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THE EVALUATION OF DETERMINANTS

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The numerical evaluation of determinants with a modern computing machine is discussed. Various methods are presented and their relations to each other are indicated. The methods presented parallel those developed in the previous paper on "The Solution of Simultaneous Equations." Especially emphasized are the Abbreviated Doolittle and the Compact methods. Additional topics include the evaluation of partially symmetric determinants by means of symmetric methods and the evaluation of determinantal ratios.

Introduction

Determinants are very useful in discovering the theoretical properties of the solutions of simultaneous equations, but they have not been found very useful in obtaining the numerical solutions. This is particularly true in least squares and correlation theory where approximate solutions only are demanded and where one usually has access to modern computing machines. Thus authors of text books on statistics frequently recommend non-determinantal methods for the numerical solution of normal equations. See, for example, (1, p. 67) (2, p. 36) (3, pp. 119-124). In a pervious article (4) the writer has indicated a number of these solutions.

It is possible to apply these methods to the evaluation of determinants. It is the purpose of this paper to show how this can be done. The reader who is familiar with the earlier paper should have little trouble in understanding the development even though the present paper is somewhat more condensed than the earlier one.

For purposes of brevity we use the fourth-order determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix}, \tag{1}$$

with the implication that the methods outlined are to be extended to the n'th order. As an illustration we take the determinant

$$\Delta = \begin{vmatrix} 1.0 & .4 & .5 & .6 \\ .4 & 1.0 & .3 & .4 \\ .5 & .3 & 1.0 & .2 \\ 6. & .4 & .2 & 1.0 \end{vmatrix}, \tag{2}$$

in which the elements are exact. It is evident from the definition of a determinant that the exact value of Δ is expressible in terms of four decimal places. This value is $\Delta = .3660$. In working with the different methods we attempt to carry the approximations to at least four places.

The determinant above is symmetric. However, it can be used to illustrate a non-symmetric determinant by assuming $a_{ij} \neq a_{ji}$. The reader will note that this determinant (2) is the determinant of the coefficients of the illustration of the paper on the solution of equations (4).

In all the methods which follow it is our plan to eliminate the first column, the second column, etc., in order. It is agreed that such an order is arbitrary. However, the technique is sufficiently general since if the rows and columns have the same order, they may be interchanged in all possible ways.

First Method. Method of Division

In this method, the elements of the first row of Δ are divided by a_{11}, the elements of the second row by a_{21}, etc. The first row of the resulting determinant is then multiplied by -1 and added to the second row, to the third row, and to the fourth row, in turn. We have then

$$\Delta = a_{11}\, a_{12}\, a_{13}\, a_{14} \begin{vmatrix} a'_{22\cdot 1} & a'_{32\cdot 1} & a'_{42\cdot 1} \\ a'_{23\cdot 1} & a'_{33\cdot 1} & a'_{43\cdot 1} \\ a'_{24\cdot 1} & a'_{34\cdot 1} & a'_{44\cdot 1} \end{vmatrix}, \tag{3}$$

where $a'_{ij\cdot 1} = \dfrac{a_{ij}}{a_{1j}} - \dfrac{a_{i1}}{a_{11}}$. If we treat the resulting determinant similarly, we get

$$\Delta = a_{11}\, a_{12}\, a_{13}\, a_{14}\, a'_{22\cdot 1}\, a'_{23\cdot 1}\, a'_{24\cdot 1} \begin{vmatrix} a'_{33\cdot 12} & a'_{43\cdot 12} \\ a'_{34\cdot 12} & a'_{44\cdot 12} \end{vmatrix}, \tag{4}$$

where

$$a'_{ij\cdot 12} = \frac{a'_{ij\cdot 1}}{a'_{2j\cdot 1}} - \frac{a'_{i2\cdot 1}}{a'_{22\cdot 1}};$$

and finally

$$\underline{\Delta} = a_{11}\, a_{12}\, a_{13}\, a_{14}\, a'_{22\cdot 1}\, a'_{23\cdot 1}\, a'_{24\cdot 1}\, a'_{33\cdot 12}\, a'_{34\cdot 12}\, a'_{44\cdot 123} \tag{5}$$

is obtained by multiplying the entries in (5) or, if one prefers, by evaluating the determinant of (4) and multiplying by the indicated values.

The method is illustrated in Table 1 where

$$\underline{\Delta} = (1.0)\,(.4)\,(.5)\,(.6)\,(2.1000)\,(.2000)\,(.2667)\,\{(7.3810)\,(3.8091) - (-.7440)\,(-1.1905)\} \underline{=} .3660\,.$$

TABLE 1

METHOD OF DIVISION

1.0	.4	.5	.6
.4	1.0	.3	.4
.5	.3	1.0	.2
.6	.4	.2	1.0
1.0000	.4000	.5000	.6000
1.0000	2.5000	.7500	1.0000
1.0000	.6000	2.0000	.4000
1.0000	.6667	.3333	1.6667
	2.1000	.2500	.4000
	.2000	1.5000	−.2000
	.2667	−.1667	1.0667
	1.0000	.1190	.1905
	1.0000	7.5000	−.1000
	1.0000	−.6250	3.9996
		7.3810	−1.1905
		−.7440	3.8091
		1.0000	−.1613
		1.0000	−5.1198
			−4.9585
			.3660

This method is the least satisfactory of the various methods presented. The symmetry of the original determinant is lost with the first set of divisions. A large number of divisions is necessary while the method demands $n(n+1)$ rows (though but $n(n+1) - 3$ rows are used if the second-order determinant is evaluated directly).

Second Method. Method of Single Division

In the method of single division, the reduction is accomplished by a division of the first row by its leading element. Thus

$$\Delta = a_{11} \begin{vmatrix} 1 & b_{21} & b_{31} & b_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix}, \tag{6}$$

where $b_{i1} = \dfrac{a_{i1}}{a_{11}}$. We multiply the first row by $-a_{12}$ and add to the second, by $-a_{13}$ and add to the third, by $-a_{14}$ and add to the fourth and get.

$$\Delta = a_{11} \begin{vmatrix} a_{22\cdot 1} & a_{32\cdot 1} & a_{42\cdot 1} \\ a_{23\cdot 1} & a_{33\cdot 1} & a_{43\cdot 1} \\ a_{24\cdot 1} & a_{34\cdot 1} & a_{44\cdot 1} \end{vmatrix}, \tag{7}$$

where $a_{ij\cdot 1} = a_{ij} - \dfrac{a_{i1}\, a_{1j}}{a_{11}}$. Similarly, we eliminate the first row of (7) and get

$$\Delta = a_{11}\, a_{22\cdot 1} \begin{vmatrix} a_{33\cdot 12} & a_{43\cdot 12} \\ a_{34\cdot 12} & a_{44\cdot 12} \end{vmatrix}, \tag{8}$$

where $a_{ij\cdot 12} = a_{ij\cdot 1} - \dfrac{a_{i2\cdot 1}\, a_{2j\cdot 1}}{a_{22\cdot 1}}$. We thus get

$$\Delta = a_{11}\, a_{22\cdot 1}\, a_{33\cdot 12}\, a_{44\cdot 123}\,. \tag{9}$$

The method is illustrated in Table 2(a). To the first four rows of Δ are added the division row. The values in the next grouping are then obtained by subtracting from the element the product of the row heading and the columnar base. The division is made and entered, the next group of rows computed, etc.

$$\Delta = (1.0)\,(.84)\,(.7381)\,(.5903) = .3660\,.$$

A somewhat shorter variation utilizes the conventional method of evaluating a second order determinant. Thus

$$\Delta = (1.0)\,(.84)\,[(.7381)\,(.6095) - (-.1190)^2] = .3660\,.$$

This method is an approximation method since divisions are introduced. It maintains the symmetry property since $a_{ij\cdot 1} = a_{ji\cdot 1}$. It demands $\dfrac{n(n+3)}{2}$ or $[(\dfrac{n(n+3)}{2} - 2)]$ rows. The method is, essen-

TABLE 2

METHOD OF SINGLE DIVISION

(a)				(b)				(c)			
1.0	.4	.5	.6	1.0	.4	.5	.6	1.0	.4	.5	.6
.4	1.0	.3	.4	—	1.0	.3	.4	.4	1.0	.3	.4
.5	.3	1.0	.2	—	—	1.0	.2	.5	.3	1.0	.2
.6	.4	.2	1.0	—	—	—	1.0	.6	.4	.2	1.0
1.0	.4	.5	.6	1.0	.4	.5	.6	1.0	.4	.5	.6
	.84	.10	.16		.84	.10	.16		.84	.10	.16
	.10	.75	−.10		—	.75	−.10		.10	—	—
	.16	−.10	.64		—	—	.64		.16	—	—
	1.0000	.1190	.1905		1.0000	.1190	.1905		1.0000	.1190	.1905
		.7381	−.1190			.7381	−.1190			.7381	−.1190
		−.1190	.6095			—	.6095			−.1190	—
		1.0000	−.1612			1.0000	−.1612			1.0000	−.1612
			.5903				.5903				.5903
			.3660				.3660				.3660

tially, that of Chiò (5). If a leading element is 0, some other order must be chosen.

Third Method. Method of Single Division-Symmetric

In case the determinant is symmetric, it is not necessary to record the values below the main diagonal. The proper entry is found by subtracting from a_{ij} the product of b_{i1} and a_{j1}. The b_{i1} term is at the bottom of the column while the a_{j1} term is obtained by moving to the left to the main diagonal and taking the columnar heading.

The illustration is given in Table 2(*b*).

Fourth Method. Abbreviated Method of Single Division

In the abbreviated method of single division, the first row and the first column only of each new grouping is computed. The proper entry is finally obtained by subtracting one product out of each grouping. Thus

$$a_{43\cdot 12} = (.2) - (.5)(.6) - (.10)(.1905) = \underline{-.1190}\,.$$

See Table 2(*c*) for illustration.

Fifth Method. Abbreviated Method of Single Division-Symmetric

The columnar entries of the fourth method may be eliminated if the determinant is symmetric, since the top rows give the multipliers. Thus

$$a_{43\cdot12} = (.2) - (.6)(.5) - (.10)(.1905) = -.1190$$

$$a_{44\cdot123} = 1.0 - (.6)^2 - (.16)(.1905) - (.1190)(-.1612) = .5903 \ .$$

See Table 3 (*a*).

TABLE 3

ABBREVIATED DOOLITTLE METHOD

(*a*)				(*b*)			
1.0	.4	.5	.6	1.0	.4	.5	.6
—	1.0	.3	.4	—	1.0	.3	.4
—	—	1.0	.2	—	—	1.0	.2
—	—	—	1.0	—	—	—	1.0
1.0	.4	.5	.6	1.0	.4	.5	.6
	.84	.10	.16	1.0	.4	.5	.6
	1.0000	.1190	.1905		.84	.10	.16
		.7381	−.1190		1.0000	.1190	.1905
		1.0000	−.1612			.7381	−.1190
			.5903			1.0000	−.1612
			.3660				.5903
							.3660

Sixth Method. Abbreviated Doolittle Determinantal Method

This method is essentially the same as the last method. The first row is repeated at the first grouping so that the technique is simply to subtract the products of elements in paired rows from the first element. This technique is easily carried out once it is understood. The final evaluation of the determinant, it is remembered, is

$$\Delta = (1.0)(.84)(.7381)(.5903) = .3660.$$

The Abbreviated Doolittle method can be expanded to give the conventional Doolittle method. It appears that Horst (6) was first to evaluate determinants with the use of the Doolittle method. He was chiefly interested in the correlation determinant but his method can be applied more generally. He derived the basic formula for evaluating determinants by any of the variations of the method of single division.

The equivalent of the formula (9) has been used more recently by Reiersøl (7) in computing all the principal minors of a determinant. His notation differs somewhat from the present notation but his technique results from an application of the methods and formulas of the method of single division.

Seventh Method. The Method of Multiplication and Subtraction

It was noticed in method one that the work was somewhat abbreviated by the use of (4) rather than (5). This leads to the suggestion as to the evaluation of the whole determinant by $ab - cd$ methods. This can be done.

The elements of the 2nd, 3rd, and 4th rows of Δ are multiplied by a_{11}, and $\dfrac{1}{a^3_{11}}$ is placed outside the determinant to compensate. The first row is then multiplied by $-a_{12}$ and added to the second, by $-a_{13}$ and added to the third, by $-a_{14}$ and added to the fourth. In the resulting determinant we have

$$\Delta = \frac{1}{a^3_{11}} \begin{vmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ 0 & A_{22\cdot 1} & A_{32\cdot 1} & A_{42\cdot 1} \\ 0 & A_{23\cdot 1} & A_{33\cdot 1} & A_{43\cdot 1} \\ 0 & A_{24\cdot 1} & A_{34\cdot 1} & A_{44\cdot 1} \end{vmatrix}, \tag{10}$$

where $A_{ij\cdot 1} = a_{ij}\, a_{11} - a_{i1}\, a_{1j}$.
Hence

$$\Delta = \frac{1}{a^2_{11}} \begin{vmatrix} A_{22\cdot 1} & A_{32\cdot 1} & A_{42\cdot 1} \\ A_{23\cdot 1} & A_{33\cdot 1} & A_{43\cdot 1} \\ A_{24\cdot 1} & A_{34\cdot 1} & A_{44\cdot 1} \end{vmatrix}. \tag{11}$$

Continuing this process we get finally

$$\Delta = \left(\frac{1}{a^2_{11}}\right)\left(\frac{1}{A_{22\cdot 1}}\right) A_{44\cdot 123}. \tag{12}$$

The successive steps in the computation are shown in Table 4(a).

This method has certain advantages. No divisions are needed for the forward solution so that the computation is eased. Also the value $\Delta = \dfrac{1}{(1.0)^2}\,\dfrac{1}{(.84)}\,(.30744)$ is exact, and the solution $\Delta = .3660$ is exact. Of course, exact values cannot be obtained beyond machine capacity. The method needs $\dfrac{n(n+1)}{2} + 1$ rows.

TABLE 4

METHOD OF MULTIPLICATION AND SUBTRACTION

(a)				(b)				(c)			
1.0	.4	.5	.6	1.0	.4	.5	.6	1.0	.4	.5	.6
.4	1.0	.3	.4	—	1.0	.3	.4	.4	1.0	.3	.4
.5	.3	1.0	.2	—	—	1.0	.2	.5	.3	1.0	.2
.6	.4	.2	1.0	—	—	—	1.0	.6	.4	.2	1.0
	.84	.10	.16		.84	.10	.16		.84	.10	.16
	.10	.75	–.10		—	.75	–.10		.10	—	—
	.16	–.10	.64		—	—	.64		.16	—	—
		.6200	–.1000			.6200	–.1000			.6200	–.1000
		–.1000	.5120			—	.5120			–.1000	—
			.30744				.30744				.30744
			.3660				.3660				.3660

Eighth Method. Method of Multiplication and Subtraction-Symmetric

In case the determinant is symmetric, the entries below the main diagonal may be left blank and the top entries used for computation as in method three. The illustration is given in Table 4(*b*).

Ninth Method. Abbreviated Method of Multiplication and Subtraction

The method can be abbreviated, as in method four, by recording only the entries of the first column and the first row in each matrix. The illustration is presented in Table 4(*c*). For example, the value of $A_{44\cdot 123}$ is

$$A_{44\cdot 123} = \{[1.0000 - (.6000)^2]\,.8400 - (.1600)^2\}\,.6200 - (-.1000)^2 = .30744.$$

Tenth Method. Abbreviated Method of Multiplication and Subtraction-Symmetric

In case symmetry is present it is possible to eliminate many of the rows of the last method. As a matter of fact, but $2n$ rows are needed to indicate the solution. The statement of the problem requires n rows. The next row gives the entries $a_{2j\cdot 1}$ (or $a_{i2\cdot 1}$), the next $a_{3j\cdot 12}$ (or $a_{i3\cdot 12}$), etc, The computational work is shown in Table 5(*a*). For example

$$A_{43\cdot 12} = [(.2000)(1.0000) - (.5000)(.6000)]\,.8400 - (.1000)(.1600) = -.1000\,.$$

A somewhat better form, though it requires an additional row, is indicated in Table 5(*b*). This differs from Table 5(*a*) only in the repetition of the first row at the end of the first n rows. It is then necessary only to take an element a_{ij}, multiply it by the element at the left of the $(n+1)$'st row and subtract the product of the i'th and j'th columnar entries in this row, multiply by the leading entry of the next row, etc. Thus

$$a_{44\cdot 123} = \{[(1.0000)(1.0000) - (.6000)^2]\,.8400 - (.1600)^2\}\,.6200 - (-.1000)^2 = .30744\,,$$

and

$$\Delta = \frac{1}{(1.0000)^2}\,\frac{1}{.8400}\,(.30744) = .3660.$$

TABLE 5

COMPACT METHOD

(a)				(b)			
1.0000	.4000	.5000	.6000	1.0000	.4000	.5000	.6000
—	1.0000	.3000	.4000	—	1.0000	.3000	.4000
—	—	1.0000	.2000	—	—	1.0000	.2000
—	—	—	1.0000	—	—	—	1.0000
	.8400	.1000	.1600	1.0000	.4000	.5000	.6000
		.6200	-.1000		.8400	.1000	.1600
			.30744			.6200	-.1000
			.3660				.30744
							.3660

This method is recommended as a most compact method in obtaining numerical approximations to the values of determinants of high order.

In all these methods a check column can be used.

The Evaluation of Partially Symmetric Determinants by Symmetric Methods

We may define a partially symmetric determinant to be one in which there is a symmetric minor, of order two or more, of any element of the principal diagonal. If this minor is of order one less than the order of the determinant, we may call the determinant "almost symmetric." Thus the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{21} & a_{j1} \\ a_{12} & a_{22} & a_{j2} \\ a_{13} & a_{23} & a_{j3} \end{vmatrix}$$

is "almost symmetric" if $a_{12} = a_{21}$ since the minor of a_{j3} is then symmetric.

We wish to use the symmetric methods in the evaluation of Δ. This is done by inserting columns, for computational use, to make the first n rows and column symmetric. Thus we might evaluate Δ by the use of

$$\begin{matrix} a_{11} & a_{21} & a_{31} & a_{j1} \\ a_{j2} & a_{22} & a_{32} & a_{j2}, \\ a_{13} & a_{23} & — & a_{j3} \end{matrix} \quad \text{with } a_{3i} = a_{i3},$$

and the solution is indicated by

$$\begin{matrix} A_{22\cdot 1} & A_{32\cdot 1} & A_{j2\cdot 1} \\ & — & A_{j3\cdot 12}, \end{matrix}$$

with $\Delta = \dfrac{A_{j3\cdot 12}}{a^2_{11} A_{22\cdot 1}}$.

If symmetry had been lacking in two columns, it would have been necessary to introduce two computational columns, etc.

This device can be adapted to the Abbreviated Doolittle method similarly.

The Simultaneous Evaluation of Almost Symmetric Determinants

It is desired to evaluate a number of determinants which are alike aside from the last column. If the number of these determinants is small, it is advised to use the method above and to add additional columns. Thus the evaluation of

$$\Delta = \begin{vmatrix} 1.0 & .4 & .5 & a_{j1} \\ .4 & 1.0 & .3 & a_{j2} \\ .5 & .3 & 1.0 & a_{j3} \\ .6 & .4 & .2 & a_{j4} \end{vmatrix} \tag{13}$$

is outlined in Table 6 by the compact method for different values of a_{ji}.

TABLE 6

ALMOST SYMMETRIC DETERMINANTS

						check
1.0	.4	.5	.6	.2	.8	3.5
.4	1.0	.3	.4	.4	.6	3.1
.5	.3	1.0	.2	.6	.4	3.0
.6	.4	.2	1.0	.8	.2	3.2
1.0	.4	.5	.6	.2	.8	3.5
	.84	.10	.16	.32	.28	.170
		.6200	−.1000	.3880	−.0280	.8800
			.30744	.36120	−.17640	.49224
			$\Delta =$ .3660	.4300	−.2100	.5860

When $a_{j1} = .6$, $a_{j2} = .4$, $a_{j3} = .2$, $a_{j4} = 1.0$, then $\Delta = .3660$.

When $a_{j1} = .2$, $a_{j2} = .4$, $a_{j3} = .6$, $a_{j4} = .8$, then $\Delta = .4300$.

When $a_{j1} = .8$, $a_{j2} = .6$, $a_{j3} = .4$, $a_{j4} = .2$, then $\Delta = -.2100$.

From the check column, when $a_{j1} = 3.5$, $a_{j2} = 3.1$, $a_{j3} = 3.0$, $a_{j4} = 3.2$, then $\Delta = .5860$.

If large numbers of these determinants are desired, however, it is preferable to divide the last column into its a_{j1}, a_{j2}, a_{j3}, a_{j4}, components, leaving a column for each component. The specified values a_{ji} can be inserted in the result. The form is shown in Table 7 where

TABLE 7 (a)

SIMULTANEOUS DETERMINANTS—ABBREVIATED DOOLITTLE METHOD

				a_{j1}	a_{j2}	a_{j3}	a_{j4}	check
1.0	.4	.5	.6	1	0	0	0	3.5
—	1.0	.3	.4	0	1	0	0	3.1
—	—	1.0	.2	0	0	1	0	3.0
—	—	—	1.0	0	0	0	1	3.2
1.0	.4	.5	.6	1	0	0	0	3.5
1.0	.4	.5	.6	1	0	0	0	3.5
	.84	.10	.16	−.40	1.00	0	0	1.70
	1.0000	.1190	.1905	−.4762	1.1905	0	0	2.0238
		.7381	−.1190	−.4524	−.1190	1.0000	.0000	1.0476
		1.0000	−.1612	−.6129	−.1612	1.3548	.0000	1.4193
			.5903	−.5966	−.2097	.1612	1.0000	.9451
			.3660	−.3699	−.1300	.1000	.6200	.5860

the fourth column is inserted to make possible symmetriic methods. The Abbreviated Doolittle method is shown in Table 7(*a*), while the "compact" method is presented in Table 7(*b*).

TABLE 7 (*b*)

SIMULTANEOUS DETERMINANTS—COMPACT

				a_{j1}	a_{j2}	a_{j3}	a_{j4}	check
1.0	.4	.5	.6	1	0	0	0	3.5
—	1.0	.3	.4	0	1	0	0	3.1
—	—	1.0	.2	0	0	1	0	3.0
—	—	—	1.0	0	0	0	1	3.2
1.0	.4	.5	.6	1	0	0	0	3.5
	.84	.10	.16	–.40	1.00	0	0	1.70
		.6200	–.1000	–.38	–.10	.84	0	.8800
			.30744	–.3108	–.1092	.0840	.5208	.49224
			.3660	–.3700	–.1300	.1000	.6200	.5860

It follows that the value of Δ is

$$\underline{\Delta} = -.3700a_{j1} - .1300a_{j2} + .1000a_{j3} + .6200a_{j4} .$$

The values of Δ, for the value of a_{ji} used in Table 6, and for other values of a_{ji}, are given in Table 8.

TABLE 8

	$i=1$	$i=2$	$i=3$	$i=4$	Δ
A_{j1}	–.3700	–.1300	.1000	.6200	–.2100
a_{ji}	.6	.4	.2	1.0	.3660
a_{ji}	.2	.4	.6	.8	.4300
a_{ji}	.8	.6	.4	.2	–.2100
a_{ji}	3.5	3.1	3.0	3.2	.5860
a_{ji}	.5	.3	1.0	.2	.0000
a_{ji}	.4	1.0	.3	.4	.0000
a_{ji}	1.0	.4	.5	.3	.0000
a_{ji}	1.0	1.0	1.0	1.0	.2200
			etc.		

In general $\Delta = \sum a_{ji} A_{ji}$, where the A_{ji} are determined. It is to be noted that the quantities A_{ji} are the co-factors of a_{ji} and hence that the method is a method of determining the co-factors of the elements of a given column.

The Evaluation of Determinantal Ratios

In the solution of equations it is the ratio of two determinants, rather than the value of a given determinant, which is desired. The methods outlined above are useful in solving for determinantal ratios and, indeed, the forward solutions of the ten methods of solving simultaneous equations are immediately obtained. It is hence improper to call these "non-determinantal solutions" since these are the solutions which improved determinantal methods indicate.

Consider the equations

$$a_{11}\,x_1 + a_{21}\,x_2 + a_{31}\,x_3 = a_{j1}$$

$$a_{12}\,x_1 + a_{22}\,x_2 + a_{32}\,x_3 = a_{j2}$$

$$a_{13}\,x_1 + a_{23}\,x_2 + a_{33}\,x_3 = a_{j3}\,.$$

If we solve for x_3, the denominator is the determinant of the coefficients while the numerator is the denominator with a_{3i} replaced by a_{ji}. It can be shown that the determinantal ratios become

(a) in the method of division: $x_3 = \dfrac{a'_{j3\cdot 12}}{a_{33\cdot 12}}$;

(b) in the methods of single division: $x_3 = \dfrac{a_{j3\cdot 12}}{a_{33\cdot 12}}$;

(c) in the methods of multiplication and subtraction: $x_3 = \dfrac{A_{j3\cdot 12}}{A_{33\cdot 12}}$;

and these agree with the results obtained in the study of the solution of equations.

It is also possible to evaluate determinatal ratios when the determinants are not of the same order. In multiple correlation theory, for example, it is desired to evaluate the ratio of a determinant to one of its principal minors. As an illustration we desire to find $\dfrac{\Delta}{\Delta_{44}}$. If the method of single division, or any of its variations, is used, we have

$$\frac{\Delta}{\Delta_{44}} = \frac{a_{11}\,a_{22\cdot 1}\,a_{33\cdot 12}\,a_{44\cdot 123}}{a_{11}\,a_{22\cdot 1}\,a_{33\cdot 1}} = a_{44\cdot 123}\,.$$

If the method of multiplication and subtraction, or any of its variations, is used, we have

$$\frac{\Delta}{\Delta_{44}} = \frac{A_{44\cdot 123}}{a_{11}\,A_{22\cdot 1}\,A_{33\cdot 12}}$$

Thus, from Table 3,

$$\frac{\Delta}{\Delta_{44}} = .5903$$

while from Table 5.

$$\frac{\Delta}{\Delta_{44}} = \frac{.30744}{(1.0000)(.8400)(.6200)} = \frac{.30744}{.5208} = .59032+.$$

In general

$$\begin{cases} \dfrac{\Delta}{\Delta_{KK}} = a_{KK\cdot 12\cdots\overline{K-1}} \\[2ex] \dfrac{\Delta}{\Delta_{KK}} = \dfrac{A_{KK\cdot 12\cdots\overline{K-1}}}{a_{11}\,A_{22\cdot 1}\,A_{33\cdot 12}\cdots A_{\overline{K-1}\,\overline{K-1}\cdot 12\cdots\overline{K-2}}}. \end{cases}$$

Conclusion

It is shown that the same methods can be used in evaluating determinants that are used in solving simultaneous equations. Especially to be recommended are the "Abbreviated Doolittle" and "Compact" methods if the determinant is symmetric or almost symmetric.

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