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Setting Single-Period Optimal Capacity Levels and Prices for Substitutable Products

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Abstract. In this paper, we consider how a company that has the flexibility to produce two substitutable products would determine optimal capacity levels and prices for these products in a single-period problem. We first consider the case where the firm is a price taker but can determine optimal capacity levels for both products. We then consider the case where the firm can set the price for one product and the optimal capacity level for the other. Finally, we consider the case where capacity is fixed for both products, but the firm can set prices. For each case, we examine the sensitivity of optimal prices and capacities to the problem parameters. Finally, we consider the case where each product is managed by a product manager trying to maximize individual product profits rather than overall firm profits and analyze how optimal price and capacity decisions are affected.

Key Words: substitutable products, capacity, determination

1. Introduction

Firms must continuously make pricing and capacity decisions to respond to market forces. Facing uncertain demand, firms must balance pricing and production decisions to respond to the market. Many firms produce a variety of products, some of which may be substitutable by consumers. Product substitutability makes pricing and capacity decisions more difficult due to the firm's needs to consider the effect that a change in the price of one product is going to have on the demand level for another. A good example is in electronics manufacturing, where a firm might produce a variety of chips. The price of a faster chip affects the demand for a slower chip as well, since, if the prices for the two types of chips are sufficiently close, many customers might opt for the faster chip, thereby significantly decreasing demand for the slower chip.

In this paper, we study single-period pricing and capacity-setting decisions for a firm that has the flexibility to produce two substitutable products. We assume that this firm produces these products on two separate production lines. Furthermore, we do not assume that these products are perfect substitutes for each other. That is, we consider the case where customers have preferences for one product or the other, but this preference is affected by the price levels of both products. (For example, even a dedicated beef eater might switch to chicken if the price of beef becomes considerably more expensive than chicken.)

Due to the difference in prices, we assume, however, that shortages in one product do not necessarily lead customers to immediately select the other. As an example, consider the case of individuals deciding what type of car to buy. If the Toyota Motor Company fixes the prices of its Lexus LS400 to be the same (or only slight above) the Camry, it is likely that many customers will switch to buying the Lexus. On the other hand, with a price difference of nearly $30,000 right now, rarely do shortages in Camries suddenly convince many potential Camry buyers to buy a Lexus LS400 instead. We therefore consider substitutability due to price differential and not due to shortages in one product line. Our aim is to build intuition on how pricing and capacity decisions change as a function of costs, demand functions, and preexisting capacity levels; we therefore focus on simple single-period models. This also enables us to contrast our results with the famous news vendor problem, which is a single-period problem where the capacity (production level) for only a single product is chosen.

We also focus on the issue of centralized versus decentralized decision making and how this affects the nature of the decisions. Many firms assign a product manager to each product the firm produces. The product manager's role is to maximize the profits made by that product. If all products are produced on separate production lines and the products are not substitutable, the decisions made by each product manager trying to maximize product profits correspond to decisions made by a centralized controller maximizing systemwide profits. The more interesting case is when products are substitutable; in that case, we analyze when the system-optimal decisions are the same as the individual product-optimal decisions.

The classic problem where price is known and capacity is uncertain is the news vendor problem. Many extensions of this classic problem have been made. For example, Ismail and Louderback (1979), Lau (1980), and Kabak and Schiff (1978) studied a one product news vendor problem where the probability of achieving a predetermined profit level is maximized. Ismail and Louderback (1979), Sankarasubramanian and Kumaraswamy (1983), and Lau and Lau (1988) studied a single product news vendor problem where the demand level is dependent on the price set. Therefore, both optimal price and optimal order quantity are determined. Li, Lau, and Lau (1990) focused on a two product news vendor problem where the probability of achieving a profit target is maximized.

A variety of attempts have been made to introduce the effects of capacity constraints on the price and production decisions in the news vendor problem. Kreps and Scheinkman (1983) examined a problem where two identical firms compete with products that are perfect substitutes. The demand for each product is a function of its own price. The two firms enter into a two-stage competition, where they set capacities in the first stage and make pricing decisions independently in the second stage. Staiger and Wolak (1992) focused on two firms that produce the same product and have the same capacity costs. They analyzed an infinitely repeated game where prices are adjusted periodically and examine the effects of having excess capacity. Lippman and McCardle (1997) considered a market for a single commodity-type product where the demand is allocated to each competitor under various allocation schemes.

Examples of research involving substitutable products include Ignall and Veinott (1969), who examine the optimality of myopic policies with several products; Bassok, Anupindi, and Akella (1993), Bitran and Dasu (1992), Hsu and Bassok (1994), and Gerchak, Tripathy, and Wang (1996) studied ordering policies with substitutable products; while Carmon and Nahmias (1994) examined lot-sizing decisions in semiconductor manufacturing where the products are substitutable. In this line of research, it is assumed that the manufacturer is a price taker for all of its products.

We study the case of a firm producing two products with price-dependent demands, where the firm has the ability to make pricing or capacity decisions for one or both of its products. We begin in Section 2 by discussing the case where a firm is a price taker for both of its products but has control over the amount of capacity to install for each product. In Section 3, we analyze the situation of a firm that needs to decide on the amount of capacity to install for one of its products and the price to set for the other product. This is a situation facing firms that introduce a new or improved product with the potential to cannibalize sales from its existing product and where the firm has only a limited amount of capacity on-line for this new product. An example of this would be a microchip manufacturer launching the next generation of microchip after its existing product has been cloned. In Section 4, we examine the case where both products have a given capacity constraint and the firm sets prices for both products. This situation is commonplace for firms that have no ability to increase capacity in the near term but can control sales and profits only by adjusting their prices.

We end in Section 5 with a discussion on how the decisions made in the previous sections differ when decisions regarding price or capacity are made sequentially instead of simultaneously. These situations can arise when a firm has different brand managers making decisions to maximize each product's profit independently rather than maximizing systemwide profits. Porteus and Whang (1991) discuss incentive strategies to induce manufacturing managers and product managers to act in the best interest of a company as a whole. Whang (1995) notes that the research to date in cross-functional coordination of manufacturing and marketing has focused on the coordination between manufacturing and marketing for either a single product or multiple independent products. In our paper, we will examine the case where the demand for both products is dependent.

We use the following notation throughout the paper:

$P_a =$ price of product A;
$P_b =$ price of product B;
$C_a =$ production capacity for product A;
$C_b =$ production capacity for product B;
$q_a =$ per unit variable cost of product A;
$q_b =$ per unit variable cost of product B;
$i_a =$ cost of adding one unit of dedicated capacity for product A;
$i_b =$ cost of adding one unit of dedicated capacity for product B;
$u_a(P_a, P_b) =$ mean demand for product A;
$u_b(P_a, P_b) =$ mean demand for product B.

For analytical simplicity, we assume throughout the paper that

1. The demands for product A and B are distributed uniformly over the intervals $[u_a(P_a, P_b) - r, u_a(P_a, P_b) + r]$ and $[u_b(P_a, P_b) - s, u_b(P_a, P_b) + s]$, where r and s are the ranges of realizable demands above and below their respective means.

We assume further that

2. The unit investment costs plus variable production costs do not exceed the product price: $i_a + q_a < P_a$ and $i_b + q_b < P_b$ with $i_a, q_a, i_b, q_b > 0$.

3. The mean demand of product A, $u_a(P_a, P_b)$, is decreasing in P_a and increasing in P_b and the mean demand of product B is increasing in P_a and decreasing in P_b.
   Note that assumption 3 follows immediately from the fact that the products are substitutable. We finally define the variable profit function for products A and B to be

$$\begin{aligned}\pi(\cdot,\cdot) \;=\; & (P_a - q_a)\cdot \int_{u_a(P_a,P_b)-r}^{C_a} x_a \cdot f_a(P_a, P_b, x_a)\, dx_a \\ & + (P_a - q_a)\cdot C_a \cdot \int_{C_a}^{u_a(P_a,P_b)+r} f_a(P_a, P_b, x_a)\, dx_a \\ & + (P_b - q_b)\cdot \int_{u_b(P_a,P_b)-s}^{C_b} x_b \cdot f_b(P_a, P_b, x_b)\, dx_b \\ & + (P_b - q_b)\cdot C_b \cdot \int_{C_b}^{u_a(P_a,P_b)+s} f_b(P_a, P_b, x_b)\, dx_b\end{aligned}$$

where

$$\begin{aligned} f_a(P_a, P_b, x_a) \;&=\; 1/2r \text{ and } x_a \text{ is the demand for product } A;\\ f_b(P_a, P_b, x_b) \;&=\; 1/2s \text{ and } x_b \text{ is the demand for product } B.\end{aligned}$$

2. Capacity decisions for a price-taking firm

We begin our discussion with a firm that manufactures two products and must decide on the amount of capacity to install to manufacture each product. In this situation, the firm is a price taker in both of the markets in which it competes. For this situation, Lau and Lau (1988) provide a solution procedure for determining the optimal capacities to achieve a given probability of obtaining a profit target. In our model, we maximize the expected profit and derive the sensitivity of the optimal capacities to changes in key parameters.

The firm's profit function is defined as

$$R(C_a, C_b) = \pi(C_a, C_b) - i_a \cdot C_a - i_b \cdot C_b. \tag{1}$$

We first show that a unique set of capacities exists that maximizes equation (1).

Proposition 1. *If $P_a > q_a > 0$ and $P_b > q_b > 0$, then there exists a unique maximum of (1) over $C_a > 0$ and $C_b > 0$. The optimal capacities are*

$$\begin{aligned} C_a \;&=\; [u_a(P_a, P_b) + r] - \frac{2\cdot i_a \cdot r}{(P_a - q_a)} \quad \text{and}\\ C_b \;&=\; [u_b(P_a, P_b) + s] - \frac{2\cdot i_b \cdot s}{(P_b - q_b)}.\end{aligned}$$

Proof: To ensure that a unique maximum exists for positive values of C_a and C_b, a solution must exist to the first order conditions and all second order conditions must be satisfied. The first order conditions are

$$\frac{\partial R(C_a, C_b)}{\partial C_a} = -i_a - \frac{C_a \cdot (P_a - q_a)}{2r} + \frac{[u_a(P_a, P_b) + r] \cdot (P_a - q_a)}{2r} = 0$$

$$\frac{\partial R(C_a, C_b)}{\partial C_b} = -i_b - \frac{C_b \cdot (P_b - q_b)}{2s} + \frac{[u_b(P_a, P_b) + s] \cdot (P_b - q_b)}{2s} = 0$$

Rearranging these terms, we obtain

$$C_a^* = [u_a(P_a, P_b) + r] - \frac{2 \cdot i_a \cdot r}{(P_a - q_a)} \geq 2r\left[1 - \frac{i_a}{(P_a - q_a)}\right] \quad (2)$$

$$C_b^* = [u_b(P_a, P_b) + s] - \frac{2 \cdot i_b \cdot s}{(P_b - q_b)} \geq 2s\left[1 - \frac{i_b}{(P_b - q_b)}\right], \quad (3)$$

which are positive by assumption 2 that some profit can be earned on each product.

The sufficient conditions for the existence of a maximum are

$$\frac{\partial^2 R(C_a, C_b)}{\partial C_a^2} = -\frac{(P_a - q_a)}{2r} < 0$$

$$\frac{\partial^2 R(C_a, C_b)}{\partial C_a^2} \cdot \frac{\partial^2 R(C_a, C_b)}{\partial C_b^2} - \left[\frac{\partial^2 R(C_a, C_b)}{\partial \partial C_b}\right]^2 = \frac{(P_a - q_a)(P_b - q_b)}{4rs} > 0,$$

which both hold by assumption 2, completing the proof. □

We now turn our attention to how the optimal capacity decisions vary with changes in the key parameters of the model.

Proposition 2. *The changes in the optimal capacity mix due to changes in the parameters are*

1. *If P_a increases, then C_a decreases if $|\frac{\partial u_a(P_a, P_b)}{\partial P_a}| > \frac{2 \cdot i_a \cdot r}{(P_a - q_a)^2}$ and C_b increases;*
2. *If P_b increases, then C_a increases and C_b decreases if $|\frac{\partial u_b(P_a, P_b)}{\partial P_b}| > \frac{2 \cdot i_b \cdot s}{(P_b - q_b)^2}$;*
3. *If q_a increases, then C_a decreases and C_b does not change;*
4. *If q_b increases, then C_a does not change and C_b decreases;*
5. *If i_a increases, then C_a decreases and C_b does not change;*
6. *If i_b increases, then C_a does not change and C_b decreases;*
7. *If r increases, then C_a increases if $i_a < (P_a - q_a)/2$ and C_b does not change;*
8. *If s increases, then C_a does not change and C_b increases if $i_b < (P_b - q_b)/2$.*

Proof: We show the methodology for change 1. The proofs of the other cases are similar.

From Eq. (2), we know that

$$C_a = [u_a(P_a, P_b) + r] - \frac{2 \cdot i_a \cdot r}{(P_a - q_a)}.$$

Taking the derivative with respect to P_a, we obtain

$$\frac{\partial C_a}{\partial P_a} = \frac{\partial u_a(P_a, P_b)}{\partial P_a} + \frac{2 \cdot i_a \cdot r}{(P_a - q_a)^2}.$$

Since $\frac{\partial u_a(P_a,P_b)}{\partial P_a} < 0$ by assumption 3, we therefore conclude that C_a will decrease if $|\frac{\partial u_a(P_a,P_b)}{\partial P_a}| > \frac{2 \cdot i_a \cdot r}{(P_a - q_a)^2}$ when P_a increases.

Similarly, from Eq. (3), we know that

$$C_b = [u_b(P_a, P_b) + s] - \frac{2 \cdot i_b \cdot s}{(P_b - q_b)}$$

and thus

$$\frac{\partial C_b}{\partial P_a} = \frac{\partial u_b(P_a, P_b)}{\partial P_a} > 0$$

by assumption 3.

Therefore, we conclude that C_b always will increase as P_a increases. □

Parts 1 and 2 of Proposition 2 point to an interesting phenomenon; namely, that an expectation of a price increase in a product that will result in a decrease in the mean demand for that product does not necessarily result in the firm decreasing the available capacity for that product. Essentially, the condition in part 1 shows that, if the mean decrease in demand is not fast enough and the cost of capacity is not too expensive, in some cases, it may be more profitable to increase capacity for that product. This is because, even though on average there is less demand for that product, the unit profit may be higher and therefore more capacity might be profitable. The following example shows the possible behaviors for C_a mentioned previously.

In the following examples, we assume both demands are uniformly distributed. The mean demand for product A is defined as $C - c_1 P_a + c_2 P_b$ and r is the range of possible demand values. For product B, the mean demand is defined as $D - d_1 P_a + d_2 P_b$ and s is the range of possible demand values.

Example 1, Case A. C_a and C_b both increase with an increase in P_a. In this case, let $q_a = 3$, $q_b = 2$, $C = 2000$, $c_1 = 60$, $c_2 = 50$, $r = 400$, $D = 3000$, $d_1 = 100$, $d_2 = 19$, $s = 250$, $i_a = 1$, $i_b = 1$, $P_a = 6$, and $P_b = 10$. Here, the optimal solution is $C_a = 2273.33$, $C_b = 2301.5$, and $R(C_a, C_b) = 18592.58$.

If we increase P_a from 6 to 7, we observe that C_a increases to 2280, C_b increases to 2320.5 and $R(C_a, C_b)$ increases to 20652.25.

Example 1, Case B. C_a decreases and C_b increases with an increase in P_a. For this example, suppose $q_a = 3$, $q_b = 2$, $C = 2000$, $c_1 = 60$, $c_2 = 50$, $r = 400$, $D = 3000$, $d_1 = 100$, $d_2 = 19$, $s = 250$, $i_a = 1$, $i_b = 1$, $P_a = 10$, and $P_b = 10$. In this case, the optimal solution is $C_a = 2185.71$, $C_b = 2377.5$, and $R(C_a, C_b) = 26168.39$.

By increasing P_a to 11, we find that the optimal C_a decreases to 2140, the optimal C_b increases to 2396.5, and $R(C_a, C_b)$ increases to 27774.25.

Similarly, parts 7 and 8 of Proposition 2 show that an increase in the variability of demand for a product may result in an increase or a decrease of the capacity for that product. If the ranges of demand increase, the firm will choose to purchase more capacity, provided the per unit investment cost for capacity is less than half the variable per unit profit made on the product. However, changes in variable production or capacity costs always have monotonic consequences on optimal capacity levels. As the production or investment costs for a product increase, the product's profit margin clearly is reduced and so the optimal strategy will be to reduce amount of capacity kept on hand to produce that product.

3. Setting the price of *A* and the capacity level of *B*

In this section, we suppose that a firm currently has fixed (limited) capacity for a new product it is introducing but must decide how much capacity to maintain for its existing product. For example, in electronics manufacturing, when a firm first introduces a new product the capacity is extremely limited due to low yields for the new product and the necessary time for the factory to ramp up for production (e.g., building a new faster chip). On the other hand, the older product that the firm produces may already be a stable product in a market with active competition. For the older product, the firm might have the option to change its capacity; however, it is a price taker for it. Therefore, the firm faces the joint problem of setting a capacity level for one product and a price level for the other.

In this situation, the firm's profit function is defined as

$$R(P_a, C_b) = \pi(P_a, C_b) - i_b \cdot C_b.$$

In this section and the next, we assume linear mean demand functions. In particular, we assume that

4. The mean demand of product A, $u_a(P_a, P_b) = C - c_1 P_a + c_2 P_b > 0$ and $u_b(P_a, P_b) = D - d_1 P_a + d_2 P_b > 0$.

The use of such linear functions to model product demands is widespread in the economics literature (see Bulow, 1982; Stokey, 1981). We also assume that

5. $c_1 > c_2 > 0$, $d_1 > d_2 > 0$, $c_1 > d_2 > 0$, and $d_1 > c_2 > 0$.

Note that these assumptions are reasonable for the following reasons:

- The assumption that $c_1 > c_2$ and $d_1 > d_2$ means that the demand function for product A is more sensitive to changes in the price of product A than changes in the price of product B (and similarly for the demand function for product B).
- The assumption that $c_1 > d_2$ (and $d_1 > c_2$) ensures that an increase in the price of a product cannot increase overall demand, since for example, increasing the price of product A would decrease the mean demand for A more than it would increase the mean demand for B (i.e., one cannot create extra demand by increasing the prices of both products).

We also note that, in this section and the next, it is easy to show that an optimal policy always will set prices (for products for which prices are being set and capacity is fixed) in such a way that the minimum realizable demand is always less than or equal to the capacity for that product. To see this, note that, if the reverse were true, increasing prices to the level where the minimum realizable demand equals capacity increases per unit revenue without affecting actual realized sales for that product, while increasing the demand for the other product.

Proposition 3. *Under assumptions* 1 *through* 5, *the function* $R(P_a, C_b)$ *achieves a unique maximum over* $P_a > q_a > 0$ *and* $C_b > 0$.

Proof: See Appendix A. □

Once again, we are interested in how the optimal decisions change as a function of the optimal parameters.

Proposition 4. *Assuming the conditions in Proposition* 3,

1. *If* q_a *increases*, *then* P_a *increases and* C_b *increases*;
2. *If* q_b *increases*, *then* P_a *decreases and* C_b *decreases*;
3. *If* i_b *increases*, *then* P_a *decreases and* C_b *decreases*;
4. *If* C *increases*, *then* P_a *increases and* C_b *increases*;
5. *If* c_1 *increases*, *then* P_a *decreases and* C_b *decreases*;
6. *If* c_2 *increases*, *then* P_a *increases and* C_b *increases*;
7. *If* D *increases*, *then* P_a *does not change and* C_b *increases*;
8. *If* d_1 *increases*, *then* P_a *does not change and* C_b *decreases*;
9. *If* d_2 *increases*, *then* P_a *increases and* C_b *increases*;
10. *If* C_a *increases*, P_a *and* C_b *decrease.*

Proof: See Appendix B for the proof of part 1. The proofs for parts 2 through 9 are analagous to the proof of part 1. In the case of part 10, where we increase C_a, we have four potential cases: (a) P_a and C_b increase, (b) P_a increases and C_b decreases, (c) P_a decreases and C_b increases, (d) P_a and C_b decrease.

Cases (a) and (b) are not possible, since given the added capacity, one would never increase price. If P_a increases, then demand will decrease for product A and increase for product B. If this strategy improved profits in case a, it would have been done at the lower level of capacity for product A. In case (b), if the price of A increased, we would not decrease the capacity of product B, since it would be in greater demand. We also may discard (c) as a plausible outcome, given that, when P_a decreases, demand for product B also will decrease. Hence, we would not want to increase the capacity of product B. If this strategy would improve the overall profits, it would have been implemented under the initial capacity level of A. Therefore, we are left with the conclusion that as C_a increases, P_a and C_b decrease. □

We now present an example showing the various potential behaviors that can occur when P_b changes.

Example 2. When P_b is changed, we may observe different behaviors in the decision variables. We now provide examples of the three possible behaviors that can occur when we increase P_b.

Example 2, Case A. P_a and C_b increase. In this case let $C_a = 500$, $q_a = 2$, $q_b = 2$, $C = 2000$, $c_1 = 100$, $c_2 = 99$, $r = 420$, $D = 3000$, $d_1 = 100$, $d_2 = 10$, $s = 1000$, $i_b = 1$, and $P_b = 5$. Here, the optimal solution is $P_a = 18.24$, $C_b = 3015.8$, and $R(P_a, C_b) = 12218.8$.

If we increase P_b from 5 to 6, we observe that P_a increases to 19.16, C_b increases to 3091.628, and $R(P_a, C_b)$ increases to 15006.61.

Example 2, Case B. P_a increases and C_b decreases. For this example, suppose $C_a = 500$, $q_a = 2$, $q_b = 2$, $C = 2000$, $c_1 = 350$, $c_2 = 18$, $r = 420$, $D = 3000$, $d_1 = 100$, $d_2 = 99$, $s = 1000$, $i_b = 1$, and $P_b = 15$. We obtain an optimal solution of $P_a = 5.29$, $C_b = 2869.44$, and $R(P_a, C_b) = 24510.26$.

By increasing P_b to 16, we find that the optimal P_a increases to 5.34, the optimal C_b decreases to 2785.52, and $R(P_a, C_b)$ increases to 25312.3.

Example 2, Case C. P_a decreases and C_b increases. In this example, let $C_a = 500$, $q_a = 2$, $q_b = 2$, $C = 2000$, $c_1 = 350$, $c_2 = 99$, $r = 420$, $D = 3000$, $d_1 = 100$, $d_2 = 5$, $s = 1000$, $i_b = 1$, and $P_b = 3$. The optimal solution is $P_a = 5.36$, $C_b = 1726.814$, and $R(P_a, C_b) = 1181.003$.

By increasing P_b to 3.3, we see that P_a decreases to 5, C_b increases to 2156.558, and $R(P_a, C_b)$ increases to 1867.28.

In Proposition 4, we discover that, when a firm has control over the price of one of its products and the capacity level of the other, changes in the production costs of either product have an impact on both decision variables. If the production cost of the product over which we have price control increases, then we should raise the price on this product and also increase the capacity of the other product. In this circumstance, the profit margin of the product over which we have price control (product A) has decreased. Consequently, to counter the erosion of the profit margin, we raise the price for product A, which in turn increases the mean demand for product B, which leads us to increase the amount of capacity purchased for that product.

In essence, the firm is shifting some of its customer demand from product A to product B. On the other hand, if the production cost increases on the product whose price we cannot control, it is best to decrease its capacity and decrease the price of the other product. In this case, the firm essentially is dropping its price of product A to entice customers away from product B, which has become relatively less profitable now that its production costs have increased. This reasoning also explains why we decrease the price of product A and the capacity of product B when the capacity investment costs for product B increase. We also discover that we will increase the price of product A and the capacity of product B if the mean demand for product A increases by a constant (i.e., the mean demand shifts upward). In this situation, the market for product A essentially has increased and the firm is able to gain additional profits by increasing its price and still sell more units. At the same time, the increased price of product A entices some of the customers to buy product B (i.e., the mean demand for product B increases) and thus the firm invests in more capacity for product B to meet the additional expected demand.

If the mean demand for product B changes in the same fashion, however, we will keep the price of product A the same and increase the capacity for product B. The additional demand for product B clearly will lead to increasing the investment in capacity for product B. However, since the firm has no control over the price of product B, the increase in the demand for product B has no impact on the demand for product A and consequently the firm has no reason to change its price for product A. When c_1, which is a measure of the price elasticity of demand for product A, increases, we will decrease both the price of product A and the capacity level for product B. In this case, the demand for product A becomes more sensitive to changes in the price of A, so the firm optimally decreases its price, which in turn leads to a decrease in the mean demand for product B and consequently causes a decrease in the capacity level for product B.

When d_1 increases, however, we will not change the price of product A but we will decrease the capacity for product B. In this situation, the demand for product B has become more sensitive to the price of product B, and since the price of product B is not controllable, the mean demand for B decreases and hence the capacity for product B decreases. The price of product A remains unchanged because the demand parameters for product B have no impact on the demand or profitability of product A because the price of B cannot be changed.

If either c_2 or d_2 (which are related to cross-price elasticities) increases, we should increases both the price of product A and the capacity for product B; the reasoning in this situation is the same as when there is an increase in the demand for product A discussed earlier. When the capacity for product A is increased, then the price of product A and the capacity for product B will decrease. In this case, the firm lowers its price of product A to take advantage of the added available capacity, which in turn entices customers away from product B and leads to a decrease in the optimal capacity of product B. We also observe that, when the price of the product in the market in which a firm is a price taker increases, the firm's optimal strategy will differ depending on the relative sizes of c_1, c_2, d_1, and d_2, as shown in Example 2.

4. Pricing decisions: Two products with capacity constraints

We now examine the case where both products being manufactured have a given capacity constraint and the firm must set prices for both of its products. The situation modeled here reflects the situation where the high investment costs for new capacity makes it unprofitable to build new capacity (e.g., because it is forecast that the products are at the end of their life cycles and therefore it is not possible to recoup the costs of new investment); therefore, the firm can exercise only pricing control. We again will assume that the demands for products A and B are distributed uniformly and that the mean demands are linear in prices. The profit function is now defined as

$$R(P_a, P_b) = \pi(P_a, P_b).$$

Proposition 5. *Under assumptions* 1 *through* 5, *the function* $R(P_a, P_b)$ *is concave in* P_a *and* P_b.

Proof: The proof of concavity for $R(P_a, P_b)$ is similar to the proof of Proposition 3 shown in Appendix A. □

Proposition 6. *Under the assumptions of Proposition* 5,

1. *If* C *increases*, *then* P_a *and* P_b *increase*;
2. *If* c_1 *increases*, *then* P_a *and* P_b *decrease*;
3. *If* c_2 *increases*, *then* P_a *and* P_b *increase.*

Given the symmetry of the problem, *the parameters* D, d_1, *and* d_2 *will exhibit the same behavior as their counterparts previously.*

Proof: The proofs of parts 1 through 3 are similar to the proof of Proposition 4 shown in Appendix B. □

We now illustrate the different cases that are possible when the capacity of A, C_a, changes.

Example 3. If C_a increases, then any of the following cases are possible.

Example 3, Case A. P_a and P_b decrease. In this example, let $C_a = 1000$, $C_b = 1000$, $q_a = 2$, $q_b = 2$, $C = 2000$, $c_1 = 50$, $c_2 = 35$, $r = 400$, $D = 3000$, $d_1 = 50$, $d_2 = 35$, and $s = 500$. In this case the optimal solution is $P_a = 98.03$, $P_b = 109.28$, and $R(P_a, P_b) = 174435.5$.

If we increase C_a from 1000 to 1001, we observe that, with the preceding set of parameters, P_a decreases to 98, P_b decreases to 109.27, and $R(P_a, P_b)$ increases to 174474.30.

Example 3, Case B. P_a and P_b increase. In this case, suppose $C_a = 1000$, $C_b = 1000$, $q_a = 2$, $q_b = 2$, $C = 2000$, $c_1 = 1000$, $c_2 = 18$, $r = 400$, $D = 3000$, $d_1 = 21$, $d_2 = 19$, and $s = 1000$. In this case, the optimal solution is $P_a = 2.24$, $P_b = 90.91$, and $R(P_a, P_b) = 72452.92$.

In this case, increasing C_a to 1001, we find that the optimal P_a increases to 3.14, the optimal P_b increases to 92.22, and $R(P_a, P_b)$ increases to 73461.97.

Example 3, Case C. P_a decreases and P_b increases. For this example, let $C_a = 1000$, $C_b = 1000$, $q_a = 2$, $q_b = 2$, $C = 2000$, $c_1 = 60$, $c_2 = 50$, $r = 400$, $D = 3000$, $d_1 = 1000$, $d_2 = 19$, and $s = 250$. In this case, the optimal solution is $P_a = 21.71$, $P_b = 2.86$, and $R(P_a, P_b) = 16293.95$.

By increasing C_a to 1001, we see that P_a decreases to 21.70, P_b increases to 2.860694, and $R(P_a, P_b)$ increases to 16299.87.

Note that the case where P_a increases and P_b decreases is not possible because this strategy would adversely affect the amount of product A being sold and thus would not help the firm take advantage of the additional capacity. In fact, if the firm were able to improve its profits using this strategy, it would have done so before adding more capacity. Since our problem is symmetric in A and B, we see that C_b can also exhibit the same behavior.

Example 4. When the production costs of the products change, we have more than one potential outcome. We next show examples of how changes in q_a affect our optimal pricing strategy. Given the symmetry of the problem, the same outcomes can occur for q_b.

As the product cost of product A (q_a) increases, we have the following possible cases.

Example 4, Case A. P_a and P_b increase. In this example, let $C_a = 1000$, $C_b = 1000$, $q_a = 2$, $q_b = 2$, $C = 2000$, $c_1 = 50$, $c_2 = 35$, $r = 400$, $D = 3000$, $d_1 = 50$, $d_2 = 35$, and $s = 500$. In this case, the optimal solution is $P_a = 98.03$, $P_b = 109.28$, and $R(P_a, P_b) = 174435.5$.

If we increase q_a from 2 to 3, we observe that P_a increases to 98.15, P_b increases to 109.35, and $R(P_a, P_b)$ decreases to 173578.5.

Example 4, Case B. P_a increases and P_b decreases. For this example, let $C_a = 1000$, $C_b = 500$, $q_a = 2$, $q_b = 1$, $C = 2000$, $c_1 = 50$, $c_2 = 49$, $r = 400$, $D = 1000$, $d_1 = 51$, $d_2 = 49$, and $s = 250$. In this case, the optimal solution is $P_a = 588.41$, $P_b = 576.96$, and $R(P_a, P_b) = 696029.8$.

In this case, increasing q_a to 3, we find that the optimal P_a increases to 588.42, the optimal P_b decreases to 576.95, and $R(P_a, P_b)$ decreases to 695218.8.

It is clear that the case where both prices decrease is not possible, since it only would adversely affect the profitability of both products. The case where P_a decreases and P_b increases also is not possible, since this strategy would have been pursued before the increase in production costs if it were possible to increase the profit function's value.

From Proposition 6, we discover that changes in the terms c_1, d_1 (related to price elasticity of demand), c_2, and d_2 (related to cross-price elasticities) have different effects on the optimal pricing strategy. When a product's demand becomes more sensitive to changes in its price, the optimal strategy is to decrease the price of both products. In this situation, since the demand for the given product is more sensitive to the price of that product, the firm will decrease the price of the product, which in turn reduces the mean demand of the other product. However, to counter this drop in demand and overall profits, the firm also will drop the price of the other product. On the other hand, if the cross-price elasticity for one of the products increases, the optimal prices of both products increase. The intuition of this result essentially is the opposite of the reasoning used to explain the case where price elasticities are increasing. When the available capacity for either product changes, the optimal pricing scheme will be different, depending on the parameters of the mean demand functions. As shown in Example 3, the values for the price elasticities, cross-price elasticities, and the ranges of demand determine the optimal strategy that should be implemented. Similarly, when production costs increase, a firm will either increase the price of the product whose cost has increased and decrease the price of the other or increase both products' prices. Example 4 illustrates that the optimal strategy depends not only on the price elasticities, cross-price elasticities, and ranges of demand but also on the available capacity and the production costs for both products.

5. Individual-product optimal decisions

In the previous sections, we examined cases where a firm makes decisions about both of its products' capacities and prices simultaneously. We also assumed that the firm was trying to maximize overall profits (from both product lines). However, often, firms have separate brand managers for each product line and these brand managers are evaluated based on the profitability of their product line alone. In that case, the brand manager will make decisions to maximize profits of his or her own product line alone rather than to maximize overall profits for the firm. In this section, we analyze how each of the decisions analyzed in Sections 2–4 would be changed by the fact that they are made by managers trying to maximize individual-product optimal decisions rather than globally optimal decisions maximizing the sum of the profits from the two product lines. We therefore differentiate between the *globally optimal* decisions in Sections 2–4 and the *individually optimal* decisions in this section.

We also note that, in this section, we pay attention to the *order* in which decisions are made and announced. We show that this order is significant when both managers can make only pricing decisions but the order does not affect the eventual decisions otherwise. We explore various assumptions about the managers' behavior as in the classical Bertrand, Stackelberg, and collusion models of duopoly.

5.1. Capacity decisions

We return to the case in Section 2 where the firm determines the optimal capacities for both of its products; however, we now assume that products A and B have their own brand managers. Regardless of whether manager A or B makes its capacity decision first; the profit function for the manager of product A is

$$R_1(C_a) = (P_a - q_a) \cdot \int_{u_a(P_a,P_b)-r}^{C_a} x_a \cdot f_a(P_a, P_b, x_a)\, dx_a + (P_a - q_a) \cdot C_a \cdot \int_{C_a}^{u_a(P_a,P_b)+r} f_a(P_a, P_b, x_a)\, dx_a - i_a \cdot C_a$$

where $f_a(P_a, P_b, x_a) = 1/2r$ and x_a is the demand for product A.

Similarly, product B's profit function is defined as

$$R_2(C_b) = (P_b - q_b) \cdot \int_{u_b(P_a,P_b)-s}^{C_b} x_b \cdot f_b(P_a, P_b, x_b)\, dx_b + (P_b - q_b) \cdot C_b \cdot \int_{C_b}^{u_a(P_a,P_b)+s} f_b(P_a, P_b, x_b)\, dx_b - i_b \cdot C_b$$

where $f_b(P_a, P_b, x_b) = 1/2s$ and x_b is the demand for product B.

Proposition 7. *The individually optimal C_a and C_b are the same as the globally optimal C_a and C_b of Proposition* 1.

Proof: The result directly follows from the fact that the first order conditions for C_a and C_b are the same as those in Section 2. □

Proposition 7 states that if brand managers can make only capacity decisions, even if they make individually optimal decisions, they will end up maximizing global profits, as one manager's capacity decision does not affect the other's decision. In the next subsections, however, we show that when pricing decisions are involved (which affect the other product's demand as well), individually optimal decisions differ from globally optimal decisions.

5.2. Deciding on the price of A and the capacity level of B

We now examine the case where the optimal price of A, P_a^*, and the optimal capacity for B, C_b^*, are determined.

The manager maximizing product A's profits has the following objective function:

$$R_1(P_a) = (P_a - q_a) \cdot \int_{u_a(P_a,P_b)-r}^{C_a} x_a \cdot f_a(P_a, P_b, x_a)\, dx_a + (P_a - q_a) \cdot C_a \cdot \int_{C_a}^{u_a(P_a,P_b)+r} f_a(P_a, P_b, x_a)\, dx_a$$

where $f_a(P_a, P_b, x_a) = 1/2r$ and x_a is the demand for product A.

Note that the manager of product A does not care about the decision that the manager of product B makes about the capacity for product B. However, the manager of product B does care about the price of product A, as this affects the demand for product B. Therefore, the optimal C_b is a function of the optimal P_a obtained by solving the previous first order condition. Product B's profit function is defined the same as $R_2(C_b)$ in the previous subsection.

The first order condition of $R_1(P_a)$ is

$$\frac{\partial R_1(P_a)}{\partial P_a} = \frac{\left[C_a^2 - (u_a(P_a, P_b) - r)^2\right]}{4r} + \frac{(P_a - q_a) \cdot [C_a - (u_a(P_a, P_b) - r)]}{2r} \cdot \frac{\partial u_a(P_a, P_b)}{\partial P_a} + \frac{C_a \cdot [u_a(P_a, P_b) + r - C_a]}{2r} = 0.$$

Rearranging terms and substituting in the functions for $u_a(P_a, P_b)$ and $u_b(P_a, P_b)$, we can obtain the optimal value of P_a.

Comparing the optimal P_a in this case with the optimal P_a derived in Section 3, we clearly see that the optimal pricing strategy for product A differs depending on whether individually optimal or globally optimal decisions are being made. The following example shows globally optimal price and capacity levels can be significantly different than individually optimal levels.

Example 5. Suppose that $C_a = 1700$, $q_a = 3$, $q_b = 2$, $C = 2000$, $c_1 = 60$, $c_2 = 50$, $r = 400$, $D = 3000$, $d_1 = 55$, $d_2 = 40$, $s = 250$, $i_b = 1$, and $P_b = 77.98$. We obtain a globally optimal solution of $P_a = 76.38$, $C_b = 2009.886$, and $R(P_a, C_b) = 228761.18$. Product A's profits in this case are 94560.83. For the same parameters, the individually optimal solutions are $P_a = 68.82$ with a profit for product A of $R(P_a) = 105701.13$, and consequently $C_b = 1707.557$ with $R(C_b) = 111224.07$, resulting in a total profit of 216925.20. We see in this case that the manager of product A can improve the profitability of product A alone by decreasing its price compared to the globally optimal price of product A; however, this decreases the sum of the profits from both products.

Interestingly, in general, it is not true that the price of Product A always will be higher and the capacity for B always will be higher when decisions are made centrally for both capacity and price at the same time. The ordering of the solutions will differ based on the values of c_1, c_2, d_1, d_2, the profit margin of B, and the capacity of A.

5.3. *Pricing decisions*

In Section 4, we discussed the case where a firm simultaneously sets the prices for both its products. We now examine the case where managers for products A and B set prices to optimize individual product profits. Let us assume that the price of product A is set first. The profit function for product A will be the same as in the previous subsection, $R_1(P_a)$. However, as we can see, the optimal price for A depends on what price will be chosen for product B. Since we assume that all pricing and capacity information is known, the decision maker for the price of product A can predict what product B's optimal pricing strategy will be after the decision is made regarding the price of A. Product B's profit function is defined as

$$R_2(P_b) = (P_b - q_b) \cdot \int_{u_b(P_a,P_b)-s}^{C_b} x_b \cdot f_b(P_a, P_b, x_b)\, dx_b + (P_b - q_b) \cdot C_b \cdot \int_{C_b}^{u_a(P_a,P_b)+s} f_b(P_a, P_b, x_b)\, dx_b - i_b \cdot C_b$$

where $f_b(P_a, P_b, x_b) = 1/2s$ and x_b is the demand for product B. Therefore, before determining product A's optimal price, we solve product B's problem.

The first order condition of $R_2(P_b)$ is

$$\frac{\partial R_2(P_b)}{\partial P_b} = \frac{\left[C_b^2 - (u_b(P_a, P_b) - s)^2\right]}{4s} + \frac{(P_b - q_b) \cdot [C_b - (u_b(P_a, P_b) - s)]}{2s} \cdot \frac{\partial u_b(P_a, P_b)}{\partial P_b} + \frac{C_b \cdot [u_b(P_a, P_b) + s - C_b]}{2s} = 0.$$

From this equation, we can obtain an expression for the optimal price for product B, which then may be used to find the optimal price for Product A. To find the optimal price for Product A, three different approaches are available: the Bertrand model, the Stackelberg model, and the collusion model. In the Bertrand model, each product manager assumes that the price of her or his product does not affect the price of the other. In the Stackelberg, one manager follows the other's lead in setting price. In the collusion model, both managers act together to establish best prices for overall profit.

Bertrand model. In this approach, the manager of product A assumes an optimal price of product B, P_b^*, as a parameter in the first order condition of $R_1(P_a)$ to obtain

$$\frac{\partial R_1(P_a)}{\partial P_a} = \frac{\left[C_a^2 - (u_a(P_a, P_b^*) - r)^2\right]}{4r} + \frac{(P_a - q_a) \cdot [C_a - (u_a(P_a, P_b^*) - r)]}{2r} \cdot \frac{\partial u_a(P_a, P_b^*)}{\partial P_a} + \frac{C_a \cdot [u_a(P_a, P_b^*) + r - C_a]}{2r} = 0.$$

The optimal solution, P_a^*, was shown in Section 5.2, where P_b now is substituted with P_b^*. The manager of product B follows the same procedure to obtain a price with P_a^* as a parameter. Solving the simultaneous equations for the two prices produces the result.

Stackelberg model. For this approach, the manager of product A assumes that the manager of product B is the price setter. We substitute the optimal P_b^* *as a function of* P_a directly into the function $R_1(P_a)$ to obtain

$$R_1(P_a) = (P_a - q_a) \cdot \int_{u_a(P_a,P_b)-r}^{C_a} x_a \cdot f_a(P_a, P_b^*, x_a)\, dx_a + (P_a - q_a) \cdot C_a \cdot \int_{C_a}^{u_a(P_a,P_b)+r} f_a(P_a, P_b^*, x_a)\, dx_a.$$

The first order condition for this function clearly differs from the Bertrand approach shown previously since P_b^* is dependent on P_a. Therefore, the optimal price for A differs from the price obtained in the Bertrand model.

We now illustrate these approaches. For analytical simplicity, we examine the case where both products have unlimited capacity and the mean demand functions for Products A and B are $u_a(P_a, P_b) = C - c_1 P_a + c_2 P_b$ and $u_b(P_a, P_b) = D - d_1 P_b + d_2 P_a$, respectively, and where $c_1 > c_2$, $d_1 > d_2$, $c_1 > d_2$, and $d_1 > c_2$.

Comparison.

Bertrand model. We begin by finding the optimal price of product B. In this case, product B's profit function is defined as

$$R_2(P_b) = (P_b - q_b) \cdot u_b(P_a, P_b).$$

The first order condition is

$$u_b(P_a, P_b) + (P_b - q_b) \cdot \frac{\partial u_b(P_a, P_b)}{\partial P_b} = 0.$$

Rearranging terms, we obtain

$$P_b^* = \frac{D + d_2 P_a + q_b d_1}{2d_1}.$$

We now solve for the optimal price for product A. The profit function for product A is

$$R_1(P_a) = (P_a - q_a) \cdot u_a(P_a, P_b).$$

The first order condition then is

$$u_a(P_a, P_b^*) + (P_a - q_a) \cdot \frac{\partial u_a(P_a, P_b^*)}{\partial P_a} = 0.$$

Substituting P_b^* into this equation and solving for P_a, we obtain the optimal price of A to be

$$P_a^* = \frac{Dc_2 + 2Cd_1 + 2c_1 d_1 q_a + c_2 d_1 q_b}{4c_1 d_1 - c_2 d_2}.$$

Stackelberg model. We again begin by finding the optimal price of product B. The profit function for product B is the same as in the Bertrand model and hence the optimal price of B, P_b^*, will be the same expression.

The profit function for Product A is now defined as

$$R_1(P_a) = (P_a - q_a) \cdot u_a(P_a, P_b^*).$$

Substituting P_b^* into the profit function and solving for P_a in the first order condition, we now have the optimal price of A to be

$$P_a^* = \frac{Dc_2 + 2Cd_1 + 2c_1 d_1 q_a + c_2 d_1 q_b - c_2 d_2 q_a}{4c_1 d_1 - 2c_2 d_2}.$$

Setting both prices simultaneously (collusion). We now return to the model presented in Section 4, where the product managers agree to set both prices simultaneously. In other words, there is collusion in setting the prices for both products to optimize systemwide profits. We present the optimal prices for the products where both products have no capacity constraints.

The profit function in this case is

$$R(P_a, P_b) = (P_a - q_a) \cdot u_a(P_a, P_b) + (P_b - q_b) \cdot u_b(P_a, P_b).$$

The first order conditions are

$$u_a(P_a, P_b) + (P_a - q_a) \cdot \frac{\partial u_a(P_a, P_b)}{\partial P_a} = 0$$
$$u_b(P_a, P_b) + (P_b - q_b) \cdot \frac{\partial u_b(P_a, P_b)}{\partial P_b} = 0.$$

Solving the equations for P_a and P_b, we obtain

$$P_a^* = \frac{C + c_1 q_a - d_2 q_b}{2c_1} + \frac{(c_2 + d_2) \cdot [2c_1(D - c_2 q_a + d_1 q_b) + (c_2 + d_2)(C + c_1 q_a - d_2 q_b)]}{2c_1[4c_1 d_1 - (c_2 + d_2)^2]}$$
$$P_b^* = \frac{(c_2 + d_2) \cdot [2c_1(D - c_2 q_a + d_1 q_b) + (c_2 + d_2)(C + c_1 q_a - d_2 q_b)]}{4c_1 d_1 - (c_2 + d_2)^2}.$$

From the expressions in the three models, we observe that the optimal prices obtained from the model depend on the price elasticity and cost parameters. We next provide examples where the optimal prices are different relative to each other based on these parameters.

Example 6. Collusion prices > Stackelberg prices > Bertrand prices. Suppose $q_a = 2$, $q_b = 2$, $C = 2000$, $c_1 = 60$, $c_2 = 30$, $D = 2000$, $d_1 = 60$, and $d_2 = 20$. In this case we obtain the following results:

	Collusion	Stackelberg	Bertrand
P_a	29.63	24	23.04
P_b	29.51	21.67	21.51

Example 7. Stackelberg prices > Bertrand prices > collusion prices. In this case, let $q_a = 100$, $q_b = 2$, $C = 2500$, $c_1 = 10$, $c_2 = 2$, $D = 2000$, $d_1 = 79$, and $d_2 = 10$. The resulting prices are

	Collusion	Stackelberg	Bertrand
P_a	167.09	177.99	177.49
P_b	20.89	24.92	24.89

Example 8. (*Collusion price of A > Bertrand price of A > Stackelberg price of A*) *and* (*Bertrand price of B > Stackelberg price of B > collusion price of B*). For this example, let $q_a = 2$, $q_b = 70$, $C = 2500$, $c_1 = 60$, $c_2 = 50$, $D = 3000$, $d_1 = 55$, and $d_2 = 40$. The optimal prices are

	Collusion	Stackelberg	Bertrand
P_a	63.64	51.40	62.14
P_b	71	80.96	84.87

From these examples, we may conclude that the timing of pricing decisions, the optimization of individual versus systemwide profits, along with the relative sizes of the price elasticities and costs can have different impacts on a firm's optimal pricing strategy.

6. Conclusions and further research

In this paper, we address joint capacity and price decisions for substitutable products. We show that pricing and capacity decisions are affected greatly by the actual parameters that the decision makers can control as well as whether decision makers are optimizing systemwide or individual channel profits. The analytical results we show are based on the assumption of demands being uniformly distributed. These same conclusions also hold if we substitute the bivariate normal distribution, provided the parameters used in the model satisfy the first and second order conditions for concavity. However, further research should focus on more general demand distributions.

Many research questions on pricing and capacity setting for substitutable products remain open. For example, returning to the model in Section 2, an interesting model would be one that assumes that product A is the higher priced product and that, if there is demand in excess of the available capacity for A, a fraction (α) of the excess demand will shift to product B. In terms of filling orders for product B, we assume that the firm will first meet the demand for product B, and if there is any excess capacity, it will be used to fulfill as much of the fraction of excess demand for product A as possible that has shifted over to B.

In this case, the firm's profit function is

$$R'(C_a, C_b) = (P_a - q_a) \cdot \int_{u_a(P_a,P_b)-r}^{C_a} x_a \cdot f_a(P_a, P_b, x_a)\, dx_a + (P_a - q_a) \cdot C_a$$

$$\cdot \int_{C_a}^{u_a(P_a,P_b)+r} f_a(P_a, P_b, x_a)\, dx_a + (P_b - q_b)$$

$$
\cdot \int_{u_b(P_a,P_b)-s}^{C_b} \left[x_b + \int_{C_a}^{\mathrm{Min}\left[\frac{C_b - x_b}{\alpha} + C_a, u_a(P_a,P_b)+r\right]} \times \alpha(x_a - C_a)\, f_a(P_a, P_b, x_a)\, dx_a \right] f_b(P_a, P_b, x_b)\, dx_b
$$

$$
+ (P_b - q_b) \cdot C_b \cdot \int_{C_b}^{u_a(P_a,P_b)+s} f_b(P_a, P_b, x_b)\, dx_b - i_a \cdot C_a - i_b \cdot C_b
$$

Initial research has shown that this extension to the model in Section 2 requires several conditions to ensure that the conditions for concavity hold. Another interesting case to explore is where a firm has control over both products' prices and capacities. (Preliminary research shows these problems to be extremely challenging.) Furthermore, we consider only a single period problem in this paper. Further research should also consider multiple period problems.

Appendix A. Proof of concavity for $R(P_a, C_b)$

It is sufficient, as in Proposition 1, to show existence of a solution to the first order conditions and to show that the Hessian, H, of $R(P_a, C_b)$ is negative definite over $P_a > q_a > 0$ and $P_b > q_b > 0$. The first order conditions are

$$
\begin{aligned}
\frac{\partial R(P_a, C_b)}{\partial P_a} &= \frac{\left[C_a^2 - (u_a(P_a, P_b) - r)^2\right]}{4r} + \frac{C_a \cdot [u_a(P_a, P_b) + r - C_a]}{2r} \\
&\quad + \frac{(P_a - q_a) \cdot [C_a - (u_a(P_a, P_b) - r)]}{2r} \cdot \frac{\partial u_a(P_a, P_b)}{\partial P_a} \\
&\quad + \frac{(P_b - q_b) \cdot [C_b - (u_b(P_a, P_b) - s)]}{2s} \cdot \frac{\partial u_b(P_a, P_b)}{\partial P_a} = 0 \qquad (4)
\end{aligned}
$$

$$
\frac{\partial R(P_a, C_b)}{\partial C_b} = -i_b + \frac{[u_b(P_a, P_b) + s - C_b] \cdot (P_b - q_b)}{2s} = 0. \qquad (5)
$$

Solving these equations, we can obtain the optimal values for P_a and C_b. For the second order conditions, the terms of H are

$$
\begin{aligned}
H_{11} = \frac{\partial^2 R(P_a, C_b)}{\partial P_a^2} &= \frac{[C_a - (u_a(P_a, P_b) - r)]}{r} \cdot \frac{\partial u_a(P_a, P_b)}{\partial P_a} \\
&\quad - \frac{(P_a - q_a)}{2r} \left[\left(\frac{\partial u_a(P_a, P_b)}{\partial P_a} \right)^2 - [C_a - (u_a(P_a, P_b) - r)) \cdot \frac{\partial^2 u_a(P_a, P_b)}{\partial P_a^2} \right] \\
&\quad - \frac{(P_b - q_b)}{2s} \left[\left(\frac{\partial u_b(P_a, P_b)}{\partial P_a} \right)^2 - [C_b - (u_b(P_a, P_b) - s)) \cdot \frac{\partial^2 u_b(P_a, P_b)}{\partial P_a^2} \right]
\end{aligned}
$$

$$H_{21} = H_{12} = \frac{\partial^2 R(P_a, C_b)}{\partial P_a \partial C_b} = \frac{(P_b - q_b)}{2s} \cdot \frac{\partial u_b(P_a, P_b)}{\partial P_a}$$
$$H_{22} = \frac{\partial^2 R(P_a, C_b)}{\partial C_b^2} = -\frac{(P_b - q_b)}{2s}$$

The sufficient conditions for concavity are

$$H_{11} < 0$$
$$H_{22} < 0 \quad (6)$$
$$H_{11}H_{22} - H_{12}^2 > 0 \quad (7)$$

For H_{11}, notice that $\partial u_a(P_a, P_b)/\partial P_a < 0$ by assumption 3 and the capacity of A is greater than the minimum realizable demand. The first term, therefore, is negative. The second derivative elements in the next two terms of H_{11} vanish, so that two terms both become negative and yield $H_{11} < 0$. For H_{22}, negativity follows by assumption 2.

Inequality (7) follows by first noting that the $[\frac{(P_b - q_b)}{2s} \cdot \frac{\partial u_b(P_a, P_b)}{\partial P_a}]^2$ terms from $H_{11}H_{22}$ and H_{12}^2 cancel. The remaining terms are

$$H_{11}H_{22} - H_{12}^2 = \left[\frac{[C_a - (u_a(P_a, P_b) - r)]}{r} \cdot \frac{\partial u_a(P_a, P_b)}{\partial P_a}\right] \cdot \left[-\frac{(P_b - q_b)}{2s}\right]$$
$$- \left[\frac{(P_a - q_a)}{2r}\left[\frac{\partial u_a(P_a, P_b)}{\partial P_a}\right]\right]^2 \cdot \left[-\frac{(P_b - q_b)}{2s}\right],$$

which is the sum of two positives by our previous arguments in the proof of (6). The result follows.

Appendix B. Proof of sensitivity of P_a and C_b to a change in q_a

The goal here is to identify the effect of changes in one parameter on the optimal solution of the first order conditions in (4) and (5). To reflect changes in parameters other than the decision variables, we add extra terms to the definition of R so that $R(P_a, C_b)$ also is written as $R(P_a, C_b, q_a)$ when we explicitly consider changes in q_a.

In this case, the first order conditions in (4) and (5) are

$$\nabla_{C_b, P_a} R(P_a, C_b, q_a) = 0,$$

where ∇_{C_b, P_a} refers to the partial derivatives with respect to C_b and P_a alone. We suppose the solution to (4) and (5) is (C_b^*, P_a^*) when $q_a = q_a^*$.

In the following, we use the notation $\nabla^2_{x,y/z}$ for the partial differential operator given by

$$\begin{pmatrix} \frac{\partial^2}{\partial x \partial z} \\ \frac{\partial^2}{\partial y \partial z} \end{pmatrix}.$$

We next consider changes from q_a^* to $q_a = q_a^* + \delta_{q_a}$. The solutions of (4) and (5) then are $P_a = P_a^* + \delta_{P_a}$ and $C_b = C_b^* + \delta_{C_b}$. We wish to find the sign of δ_{P_a} and δ_{C_b} given $\delta_q > 0$.

First, using the first order properties, implicit function theorem, and that $\nabla_{C_b,P_a} R(P_a^*, C_b^*, q_a^*) = 0$, we must have

$$\begin{aligned} 0 &= \nabla_{C_b,P_a} R(P_a, C_b, q_a) \\ &= \nabla_{C_b,P_a/P_a} R(P_a^*, C_b^*, q_a^*)\delta_{P_a} + \nabla_{C_b,P_a/C_b} R(P_a^*, C_b^*, q_a^*)\delta_{C_b} \\ &\quad + \nabla_{C_b,P_a/q_a} R(P_a^*, C_b^*, q_a^*)\delta_{q_a} + \epsilon_{P_a}\delta_{P_a} + \epsilon_{C_b}\delta_{C_b} + \epsilon_{q_a}\delta_{q_a} \end{aligned}$$

where ϵ_{P_a}, ϵ_{C_b}, and ϵ_{q_a} all approach 0 as δ_{q_a} approaches 0. For small changes δ_{q_a}, we therefore seek δ_{P_a} and δ_{C_b} to solve

$$\begin{aligned} &\nabla_{C_b,P_a/P_a} R(P_a^*, C_b^*, q_a^*)\delta_{P_a} + \nabla_{C_b,P_a/C_b} R(P_a^*, C_b^*, q_a^*)\delta_{C_b} \\ &\quad = -\nabla_{C_b,P_a/q_a} R(P_a^*, C_b^*, q_a^*)\delta_{q_a}. \end{aligned}$$

To simplify the notation, let

$$\begin{aligned} H_{q_a} &= \left[\nabla^2_{C_b,P_a} R(P_a, C_b, q_a)\right] \\ &= \begin{bmatrix} \frac{\partial^2 R(P_a,C_b)}{\partial P_a^2} & \frac{\partial^2 R(P_a,C_b)}{\partial P_a \partial C_b} \\ \frac{\partial^2 R(P_a,C_b)}{\partial P_a \partial C_b} & \frac{\partial^2 R(P_a,C_b)}{\partial C_b^2} \end{bmatrix}. \end{aligned}$$

It is straightforward to show that the determinant of the matrix H_{q_a} has the same sign as the Hessian of $R(P_a, C_b)$ in the proof of Proposition 3, since the only addition has been δ to the P_a terms of that matrix.

Therefore, we have

$$\begin{aligned} \delta_{P_a} &= \frac{\det \begin{bmatrix} -\delta_{q_a} \frac{\partial^2 R(P_a,C_b)}{\partial P_a \partial q_a} & \frac{\partial^2 R(P_a,C_b)}{\partial P_a \partial C_b} \\ -\delta_{q_a} \frac{\partial^2 R(P_a,C_b)}{\partial C_b \partial q_a} & \frac{\partial^2 R(P_a,C_b)}{\partial C_b^2} \end{bmatrix}}{\det H_{q_a} > 0} \\ &= \frac{\det \begin{bmatrix} \delta_{q_a}\left[\frac{[C_a-(u_a(P_a,P_b)-r)]}{2r} \cdot \frac{\partial u_a(P_a,P_b)}{\partial P_a}\right] & \frac{(P_b-q_b)}{2s} \cdot \frac{\partial u_b(P_a,P_b)}{\partial P_a} \\ 0 & -\frac{(P_b-q_b)}{2s} \end{bmatrix} > 0}{\det H_{q_a} > 0}. \end{aligned}$$

Since $\partial u_a(P_a, P_b)/\partial P_a < 0$, the preceding numerator always will be positive. Therefore, as q_a increases, P_a increases.

The change in C_b is

$$\delta_{C_b} = \frac{\det \begin{bmatrix} \frac{\partial^2 R(P_a,C_b)}{\partial P_a^2} & -\delta_{q_a} \frac{\partial^2 R(P_a,C_b)}{\partial P_a \partial q_a} \\ \frac{\partial^2 R(P_a,C_b)}{\partial P_a \partial C_b} & -\delta_{q_a} \frac{\partial^2 R(P_a,C_b)}{\partial C_b \partial q_a} \end{bmatrix}}{\det H_{q_a} > 0}$$

$$= \frac{\det \begin{bmatrix} \frac{[C_a-(u_a(P_a,P_b)-r)]}{r} \cdot \frac{\partial u_a(P_a,P_b)}{\partial P_a} & \\ -\frac{(P_a-q_a)}{2r}\left(\frac{\partial u_a(P_a,P_b)}{\partial P_a}\right)^2 & \delta_{q_a}\left[\frac{[C_a-(u_a(P_a,P_b)-r)]}{2r} \cdot \frac{\partial u_a(P_a,P_b)}{\partial P_a}\right] \\ -\frac{(P_b-q_b)}{2s}\left(\frac{\partial u_b(P_a,P_b)}{\partial P_a}\right)^2 & \\ \frac{(P_b-q_b)}{2s} \cdot \frac{\partial u_b(P_a,P_b)}{\partial P_a} & 0 \end{bmatrix}}{\det H_{q_a} > 0}.$$

Since $\partial u_a(P_a, P_b)/\partial P_a < 0$ and $\partial u_a(P_a, P_b)/\partial P_b > 0$, the preceding numerator again is positive by assumption 3. Therefore, as q_a increases, C_b increases.

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