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Coinitial sets and fixed points in partially ordered sets

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A partially ordered set P has the fixed point property if for every order-preserving map $f: P \to P$ there is $x \in P$ such that $f(x) = x$. If P is a lattice, then P has the fixed point property if and only if P is complete (A. Davis [4], A. Tarski [7]). For partially ordered sets that are not lattices, the problem of characterizing those having the fixed point property remains unsolved. In the case that in P at least suprema and infima of chains exist (P is chain-complete), various sufficient conditions for the fixed point property are known. Most of these are based on the results of [2].

The authors proved in [6, Corollary to Theorem 2] that any chain-complete partially ordered set with finitely many minimal elements, where each element is above a minimal element, and where each non-empty set of minimal elements has a least upper bound, has the fixed point property. This result has been extended and sharpened by several authors, among others K. Baclawski and A. Björner [3, Corollary 4.3 and Corollary 5.3] and P. Edelman [5, Theorem]. In all these extensions either finiteness conditions or antichain conditions continue to play an essential role. In this note we shall show that both types of conditions can be eliminated. This result was inspired by the notion of a coinitial set used by A. Abian [1].

Let P be a partially ordered set. A non-empty subset $M \subset P$ is a *complete coinitial* set if

(1) for each $p \in P$ there is $m \in M$ such that $m \leq p$,
(2) for each non-empty $A \subset M$, $\sup A$ exists in P,
(3) each non-empty chain C in $\bar{M} = \{\sup A \mid A \subset M, A \neq \emptyset\}$ has an infinum in $\bar{M}$.

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The dual notion of a *complete cofinal* set has its obvious definition. If P is chain-complete, then condition (3) is a consequence of (1) and (2). Furthermore, if each element of P is above a minimal (below a maximal) element of P, then condition (1) is satisfied, if M is chosen to be the set of all minimal elements of P. Note, that a coinitial set does not have to be an antichain, in fact, it can be a chain.

THEOREM. *Let P be a partially ordered set, and let $f: P \to P$ be an order-preserving mapping. If P contains a complete coinitial set, then there is $x \in P$ such that $x \leq f(x)$.*

Proof. Let M be a complete coinitial set in P. We define a map $\phi: P \to P$ by $\phi(y) = \sup\{m \in M \mid m \leq y\}$. ϕ is order-preserving, $\phi(y) \leq y$ for all $y \in P$, and $\phi P = \bar{M} = \{\sup A \mid A \subset M,\ A \neq \emptyset\}$. $\sup M$ is the largest element of $\bar{M}$ and by property (3), infima of chains exist in $\bar{M}$. Theorem 2 in [2] implies that $\bar{M}$ has the fixed point property. For an order-preserving $f: P \to P$ we restrict f to $\bar{M}$ and note that this restriction provides an order-preserving map $\phi \circ f: \bar{M} \to \bar{M}$. This map has a fixed point x. Now $x = \phi \circ f(x) \leq f(x)$.(1)

As an immediate consequence we get an infinite version of Theorem 2 in [6].

COROLLARY 1. *Suppose that the set M of minimal elements of P satisfies conditions (1), (2), and (3). then there is $x \in P$ such that $x \leq f(x)$.*

Theorem 2 in [2] and the Theorem above result in the following fixed point theorem.

COROLLARY 2. *Let P be chain-complete. If P contains a complete coinitial (cofinal) set, then P has the fixed point property.*

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1 This version of the proof is due to the referee. Our version made use of Zorn's Lemma, we thank the referee for providing this simpler argument.

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