PSYCHOMETRIKA—VOL. 32, NO. 2
JUNE, 1967

THE NUMBER OF WAYS TO LABEL A STRUCTURE*

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It has been observed that the number of different ways in which a graph with p points can be labelled is $p!$ divided by the number of symmetries, and that this holds regardless of the species of structure at hand. In this note, a simple group-theoretic proof is provided.

The article by Harary and Read [1966] concluded with a table listing the probabilities $P(n, k)$ that a connected functional digraph with n points has a cycle of length k, for $n = 2$ to 7. We wish to acknowledge that the entries in this table are given by the formula

$$P(n, k) = \frac{(n-1)!}{(n-k)!}\frac{n^{n-k}}{(n-1)^n} \tag{1}$$

in accordance with the theorem in Katz [1955]. This result was anticipated in turn by Rubin and Sitgreaves in an unpublished memorandum cited in Katz [1955].

In order to contribute something positive in this note, we now prove the theorem about graphs and groups which justifies the formula given in Harary and Read [1966] for the number of ways to label a structure. Since this is a sequel to Harary and Read [1966], its notation and terminology will be used. Thus we write $s(G)$ for the symmetry number of graph G (the order of its automorphism group $\Gamma(G)$) and $l(G)$ for the number of labelings of G. As usual we denote the number of points of G by p.

The notation used in the following proof follows that in Harary [in press] and Harary and Palmer [1965]. Accordingly, S_p is the symmetric group of degree p acting on $X = \{1, 2, \cdots, p\}$; $X^{(2)}$ is the set of unordered pairs of the objects in X; $S_p^{(2)}$ is the *pair group* acting on $X^{(2)}$ as induced by S_p; and E_2 is the identity group on $Y = \{0, 1\}$. The *power group* (introduced in Harary and Palmer [1965]) $E_2^{S_p^{(2)}}$ acts on $Y^{X^{(2)}}$ and each function f from $X^{(2)}$ into Y represents a labeled graph with point set X. Two points $i, j \in X$ are considered adjacent in the graph of f whenever $f(\{i, j\}) = 1$.

*This work was supported by Grant MH 10834 from the National Institute of Mental Health.

In order to present this proof concisely, we assume the basic properties of permutation groups A acting on X. These include the "stabilizer" of an object $x \varepsilon X$ (the subgroup of A which fixes x), the "orbit" of A which contains x (the set of all objects to which x can be mapped by permutations in A), and the "index" of a subgroup B of A (the ratio of the order of A to that of B). We also recall the well known result:

Lemma. The index in the group A of the stabilizer A_x of an object $x \varepsilon X$ is the number of objects in the orbit of A which contains x.

The theorem is stated for graphs, but is easily modified to apply to any type of structure, e.g., trees, directed graphs, tournaments, relations, 1-choice structures (functional digraphs), and nets.

Theorem. The number of different ways in which the points of G can be labeled is:

$$l(G) = \frac{p!}{s(G)}. \tag{2}$$

Proof. Since the theorem is obvious for $p = 1, 2$, we assume $p \geq 3$.

Now let G be the unlabeled graph on p points which corresponds to the function f mentioned above. It is clear that the number of ways in which G can be labeled is simply the number of functions in the orbit of f regarded as an element in the object set of the power group $E_2^{S_p^{(2)}}$. Furthermore, the stabilizer of f in $E_2^{S_p^{(2)}}$ is obviously isomorphic to $\Gamma(G)$. Applying the lemma to this power group, we have the result that the number of ways of labeling G is the order of $E_2^{S_p^{(2)}}$ divided by the order of $\Gamma(G)$, i.e., the index of $\Gamma(G)$ regarded as a subgroup of the power group. The proof is completed by observing that the order of this power group is $p!$ when $p \geq 3$.

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Manuscript received 7/25/66