquad x_1 \in \mathbb{R} \tag{3.3}$$

Proof. For $k \geqslant 1$ and $x_1 \in \mathbb{R}$ let $q_k(x_1)$ be the probability that the diffusion started at the point $(x_1, g_k(x_1))$ hits the graph of g_{k+1} before hitting the graph of g_0. Let Q be the rectangle

$$Q = \left\{ (y_1, y_2) : |y_1 - x_1| < \frac{1}{M} 2^{k-2} r, \ -2^{k-2} r < y_2 - g_k(x_1) < 3 \cdot 2^{k-1} r \right\}$$

It is easy to see from the fact that $\|g'\|_\infty \leqslant M$ that the boundary $y_2 - g_k(x_1) = 3 \cdot 2^{k-1} r$ of Q lies above the graph of g_{k+1} and the boundary $y_2 - g_k(x_1) = -2^{k-2} r$ lies above the graph of g_0. Hence if p_Q denotes the probability that the diffusion process started at $(x_1, g_k(x_1))$ exits Q through the boundary $y_2 - g_k(x_1) = 3 \cdot 2^{k-1} r$ then it follows that $q_k(x_1) \geqslant p_Q$. We can similarly define a second rectangle Q' with the property that the uppermost boundary lies below the graph of g_{k+1} and the lowermost boundary lies below the graph of g_0. If $q_{Q'}$ is the probability that the diffusion started at $(x_1, g_k(x_1))$ exits Q' through the lowermost boundary then we have $1-q_k(x_1)\geqslant q_{Q'}$. Now by Lemma 3.1 there is a number γ, $0<\gamma<1$, depending only on λ, Λ, M such that p_Q, $q_{Q'}\geqslant\gamma$. We conclude then that

$$\gamma\leqslant q_k(x_1)\leqslant 1-\gamma, \qquad k=1,2,..., x_1\in\mathbb{R} \tag{3.4}$$

We can improve the inequality (3.4) when $r2^k \ll R$. Let N' be the largest integer such that $r2^{N'}<R$. For $k<N'$ we let L be the tangent line to the graph of g at the point x_1. Let L_1 be the line parallel to L through the point

$$(x_1,\, g_k(x_1)+r-2^k r[1-2^{(k-N')\,\alpha/2}])$$

Similarly let L_2 be the line parallel to L through the point

$$(x_1,\, g_k(x_1)+2^k r[1+2^{(k-N')\,\alpha/2}])$$

Consider now the parallelogram Q consisting of points (y_1, y_2) lying in the strip between L_1 and L_2 which satisfy $|y_1-x_1|<(N'-k)\,2^k r$. In view of (3.1) there is a constant N_0 depending only on M, α such that if $1\leqslant k\leqslant N'-N_0$ then the boundary of Q contained in L_1 lies above the graph of g_0 and the boundary of Q contained in L_2 lies above the graphs of g_{k+1}. Hence if p_Q denotes the probability that the diffusion started at $(x_1, g_k(x))$ exits Q through the boundary contained in L_2 then it follows that $q_k(x_1)\geqslant p_Q$.

We can use the upper bound in Lemma 3.1 to estimate p_Q from below. Observe that if the diffusion is started at $(x_1, g_k(x_1))$ then the probability it exits the strip between L_1 and L_2 through L_2 is given by

$$\begin{aligned}\{2^k r[1-2^{(k-N')\,\alpha/2}]-r\}/\{&2^k r[1-2^{(k-N')\,\alpha/2}]\\ &-r+2^k r[1+2^{(k-N')\,\alpha/2}]\}\\ =\{1-2^{(k-N')\,\alpha/2}&-2^{-k}\}/\{2-2^{-k}\}\end{aligned}$$

From Lemma 3.1 the probability of the diffusion started at $(x_1, g_k(x_1))$ of exiting Q through the sides $|y_1-x_1|=(N'-k)\,2^k r$ is bounded above by $2^{\varepsilon(k-N')}$ for some positive constant ε depending only on λ, Λ. We conclude therefore that

$$\begin{aligned}&q_k(x_1)\geqslant p_Q\geqslant\{1-2^{(k-N')\,\alpha/2}-2^{-k}\}/\{2-2^{-k}\}-2^{\varepsilon(k-N')},\\ &\quad 1\leqslant k\leqslant N'-N_0\end{aligned}$$

By constructing a parallelogram Q' such that the uppermost boundary lies below the graph of g_{k+1} and the lowermost boundary below the graph of g_0 we can obtain a similar lower bound on $1-q_k(x_1)$. We conclude therefore that there are constants δ, $A>0$ depending only on λ, Λ, M, α such that

$$\frac{1}{2}-A[2^{-\delta k}+2^{-\delta(N'-k)}]\leqslant q_k(x_1)\leqslant \frac{1}{2}+A[2^{-\delta k}+2^{-\delta(N'-k)}],$$
$$1\leqslant k\leqslant N'-N_0 \tag{3.5}$$

The result follows from (3.4), (3.5) in using the fact that

$$\prod_{k=1}^{N-1}\inf_{x_1} q_k(x_1)\leqslant p_N(x_1)\leqslant \prod_{k=1}^{N-1}\sup_{x_1} q_k(x_1) \qquad \text{QED}$$

Lemma 3. Let N be an integer, $N\geqslant 2$, and suppose that S_0, S_1, S_N are Jordan curves in $\mathbb{R}^2$ with the property that S_0 lies inside S_1 which in turn lies inside S_N. Let X be a stochastic process in $\mathbb{R}^2$ with continuous sample paths. For $x\in S_1$ let $p_N(x)$ be the probability that the process started at x hits S_N before S_0. Let W be a set inside S_0 and suppose that for $y\in S_0$, τ_y is the time the process spends in W before hitting S_1, and T_y^N is the time the process spends in W before hitting S_N. Assume p_N satisfies the inequality

$$\frac{c}{2^N}\leqslant p_N(x)\leqslant\frac{C}{2^N}, \qquad x\in S_1 \tag{3.6}$$

and τ_y satisfies the inequality

$$c_1 r^2\leqslant E[\tau_y]\leqslant C_1 r^2, \qquad y\in S_0 \tag{3.7}$$

where c, C, c_1, C_1 are positive constants. Then T_y^N satisfies the inequality

$$\frac{c_1 r^2}{C}2^N\leqslant E[T_y^N]\leqslant\frac{C_1 r^2}{c}2^N, \qquad y\in S_0$$

Proof. For $n=1, 2,...$ and $x\in S_1$, $y\in S_0$ let $k_n(x, y)$ be the probability density function for the process started at x on the nth hit of S_0. Let $\rho(y, x)$ be the first hitting probability density on S_1 for the process started at y. Then we have

$$E[T_y^N]=E[\tau_y]+\sum_{n=1}^{\infty}\int_{S_1}dx'\int_{S_0}dy'\,\rho(y, x')\,k_n(x', y')\,E[\tau_{y'}]$$

From (3.6) it follows that

$$\left(1-\frac{C}{2^N}\right)^n \leqslant \int_{S_0} k_n(x', y')\, dy' \leqslant \left(1-\frac{c}{2^N}\right)^n, \qquad x' \in S_1$$

The result follows from this last inequality, (3.7) and the obvious fact that

$$\int_{S_1} \rho(y, x')\, dx' = 1, \qquad y \in S_0 \qquad \text{QED}$$

For $k = 0, 1, 2,...$ let g_k^* be defined analogously to g_k by $g_k^*(x_1) = g(x_1) - r2^k$. The curves S_k, $k = 0, 1, 2,...$ are defined as the union of the graphs of g_k, g_k^*. The set W is the inside of S_0. With this definition we then have the following:

Lemma 4. The inequality (3.7) holds.

Proof. We consider first the lower bound. Observe that for every x on the graph of g, the disc $\{y : |y - x| < \varepsilon r\}$ is contained in W provided ε is sufficiently small, depending on α, M. Lemma 2.1 then implies that $E[\tau_x] \geqslant \delta r^2$ for some $\delta > 0$ depending only on α, M, λ, Λ if x is in the graph of g. By Lemma 3.1 the diffusion started at $y \in S_0$ hits the graph of g before hitting S_1 with probability $\gamma > 0$ depending only on λ, Λ, M, α. Hence $E[\tau_y] \geqslant \gamma \delta r^2$, $y \in S_0$.

To get the upper bound let $x = (x_1, x_2)$ be inside S_1 and Q the rectangle centered at x given by $Q = \{(y_1, y_2) : |y_1 - x_1| < r,\ |y_2 - x_2| < 2(M+4)\, r\}$. Clearly the uppermost boundary of Q lies above the graph of g_1 and the lowermost boundary below the graph of g_1^*. Let τ_x^* be the time taken for the diffusion started at x to exit Q. By Lemma 2.1 it follows that $E[\tau_x^*] \leqslant K r^2$ for some constant K depending only on λ, Λ, M. Chebyshev's inequality yields therefore

$$P(\tau_x^* > m r^2) \leqslant K/m, \qquad m > 0$$

From Lemma 3.1 we have that the probability the diffusion started at x exits Q through the boundary $|y_2 - x_2| = 2(M+4)\, r$ is bounded below by $\gamma > 0$ depending only on λ, Λ, M. Hence

$$P(\tau_x < m r^2) \geqslant P(\tau_x^* < m r^2) - [1 - \gamma] \geqslant \gamma - \frac{K}{m}$$

Hence for $m \geqslant 2K/\gamma$ we have

$$P(\tau_x > m r^2) \leqslant 1 - \gamma/2, \qquad x \text{ inside } S_1$$

We conclude from this that

$$E[\tau_x] \leqslant \sum_{k=0}^{\infty} \left(1 - \frac{\gamma}{2}\right)^k (k+1)\, mr^2 \leqslant Cr^2$$

for any x inside S_1. QED

Proof of Theorem 3.1. The upper bound follows immediately from Lemmas 3.2, 3.3, 3.4. We simply choose N so that the graphs of g_N and g_N^* lie outside the disc $|x| < R$. Evidently N can be chosen to satisfy $2^N \leqslant C(R/r)$, where C depends only on M.

To get the lower bound let N_0 be the smallest integer such that $2^{N_0} r \geqslant R$. We consider $N \ll N_0$. Observe first that by Lemmas 3.3, 3.4 there is a constant C depending only on λ, Λ, M, α such that

$$E[T_x^N] \leqslant C 2^N r^2, \qquad x \text{ inside } S_1$$

whence by Chebyshev

$$P(T_x^N > 2C2^N r^2) \leqslant 1/2, \qquad x \text{ inside } S_1$$

Hence

$$E[T_x^N;\, T_x^N > 2C2^N r^2 m] \leqslant \sum_{k=m}^{\infty} (\tfrac{1}{2})^k\, 2C2^N r^2 (k+1) \leqslant C_1 2^{-m/2} 2^N r^2$$

for any integer $m \geqslant 1$. Again by Lemmas 3.3, 3.4 there is a constant $C_2 > 0$ such that

$$E[T_x^N] \geqslant C_2 2^N r^2, \qquad x \in S_0$$

Hence from the last two inequalities we have

$$E[T_x^N;\, T_x^N < 2C2^N r^2 m] \geqslant 2^N r^2 [C_2 - C_1 2^{-m/2}]$$

We also have for any δ, $0 < \delta < 1$,

$$E[T_x^N;\, T_x^N < 2C2^N r^2 m] \leqslant 2C2^N r^2 m [\delta + P(T_x^N > 2C2^N r^2 m\delta)]$$

It follows from these last two inequalities that there is a constant $\varepsilon > 0$ depending only on λ, Λ, M, α such that

$$P(T_x^N > 2C2^N r^2 m\delta) > \varepsilon, \qquad x \in S_0$$

Let D_N be the intersection of the interior of S_N with the disc $\{|x| < R\}$. For $x \in S_0$ with $|x| < R/2$ let $q_N(x)$ be the probability that the diffusion started at x exits D_N through the boundary where $|y| = R$. By Lemma 3.1 we can choose a constant $K > 0$ depending only on λ, Λ, α, M such that if $N \leqslant N_0 - K$ then $q_N(x) < \varepsilon/2$, $x \in S_0$, $|x| < R/2$. Hence

$$\begin{aligned} w_r(x) &\geqslant 2C2^N r^2 m\delta[P(T_x^N > 2C2^N r^2 m\delta) - q_N(x)] \\ &\geqslant C2^N r^2 m\delta\varepsilon, \qquad x \in S_0, \quad |x| < R/2 \end{aligned}$$

To complete the result we need just to observe that for any x with $|x| < R/2$ the probability that the diffusion started at x hits S_0 on $\{|y| < R/2\}$ before exiting the disc $\{|y| < R\}$ is bounded below by $\gamma > 0$ depending only on λ, Λ. Hence we have

$$w_r(x) \geqslant \gamma C2^N r^2 m\delta\varepsilon \geqslant crR, \qquad |x| < R/2 \qquad \text{QED}$$

4. ESCAPE PROBABILITY FROM A LIPSCHITZ CURVE

Our goal in this section is to obtain a generalization of the lower bound in Lemma 3.2 to Lipschitz curves. Let $g: (-\infty, \infty) \to \mathbb{R}$ be a differentiable function satisfying $\|g'\|_\infty \leqslant M$. Let r satisfy $0 < r < 1$ and for $k = 0, 1, 2,...$ let g_k denote the functions $g_k(x) = g(x) + r2^k$, $x \in \mathbb{R}$. As in Section 3 we view the diffusion as a random walk on the graphs g_k. For the diffusion started at a point (x_1, x_2) on the graph of g_1 we again denote by $p_N(x_1)$ the probability that it hits the graph of g_N before hitting the graph of g_0.

Theorem 1. Suppose N is an integer satisfying $N \geqslant 2$ and the inequality $r2^N \leqslant K$, for some constant K. Then there is a constant C depending only an λ, Λ, M, K such that

$$\int_0^1 p_N(x_1)\, dx_1 \geqslant \frac{1}{2^N} \exp[-C\sqrt{N}]$$

The proof of Theorem 4.1 is based on the expansion of g' in a Haar basis. We write

$$g'(x) = \sum_{j=0}^{\infty} \psi_j(x), \qquad x \in \mathbb{R} \tag{4.1}$$

The functions ψ_j have the property that for any $m \in \mathbb{Z}$, ψ_j is constant on the interval $(m2^{-j}, (m+1)\,2^{-j})$. In addition if $j \geqslant 1$ and m is even then

$\psi_j(x) = -\psi_j(x')$ for $m2^{-j} < x < (m+1)\,2^{-j}$, $(m+1)\,2^{-j} < x' < (m+2)\,2^{-j}$. It is clear that the ψ_j form an orthogonal set and $\|\psi_j\|_\infty \leqslant \|g'\|_\infty \leqslant M$.

Just as in Lemma 3.2 we consider the probabilities $q_k(x_1)$. Thus $q_k(x_1)$ is the probability that the diffusion started at the point $(x_1, g_k(x_1))$ hits the graph of g_{k+1} before hitting the graph of g_0. We shall bound q_k below in terms of the functions ψ_j. The bound will be dominated by those ψ_j with $j \sim N-k$. The reason for this is as follows: Normalizing g so that $g(0)=0$ we have

$$g(x) = \sum_{j=0}^{\infty} \int_0^x \psi_j(x')\,dx'$$

The function ψ_j contributes therefore at most $2^{-j}\,\|\psi_j\|_\infty \leqslant M2^{-j}$ to g if $j \geqslant 1$. In estimating q_k the relevant length scale is $r2^k \sim K2^{k-N}$. Hence if $j \gg N-k$ the contribution of ψ_j is much less than this length scale. For $j < N-k$ a different mechanism is at work. The contribution of ψ_j is a function linear on a length scale 2^{-j}. Since the addition of a linear function to g does not alter the probability q_k the contribution of ψ_j to q_k is again small if $j \ll N-k$.

To illustrate the differing mechanisms for estimating q_k we shall first consider the situation when g' is given by a single function ψ_j. The following two lemmas enable us to get a lower bound on q_k in the case when $j > N-k$.

Lemma 1. Let (ρ, θ) be polar co-ordinates in the (x_1, x_2) plane, $\rho^2 = x_1^2 + x_2^2$, $x_1 = \rho\cos\theta$. For $0 < \theta_0 < \pi/2$ let $W(\theta_0)$ be the wedge consisting of points satisfying $0 < \theta < \theta_0$. Let $u(x_1, x_2)$ be the solution of the Dirichlet problem $Lu(x) = 0$, $x \in W(\theta_0)$,

$$u = 0 \quad \text{if} \quad \theta = 0, \qquad u = 1 \quad \text{if} \quad \theta = \theta_0$$

Then there are constants C, $c > 0$ depending only on λ, Λ such that

$$\frac{c\theta}{\theta_0} \leqslant u(x) \leqslant \frac{C\theta}{\theta_0}, \qquad x \in W(\theta_0) \qquad \text{with polar coordinates } (\rho, \theta) \tag{4.2}$$

Proof. We consider functions $w(x_1, x_2)$ of the form $w(x_1, x_2) = h(x_2/x_1)$. Then

$$Lw(x) = \frac{1}{x_1^2}\,[\{a_{11}(x)\,z^2 - 2a_{12}(x)\,z + a_{22}(x)\}\,h''(z) + \{2a_{11}(x)\,z - 2a_{12}(x)\}\,h'(z)]$$

where $z = x_2/x_1$. If we take $h(z)$ to be the function given by

$$h'(z) = \frac{A}{(z+\sqrt{\varepsilon})^2+\varepsilon}, \qquad h(0)=0$$

where A, $\varepsilon > 0$ are parameters, then it follows that

$$h''(z) = \frac{-2(z+\sqrt{\varepsilon})}{(z+\sqrt{\varepsilon})^2+\varepsilon}\, h'(z)$$

Hence with this choice of h, we have

$$Lw(x) \leqslant \frac{1}{x_1^2}\,[\lambda(z^2+1)\,h''(z) + 2\Lambda(z+1)\,h'(z)] \leqslant 0$$

if $0 < z < \sqrt{\varepsilon}$ and ε is sufficiently small depending only on λ, Λ. Choosing A so that $h(\sqrt{\varepsilon}) = 1$ it follows by the maximum principle that if θ_0 is defined by $\tan\theta_0 = \sqrt{\varepsilon}$ then $u(x) \leqslant w(x)$, $x \in W(\theta_0)$. The upper bound in (4.2) follows then provided $\theta_0 \leqslant \Phi_0$, where $\Phi_0 > 0$ depends only on λ, Λ. To get the bound for $\Phi_0 < \theta_0 < \pi/2$ observe that if we denote by u_{θ_0} the solution of the Dirichlet problem for the wedge $W(\theta_0)$ then $u_{\theta_0}(x) \leqslant u_{\Phi_0}(x)$, $x \in W(\Phi_0)$.

To get the lower bound in (4.2) we choose h to be given by

$$h'(z) = Ae^{\gamma z}, \qquad h(0)=0$$

where A, $\gamma > 0$ are parameters. Hence with this choice of h, we have

$$Lw(x) \geqslant \frac{1}{x_1^2}\,[\lambda(z^2+1)\,h''(z) - 2\Lambda(z+1)\,h'(z)] \geqslant 0$$

for all $z > 0$ provided γ is chosen large enough, depending on λ, Λ. If $0 < \theta_0 < \pi/4$ we choose A to satisfy $h(\tan\theta_0) = 1$. By the maximum principle $u(x) \geqslant w(x)$, $x \in W(\theta_0)$, whence the lower bound follows if $0 < \theta_0 < \pi/4$. To deal with the case $\pi/4 < \theta_0 < \pi/2$ let p be the infimum of the probabilities that the diffusion started on the line segment $\theta = \pi/4$ exits $W(\theta_0)$ through the boundary $\theta = \theta_0$. By Lemma 3.1 p is bounded below by a positive constant depending only on λ, Λ. The lower bound (4.2) follows now from the inequality $u_{\theta_0}(x) \geqslant p u_{\pi/4}(x)$, $x \in W(\pi/4)$. QED

Lemma 2. Suppose the diffusion is started at the point $(0, 1)$ in the (x_1, x_2) plane. For $0 < \varepsilon < 1$, let P_ε be the probability that it first hits the

x_1 axis in the interval $-\varepsilon < x_1 < \varepsilon$. Then there are constants C, $\alpha > 0$ depending only on λ, Λ such that $P_\varepsilon \leqslant C\varepsilon^\alpha$.

Proof. Let N be the largest integer such that $2^{-N} \geqslant \varepsilon$. For $m = 0, 1, 2, \ldots$ let S_m be the circle centered at the origin with radius 2^{-m}.

Suppose the diffusion starts at a point $x \in S_m$. Then the probability that it exits the annulus bounded by S_{m-1} and S_{m+1} without hitting the x_1 axis is bounded above by $1 - \delta$, where $\delta > 0$ depends only on λ, Λ. This follows from Lemma 3.1. Hence

$$P_\varepsilon \leqslant (1-\delta)^N \leqslant C\varepsilon^\alpha$$

for suitable constants C, α depending only on λ, Λ. QED

For $x \in \mathbb{R}$, m, $j \in \mathbb{Z}$, we define the interval $I(x, m, j)$ by $I(x, m, j) = (x + m2^{-j}, x + (m+1)\, 2^{-j})$.

Lemma 3. Suppose N is a given in the statement of Theorem 4.1 and k, j are integers satisfying $1 \leqslant k < N$, $j \geqslant N - k$. Assume the representation (4.1) for g' has the simple form $g' = \psi_j$. Let $q_k(x_1)$ be the probability that the diffusion started at $(x_1, g_k(x_1))$ hits the graph of g_{k+1} before hitting the graph of g_0. Then there are positive constants γ, C, c with $0 < \gamma < 1$, depending only on λ, Λ, M such that

$$q_k(x_1) \geqslant 1 \Big/ \left[2 + C2^{-k} + C\gamma^{j+k-N} \sum_{m=-\infty}^{\infty} e^{-c\,|m|} Av_{I(x_1, m, N-k)}\, |\psi_j| \right]$$

Proof. Observe that the inequality (3.4) continues to hold for Lipschitz curves. Hence it is sufficient to show that

$$q_k(x_1) \geqslant \tfrac{1}{2} - C2^{-k} - C\gamma^{j+k-N} \sum_{m=-\infty}^{\infty} e^{-c\,|m|} Av_{I(x_1, m, N-k)}\, |\psi_j| \tag{4.3}$$

Wlog we shall assume $x_1 = 0$ in (4.3), and that $g(0) = 0$. Let P be the probability that the diffusion started at $(0, 2^k r)$ hits the graph of g_0 and then exits the strip $\{(x_1, x_2) : 0 < x_2 < 2^{k+1} r\}$ through the boundary $x_2 = 2^{k+1} r$.

Similarly let Q be the probability that the diffusion started at $(0, 2^k r)$ exits the strip $\{(x_1, x_2) : 0 < x_2 < 2^{k+1} r\}$ through the boundary $x_2 = 2^{k+1} r$ but fails then to hit the graph of g_{k+1} before hitting the graph of g_0. It is evident that

$$q_k(0) \geqslant \tfrac{1}{2} - P - Q \tag{4.4}$$

We shall obtain upper bounds on P and Q which will imply (4.3). First we write P as a sum

$$P = \sum_{m=-\infty}^{\infty} \sum_{i=1}^{2^{j+k-N}} P_{m,i}$$

where $P_{m,i}$ is the probability that the diffusion started at $(0, 2^k r)$ hits the graph of g_0 first in the interval $m2^{k-N} + (i-1)\, 2^{-j} < x_1 < m2^{k-N} + i2^{-j}$, and then exits the strip $\{(x_1, x_2) : 0 < x_2 < 2^{k+1} r\}$ through the boundary $x_2 = 2^{k+1} r$.

Let $\xi_{m,i}$ be the probability that the diffusion started at $(0, 2^k r)$ hits the graph of g_0 first in the interval $m2^{k-N} + (i-1)\, 2^{-j} < x_1 < m2^{k-N} + i2^{-j}$ before exiting the strip $0 < x_2 < 2^{k+1} r$. Let $\zeta_{m,i}$ be the supremum over x_1 in the interval $m2^{k-N} + (i-1)\, 2^{-j} < x_1 < m2^{k-N} + i2^{-j}$ of the probability that the diffusion started at $(x_1, g_0(x_1))$ exits the strip $0 < x_2 < 2^{k+1} r$ through the boundary $x_2 = 2^{k+1} r$. It is clear that $P_{m,i} \leqslant \xi_{m,i} \zeta_{m,i}$. If x_i satisfies the inequality $m2^{k-N} + (i-1)\, 2^{-j} < x_1 < m2^{k-N} + i2^{-j}$ then $g_0(x_1)$ is bounded by

$$|g_0(x_1)| \leqslant r + \int_{m2^{k-N} + (i-1)\, 2^{-j}}^{m2^{k-N} + i2^{-j}} |\psi_j(y_1)|\, dy_1$$

We conclude therefore that

$$\zeta_{m,i} \leqslant 2^{-k-1} + \frac{2^{-k-1}}{r} \int_{m2^{k-N} + (i-1)\, 2^{-j}}^{m2^{k-N} + i2^{-j}} |\psi_j(y_1)|\, dy_1 \tag{4.5}$$

Let η_m be the probability that the diffusion started at $(0, 2^k r)$ exits the rectangle $\{(x_1, x_2) : |x_1| < (|m| - 1)\, 2^{k-N}, 0 < x_2 < 2^{k+1} r\}$ through the boundaries $|x_1| = (|m| - 1)\, 2^{k-N}$. It is evident that

$$\sum_{i=1}^{2^{j+k-N}} \xi_{m,i} \leqslant \eta_m \tag{4.6}$$

Further, we have by Lemma 3.1 that $\eta_m \leqslant \exp[-c\, |m|]$ for some $c > 0$ depending only on λ, Λ. Let $\rho_{m,i}$ be the supremum over $x = (x_1, x_2)$ on the boundaries $|x_1| = (|m| - 1)\, 2^{k-N}$, $0 < x_2 < 2^{k+1} r$, of the probability that the diffusion started at x first hits the graph of g_0 in the interval $m2^{k-N} + (i-1)\, 2^{-j} < x_1 < m2^{k-N} + i2^{-j}$ before exiting the strip $0 < x_2 < 2^{k+1} r$. Then $\xi_{m,i} \leqslant \eta_m \rho_{m,i}$. Observe now that the interval $m2^{k-N} + (i-1)\, 2^{-j} < x_1 < m2^{k-N} + i2^{-j}$ has length 2^{-j} and the point x is a distance of order

2^{k-N} from the part of the graph of g_0 corresponding to this interval. Hence we may apply Lemma 4.2 with $\varepsilon = 2^{-j}/2^{k-N}$ to obtain the inequality, $\rho_{m,i} \leqslant \gamma^{j+k-N}$ for some γ, $0 < \gamma < 1$, depending only on λ, Λ. Hence we have the inequality

$$\xi_{m,i} \leqslant \gamma^{j+k-N} \exp[-c\,|m|] \tag{4.7}$$

From (4.5), (4.6), (4.7) we can estimate

$$\begin{aligned}
\sum_{i=1}^{2^{j+k-N}} P_{m,i} &\leqslant \sum_{i=1}^{2^{j+k-N}} \xi_{m,i} \zeta_{m,i} \\
&\leqslant \left\{ \sum_{i=1}^{2^{j+k-N}} \xi_{m,i} \right\} 2^{-k-1} + \gamma^{j+k-N} \\
&\quad \times \exp[-c\,|m|] \frac{2^{-k-1}}{r} \int_{m2^{k-N}}^{(m+1)\,2^{k-N}} |\psi_j(y_1)|\, dy_1 \\
&\leqslant 2^{-k-1} \exp[-c\,|m|] + C\gamma^{j+k-N} \\
&\quad \times \exp[-c\,|m|]\, Av_{I(0,m,N-k)} |\psi_j|
\end{aligned} \tag{4.8}$$

since $r \sim 2^{-N}$. Summing this last inequality with respect to m yields an upper bound on P which is consistent with (4.4) and (4.3).

We bound Q in a similar way to the method we used for P. We write

$$Q = \sum_{m=-\infty}^{\infty} \sum_{i=1}^{2^{j+k-N}} Q_{m,i}$$

where $Q_{m,i}$ is the probability that the diffusion started at $(0, 2^k r)$ exits the strip $\{(x_1, x_2) : 0 < x_2 < 2^{k+1} r\}$ through the line segment $m2^{k-N} + (i-1)\, 2^{-j} < x_1 < m2^{k-N} + i2^{-j}$, $x_2 = 2^{k+1} r$, but fails then to hit the graph of g_{k+1} before hitting the graph of g_0.

Let $\xi_{m,i}$ be the probability that the diffusion started at $(0, 2^k r)$ exits the strip $\{(x_1, x_2) : 0 < x_2 < 2^{k+1} r\}$ through the line segment $m2^{k-N} + (i-1)\, 2^{-j} < x_1 < m2^{k-N} + i2^{-j}$, $x_2 = 2^{k+1} r$. Then $\xi_{m,i}$ satisfies the inequalities (4.6), (4.7). Let $\zeta_{m,i}$ be the supremum over $x = (x_1, x_2)$ in the line segment $m2^{k-N} + (i-1)\, 2^{-j} < x_1 < m2^{k-N} + i2^{-j}$, $x_2 = 2^{k+1} r$ of the probability that the diffusion started at x hits the graph of g_0 before hitting the graph of g_{k+1}. It is clear that $Q_{m,i} \leqslant \xi_{m,i} \zeta_{m,i}$. Further, by Lemma 4.1, we have the inequality, Probability [the diffusion started at $x=(x_1, x_2)$ with $m2^{k-N}+(i-1)\,2^{-j}<x_1<m2^{k-N}+i2^{-j}$, $x_2=2^{k+1}r$ of not exiting the rectangle

$$\Big\{(y_1, y_2) : m2^{k-N}+(i-2)\,2^{-j}<y<m2^{k-N}+(i+1)\,2^{-j}$$

$$-2^{-j}<y_2-2^{k+1}r<\int_{m2^{k-N}(i-2)\,2^{-j}}^{m2^{k-N}+(i+1)\,2^{-j}} |\psi_j(y)|\, dy\Big\}$$

through the boundary

$$y_2-2^{k+1}r=\int_{m2^{k-N}+(i-2)\,2^{-j}}^{m2^{k-N}+(i+1)\,2^{-j}} |\psi_j(y)|\, dy\Big]$$

$$\leqslant C2^j\int_{m2^{k-N}+(i-2)\,2^{-j}}^{m2^{k-N}+(i+1)\,2^{-j}} |\psi_j(y)|\, dy$$

where C is a constant depending only on λ, Λ. It is also clear that

Probability [the diffusion started at $x=(x_1, x_2)$ with $x_2=2^{k+1}r-2^{-j}$ of hitting the graph of g_0 before hitting the graph of $g_{k+1}]\leqslant C2^{N-k-j}$, where the constant C depends only on the Lipschitz constant M. From these last two inequalities we obtain

$$\zeta_{m,i}\leqslant C2^{N-k}\int_{m2^{k-N}+(i-2)\,2^{-j}}^{m2^{k-N}+(i+1)\,2^{-j}} |\psi_j(y)|\, dy$$

Since this last inequality is analogous to (4.5) we can obtain a bound on $\sum_{i=1}^{2^{j+k-N}} Q_{m,i}$ which is similar to the estimate (4.8) on $\sum_{i=1}^{2^{j+k-N}} P_{m,i}$. Consequently we obtain an upper bound on Q which is consistent with (4.4), (4.3). QED

Lemma 4. Suppose N is as given in the statement of Theorem 4.1 and k, j are integers satisfying $1\leqslant k<N$, $0\leqslant j<N-k$. Assume the representation (4.1) for g' has the simple form $g'=\psi_j$. For $x\in\mathbb{R}$ let $d(x,\mathbb{Z})$ be the distance from x to the integers $\mathbb{Z}$. Then there are positive constants C, c depending only on λ, Λ, M such that

$$q_k(x_1)\geqslant 1\Big/\Big[2+C2^{-k}+C\sum_{m=-\infty}^{\infty}\exp[-c\{|m|+2^{N-k-j}\,d(2^jx_1,\mathbb{Z})\}]$$
$$\times Av_{I(x_1, m, N-k)}\,|\psi_j|\Big] \tag{4.9}$$

Proof. Let us first assume that $0 < x_1 < 2^{-j-1}$, whence the nearest point in $\mathbb{Z}$ to $2^j x_1$ is 0. We shall first assume that $\psi_j(x_1) = 0$. Since $g(0) = 0$ and ψ_j is constant on the interval $(0, 2^{-j})$ it follows that $g(y_1) = 0$, $0 < y_1 < 2^{-j}$. We proceed now in a similar way to Lemma 4.3. Let P be the probability that the diffusion started at $(x_1, 2^k r)$ hits the graph of g_0 and then exits the strip $\{(y_1, y_2) : r < y_2 < 2^{k+1} r\}$ through the boundary $y_2 = 2^{k+1} r$. Similarly Q is the probability that the diffusion started at $(x_1, 2^k r)$ exits the strip $\{(y_1, y_2) : r < y_2 < 2^{k+1} r\}$ through the boundary $y_2 = 2^{k+1} r$ but fails then to hit the graph of g_{k+1} before hitting the graph of g_0. It is clear that

$$q_k(x_1) \geqslant [2^k - 1]/[2^{k+1} - 1] - P - Q \tag{4.10}$$

We write P as a sum

$$P = \sum_{m=-\infty}^{\infty} P_m$$

where P_m is the probability that the diffusion started at $(x_1, 2^k r)$ hits the graph of g_0 first in the interval $x_1 + m 2^{k-N} < y_1 < x_1 + (m+1)\, 2^{k-N}$ before exiting the strip through the boundary $y_2 = 2^{k+1} r$. Since the graph of g_0 coincides with the line $y_2 = r$ in the region $0 < y_1 < 2^{-j}$ we conclude that

$$P_m = 0, \qquad |m| < 2^{N-k-j} d(2^j x_1, \mathbb{Z}) - 1$$

Now for $|m| > 2^{N-k-j} d(2^j x_1, \mathbb{Z}) - 1$ let ξ_m be the probability that the diffusion started at $(x_1, 2^k r)$ hits the graph of g_0 first in the interval $x_1 + m 2^{k-N} < y_1 < x_1 + (m+1)\, 2^{k-N}$ before exiting the strip. Then there is a constant $c > 0$ depending only on λ, Λ such that $\xi_m \leqslant \exp[-c\, |m|]$. Let ζ_m be the supremum over y_1 in the interval $x_1 + m 2^{k-N} < y_1 < x_1 + (m+1)\, 2^{k-N}$ that the diffusion started at $(y_1, g_0(y_1))$ exits the strip through the boundary $y_2 = 2^{k+1} r$. We have

$$|g_0(y_1) - r| \leqslant \sum_{|m'| \leqslant |m|} 2^{k-N} Av_{I(x_1, m', N-k)}\, |\psi_j|$$

Hence

$$\zeta_m \leqslant \frac{1}{(2^{k+1} - 1)\, r}\, |g_0(y_1) - r| \leqslant C \sum_{|m'| \leqslant |m|} Av_{I(x_1, m', N-k)}\, |\psi_j|$$

where C is a constant. In view of the fact that $P_m \leqslant \xi_m \zeta_m$ we conclude that

$$
\begin{aligned}
P &\leqslant \sum_{\{|m| > 2^{N-k-j} d(2^j x_1, \mathbb{Z}) - 1\}} P_m \\
&\leqslant \sum_{\{|m| > 2^{N-k-j} d(2^j x_1, \mathbb{Z}) - 1\}} \exp[-c\,|m|]\, C \sum_{|m'| \leqslant |m|} Av_{I(x_1, m', N-k)}\, |\psi_j| \\
&\leqslant C_1 \sum_{m=-\infty}^{\infty} \exp[-c_1\{|m| + 2^{N-k-j} d(2^j x_1, \mathbb{Z})\}]\, Av_{I(x_1, m, N-k)}\, |\psi_j|
\end{aligned} \tag{4.11}
$$

for suitable constants C_1, $c_1 > 0$.

We can get a similar estimate to (4.11) on Q. The result follows then from (4.10) in the case when $0 < x_1 < 2^{-j-1}$ and $\psi_j(x_1) = 0$. To deal with the case when $\psi_j(x_1) \neq 0$ observe that we can restrict ourselves to the situation where $|\psi_j(x_1)| \leqslant \varepsilon$ and $\varepsilon > 0$ is a small number depending only on the Lipschitz constant M. Indeed if $|\psi_j(x_1)| \geqslant \varepsilon$ then (4.9) gives

$$
q_k(x_1) \geqslant \tfrac{1}{2} - C_1 2^{-k} - C_2 \exp[-c 2^{N-k-j} d(2^j x_1, \mathbb{Z})]
$$

where C_1, $C_2 > 0$ are bounded below. This inequality can be obtained from the estimate

$$
q_k(x_1) \geqslant [2^k - 1]/[2^{k+1} - 1] - p
$$

where p is the probability that the diffusion started at $(x_1, g_k(x_1))$ exits the region between the graphs of g_0 and g_{k+1} and the lines $y_1 = 0$, $y_1 = 2^{-j}$, through the boundaries $y_1 = 0$, $y_1 = 2^{-j}$. Since this region is a parallelogram it follows from Lemma 3.1 that there is a constant $c > 0$ depending only on λ, Λ such that

$$
p \leqslant \exp[-c 2^{N-k-j} d(2^j x_1, \mathbb{Z})]
$$

Let us assume now that $|\psi_j(x_1)| \leqslant \varepsilon$ and ε is sufficiently small so that the graph of g is still Lipschitz when the axis is rotated so that $g(y_1) = 0$, $0 < y_1 < 2^{-j}$. The inequality (4.11) then becomes

$$
\begin{aligned}
P &\leqslant \sum_{\{|m| > 2^{N-k-j} d(2^j x_1, \mathbb{Z}) - 1\}} \exp[-c\,|m|]\, C \\
&\quad \times \sum_{|m'| \leqslant |m|} Av_{I(x_1, m', N-k)} [\,|\psi_j| + |\psi_j(x_1)|\,] \\
&\leqslant C_1 \sum_{m=-\infty}^{\infty} \exp[-c_1\{|m| + 2^{N-k-j} d(2^j x_1, \mathbb{Z})\}]\, Av_{I(x_1, m, N-k)}\, |\psi_j|
\end{aligned}
$$

as before. The result follows as before from (4.10) and a similar estimate on Q. QED

Next we combine the method of Lemmas 4.3, 4.4 to obtain a lower bound on $q_k(x_1)$ for general g' given by (4.1)

Proposition 1. Suppose N is as given in the statement of Theorem 4.1, g' is given by the representation (4.1) and k is an integer satisfying $1 \leqslant k < N$. Then there are positive constants C, c, γ with $0 < \gamma < 1$, depending only on λ, Λ, M such that

$$q_k(x_1) \geqslant 1 \Bigg/ \Bigg[2 + C 2^{-k} + C \sum_{j=0}^{N-k} \sum_{m=-\infty}^{\infty} \times \exp[-c\{|m| + 2^{N-k-j} d(2^j x_1, \mathbb{Z})\}]\, Av_{I(x_1, m, N-k)} |\psi_j| + C \sum_{j=N-k+1}^{\infty} \gamma^{j+k-N} \sum_{m=-\infty}^{\infty} e^{-c|m|} Av_{I(x_1, m, N-k)} |\psi_j| \Bigg]$$

Proof. We first consider the case where $x_1 = 0$. We write $g(x) = g^1(x) + g^2(x)$, where

$$dg^1(x)/dx = \sum_{j=N-k+1}^{\infty} \psi_j(x), \qquad g^1(0) = 0$$

We define a dyadic decomposition of $\mathbb{R}$ into intervals of length 2^{-j}, $j \geqslant N-k$ where each interval has the form $(m 2^{-j}, (m+1) 2^{-j})$, $m \in \mathbb{Z}$. Denote by I_j an interval in this decomposition which has length $|I_j| = 2^{-j}$. Let ε satisfy $0 < \varepsilon < 1$. We define a function N_1,

$$N_1 : \mathbb{R} \to \{ j \in \mathbb{Z} \cup \{\infty\} : j \geqslant N-k+1 \}$$

by

(a) $N_1(x) = \infty$ if for all dyadic intervals I_j with $j > N-k$ and $x \in I_j \subset I_{N-k}$ we have

$$|I_j|^{1-\varepsilon} Av_{I_j} |\psi_j| \leqslant |I_{N-k}|^{1-\varepsilon} \sup_{I_{N-k}} |g^2| / |I_{N-k}|$$

(b) Otherwise $2^{-N_1(x)}$ is the length of the side of the largest dyadic interval, I_j, $j > N-k$, $x \in I_j \subset I_{N-k}$ with

$$|I_j|^{1-\varepsilon} Av_{I_j} |\psi_j| > |I_{N-k}|^{1-\varepsilon} \sup_{I_{N-k}} |g^2| / |I_{N-k}|$$

Observe that ψ_j is constant on I_j whence $Av_{I_j}\,|\psi_j|$ is just the absolute value of this constant.

Next we define the set G_1 to be $G_1 = \{x \in \mathbb{R} : N_1(x) = \infty\}$. It is clear there is a unique family $\mathscr{F}_1$ of disjoint dyadic intervals with the property that

$$\bigcup_{I \in \mathscr{F}_1} I = \mathbb{R} \backslash G_1$$

If $\mathscr{F}_1$ is nonempty we define a function N_2 on $\mathbb{R}$ which is analogous to N_1. Thus

(a) $N_2(x) = \infty$ if $x \in G_1$.

(b) $N_2(x) = \infty$ if $x \in \mathbb{R}\backslash G_1$ and for any $j' > j$ with $x \in I_{j'} \subset I_j \in \mathscr{F}_1$ we have

$$|I_{j'}|^{1-\varepsilon}\,Av_{I_{j'}}\,|\psi_{j'}| \leqslant |I_j|^{1-\varepsilon}\,Av_{I_j}\,|\psi_j|$$

(c) Otherwise $2^{-N_2(x)}$ is the length of the side of the largest dyadic interval $I_{j'}$, $x \in I_{j'} \subset I_j \in \mathscr{F}_1$ such that

$$|I_{j'}|^{1-\varepsilon}\,Av_{I_{j'}}\,|\psi_{j'}| > |I_j|^{1-\varepsilon}\,Av_{I_j}\,|\psi_j|$$

Evidently we have $N_2(x) \geqslant N_1(x) + 1$, a.e. $x \in \mathbb{R}$. Now define G_2 to be the set $G_2 = \{x \in \mathbb{R}\backslash G_1 : N_2(x) = \infty\}$. Then, as with N_1, there is a unique family $\mathscr{F}_2$ of disjoint dyadic intervals with the property

$$\bigcup_{I \in \mathscr{F}_2} I = \mathbb{R}\backslash G_1 \backslash G_2$$

One can continue this procedure inductively to construct a sequence of functions N_t, $t \geqslant 1$, on $\mathbb{R}$, a sequence of disjoint subsets G_t, $t \geqslant 1$, of $\mathbb{R}$, and a sequence of families $\mathscr{F}_t$, $t \geqslant 1$, with the properties:

(a) $\bigcup_{t=1}^{\infty} G_t = \mathbb{R}$.

(b) $\mathscr{F}_t$ is a collection of disjoint dyadic intervals such that

$$\bigcup_{I \in \mathscr{F}_t} I = \mathbb{R} \Big\backslash \bigcup_{t'=1}^{t} G_{t'}$$

(c) For any $I_{j'} \in \mathscr{F}_t$, $t \geqslant 2$, let $I_j \in \mathscr{F}_{t-1}$ be the unique dyadic interval containing $I_{j'}$. Then

$$|I_{j'}|^{1-\varepsilon}\,Av_{j'}\,|\psi_{j'}| > |I_j|^{1-\varepsilon}\,Av_{I_j}\,|\psi_j|$$

(d) $N_t(x) = \infty$ for $x \in \bigcup_{t'=1}^{t} G_{t'}$.

Otherwise $N_t(x)$ is defined by $2^{-N_t(x)} = |I|$ where I is the unique interval in $\mathscr{F}_t$ with $x \in I$.

We have defined a Calderon–Zygmund decomposition just as was done in Ref. 5. The purpose of this decomposition is to estimate the value of $g(x)$ in terms of the functions ψ_j. To see this let us first consider $x \in G_1$. Hence

$$|g(x)| \leqslant |g^2(x)| + \sum_{j=N-k+1}^{\infty} 2^{-j} Av_{I_j} |\psi_j|$$

where the I_j are the unique dyadic intervals of length 2^{-j} with $x \in I_j$. Since $x \in G_1$ we have

$$\begin{aligned} |g(x)| &\leqslant \sup_{I_{N-k}} |g^2| + \sup_{I_{N-k}} |g^2| \sum_{j=N-k+1}^{\infty} 2^{\varepsilon(N-k-j)} \\ &= \sup_{I_{N-k}} |g^2| / [1 - 2^{-\varepsilon}] \end{aligned}$$

Suppose next that $x \in G_2$ and that $x \in I_j \in \mathscr{F}_1$ where $j > N-k$. Then if $I_j \subset I_{N-k}$ we have

$$\begin{aligned} |g(x)| &\leqslant \sup_{I_{N-k}} |g^2| + \sum_{j'=N-k+1}^{j-1} 2^{-j'} Av_{I_{j'}} |\psi_{j'}| + \sum_{j'=j}^{\infty} 2^{-j'} Av_{I_{j'}} |\psi_{j'}| \\ &\leqslant \sup_{I_{N-k}} |g^2| + \sup_{I_{N-k}} |g^2| \sum_{j'=N-k+1}^{j-1} 2^{\varepsilon(N-k-j')} \\ &\quad + 2^{-j} Av_{I_j} |\psi_j| \sum_{j'=j}^{\infty} 2^{-\varepsilon(j'-j)} \\ &\leqslant [\sup_{I_{N-k}} |g^2| + 2^{-j} Av_{I_j} |\psi_j|] / [1 - 2^{-\varepsilon}] \\ &\leqslant 2^{-j} Av_{I_j} |\psi_j| \, [2^{\varepsilon(j+k-N)} + 1] / [1 - 2^{-\varepsilon}] \end{aligned}$$

More generally it is easy to see that for $t \geqslant 2$, $x \in G_t$ with $x \in I_j \in \mathscr{F}_{t-1}$ then

$$|g(x)| \leqslant 2^{-j} Av_{I_j} |\psi_j| \, 2^{\varepsilon(j+k-N)} / [1 - 2^{-\varepsilon}]^2$$

We proceed now as in the proof of Lemma 4.3. Let P be the probability that the diffusion started at $(0, 2^k r)$ hits the graph of g_0 and then exits the strip$\{(x_1, x_2) : r < x_2 < 2^{k+1} r\}$ through the boundary $x_2 = 2^{k+1} r$. Similarly let Q be the probability that the diffusion started at $(0, 2^k r)$ exits the strip $\{(x_1, x_2) : r < x_2 < 2^{k+1}r\}$ through the boundary $x_2 = 2^{k+1}r$ but fails then to hit the graph of g_{k+1} before hitting the graph of g_0. It is clear that we have as in (4.10),

$$q_k(0) \geqslant [2^k - 1]/[2^{k+1} - 1] - P - Q \tag{4.12}$$

We write P as a sum,

$$P = \sum_{m=-\infty}^{\infty} P_m$$

where P_m is the probability that the diffusion started at $(0, 2^k r)$ hits the graph of g_0 first in the interval $m2^{k-N} < x_1 < (m+1)\, 2^{k-N}$ before exiting the strip through the boundary $x_2 = 2^{k+1}r$. We can further write

$$P_m = \sum_{t=1}^{\infty} P_{m,t}$$

where $P_{m,t}$ is the probability the diffusion hits the graph of g_0 first in the set $(m2^{k-N}, (m+1)\, 2^{k-N}) \cap G_t$ before exiting through the boundary $x_2 = 2^{k+1}r$. Arguing as in Lemma 4.3 it is clear that

$$2^k r P_{m,1} \leqslant C_\varepsilon e^{-c|m|} \sup_{I(0,m,N-k)} |g^2|$$

where the constant C_ε depends only on $\varepsilon > 0$. Similarly there is a constant γ_1, $0 < \gamma_1 < 1$, such that if $t \geqslant 2$, then

$$2^k r P_{m,t} \leqslant C_\varepsilon e^{-c|m|} \sum_{I_j \in \mathscr{F}_{t-1}} (\gamma_1 2^\varepsilon)^{j+k-N}\, 2^{-j} Av_{I_j} |\psi_j|$$

We can choose now $\varepsilon > 0$ to be sufficiently small so that $\gamma_1 2^\varepsilon = \gamma < 1$. Since the families $\mathscr{F}_t$, $t \geqslant 1$, are disjoint it follows then that

$$2^k r P_m \leqslant C_\varepsilon e^{-c|m|} \left\{ \sup_{I_{N-k}} |g^2| + \sum_{j=N-k+1}^{\infty} \gamma^{j+k-N} \sum_{I_j \subset I_{N-k}} 2^{-j} Av_{I_j} |\psi_j| \right\}$$

If we use now the fact that

$$\begin{aligned} &2^{N-k} \sum_{I_j \subset I_{N-k}} 2^{-j} Av\, |\psi_j| = Av_{I_{N-k}} |\psi_j| \\ &\sup_{I_{N-k}} |g^2| = \sup_{I(0,m,N-k)} |g^2| \leqslant 2^{k-N} \sum_{j=0}^{N-k} \sum_{|m'| < |m|+1} Av_{I(0,m',N-k)} |\psi_j| \end{aligned} \tag{4.13}$$

we can conclude that

$$P \leqslant C \sum_{j=0}^{N-k} \sum_{m=-\infty}^{\infty} e^{-c\,|m|} Av_{I(0,\,m,\,N-k)}\, |\psi_j| + C \sum_{j=N-k+1}^{\infty} \gamma^{j+k-N} \sum_{m=-\infty}^{\infty} e^{-c\,|m|} Av_{I(0,\,m,\,N-k)}\, |\psi_j| \tag{4.14}$$

To estimate Q we slightly modify the Calderon–Zygmund decomposition we employed in the estimate of P. For I_j a dyadic interval, $I_j = (m2^{-j}, (m+1)\,2^{-j})$ for some $m \in \mathbb{Z}$, and $\alpha \in \mathbb{Z}$, let $I_j(\alpha)$ be the translate of I_j by a distance $\alpha 2^{-j}$. Thus $I_j(\alpha) = ((m+\alpha)\,2^{-j}, (m+1+\alpha)\,2^{-j})$. Let a, $2 > a > 1$ be a fixed number. For our new decomposition we define the function N_1 by

(a′) $N_1(x) = \infty$ if for all dyadic intervals I_j with $j > N-k$ and $x \in I_j \subset I_{N-k}$ we have

$$|I_j|^{1-\varepsilon} \sup\{Av_{I_j(\alpha)}\, |\psi_j| : |\alpha| \leqslant 2a^{j+k-N}\} \leqslant |I_{N-k}|^{1-\varepsilon} \sup\{\sup_{I_{N-k}(\alpha)} |g^2| : |\alpha| \leqslant 2\}/|I_{N-k}|$$

(b′) Otherwise $2^{-N_1(x)}$ is the length of the side of the largest dyadic interval I_j, $j > N-k$, $x \in I_j \subset I_{N-k}$ with

$$|I_j|^{1-\varepsilon} \sup\{Av_{I_j(\alpha)}\, |\psi_j| : |\alpha| \leqslant 2a^{j+k-N}\} > |I_{N-k}|^{1-\varepsilon} \sup\{\sup_{I_{N-k}(\alpha)} |g^2| : |\alpha| \leqslant 2\}/|I_{N-k}|$$

Similarly to before we can define functions N_t, $t \geqslant 2$, sets G_t and families $\mathscr{F}_t$, $t \geqslant 1$ with the properties (a), (b), (d) as before but with (c) replaced by

(c′) For any $I_{j'} \in \mathscr{F}_t$, $t \geqslant 2$, let $I_j \in \mathscr{F}_{t-1}$ be the unique dyadic interval containing $I_{j'}$. Then

$$|I_{j'}|^{1-\varepsilon} \sup\{Av_{I_{j'}(\alpha)}\, |\psi_{j'}| : |\alpha| \leqslant 2a^{j'+k-N}\} > |I_j|^{1-\varepsilon} \sup\{Av_{I_j(\alpha)}\, |\psi_j| : |\alpha| \leqslant 2a^{j+k-N}\}$$

We write Q as a sum, $Q = \sum_{m=-\infty}^{\infty} Q_m$, where Q_m is the probability that the diffusion started at $(0, 2^k r)$ exits the strip $\{(x_1, x_2) : r < x_2 < 2^{k+1} r\}$ through the boundary $x_2 = 2^{k+1} r$, $m2^{k-N} < x_1 < (m+1)\,2^{k-N}$, but fails then to hit the graph of g_{k+1} before hitting the graph of g_0. We can further decompose Q_m as

$$Q_m = \sum_{t=1}^{\infty} Q_{m,t}$$

where $Q_{m,t}$ is the probability it exits through the boundary $x_2 = 2^{k+1}r$, $\{x_1 \in G_t : m2^{k-N} < x_1 < (m+1)\, 2^{k-N}\}$ but fails to hit the graph of g_{k+1} before hitting the graph of g_0. Let $\zeta_{m,t}(x_1)$ be defined as the probability that the diffusion started at (x_1, x_2) with $x_2 = 2^{k+1}r$, $x_1 \in G_t$ and $m2^{k-N} < x_1 < (m+1)\, 2^{k-N}$, hits the graph of g_0 before hitting the graph of g_{k+1}. Evidently we have $Q_{m,t} \leqslant \exp[-c\,|m|] \sup \zeta_{m,t}$ for some constant $c > 0$ depending only on λ, Λ.

We first estimate $\zeta_{m,1}$. We shall show that

$$\sup \zeta_{m,1} \leqslant C \sup\{ \sup_{I_{N-k}(\alpha)} |g^2| : |\alpha| \leqslant 2\} / |I_{N-k}| \tag{4.15}$$

for some constant C. We can assume without loss of generality that

$$\sup\{ \sup_{I_{N-k}(\alpha)} |g^2| : |\alpha| \leqslant 2\} = 2^{-j}$$

for some $j > N-k$. Suppose now $x_1 \in G_1 \cap I_{N-k}$ and $|y_1 - x_1| < [2a^{j+k-N} - 1]\, 2^{-j}$. Then

$$\begin{aligned} |g(y_1)| &\leqslant |g^2(y_1)| + \sum_{j'=N-k+1}^{\infty} 2^{-j'} Av_{I_j} |\psi_{j'}| \\ &\leqslant 2^{-j} + \sum_{j'=N-k+1}^{j} 2^{-\varepsilon(j'+k-N)} 2^{-j} + M \sum_{j'=j+1}^{\infty} 2^{-j'} \leqslant C2^{-j} \end{aligned} \tag{4.16}$$

where M is the Lipschitz constant for g. We can use a similar argument to obtain a weaker estimate on $g(y_1)$ than in (4.16), but on a wider range of values of y_1. To see this let $a = 2^{\mu}$ where $0 < \mu < 1$. It is evident that for j' satisfying $j' > N-k$, then

$$a^{(j'+k-N)/2} 2^{-(j'+k-N)} < a^{(j''+k-N)} 2^{-(j''+k-N)}$$

provided j'' satisfies the inequality

$$j'' + k - N < (j' + k - N)(1 - \mu/2)/(1 - \mu) \tag{4.17}$$

Hence if $x_1 \in G_1 \cap I_{N-k}$ and $|y_1 - x_1| < a^{(j'+k-N)/2} 2^{-j'}$, then one also has

$$|y_1 - x_1| < a^{(j''+k-N)} 2^{-j''} < [2a^{(j''+k-N)} - 1]\, 2^{-j''}$$

provided $j' + k - N$ is sufficiently large, depending only on a. Letting j'' be the largest integer satisfying (4.17) we conclude that for

$$|y_1 - x_1| < a^{(j'+k-N)/2} 2^{-j'}$$

then

$$|g(y_1)| \leqslant 2^{-j} \sum_{i=N-k}^{j''} 2^{-\varepsilon(i+k-N)} + M \sum_{i=j''+1}^{\infty} 2^{-i} \leqslant C[2^{-j} + 2^{-j''}]$$

whence

$$\begin{aligned} &|g(y_1)| \leqslant C 2^{k-N} [2^{-(1+\delta)(j'+k-N)} + 2^{-(j+k-N)}], \\ &|y_1 - x_1| < a^{(j'+k-N)/2} 2^{-j'} \end{aligned} \tag{4.18}$$

where $\delta > 0$.

To estimate $\zeta_{m,1}(x_1)$ we consider regions $R_{j'}$, $j' \leqslant j$, defined as the points (y_1, y_2) which lie between the graph of g_{k+1} and the line $y_2 = 2^{k+1} r - 2^{-j'}$ and satisfy the inequality,

$$|y_1 - x_1| < \sum_{i=j'}^{j} (i + k - N)^2\, 2^{-i}$$

Evidently the $R_{j'}$ are an increasing set of regions with R_j being the smallest. For $j' \leqslant j$ let $\eta_{j'}$ be the probability that the diffusion started at $(x_1, 2^{k+1} r)$ exits R_i through the boundary $y_2 = 2^{k+1} r - 2^{-i}$ for $i > j'$ but exits $R_{j'}$ through the boundary

$$|y_1 - x_1| = \sum_{i=j'}^{j} (i + k - N)^2\, 2^{-i}$$

Then it is clear that

$$\zeta_{m,1}(x_1) \leqslant \sum_{j'=N-k-\ell}^{j} \eta_{j'}$$

provided ℓ is chosen so that the graph of g_0 lies above the line $y_2 = 2^{k+1} r - 2^{-(N-k-\ell)}$.

By Lemma 3.1 and (4.16) we have that $\eta_j \leqslant e^{-c(j+k-N)^2}$, for some constant $c>0$ depending only on λ, Λ, M. To estimate η'_j for $j'<j$ we shall assume that

$$\sum_{i=j''}^{j} (i+k-N)^2\, 2^{-i} < a^{(j''+k-N)/2} 2^{-j''}, \qquad j' \leqslant j'' \leqslant j$$

This inequality clearly holds if $j'+k-N$ is bounded below by a sufficiently large constant. Arguing then as in Lemma 3.2 we have from (4.18)

$$\begin{aligned}\eta_{j'} &\leqslant e^{-c(j'+k-N)^2} \prod_{i=j'+1}^{j} \{2^{-i} + C2^{k-N}[2^{-(1+\delta)(i+k-N)} + 2^{-(j+k-N)}]\}\\ &\quad /\{2^{-i+1} + C2^{k-N}[2^{-(1+\delta)(i+k-N)} + 2^{-(j+k-N)}]\}\\ &\leqslant C_1 2^{-(j-j')} e^{-c(j'+k-N)^2}\end{aligned}$$

for some constant C_1. We conclude that

$$\zeta_{m,1}(x_1) \leqslant \sum_{j'=N-k-\ell}^{j} C_1 2^{-(j-j')} e^{-c(j'+k-N)^2} \leqslant C_2 2^{-(j+k-N)} \tag{4.19}$$

which yields the inequality (4.15).

Next we consider the case of $\zeta_{m,t}$ with $t \geqslant 2$. We shall show that

$$\begin{aligned}&\zeta_{m,t}(x_1) \leqslant 2^{(\varepsilon-1)(j+k-N)} \sup\{Av_{I_j(\alpha)} |\psi_j| : |\alpha| \leqslant 2a^{j+k-N}\},\\ &x_1 \in I_j \cap G_t, \quad I_j \in \mathscr{F}_{t-1}\end{aligned} \tag{4.20}$$

First we prove the inequality in the case when

$$\sup\{Av_{I_j(\alpha)} |\psi_j| : |\alpha| \leqslant 2a^{j+k-N}\} \geqslant c_1 > 0 \tag{4.21}$$

for some constant c_1. Define an integer j_ε as the largest integer such that

$$2^{(1-\varepsilon)(j+k-N)} > 2^{j_\varepsilon+k-N}$$

Evidently we have $j > j_\varepsilon > N-k$. Suppose now y_1 satisfies the inequality $|y_1 - x_1| < [2a^{j_\varepsilon+k-N} - 1]\, 2^{-j_\varepsilon}$. Then, since $x_1 \in I_j \cap G_t$, $I_j \in \mathscr{F}_{t-1}$, we have

$$\begin{aligned}|g(y_1)| &\leqslant M \sum_{i=N-k}^{j_\varepsilon} 2^{-(1-\varepsilon)(j-i)} 2^{-i} + M \sum_{i=j_\varepsilon+1}^{\infty} 2^{-i}\\ &= M \sum_{i=N-k}^{j_\varepsilon} 2^{-\varepsilon(i+k-N)} 2^{-j_\varepsilon} + M \sum_{i=j_\varepsilon+1}^{\infty} 2^{-i}\\ &\leqslant 2M 2^{-j_\varepsilon}/[1-2^{-\varepsilon}]\end{aligned} \tag{4.22}$$

The inequality (4.22) is analogous to (4.16). We can similarly obtain an inequality analogous to (4.18). In fact for $N-k<j'<j_\varepsilon$ let j'' be defined by (4.17). Then if $|y_1-x_1|<a^{(j'+k-N)/2}2^{-j'}$ we have the inequality

$$|g(y_1)| \leqslant M \sum_{i=N-k}^{j_\varepsilon \wedge j''} 2^{-(1-\varepsilon)(j-i)} 2^{-i} + M \sum_{i=j''+1}^{\infty} 2^{-i}$$
$$\leqslant C[2^{-j_\varepsilon} + 2^{-j''}] \tag{4.23}$$

We argue now exactly as before, replacing the inequalities (4.16), (4.18) by (4.22), (4.23). The inequality corresponding to (4.19) is given by

$$\zeta_{m,t}(x_1) \leqslant C2^{-(j_\varepsilon+k-N)} \leqslant C2^{(\varepsilon-1)(j+k-N)} \tag{4.24}$$

which proves (4.20) in the case when (4.21) holds.

More generally we define j_ε as the largest integer such that

$$2^{-(j_\varepsilon+k-N)} > 2^{-(1-\varepsilon)(j+k-N)} \sup\{Av_{I_j(\alpha)} |\psi_j| : |\alpha| \leqslant 2a^{j+k-N}\} \tag{4.25}$$

It is now clearly possible to have $j_\varepsilon<j$ or $j_\varepsilon>j$. In the case when $j_\varepsilon<j$ we can proceed in a similar manner to the derivation of (4.24). In fact the analogue of the inequality (4.22) is given by

$$|g(y_1)| \leqslant \sum_{i=N-k}^{j_\varepsilon} 2^{-(1-\varepsilon)(j-i)} 2^{-i} \sup\{Av_{I_j(\alpha)} |\psi_j| : |\alpha| \leqslant 2a^{j+k-N}\}$$
$$+ M \sum_{i=j_\varepsilon+1}^{\infty} 2^{-i}$$

provided

$$|y_1-x_1| < [2a^{j_\varepsilon+k-N}-1]\, 2^{-j_\varepsilon}$$

Using the definition (4.25) this yields the estimate $|g(y_1)| \leqslant C2^{-j_\varepsilon}$ for some constant C, which is the same as (4.22), up to a constant. In a similar way we obtain (4.23). The inequality (4.24) then yields (4.20).

Consider next the case $j_\varepsilon>j$. Then

$$|g(y_1)| \leqslant \sum_{i=N-k}^{j} 2^{-(1-\varepsilon)(j-i)} 2^{-i} \sup\{Av_{I_j(\alpha)} |\psi_j| : |\alpha| \leqslant 2a^{j+k-N}\}$$
$$+ \sum_{i=j+1}^{j_\varepsilon} 2^{(1-\varepsilon)(i-j)} 2^{-i} \sup\{Av_{I_j(\alpha)} |\psi_j| : |\alpha| \leqslant 2a^{j+k-N}\}$$
$$+ M \sum_{i=j_\varepsilon+1}^{\infty} 2^{-i}$$

provided

$$|y_1 - x_1| < [2a^{j_\varepsilon + k - N} - 1]\, 2^{-j_\varepsilon}$$

It is easy to see from the definition (4.25) again of j_ε that $|g(y_1)| \leqslant C2^{-j_\varepsilon}$ for some constant C. Similarly to before, one obtains the estimate (4.23) again, whence (4.20) follows again from (4.24).

We use (4.15), (4.20) to bound $Q_{m,t}$. Thus there exists γ_1, $0 < \gamma_1 < 1$, and a constant C such that for $t \geqslant 2$,

$$Q_{m,t} \leqslant \exp[-c\,|m|]\, C \sum_{[I_j \in \mathscr{F}_{t-1},\, I_j \subset I(0,m,N-k)]} \gamma_1^{j+k-N} \times \sup\{\zeta_{m,t}(x_1) : x_1 \in G_t \cap I_j\}$$

whence (4.20) yields

$$Q_{m,t} \leqslant \exp[-c\,|m|]\, C \sum_{\{I_j \in \mathscr{F}_{t-1},\, I_j \subset I(0,m,N-k)\}} 2^{-(j+k-N)} \times (\gamma_1 2^\varepsilon)^{(j+k-N)} \sup\{Av_{I_j(\alpha)}\, |\psi_j| : |\alpha| \leqslant 2a^{j+k-N}\}$$

Evidently (4.15) yields

$$Q_{m,1} \leqslant \exp[-c\,|m|]\, C \sup\{\sup_{I_{N-k}(\alpha)} |g^2| : |\alpha| \leqslant 2\}/|I_{N-k}|$$

where $I_{N-k} = (m2^{k-N}, (m+1)\, 2^{k-N})$. Choosing $a > 1$ now to be sufficiently small so that $\gamma = \gamma_1 2^\varepsilon a < 1$, we can conclude that

$$\begin{aligned} Q &= \sum_{m=-\infty}^{\infty} \sum_{t=1}^{\infty} Q_{m,t} \\ &\leqslant C \sum_{m=-\infty}^{\infty} e^{-c\,|m|}\, 2^{N-k} \sup_{I(0,m,N-k)} |g^2| \\ &\quad + C \sum_{m=-\infty}^{\infty} e^{-c\,|m|} \sum_{j=N-k+1}^{\infty} \gamma^{j+k-N} Av_{I(0,m,N-k)}\, |\psi_j| \end{aligned} \tag{4.26}$$

for suitable constants c, $C > 0$. Using now the estimate (4.13) we see that the previous estimate on Q yields an estimate of the same form as for P in (4.14). These inequalities together with (4.12) yield the result of Proposition 4.1 in the case when $x_1 = 0$.

If g^2 is the integral of a single function ψ_j, $0 \leqslant j \leqslant N-k$, then the method of Lemma 4.4 combined with the inequalities (4.26) on Q and the corresponding inequality for P yield the result of the Proposition for arbitrary x_1. More generally we shall show that P and Q satisfy the inequality

$$\begin{aligned} P, Q \leqslant C &\sum_{m=-\infty}^{\infty} e^{-c\,|m|} \sum_{j=N-k+1}^{\infty} \gamma^{j+k-N} Av_{I(x_1,\, m,\, N-k)}\, |\psi_j| \\ &+ C \sum_{|m| \geqslant m(x_1)} e^{-c\,|m|}\, 2^{N-k} \sup_{I(x_1,\, m,\, N-k)} |g^2| \\ &+ C \sum_{m=1}^{m(x_1)} e^{-cm} \sup_{[\,|y_1-x_1| \leqslant m 2^{k-N}]} |dg^2/dy_1 - dg^2/dy_1(x_1+)| \end{aligned} \tag{4.27}$$

where $m(x_1)$ is defined as the smallest positive integer such that

$$m(x_1)\, 2^{k-N} > d(x_1, \mathbb{Z})$$

and

$$dg^2/dy_1(x_1+) = \lim_{y_1 \to x_1+} dg^2/dy_1(y_1)$$

To see that (4.27) implies the result, observe that

$$dg^2/dy_1 = \sum_{j=0}^{N-k} \psi_j(y_1)$$

whence

$$\begin{aligned} &\sup_{|y_1-x_1| \leqslant m 2^{k-N}} |dg^2/dy_1 - dg^2/dy_1(x_1+)| \\ &\quad \leqslant 2 \sum_{j=j_0+1}^{N-k} \sup_{|y_1-x_1| \leqslant m 2^{k-N}} |\psi_j(y_1)| \end{aligned}$$

where j_0 is the largest integer such that

$$d(2^{j_0} x_1, \mathbb{Z})/2^{j_0} > m 2^{k-N}$$

To prove (4.27) first consider the case when $dg^2/dy_1(x_1+) = 0$. Then the inequality follows in exactly the same way as (4.26). For the case when $dg^2/dy_1(x_1+) \neq 0$ we regard the line with slope $dg^2/dy_1(x_1+)$ as a new axis. The argument then goes through as before. QED

For the diffusion started at (x_1, x_2) on the graph of g_1, we consider only paths which hit the graph of g_k before hitting the graph of g_0, $k \geqslant 2$. Let X_{k, x_1} be the random variable defined as the first hitting point on the graph of g_k is $(y_1, g_k(y_1))$ where $y_1 = X_{k, x_1}$.

Lemma 5. There is a constant α, $0 < \alpha < 1$, depending only on λ, Λ, M such that

$$P(|X_{k, x_1} - x_1| > m 2^{k-N}) \leqslant \alpha^m, \qquad m = 0, 1, 2,...$$

Proof. Let X^n, $n = 2, 3,...$ be the random variable which has the property that $(X^n, g_n(X^n))$ is the first hitting position on the graph of g_n. Thus $X_{k, x_1} = X^k$. Letting $X^1 = x_1$ we have for any $\varepsilon > 0$, the inequality

$$E[\exp\{\varepsilon 2^{N-k} |X_{k, x_1} - x_1|\}] \leqslant E\left[\exp\left\{\varepsilon 2^{N-k} \sum_{n=2}^{k} |X^n - X^{n-1}|\right\}\right] \tag{4.28}$$

It follows from Lemma 3.1 that there is a constant β, $0 < \beta < 1$, depending only on λ, Λ, M such that

$$P(|X^k - X^{k-1}| > m 2^{k-N} \mid X^2,..., X^{k-1}) \leqslant \beta^m, \qquad m = 0, 1, 2,...$$

It follows also by the same argument that for $n = 2,..., k$

$$P_{uc}(|X^n - X^{n-1}| > m 2^{n-N} \mid X^2,..., X^{n-1}) \leqslant \beta^m, \qquad m = 0, 1, 2,...$$

where P_{uc} denotes the unconditioned probability of paths starting at $(X^{n-1}, g_{n-1}(X^{n-1}))$ which are conditioned only on hitting the graph of g_n before the graph of g_0. Thus we drop the condition that the diffusion started at $(X^n, g_n(X^n))$ must hit the graph of g_k before hitting the graph of g_0. It is clear again from Lemma 3.1 that there is a number $a > 1$, depending only on λ, Λ, M such that

$$\begin{aligned} &P(|X^n - X^{n-1}| > m 2^{n-N} \mid X^2,..., X^{n-1}) \\ &\quad \leqslant a^{k-n} P_{uc}(|X^n - X^{n-1}| > m 2^{n-N} \mid X^2,..., X^{n-1}), \qquad n = 2,..., k \end{aligned}$$

whence we have

$$\begin{aligned} &P(|X^n - X^{n-1}| > m 2^{n-N} \mid X^2,..., X^{n-1}) \\ &\quad \leqslant a^{k-n} \beta^m, \qquad n = 2,..., k, \quad m = 0, 1, 2,... \end{aligned} \tag{4.29}$$

We shall use the inequality (4.29) to bound the RHS of (4.28). Thus applying (4.29) with $n=k$ and using Bayes theorem we have

$$E\left[\exp\left\{\varepsilon 2^{N-k}\sum_{n=2}^{k}|X^n-X^{n-1}|\right\}\right]$$

$$\leqslant\left[\sum_{m=0}^{\infty}e^{\varepsilon(m+1)}\beta^m\right]E\left[\exp\left\{\varepsilon 2^{N-k}\sum_{n=2}^{k-1}|X^n-X^{n-1}|\right\}\right]$$

$$=e^{\varepsilon}[1-\beta e^{\varepsilon}]^{-1}E\left[\exp\left\{\varepsilon 2^{N-k}\sum_{n=2}^{k-1}|X^n-X^{n-1}|\right\}\right]$$

provided $\varepsilon>0$ is chosen sufficiently small so that $\beta e^{\varepsilon}=\delta<1$. For $2\leqslant n<k$, we have from (4.29),

$$E[\exp\{\varepsilon 2^{N-k}|X^n-X^{n-1}|\}\mid X^2,..., X^{n-1}]$$

$$\leqslant\exp[\varepsilon 2^{n-k}(m_0+1)]+\sum_{m=m_0}^{\infty}\exp[\varepsilon 2^{n-k}(m+1)]\,a^{k-n}\beta^m$$

for any integer $m_0\geqslant 0$. Hence

$$E[\exp\{\varepsilon 2^{N-k}|X^n-X^{n-1}|\}\mid X^2,..., X^{n-1}]$$

$$\leqslant\exp[\varepsilon 2^{n-k}(m_0+1)]\{1+a^{k-n}\beta^{m_0}/[1-\beta\exp[\varepsilon 2^{n-k}]]\}$$

Choosing $m_0=(n-k)^2$ we conclude that there is a constant C_1 such that

$$E[\exp\{\varepsilon 2^{N-k}|X^n-X^{n-1}|\}\mid X^2,..., X^{n-1}]$$

$$\leqslant 1+C_1 2^{(n-k)/2},\qquad n=2,..., k$$

It follows now from (4.28) that

$$E[\exp\{\varepsilon 2^{N-k}|X_{k,x_1}-x_1|\}]\leqslant\prod_{n=2}^{k}[1+C_1 2^{(n-k)/2}]\leqslant C_2$$

for some constant C_2. QED

Proof of Theorem 4.1. We define conditional probability measures $d\mu^k_{y_{k-1}}(y_k)$ for $k=2, 3,...$. Thus we consider the diffusion starting at the point $(y_{k-1}, g_{k-1}(y_{k-1}))$ on the graph of g_{k-1} and consider only paths which hit the graph of g_k before hitting the graph of g_0. Then for any open set $O \subset \mathbb{R}$,

$$\int_0 d\mu^k_{y_{k-1}}(y_k) = \text{probability the diffusion first hits the graph of } g_k \text{ at points } (z, g_k(z)) \text{ with } z \in O$$

It is clear now from the definition of the probabilities p_N and q_k, $k = 1, 2,...$ that

$$p_N(y_1) = \int_{\mathbb{R}^{N-1}} q_1(y_1)\, d\mu^2_{y_1}(y_2)\, q_2(y_2) \times d\mu^3_{y_2}(y_3) \cdots q_{N-1}(y_{N-1})\, d\mu^N_{y_{N-1}}(y_N) \tag{4.30}$$

Let a_k, $k = 1,..., N-1$ be functions such that

$$q_k(y_k) \geqslant \tfrac{1}{2} \exp[-a_k(y_k)], \qquad y_k \in \mathbb{R}$$

Then, upon using Jensen's inequality in (4.30), we have

$$2^{N-1} p_N(y_1) \geqslant \exp\Big[-a_1(y_1) - \int_{\mathbb{R}} a_2(y_2)\, d\mu^2_{y_1}(y_2) - \int_{\mathbb{R}^2} a_3(y_3)\, d\mu^2_{y_1}(y_2)\, d\mu^3_{y_2}(y_3) \cdots - \int_{\mathbb{R}^{N-1}} a_{N-1}(y_{N-1})\, d\mu^2_{y_1}(y_2)\, d\mu^3_{y_2}(y_3) \cdots d\mu^{N-1}_{y_{N-2}}(y_{N-1}) \Big]$$

Using the random variables X_{k, x_1}, $k = 2, 3,...$ of Lemma 4.5 this last inequality can be written as

$$2^{N-1} p_N(x_1) \geqslant \exp[-a_1(x_1) - E[a_2(X_{2, x_1})] - E[a_3(X_{3, x_1})] \cdots - E[a_{N-1}(X_{N-1, x_1})]]$$

Using Jensen's inequality again we have that

$$2^{N-1} \int_0^1 p_N(x_1)\, dx_1 \geqslant \exp\Big[-\int_0^1 a_1(x_1)\, dx_1 - \int_0^1 E[a_2(X_{2, x_1})]\, dx_1 \cdots - \int_0^1 E[a_{N-1}(X_{N-1, x_1})]\, dx_1 \Big] \tag{4.31}$$

From Proposition 4.1 we see that the functions a_k are bounded by

$$a_k(z) \leqslant C2^{-k} + C \sum_{j=0}^{N-k} \sum_{m=-\infty}^{\infty} \exp[-c\{|m| + 2^{N-k-j} d(2^j z, \mathbb{Z})\}]$$

$$\times Av_{I(z,m,N-k)} |\psi_j| + C \sum_{j=N-k+1}^{\infty} \gamma^{j+k-N}$$

$$\times \sum_{m=-\infty}^{\infty} e^{-c|m|} Av_{I(z,m,N-k)} |\psi_j|, \qquad z \in \mathbb{R}$$

In view of Lemma 4.5 we have from the previous inequality

$$E[a_k(X_{k,x_1})] \leqslant C2^{-k} + C \sum_{j=0}^{N-k} \sum_{m=-\infty}^{\infty} \exp[-c\{|m| + 2^{N-k-j} d(2^j x_1, \mathbb{Z})\}]$$

$$\times Av_{I(x_1,m,N-k)} |\psi_j| + C \sum_{j=N-k+1}^{\infty} \gamma^{j+k-N}$$

$$\times \sum_{m=-\infty}^{\infty} e^{-c|m|} Av_{I(x_1,m,N-k)} |\psi_j| \tag{4.32}$$

for suitable constants C, c depending only on λ, Λ, M.

Observe now that

$$\int_0^1 dx_1\, Av_{I(x_1,m,N-k)} |\psi_j| = 2^{N-k} \int_0^1 dx_1 \int_0^{2^{k-N}} dy\, |\psi_j|\, (x_1 + m2^{k-N} + y) \leqslant \int_0^2 dz\, |\psi_j|\, (z + m2^{k-N}) \tag{4.33}$$

Next we wish to bound

$$\int_0^1 dx_1 \exp[-c2^{N-k-j} d(2^j x_1, \mathbb{Z})]\, Av_{I(x_1,m,N-k)} |\psi_j| \tag{4.34}$$

where $0 \leqslant j \leqslant N-k$. Let n be the unique integer such that $n2^{-j} \leqslant m2^{k-N} < (n+1)\,2^{-j}$. Then, in view of the fact that ψ_j is constant on intervals $(n'2^{-j}, (n'+1)\,2^{-j})$, $n' \in \mathbb{Z}$, we have that the expression (4.34) is bounded by

$$\sum_{n'=n}^{n+2^j+1} 2^j \int_{n'2^{-j}}^{(n'+1)\,2^{-j}} |\psi_j(z)|\, dz \int_{U_{n'}} 2^{N-k} \exp[-c2^{N-k-j} d(2^j x_1, \mathbb{Z})]\, dx_1\, dy$$

where $U_{n'}$ is the set,

$$U_{n'} = \{0 < x_1 < 1, 0 < y < 2^{k-N}, n' 2^{-j} < x_1 + y + m2^{k-N} < (n'+1) 2^{-j}\}$$

It is easy to see that there is a constant C_1, depending only on c such that

$$\int_{U_{n'}} 2^{N-k} \exp[-c 2^{N-k-j} d(2^j x_1, \mathbb{Z})] \, dx_1 \, dy \leqslant C_1 2^{k-N}$$

We conclude therefore that (4.34) is bounded by

$$C_1 2^{k-N+j} \int_{n2^{-j}}^{n2^{-j}+1+2^{-j+1}} |\psi_j(z)| \, dz$$

$$\leqslant C_1 2^{k-N+j} \int_{-3}^{3} dz \, |\psi_j| \, (z + m2^{k-N})$$

It follows now from this last inequality, (4.32) and (4.33) that

$$\int_0^1 dx_1 \, E[a_k(X_{k, x_1})]$$

$$\leqslant C_2 2^{-k} + C_2 \sum_{j=0}^{N-k} 2^{k-N+j} \sum_{m=-\infty}^{\infty} e^{-c|m|} \int_{-3}^{3} dz \, |\psi_j| \, (z + m2^{k-N})$$

$$+ C_2 \sum_{j=N-k+1}^{\infty} \gamma^{j+k-N} \sum_{m=-\infty}^{\infty} e^{-c|m|} \int_{-3}^{3} dz \, |\psi_j| \, (z + m2^{k-N})$$

for some constant C_2 depending only on λ, Λ, M. Hence from (4.31)

$$-\ell n \left[2^{N-1} \int_0^1 p_N(x_1) \, dx_1 \right]$$

$$\leqslant C_3 + C_3 \sum_{j=0}^{N} \sum_{m=-\infty}^{\infty} e^{-c|m|} \int_{-3}^{3} dz \, |\psi_j| \, (z + m2^{k-N})$$

$$+ C_3 \sum_{j=N+1}^{\infty} \gamma^{j-N} \sum_{m=-\infty}^{\infty} e^{-c|m|} \int_{-3}^{3} dz \, |\psi_j| \, (z + m2^{k-N}) \tag{4.35}$$

where $C_3 > 0$, γ satisfying $0 < \gamma < 1$ depend only on λ, Λ, M. We have now from the Schwarz inequality that

$$\sum_{j=0}^{N} \int_{-3}^{3} dz \, |\psi_j| \, (z + m2^{k-N}) \leqslant \sqrt{6N} \left\{ \sum_{j=0}^{N} \int_{-3}^{3} dz \, |\psi_j|^2 \, (z + m2^{k-N}) \right\}^{1/2}$$

Let $n \in \mathbb{Z}$ be the unique integer such that $n \leqslant m2^{k-N} < n+1$. Then

$$\sum_{j=0}^{N} \int_{-3}^{3} dz \, |\psi_j|^2 \, (z + m2^{k-N}) \leqslant \sum_{j=0}^{N} \int_{n-3}^{n+4} dz \, |\psi_j|^2 \, (z)$$
$$\leqslant \int_{n-3}^{n+4} dz \; g'(z)^2 \leqslant 7M^2$$

where we have used the orthogonality of the representation (4.1). Using also the fact that $\|\psi_j\|_\infty \leqslant M$ we conclude from (4.35) that

$$-\ell n \left[2^{N-1} \int_0^1 p_N(x_1) \, dx_1 \right]$$
$$\leqslant C_3 + C_3 M \sqrt{42N} \sum_{m=-\infty}^{\infty} e^{-c\,|m|} + \frac{6C_3 M_\gamma}{1-\gamma} \sum_{m=-\infty}^{\infty} e^{-c\,|m|}$$
$$\leqslant C_4 \sqrt{N}$$

where the constant C_4 depends only on λ, Λ, M. QED

5. RANDOM WALK ON A LINE

In Section 2 we saw that the expected time the diffusion process generated by uniformly elliptic operator L spends in a neighborhood of a line is comparable to the corresponding Brownian time. In this section we shall see that this comparison goes beyond the expected time. Let (x, y) denote co-ordinates in the plane $\mathbb{R}^2$, and $O \subset \mathbb{R}$ be an arbitrary open set. We define a kernel $k(x, x')$, $x, x' \in \mathbb{R}$ by

$$\int_O k(x, x') \, dx' = \text{probability that the diffusion started at } (x, 0) \text{ first hits the line } y = 0 \text{ in the set } \{(z, 0) : z \in O\} \text{ after hitting the line } y = 1 \tag{5.1}$$

The kernel k induces a random walk on the line $y = 0$ with transition probability given by k. Here we shall study this random walk for general uniformly elliptic L and compare it to the corresponding walk when $L \equiv \Delta$.

It is possible to explicitly compute k in the case $L \equiv \Delta$. In fact k is given by the well known formula for the Cauchy distribution,

$$k(x, x') = \frac{2}{\pi} \frac{1}{(x - x')^2 + 4}, \qquad x, x' \in \mathbb{R}$$

We denote the position of the walk by random variables $Y(0)$, $Y(1)$, $Y(2)$,..., in $\mathbb{R}$, where $(Y(n), 0)$ is the position of the walk on the nth hit of the x axis. The kernel $k(x, x')$ is the probability density function for the random variable $Y(1)$ conditioned on $Y(0)=x$. Let $k_n(x, x')$ be the probability density function for $Y(n)$ conditioned on $Y(0)=x$. Then in the case when $L \equiv \Delta$, k_n is given by the formula,

$$k(x, x') = \frac{2}{\pi} \frac{n}{(x-x')^2+4n^2}, \qquad x, x' \in \mathbb{R}$$

It is clear that we have the following:

Lemma 1. Suppose $L \equiv \Delta$ and m, n are nonnegative integers with $n \geqslant 1$. Then

(a) There exists universal constants C, $c>0$ such that

$$c/2^m \leqslant P_x(n[2^m-1] \leqslant |Y(n)-x| < n[2^{m+1}-1]) \leqslant C/2^m$$

(b) For $p>1$ there exists a constant C_p depending only on p such that

$$\int_{|x-x'|<n[2^{m+1}-1]} k_n(x, x')^p \, dx' \leqslant C_p/n^{p-1} 2^{m(2p-1)}$$

The inequality (a) tells us that the walk is spread out on a length scale n at time n. The inequality (b) tells us that the fluctuations of the walk on length scales smaller than n cannot be too large. We shall see in the following that Lemma 5.1(a) generalizes to uniformly elliptic L but Lemma 5.1(b) does not.

First we consider the analogue of Lemma 5.1(a). We have the following:

Theorem 1. Suppose L is uniformly elliptic and m, n are nonnegative integers with $n \geqslant 1$. Then there exists constants C, $c>0$ depending only on λ, Λ such that

$$c/2^m \leqslant P_x(n[2^m-1] \leqslant |Y(n)-x| < n[2^{m+1}-1]) \leqslant C/2^m$$

We prove Theorem 5.1 in a number of steps. First we consider the case when $n=1$.

Lemma 2. There exist constants c, C depending only on λ, Λ, such that for $m \geqslant 0$ an integer,

$$c/2^m \leqslant P_x(2^m - 1 \leqslant |Y(1) - x| < 2^{m+1} - 1) \leqslant C/2^m$$

Proof. We first prove the lower bound. Let q_x be the probability that the diffusion started at $(x, 0)$ first hits the line $y = 1$ in the set $\{(z, 1) : |z - x| < 1\}$ and then subsequently hits the lines $y = 2^k$, $k = 1,..., m$, in the sets

$$\left\{(z, 2^k) : |z - x| < \sum_{j=1}^{k} (m + 1 - j)^2\, 2^j\right\}$$

before hitting the axis $y = 0$ again. It follows from Lemma 3.1 and Lemma 3.2 that there is a constant $c_1 > 0$ depending only on λ, Λ such that $q_x \geqslant c_1/2^m$. Next let ξ be the infimum of the probabilities that the diffusion started at $(z, 2^m)$ with

$$|z - x| < \sum_{j=1}^{m} (m + 1 - j)^2\, 2^j \tag{5.2}$$

first hits the axis $y = 0$ in the set $\{(z, 0) : 2^m - 1 \leqslant |z - x| < 2^{m+1} - 1\}$. Since the sum on the right in (5.2) is bounded by a constant times 2^m it follows from Lemma 3.1 that there is a constant $c_2 > 0$ depending only on λ, Λ such that $\xi \geqslant c_2$. Hence

$$P_x(2^m - 1 \leqslant |Y(1) - x| < 2^{m+1} - 1) \geqslant \xi q_x \geqslant c_2 c_1/2^m$$

We use Lemma 4.1 to obtain the upper bound. For the diffusion started at $(x, 0)$ let $(Z, 1)$ be the first hitting position on the line $y = 1$. Then

$$\begin{aligned} &P_x(2^m - 1 \leqslant |Y(1) - x| < 2^{m+1} - 1) \\ &\qquad \leqslant P_x(|Z - x| > [2^m - 1]/2) + P_x(|Y(1) - Z| > [2^m - 1]/2) \end{aligned} \tag{5.3}$$

Let W be the wedge with vertex at the point $(x - [2^m - 1]/2, 1)$ and with boundaries given by the two line segments $L_1 = \{(z, 1) : z > x - [2^m - 1]/2\}$, $L_2 = \{(x - [2^m - 1]/2, z) : z \leqslant 1\}$. Similarly let W' be the wedge with vertex at the point $(x + [2^m - 1]/2, 1)$ and with boundaries given by the two line segments $L_1' = \{(z, 1) : z < x + [2^m - 1]/2\}$, $L_2' = \{(x + [2^m - 1]/2, z) : z \leqslant 1\}$. Let $p(W)$ be the probability that the diffusion started at $(x, 0)$ exits W through the boundary L_2. Similarly let $p(W')$ be the probability that the diffusion started at $(x, 0)$ exits W' through the boundary L_2'. Then we have

$$P_x(|Z-x| > [2^m - 1]/2) \leqslant p(W) + p(W')$$

Now by Lemma 4.1 it follows that there is a constant C_1 depending only on λ, Λ such that $p(W)$, $p(W') \leqslant C_1/2^m$. Hence the first term on the RHS of (5.3) is bounded by $2C_1/2^m$. Since we can make an exactly similar argument for the second term on the RHS of (5.3) we have completed the proof of the upper bound. QED

Next we consider the case of Theorem 5.1 when $m = 0$.

Lemma 3. Let $\alpha > 0$. Then there is a constant c depending only on α, λ, Λ, such that

$$P_x(\sup_{0 \leqslant j \leqslant n} |Y(j) - x| < \alpha n) \geqslant c > 0$$

Proof. We first show that by picking α large enough, depending only on λ, Λ, c can be taken to be 1/4. To see this let q be the probability that the diffusion started at $(x, 0)$ hits one of the lines $y = \pm 4n$ before the random walk Y hits the axis $y = 0$ for the nth time. Now, for the diffusion started on the line $y = 1$, the probability it hits the line $y = 4n$ before hitting the axis $y = 0$ is $1/4n$. Hence $q \leqslant 2n/4n = 1/2$. Thus the diffusion started at $(x, 0)$ stays within the strip $|y| < 4n$ up to the nth hit of Y with probability at least 1/2. Observe next from Lemma 3.1 that there exists $\alpha > 0$, depending only on λ, Λ, such that the diffusion started at $(x, 0)$ exits the rectangle $\{(z, y) : |z - x| < \alpha n, |y| < 4n\}$ through the boundaries $|z - x| = \alpha n$ with probability less than 1/4. We conclude then that

$$P_x(\sup_{0 \leqslant j \leqslant n} |Y(j) - x| < \alpha n) \geqslant \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}$$

We obtain the result of the lemma by modifying the above argument. Let $R_{\alpha, \beta}$ be the rectangle $\{(z, y) : |z - x| < \alpha n, |y| < \beta n\}$. Suppose the diffusion starts at a point $(x', 0)$ with $|x - x'| < \alpha n/2$. Then, by Lemma 3.1, we can choose β sufficiently small, depending only on α, λ, Λ, so that the probability the diffusion exits $R_{\alpha, \beta}$ through the boundaries $|z - x| = \alpha n$ is less than 1/4. Next we choose $\varepsilon > 0$, depending only on β, λ, Λ, such that the probability the diffusion started at $(x', 0)$ with $|x - x'| < \alpha n/2$ stays within the strip $|y| < \beta n$ for the first εn steps of the walk Y exceeds 1/2. We conclude therefore that

$$\begin{aligned}&\text{Prob[diffusion started at } (x', 0) \text{ with } |x-x'| < \alpha n/2, \text{ exits } R_{\alpha,\beta} \text{ through}\\&\quad \text{the boundaries } |y| = \beta n \text{ after at least } \varepsilon n \text{ steps of the walk } Y]\\&\geqslant \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}\end{aligned} \tag{5.4}$$

We can assume that $\varepsilon < 1$ with $1/\varepsilon$ an integer. Let q be defined by

$$\begin{aligned}q = \inf\{&\text{probability that the diffusion started at a point } (z, y)\\&\text{with } |z-x| < \alpha n, |y| = \beta n, \text{ first hits the axis } y=0 \text{ in the segment}\\&(x', 0), |x-x'| < \alpha n/2\}\end{aligned}$$

It follows from Lemma 3.1 that $q > 0$ and depends only on α, β, λ, Λ. In view of (5.4) one has

$$P_x\big(\sup_{0 \leqslant j \leqslant n} |Y(j) - x| < \alpha n\big) \geqslant (q/4)^{1/\varepsilon}$$

We have proved the lemma with $c = (q/4)^{1/\varepsilon}$. QED

The upper bound in Theorem 5.1 in the general case $n \geqslant 1$, $m \geqslant 0$ follows from:

Lemma 4. There is a constant C depending only on λ, Λ such that

$$\begin{aligned}&\text{Prob(diffusion started at } (x, 0) \text{ crosses one of lines } (x \pm n2^m, y), y \in \mathbb{R},\\&\quad \text{before the } n\text{th step of the walk } Y) \leqslant C/2^m\end{aligned} \tag{5.5}$$

Proof. We first show how to obtain a slightly weaker inequality, namely with the RHS of (5.5) replaced by $Cm/2^m$. To see this let R_α be the rectangle $R_\alpha = \{(z, y) : |z-x| < n2^m, |y| < \alpha n 2^m/m\}$. By Lemma 3.1 we can choose $\alpha > 0$ depending only on λ, Λ such that

$$\begin{aligned}&\text{Prob[diffusion started at } (x, 0) \text{ exits } R_\alpha \text{ through the boundaries}\\&\quad |z-x| = n2^m] \leqslant 1/2^{2m}\end{aligned} \tag{5.6}$$

We also have just as in Lemma 5.3,

$$\begin{aligned}&\text{Prob[diffusion started at } (x, 0) \text{ hits one of the lines } y = \pm \alpha n 2^m/m\\&\quad \text{before the } n\text{th step of the walk } Y] \leqslant 2n[m/\alpha n 2^m] = 2m/\alpha 2^m\end{aligned} \tag{5.7}$$

We conclude then from the previous two inequalities that

$$\begin{aligned}&\text{Prob[diffusion started at } (x,0) \text{ exits } R_\alpha\\&\qquad \text{before the } n\text{th step of the walk } Y] \leqslant Cm/2^m\end{aligned}$$

where C depends only on α, λ, Λ. This last inequality evidently implies the weaker version of (5.5).

To establish (5.5) we combine this with the estimate for the $n=1$ case given in Lemma 5.2. Thus if $j\geqslant 1$ and $|Y(j-1)-x|<n2^m$ then by the argument of Lemma 5.2 we have

$$\begin{aligned}&\text{Prob(diffusion started at } (Y(j-1),0) \text{ crosses one of the lines}\\&\qquad (x\pm n2^m, y),\ y\in\mathbb{R}, \text{ before the } j\text{th step of the walk } Y)\\&\quad \leqslant C/[n2^m+1-|Y(j-1)-x|]\end{aligned}$$

where C is a constant depending only on λ, Λ. Hence

$$\begin{aligned}&\text{Prob(diffusion started at } (x,0) \text{ crosses one of lines } (x\pm n2^m, y),\ y\in\mathbb{R},\\&\qquad \text{before the } n\text{th step of the walk } Y)\\&\quad \leqslant \sum_{j=1}^{n} CE[(n2^m+1-|Y(j-1)-x|)^{-1};\ |Y(j-1)-x|<n2^m] \qquad (5.8)\end{aligned}$$

Let k be an integer, $1\leqslant k<m$. We shall use the weak version of (5.5) we have already proved to show that

$$\begin{aligned}&P_x[n(2^{k-1}-1)\leqslant ||Y(n)-x|-n2^m|<n(2^k-1)]\\&\qquad \leqslant Cm^{1+\eta}/2^{m+(m-k)\eta} \qquad (5.9)\end{aligned}$$

where the constants C, $\eta>0$ depend only on λ, Λ. From this last inequality it follows that

$$\begin{aligned}&E[(n2^m+1-|Y(j-1)-x|)^{-1};\ |Y(j-1)-x|<n2^m]\\&\qquad \leqslant C_1\sum_{k=1}^{m+r}\frac{(m+r)^{1+\eta}}{2^{(m+r)+(m+r-k)\eta}}\,\frac{1}{(j-1)\,2^k}+\frac{C_2}{n2^m}\end{aligned}$$

on writing $n=(j-1)\,2^r$. It is clear the sum on the RHS of the last inequality is bounded by $C_3/n2^m$ for some constant C_3 depending only on λ, Λ. The result of the lemma follows from this and (5.8).

We are left to prove (5.9). Let R_α be the rectangle defined previously with n replaced by $n/2$, such that (5.6) holds. For integer k' satisfying

$1 \leqslant k' < m$, let $q_{k'}$ be the supremum of the probabilities that the diffusion started on one of the lines $y = \pm \alpha n 2^m/m$ first hits the axis $y=0$ in the set $\{(z, 0) : n(2^{k'-1} - 1) \leqslant ||z-x| - n2^m| < n(2^{k'} - 1)\}$. From Lemma 4.2 it follows that there is a constant $C_1 > 0$ depending only on λ, Λ, such that

$$q_{k'} \leqslant C_1 [m/2^{m-k'}]^\eta \tag{5.10}$$

for some $\eta > 0$ depending only on λ, Λ. We also have from the weak version of (5.5) that for any $n^* < n$, $1 \leqslant k < k' < m$, the conditional probability,

$$\begin{aligned} &\mathrm{Prob}[n(2^{k-1}-1) \leqslant ||Y(n) - x| - n2^m| < n(2^k - 1) \mid \\ &\quad n(2^{k'-1} - 1) \leqslant ||Y(n^*) - x| - n2^m| < n(2^{k'} - 1)] \leqslant C_2 k'/2^{k'} \end{aligned} \tag{5.11}$$

Let q_m be the supremum of the probabilities that the diffusion started on one of the lines $y = \pm \alpha n 2^m/m$ first hits the axis $y=0$ in the set $\{(z, 0) : ||z-x| - n2^m| \geqslant n(2^{m-1} - 1)\}$. It is clear now from (5.6), (5.7), (5.11) that

$$\begin{aligned} &P_x[n(2^{k-1}-1) \leqslant ||Y(n) - x| - n2^m| < n(2^k - 1)] \\ &\quad \leqslant 1/2^{2m} + \frac{2m}{\alpha 2^m} \left[q_k + \sum_{k'=k+1}^{m} C_2 q_{k'} k'/2^{k'} \right] \end{aligned}$$

Now, using (5.10) in the above inequality and the obvious fact that $q_m \leqslant 1$, we obtain (5.9). QED

Proof of Theorem 5.1. We have already observed that the upper bound is a direct consequence of Lemma 5.4. To get the lower bound observe that

$$\begin{aligned} &\mathrm{Prob}[\text{diffusion started at } (x, 0) \text{ hits the line } y = n2^m \\ &\qquad \text{before the } n\text{th step of the walk } Y] \\ &\quad \geqslant 1 - \left(1 - \frac{1}{n2^m}\right)^n \geqslant \frac{c_1}{2^m} \end{aligned}$$

for some universal constant $c_1 > 0$. In view of Lemma 5.4 it follows that there is a constant $\gamma > 0$, depending only on λ, Λ such that

$$\begin{aligned} &\mathrm{Prob}(\text{diffusion started at } (x, 0) \text{ hits the line } y = n2^m \text{ in the segment} \\ &\qquad (z, n2^m),\ |z-x| < \gamma n 2^m, \text{ before the } n\text{th step of the walk } Y) \\ &\quad \geqslant c_1/2^{m+1} \end{aligned}$$

Now by Lemma 3.1 there exists $c_2>0$ depending only on λ, Λ such that

$$\begin{aligned}&\text{Prob(diffusion started at a point } (x,0) \text{ in the line segment } (z, n2^m),\\ &\quad |z-x|<\gamma n2^m, \text{ first hits the axis } y=0 \text{ in the line segment}(z,0),\\ &\quad n[2^m-1]+n2^m/5<z-x<n[2^m-1]+3n2^m/5]\geqslant c_2\end{aligned}$$

We conclude from these last two inequalities that

$$\begin{aligned}&P_x(n[2^m-1]\leqslant |Y(n)-x|<n[2^{m+1}-1])\\ &\quad\geqslant \frac{c_1c_2}{2^{m+1}}\inf\{P_y(\sup_{0\leqslant j\leqslant n}|Y(j)-y|<n/5):y\in\mathbb{R}\}\end{aligned}$$

where P_y denotes the probability given $Y(0)=y$. The lower bound follows now from Lemma 5.3. QED

Next we show that Lemma 5.1(b) does not hold for general uniformly elliptic L.

Lemma 5. Suppose $p>1$. Then there exists uniformly elliptic L such that the transition probability $k(x,x')$ defined by (5.1) satisfies

$$\int_{-1}^{1} k(0,x')^p\,dx'=\infty$$

Proof. We take L to be the operator L_ε as defined by (2.6). Let (r,θ) be the polar co-ordinates for a point (x,y) in the plan. Thus $x=r\cos\theta$, $y=r\sin\theta$. Then from (2.6) it follows that for a function u of r, θ one has

$$L_\varepsilon u=(1+\varepsilon)\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} \tag{5.12}$$

If we put $u(r,\theta)=r^{-\alpha}\sin\theta$ with $(1+\varepsilon)\alpha=1$ it easily follows that $L_\varepsilon u\equiv 0$. Let S_δ be the curve consisting of points $(x,0)$ with $|x|\geqslant\delta$ and points with polar co-ordinates (r,θ) satisfying $r=\delta$, $0\leqslant\theta\leqslant\pi$. Now for any point with polar co-ordinates (r,θ), $r>\delta$, $0<\theta<\pi$, let $q_\delta(r,\theta)$ be the probability that the diffusion started at this point first hits the set S_δ in the semi-circle $r=\delta$, $0\leqslant\theta\leqslant\pi$. It follows from the maximum principle that

$$q_\delta(r,\theta)\geqslant\left(\frac{\delta}{r}\right)^\alpha\sin\theta$$

Now for z satisfying $|z|<1$, let $p_\delta(z)$ be the probability that the diffusion started at the point $(z, 1)$ first hits the axis $y=0$ in the set $(x, 0)$ with $|x|<\delta$. From Lemma 3.1 there is a constant $c_2>0$, depending only on ε such that

$$p_\delta(z) \geqslant c_2 q_{\delta/2}(r, \theta)$$

where (r, θ) are the polar co-ordinates of the point $(z, 1)$. Hence there is a constant $c_3>0$ depending only on ε such that

$$p_\delta(z) \geqslant c_3 \delta^\alpha, \qquad |z| \leqslant 1$$

Using Lemma 3.1 again, it is easy to see that there is a constant $c_4>0$ depending only on ε such that

$$\int_{-\delta}^{\delta} k(0, x')\, dx' \geqslant c_4 \inf_{|z| \leqslant 1} p_\delta(z)$$

Combining these last two inequalities with Holder's inequality we obtain the lower bound,

$$\left[\int_1^1 k(0, x')^p\, dx' \right]^{1/p} \geqslant \frac{c_3 c_4}{2^{1-1/p}} \delta^{\alpha + 1/p - 1}$$

Now we choose ε sufficiently large so that $\alpha = (1+\varepsilon)^{-1}$ satisfies $\alpha + 1/p - 1 < 0$. The result follows by letting $\delta \to 0$ in the last inequality. QED

In the previous lemma the operator L was chosen depending on p. This is in fact not necessary.

Proposition 1. There exists uniformly elliptic L such that the transition probability defined by (5.1) satisfies

$$\int_1^1 k(0, x')^p\, dx' = \infty, \qquad p>1$$

We prove proposition 5.1 in a sequence of lemmas. First we generalize Lemma 5.5. For integer n, m with $n \geqslant 0$, $m \geqslant 1$ let $E_{m,n}$ be the set

$$E_{m,n} = \bigcup \{ (k2^{-n} - 2^{-m-n}, k2^{-n} + 2^{-m-n}) : k \in \mathbb{Z} \} \tag{5.13}$$

It is clear that the measure of $E_{m,n} \cap (-1, 1)$ is 2^{-m+2}. Let α satisfy $0 < \alpha < 1$. In Lemma 5.5 we constructed a uniformly elliptic operator L_0 and a positive constant c_α, depending only on α, such that

$$\int_{(-1,1) \cap E_{m,0}} k(0, x')\, dx' \geqslant c_\alpha 2^{-m\alpha}, \qquad m \geqslant 1$$

Lemma 6. Let α satisfy $0 < \alpha < 1$. For $n \geqslant 0$ there exists uniformly elliptic operators L_n, with uniform ellipticity bounds λ, Λ depending only on α, and constants C_α, c_α depending only on α, such that

$$C_\alpha 2^{-m\alpha} \geqslant \int_{(-1,1) \cap E_{m,n}} k(0, x')\, dx' \geqslant c_\alpha 2^{-m\alpha}, \qquad m \geqslant 1$$

where k is the probability density (5.1) associated with the elliptic operator L_n.

Proof. Our definition of L_n is a generalization of the definition of L_0. Suppose the point $(x, y) \in \mathbb{R}^2$ is a distance less than 2^{-n-1} from the point $(k2^{-n}, 0)$ where $k \in \mathbb{Z}$. Then we define L_n at (x, y) by the operator L_ε of (5.12), where now the polar co-ordinates (r, θ) are centered at the point $(k2^{-n}, 0)$. If (x, y) is further than 2^{-n-1} from all points $(k2^{-n}, 0)$, $k \in \mathbb{Z}$, we define L_n at (x, y) to be simply the Laplacian.

For $k \in \mathbb{Z}$ let Ω_k be the set of points with polar co-ordinates (r, θ) centered at $(k2^{-n}, 0)$ satisfying $2^{-m-n-1} < r < 2^{-n-1}$, $0 < \theta < \pi$. Consider the function $u(r, \theta)$ on Ω_k defined by

$$u(r, \theta) = (2^{-m-n-1}/r)^\alpha \sin\theta - 2^{-m\alpha}(2^{n+1} r)^{1-\alpha} \tag{5.14}$$

Since $(1+\varepsilon)\,\alpha = 1$ in (5.12) it follows that $L_n u \equiv 0$. It is easy to check further that $u \leqslant 0$ on $\{(r, \theta) : r = 2^{-n-1}, 0 < \theta < \pi\}$, and $u \leqslant 1$ on $\{(r, \theta) : r = 2^{-m-n-1}, 0 < \theta < \pi\}$. We also have

$$u(r, \theta) \geqslant 2^{-(m-1)\alpha}(\sqrt{2} - 1)/2, \qquad r = 2^{-n-2}, \quad \frac{\pi}{4} < \theta < \frac{3\pi}{4} \tag{5.15}$$

Suppose now the diffusion starts at $(0, 0)$. By Lemma 3.1 it hits the line $y = 1$ in the interval $-1 < x < 1$ with probability larger than $c_{1,\alpha}$ where $c_{1,\alpha} > 0$ depends only on α. Next suppose the diffusion starts at a point $(x, 1)$ with $-1 < x < 1$. Again by Lemma 3.1 it first hits the line $y = 2^{-n-1}$ in the interval $-1/2 < x < 1/2$ with probability larger than $c_{2,\alpha}$, where $c_{2,\alpha} > 0$ depends only on α. Finally suppose the diffusion starts at a point

$(x, 2^{-n-1})$ with $-1/2 < x < 1/2$. Lemma 3.1 implies that it hits one of the circles centered at points $(k2^{-n}, 0)$, radius 2^{-n-2}, $k \in \mathbb{Z}$, $|k| < 2^n$, in the arc $\pi/4 < \theta < 3\pi/4$ before hitting the line $y = 0$ with probability larger than $c_{3,\alpha}$, where $c_{3,\alpha} > 0$ depends only on α. We have therefore shown that there is a constant $c_\alpha > 0$ depending only on α such that

$$\begin{aligned}&\text{Prob}[\text{diffusion started at } (0, 0)\text{, after hitting the line } y = 1\text{, hits one of}\\ &\quad \text{the circles centered at points } (k2^{-n}, 0)\text{, radius } 2^{-n-2},\ k \in \mathbb{Z},\\ &\quad |k| < 2^n\text{, in the arc } \pi/4 < \theta < 3\pi/4 \text{ before hitting the line } y = 0] \geqslant c_\alpha\end{aligned}$$

It follows from the last two inequalities that

$$\int_{(-1,1) \cap E_{m,n}} k(0, x')\, dx' \geqslant c_\alpha 2^{-(m-1)\alpha}(\sqrt{2} - 1)/2$$

which is the lower bound of the lemma.

Next we turn to the upper bound. For $k \in \mathbb{Z}$ let D_k be the set of points with polar co-ordinates (r, θ) centered at $(k2^{-n}, 0)$ satisfying $0 < r < 2^{-n-1}$, $0 < \theta < \pi$. Let u satisfy the Dirichlet problem on D_k,

$$\begin{aligned} L_n u(x, y) &= 0, & (x, y) &\in D_k \\ u(x, 0) &= 1, & |x - k2^{-n}| &< 2^{-m-n} \\ u(x, y) &= 0, & (x, y) &\in \partial D_k \backslash \{(z, 0) : |z - k2^{-n}| < 2^{-m-n}\} \end{aligned}$$

Evidently $u(x, y)$ is the probability that the diffusion started at $(x, y) \in D_k$ exits ∂D_k through the set $\partial D_k \cap E_{m,n}$. We consider the values of u on the semi circle $r = 2^{-m-n+1}$, $0 < \theta < \pi$. It follows from the maximum principle that $u \leqslant 1$ at all points on the semi circle. Let E_k be the semi circle centered at the point $r = 2^{-m-n+1}$, $\theta = 0$, with radius 2^{-m-n}. It is clear that

$$\begin{aligned} L_n u(x, y) &= 0, & (x, y) &\in E_k \\ u(x, y) &\leqslant 1, & (x, y) &\in \partial E_k \\ u(x, 0) &= 0, & (x, 0) &\in \partial E_k \end{aligned}$$

Observe that the coefficients of the elliptic operator L_n are C^∞ inside E_k. It follows now by standard elliptic regularity theory that there is a constant C_α depending only on α such that

$$u(x, y) \leqslant C_\alpha 2^{m+n} y$$

if the point (x, y) lies within a distance 2^{-m-n-1} from the center of E_k. We can similarly construct a semi circle F_k centered at the point $r=2^{-m-n+1}$, $\theta=\pi$, with radius 2^{-m-n}. By the same argument as above we also have that

$$u(x, y) \leqslant C_\alpha 2^{m+n} y$$

if the point (x, y) lies within a distance 2^{-m-n-1}, from the center of F_k. Writing now u as a function of polar co-ordinates (r, θ), it follows from the previous two inequalities that there is a constant C_α depending only on α such that $u(2^{-m-n+1}, \theta) \leqslant C_\alpha \sin\theta$, $0<\theta<\pi$. Hence by the maximum principle

$$u(r, \theta) \leqslant C_\alpha (2^{m+n-1} r)^{-\alpha} \sin\theta, \qquad 0<\theta<\pi, \quad 2^{-m-n+1}<r<2^{-n-1} \tag{5.16}$$

There exists a constant γ depending only on α with $0<\gamma<1$ such that

$$\begin{aligned}&\text{Prob[diffusion started at a point in } D_k \text{ with polar coordinates } (r, \theta),\\ &\quad r=2^{-n-2}, 0<\theta<\pi, \text{ exits } D_k \text{ through the boundary } r=2^{-n-1}] \leqslant \gamma\end{aligned} \tag{5.17}$$

This follows from Lemma 3.1. Now from (5.16), (5.17) we have that

$$\begin{aligned}&\text{Prob[diffusion started on the line } y=1 \text{ first hits the line } y=0\\ &\quad \text{in the set } E_{m,n}] \leqslant \sum_{k=0}^{\infty} \gamma^k C_\alpha 2^{-(m-3)\alpha} = C_\alpha 2^{-(m-3)\alpha}/(1-\gamma)\end{aligned}$$

The upper bound follows immediately from this last inequality. QED

For $n \geqslant 0$, $m \geqslant 1$, $j \geqslant 0$ we define sets $E_{m,n,j}$ by $E_{m,n,0}=E_{m,n}$ as in (5.13) and for $j \geqslant 1$,

$$E_{m,n,j} = \bigcup \{(2k 2^{-n-jm} - 2^{-n-(j+1)m}, 2k 2^{-n-jm} + 2^{-n-(j+1)m}) : k \in \mathbb{Z}\} - \bigcup_{0 \leqslant j' < j} E_{m,n,j'}$$

It is easy to see that the measure of $E_{m,n,j} \cap (-1, 1)$ satisfies the inequality,

$$2^{-m+2} \geqslant |E_{m,n,j} \cap (-1, 1)| \geqslant 2^{-m+2} - j 2^{-2m+2} \tag{5.18}$$

Lemma 7. Let α satisfy $0<\alpha<1$. For $n\geqslant 0$, $m\geqslant 1$ there exists uniformly elliptic operators $L_{n,m}$, with uniform ellipticity bounds λ, Λ depending only on α, and constant $c_\alpha>0$ depending only on α, such that

$$\int_{(-1,1)\cap E_{m,n,j}} k(0,x')\,dx' \geqslant c_\alpha 2^{-m\alpha}, \qquad j\leqslant c_\alpha 2^{m\alpha}, \quad m\geqslant 1 \tag{5.19}$$

where k is the probability density of (5.1) associated with the elliptic operator $L_{n,m}$.

Proof. Our definition of $L_{n,m}$ generalizes the definition of L_n. Suppose the point $(x,y)\in\mathbb{R}^2$ is a distance less than 2^{-n-1} from the point $(k2^{-n},0)$ where $k\in\mathbb{Z}$ but also $y>2^{-m-n}$. Then we define $L_{n,m}$ at (x,y) by the operator L_ε of (5.12), where now the polar co-ordinates (r,θ) are centered at the point $(k2^{-n},0)$. Suppose for some $j\geqslant 1$, the point (x,y) is a distance less than 2^{-n-jm} from the point $(2k2^{-n-jm},0)\in\mathbb{R}^2$ where $k\in\mathbb{Z}$ but also $y>2^{-n-(j+1)m}$. Then we define $L_{n,m}$ at (x,y) by the operator L_ε of (5.12), where now the polar co-ordinates (r,θ) are centered at the point $(2k2^{-n-jm},0)$. For all other points (x,y) we define $L_{n,m}$ at (x,y) to be simply the Laplacian.

We first show the inequality (5.19) holds for $j=0$. It is clear from Lemma 3.1 that it follows immediately from the inequality,

$$\begin{aligned}&\text{Prob}[\text{diffusion started at a point } (x,1)\in\mathbb{R}^2 \text{ with } |x|<1, \text{ hits the line}\\ &\quad y=2^{-n-m} \text{ in the set } \{(z,2^{-n-m}) : |z|<1,\ z\in E_{m,n,0}\}]\geqslant c_\alpha 2^{-m\alpha}\end{aligned}$$

Observe that (5.20) does not follow from the lower bound argument of Lemma 5.6. In fact to prove it we need to combine both the upper and lower bound arguments of Lemma 5.6. First consider a point $(k2^{-n},0)\in\mathbb{R}^2$ with $k\in\mathbb{Z}$ and let (r,θ) be polar coordinates with respect to this point. We consider the function $u(r,\theta)$ defined by (5.14). It follows from the definition of $L_{n,m}$ that $L_{n,m}u(r,\theta)=0$ provided $r<2^{-n-1}$ and $r\sin\theta>2^{-m-n}$. Further, $u\leqslant 0$ on the set $\{(r,\theta : r=2^{-n-1}, 0<\theta<\pi\}$. We consider the values of u on the line $r\sin\theta=2^{-m-n+1}$. It is clear from (5.14) that

$$u(r,\theta)\leqslant 4^{-\alpha}(2^{-m-n+1}/r)^{1+\alpha}, \qquad r\sin\theta=2^{-m-n+1} \tag{5.21}$$

For $j=1,2,...$ let $p_j(\theta)$ be the probability that the diffusion started at the point with polar coordinates $(2^{-n-2},\theta)$ first hits the line $r\sin\theta=2^{-m-n+1}$ in the set $j2^{-m-n+1}\leqslant r<(j+1)\,2^{-m-n+1}$ while remaining in the disc $r<2^{-n-1}$. In view of (5.21) we have

$$\sum_{j=1}^{\infty}\frac{4^{-\alpha}}{j^{1+\alpha}}\,p_j(\theta)\geqslant u(2^{-n-2},\theta), \qquad 0<\theta<\pi$$

Hence from (5.15) and the previous inequality we conclude that

$$\sum_{j=1}^{\infty} \frac{4^{-\alpha}}{j^{1+\alpha}} p_j(\theta) \geqslant 2^{-(m-1)\alpha}(\sqrt{2}-1)/2, \qquad \frac{\pi}{4} < \theta < \frac{3\pi}{4} \tag{5.22}$$

Next we use the upper bound argument of Lemma 5.6 to show that there exists J depending only on α such that

$$\sum_{j=J}^{\infty} \frac{4^{-\alpha}}{j^{1+\alpha}} p_j(\theta) \leqslant 2^{-(m-1)\alpha}(\sqrt{2}-1)/4, \qquad \frac{\pi}{4} < \theta < \frac{3\pi}{4} \tag{5.23}$$

To do this we show that there is a constant C_α depending only on α such that

$$p_j(\theta) \leqslant C_\alpha (j2^{-m})^\alpha / j, \qquad j \leqslant 2^{m-10} \tag{5.24}$$

The inequality (5.23) follows from (5.24) and the fact that

$$\sum_{j=1}^{\infty} p_j(\theta) \leqslant 1, \qquad \frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

The bound (5.24) is proved exactly like the upper bound in Lemma 5.6. In fact let S_j be the union of the circular segment $r = j2^{-m-n+2}$, $0 < \theta < \pi$, $r \sin\theta > 2^{-m-n+1}$, and the line segments $r \sin\theta = 2^{-m-n+1}$, $j2^{-m-n+2} < r < 2^{-n-1}$. Let $q_j(\theta)$ be the probability that the diffusion started at the point with polar co-ordinates $(2^{-n-2}, \theta)$ first hits S_j in the circular segment while remaining in the disc $r < 2^{-n-1}$. By the upper bound argument of Lemma 5.6 one has

$$q_j(\theta) \leqslant C_{\alpha,1} (j2^{-m})^\alpha, \qquad j \leqslant 2^{m-10}$$

Next let T_j be the semi circle centered at the point $r = j2^{-m-n+1}$, $\theta = 0$ with radius $j2^{-m-n}$. Then by elliptic regularity theory it follows there is such a constant $C_{\alpha,2}$ depending only on α such that

$$\begin{aligned}&\text{Prob}[\text{diffusion started at a point on } T_j \text{ exits the region bounded}\\ &\quad \text{by the line } r\sin\theta = 2^{-m-n+1} \text{ and the semi circle } r = j2^{-m-n+2},\\ &\quad 0<\theta<\pi, \text{ through the segment } \{(r,\theta) : r\sin\theta = 2^{-m-n+1},\\ &\quad j2^{-m-n+1} \leqslant r < (j+1)\,2^{-m-n+1}\}] \leqslant C_{\alpha,2}/j\end{aligned}$$

There is also a constant $\gamma<1$ depending only on α such that

$$\begin{aligned}&\text{Prob[diffusion started at a point on } T_j \text{ exits the region bounded}\\ &\quad \text{by the line } r\sin\theta=2^{-m-n+1} \text{ and the semi circle } r=j2^{-m-n+2},\\ &\quad 0<\theta<\pi, \text{ through the semicircle]} \leqslant \gamma\end{aligned}$$

This is a consequence of Lemma 3.1. It follows now from the last three inequalities that

$$\begin{aligned}p_j(\theta) &\leqslant q_j(\theta)\, C_{\alpha,2}/j(1-\gamma)\\ &\leqslant C_{\alpha,1} C_{\alpha,2}(j2^{-m})^{\alpha}/j(1-\gamma)\end{aligned}$$

The proof of (5.20) follows in a straight forward way from (5.22), (5.23). In fact from these inequalities we have that there exists J depending only on α such that

$$\sum_{j=1}^{J} p_j(\theta) \geqslant 2^{-(m-1)\alpha}(\sqrt{2}-1)/4, \qquad \frac{\pi}{4}<\theta<\frac{3\pi}{4}$$

Now by Lemma 3.1 there exists $C_\alpha>0$ depending only on α such that

$$\begin{aligned}&\text{Prob[diffusion started at a point on the line segment } r\sin\theta=2^{-m-n+1},\\ &\quad 2^{-m-n+1}\leqslant r<(J+1)\,2^{-m-n+1}, \text{ first hits the line } r\sin\theta=2^{-m-n}\\ &\quad \text{in the set } \{(z,2^{-n-m})\in\mathbb{R}^2 : |z|<1, z\in E_{m,n,0}\}\,] \geqslant c_\alpha\end{aligned}$$

The inequality (5.20) follows from these last two inequalities.

Next we prove an upper bound analogous to (5.20). We show there is a constant $C_\alpha>0$ depending only on α such that

$$\begin{aligned}&\text{Prob[diffusion started at a point } (x,1)\in\mathbb{R}^2 \text{ first hits the line}\\ &\quad y=y_0 \text{ in the set } \{(z,y_0) : |z|<1, z\in E_{m,n,0}\}\,]\\ &\quad \leqslant C_\alpha 2^{-m\alpha}, \text{ provided } 0\leqslant y_0\leqslant 2^{-n-m}\end{aligned} \tag{5.25}$$

To prove (5.25) we use (5.24) and Lemma 4.1. Thus we have chosen a point $(k2^{-n},0)\in\mathbb{R}^2$, $k\in\mathbb{Z}$, and taken (r,θ) to be polar co-ordinates with respect to this point. Let $Q(\theta)$ be the probability that the diffusion started at the point with polar co-ordinates $(2^{-n-2},\theta)$ first hits the line $r\sin\theta=y_0$ in the set $\{(z,y_0) : |z|<1, z\in E_{m,n,0}\}$ while remaining in the disc $r<2^{-n-1}$.

By Lemma 4.1 it follows that there is a constant $C_{1,\alpha} > 0$ depending only on α such that

$$Q(\theta) \leqslant \sum_{j=1}^{\infty} C_{1,\alpha} j^{-1} p_j(\theta) \leqslant C_{2,\alpha} 2^{-m\alpha}$$

by (5.24). Next observe that there exists $\gamma < 1$, depending only on α such that

$$\begin{aligned}&\text{Prob[diffusion started at a point } (2^{-n-2}, \theta) \text{ exits the region bounded}\\ &\quad \text{by the line } r \sin\theta = y_0 \text{ and the circle } r = 2^{-n-1},\\ &\quad \text{through the circular boundary]} \leqslant \gamma\end{aligned}$$

It follows from the last two inequalities that the LHS of (5.25) is bounded above by $C_{2,\alpha} 2^{-m\alpha}/(1-\gamma)$.

We turn to the proof of (5.19) in the case $j \geqslant 1$. Observe that the proof of (5.25) directly generalizes to

$$\begin{aligned}&\text{Prob[diffusion started at a point } (x, 1) \in \mathbb{R}^2 \text{ first hits the line } y = y_0\\ &\quad \text{in the set } \{(z, y_0) : |z| < 1, z \in E_{m,n,j}\}]\\ &\leqslant C_\alpha 2^{-m\alpha} \qquad \text{provided} \quad 0 \leqslant y_0 \leqslant 2^{-m-(j+1)m}\end{aligned}$$

We conclude from this last inequality that

$$\begin{aligned}&\text{Prob}\Bigg[\text{diffusion started at a point } (x, 1) \in \mathbb{R}^2 \text{ first hits the line}\\ &\quad y = 2^{-n-jm} \text{ in the set } \Big\{(z, 2^{-n-jm}) : |z| < 1, z \in \bigcup_{i=0}^{j-1} E_{m,n,i}\Big\}\Bigg]\\ &\leqslant C_{\alpha,2}\, j 2^{-m\alpha}\end{aligned}$$

Observe next that the proof of (5.20) directly generalizes to

$$\begin{aligned}&\text{Prob}\Bigg[\text{diffusion started at a point } (x, 2^{-n-jm}) \in \mathbb{R}^2 \text{ with } |x| < 1,\\ &\quad x \notin \bigcup_{i=0}^{j-1} E_{m,n,i} \text{ first hits the line } y = 2^{-n-(j+1)m} \text{ in the set}\\ &\quad \{(z, 2^{-n-(j+1)m}) : |z| < 1, z \in E_{m,n,j}\}\Bigg] \geqslant c_{\alpha,2} 2^{-m\alpha}\end{aligned}$$

We conclude from these last two inequalities that

$$\int_{(-1,1)\cap E_{m,n,j}} k(0,x')\,dx' \geqslant [p_\alpha - C_{\alpha,2}\,j2^{-m\alpha}]\,c_{\alpha,2}2^{-m\alpha}q_\alpha \tag{5.26}$$

where p_α is the probability that the diffusion started at $(0, 0)\in\mathbb{R}^2$, after hitting the line $y=1$, hits the line $y=2^{-n-jm}$ in the set $\{(z, 2^{-n-jm}) : |z|<1\}$. The quantity q_a is a lower bound on the probability that the diffusion started at a point $(z, 2^{-n-(j+1)m})$ with $z\in(-1,1)\cap E_{m,n,j}$ first hits the line $y=0$ in the set $\{(z,0) : z\in(-1,1)\cap E_{m,n,j}\}$. Evidently p_α, q_α are bounded below by positive constants which depend only on α. The result follows now from (5.26). QED

Proof of Proposition 5.1. It follows from Lemma 5.7, (5.18) and Holder's inequality that for any $p>1$, one has

$$\int_{(-1,1)\cap E_{m,n,j}} k(0,x')^p\,dx' \geqslant c_\alpha 2^{-mp\alpha}2^{m(p-1)}, \qquad j\leqslant c_\alpha 2^{m\alpha}$$

where $c_\alpha>0$ depends only on α. Setting

$$E_m = \bigcup_{j=0}^{c\,2^{m\alpha}} (-1,1)\cap E_{m,n,j}$$

we conclude that

$$\int_{E_m} k(0,x')^p\,dx' \geqslant c_\alpha^2 2^{m(p-1)(1-\alpha)}$$

Observe that this last inequality almost implies the result of the proposition since we can take $m\to\infty$ while the uniform ellipticity bounds λ, Λ depend only on α. To construct a concrete operator L for which Proposition 5.1 holds let $M\geqslant 1$ be a large integer. We define a sequence J_j, $j\geqslant M$, by

$$J_M = [2^{M\alpha}/M^2], \qquad J_{j+1} = J_j + [2^{(j+1)\alpha}/(j+1)^2], \qquad j\geqslant M$$

Here [ ] denotes the integer part. We put $J_{M-1}=0$, and define a new sequence N_i, $i\geqslant 0$, by $N_0=0$ and

$$N_i = N_{J_j} + (i-J_j)(j+1), \qquad J_j\leqslant i\leqslant J_{j+1}, \quad j\geqslant M-1$$

Evidently the N_i form an increasing sequence of integers. We define L now analogously to the operator defined in Proposition 5.1. Suppose the point

$(x, y) \in \mathbb{R}^2$ is a distance less than 2^{-N_i} from the point $(2k2^{-N_i}, 0) \in \mathbb{R}^2$ where $k \in \mathbb{Z}$ but also $y > 2^{-N_{i+1}}$. Then we define L at (x, y) by the operator L_ε of (5.12), where now the polar co-ordinates (r, θ) are centered at the point $(2k2^{-N_i}, 0)$. We do this for all $i = 0, 1, 2, \ldots$ and assign L to be the Laplacian at other points $(x, y) \in \mathbb{R}^2$. For $i = 0, 1, 2, \ldots$ let E_i be the set

$$E_i = \bigcup \{(2k2^{N_i} - 2^{-N_{i+1}}, 2k2^{-N_i} + 2^{-N_{i+1}}) : k \in \mathbb{Z}\} - \bigcup_{0 \leqslant i' < i} E_{i'}$$

It is easy to see that if M is sufficiently large, then the measure of $E_i \cap (-1, 1)$ satisfies

$$2^{N_i - N_{i+1}} \geqslant |E_i \cap (-1, 1)| \geqslant 2^{N_i - N_{i+1} - 1}$$

We also have by exactly the same argument as in Proposition 5.1 that

$$\int_{(-1,1) \cap E_i} k(0, x')\, dx' \geqslant c_\alpha 2^{(N_i - N_{i+1})\alpha}$$

for some constant $c_\alpha > 0$ depending only on α. Hence Holder's inequality implies from the last two inequalities that there is a constant $C_\alpha > 0$ depending only on α such that

$$\int_{(-1,1) \cap E_i} k(0, x')^p\, dx' \geqslant c_\alpha 2^{(N_i - N_{i+1})\, p\alpha} 2^{(N_{i+1} - N_i)(p-1)}$$

We conclude therefore that

$$\begin{aligned} \int_{(-1,1) \cap E_i} k(0, x')^p\, dx' &\geqslant \sum_{i=0}^{\infty} c_\alpha 2^{(N_i - N_{i+1})\, p\alpha} 2^{(N_{i+1} - N_i)(p-1)} \\ &\geqslant c_{\alpha, 1} \sum_{m=M}^{\infty} 2^{m(p-1)(1-\alpha)}/m^2 = \infty \end{aligned}$$

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