Further Results Concerning the Simple Queueing Loss System

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Summary: The output or departure process from a single channel queue with independent arrival intervals, exponential service times and no storage is considered. The main result obtained is that successive departure intervals are uncorrelated if and only if the arrival process is Poisson.

Zusammenfassung: Der output bzw. der Abgang eines einzelnen Wartekanals mit unabhängigen Ankunftsintervallen, exponentieller Bedienungszeit und ohne Speicherung wird untersucht. Als wesentliches Resultat ergibt sich, daß sukzessive Abgangsintervalle genau dann unkorreliert sind, wenn der Ankunftsprozeß Poisson-verteilt ist.

Schmidt proves that in the simple queueing loss system GI/G/1/0 the output or departure process is a renewal process if and only if GI $=$ M, that is the arrival process is Poisson. In order to obtain this result, however, it must be assumed that the probability density functions of the arrival and service processes are both continuous and strictly positive on $[0, \infty]$. These restrictions eliminate from consideration a large class of important density functions including the gamma, uniform and deterministic.

We consider a less general queue, the GI/M/1/0 and without the above restrictions prove that the departure process is uncorrelated if and only if GI $=$ M. We assume arrivals occur at epochs $\{t_n; n \geq 0\}$. Let $Y_n = t_n - t_{n-1}$, $n > 0$,

$A(x) = P[Y_n \leq x] \forall_n$, $A(0+) = 0$ and $a^*(\theta) = \int\limits_0^\infty e^{-\theta x} \, dA(x)$. Let $a_j(x)$ be the

probability density function of the sum of j consecutive arrival intervals, $j = 1, 2, \ldots$. The service times $\{S_n; n \geq 0\}$ form a sequence of independent, identically distributed random variables independent of the arrival process with distribution

function $B(x) = 1 - e^{-\mu x}$, $b^*(\theta) = E[e^{-\theta S_n}] = \dfrac{\mu}{\mu + \theta}$. Let $\{I_n; n \geq 1\}$ be the

sequence of idle periods that occur immediately after each departure, $F_I(x) =$

$P[I_n \leq x] \forall_n, f_I^*(\theta) = \int\limits_0^\infty e^{-x} \, dF_I(x)$. Let $\{U_n; n \geq 0\}$ be the sequence of periods

representing the elapsed time since the last arrival measured from the nth departure epoch, $F_U(x) = P[U_n \leq x] \forall_n$. Finally let $\{X_n; n \geq 0\}$ be the sequence of departure intervals from the queue with $F(x) = P[X_n \leq x] \forall_n$.

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Theorem:

The sequence $\{X_n; n \geq 0\}$ is uncorrelated if and only if GI $= M$, (Poisson arrivals).

Proof:

If GI $= M$ then clearly the sequence $\{X_n; n \geq 0\}$ is uncorrelated.

If the sequence $\{X_n; n \geq 0\}$ is uncorrelated then $E[X_{n+1}X_n] = E[X_{n+1}]E[X_n]$, $n \geq 0$. But $E[X_{n+1}X_n] = E[(S_{n+1} + I_{n+1})X_n] = E[S_{n+1}]E[X_n] + E[I_{n+1}X_n]$ since S_{n+1} and X_n are independent. Also $E[X_{n+1}]E[X_n] = E[S_{n+1}]E[X_n] + E[I_{n+1}]E[X_n]$. Thus the condition for X_n and X_{n+1} to be uncorrelated is $E[I_{n+1}X_n] = E[I_{n+1}]E[X_n]$.

From SCHMIDT

$$[1] \quad F_I(x) = \int_0^\infty \int_0^\infty \frac{e^{-\mu y} dA(y + w) dw}{1 - a^*(\mu)}$$

and thus

$$E[I_{n+1}] = \frac{\dfrac{\mu}{\lambda} - (1 - a^*(\mu))}{(1 - a^*(\mu))},$$

and

$$E[X_n] = \frac{1}{\mu} + E[I_{n+1}] = \frac{1}{\lambda(1 - a^*(\mu))}.$$

Finally,

$$E[I_{n+1}]E[X_n] = \frac{\dfrac{\mu}{\lambda} - (1 - a^*(\mu))}{\mu\lambda[1 - a^*(\mu)]^2}. \tag{1}$$

In order to find $E[I_{n+1}X_n]$ we first find the joint probability density function of I_{n+1} and X_n. Let $f(x,w)$ be the joint probability density function of I_{n+1} and X_n and let $g(y,w)$ be the joint probability density function of U_n and X_n.

Then

$$f(x,w) = \int_0^w \frac{a(x + y)}{1 - A(y)} g(y, w) dy. \tag{2}$$

In order to find $g(y,w)$ let $g(y,w/t)$ be the conditional joint probability density function of U_n and X_n given $U_{n-1} = t$, $n \geq 1$. Then

$$g(y,w/t) = \frac{[1 - A(y)]e^{-\mu y}}{1 - A(t)} \left\{ a(w - y + t) + \int_0^{w-y} e^{-\mu\xi} a(w - y - \xi + t) \sum_{j=1}^\infty a_j(\xi) d\xi \right\} \tag{3}$$

and

$$g(y,w) = \int_0^\infty g(y,w/t) dF_U(t).$$

But

$$
\begin{aligned}
F_U(t) &= P[U_n \le t] = P[S_n \le t, S_n < Y_{n+1}] \\
&\quad + P[S_n \le Y_{n+1} + t, Y_{n+1} < S_n < Y_{n+1} + Y_{n+2}] \\
&\quad + P[S_n < Y_{n+1} + Y_{n+2} + t, Y_{n+1} + Y_{n+2} < S_n < Y_{n+1} + Y_{n+2} + Y_{n+3}] \\
&\quad + \cdots = \int_0^t \mu e^{-\mu y}[1 - A(y)]\,dy + \int_0^\infty \int_0^t \mu e^{-\mu(y_1 + y_2)}[1 - A(y_1)]\,dy_1\,dA(y_2) \\
&\quad + \int_0^\infty \int_0^\infty \int_0^x \mu e^{-\mu(y_1 + y_2 + y_3)}[1 - A(y_1)]\,dy_1\,dA(y_2)\,dA(y_3) \\
&= \frac{\displaystyle\int_0^t \mu e^{-\mu y}[1 - A(y)]\,dy}{1 - a^*(\mu)} .
\end{aligned}
\tag{4}
$$

Combining (2), (3) and (4),

$$
E[e^{-\theta_1 I_{n+1}} e^{-\theta_2 X_n}] = \int_0^\infty \int_0^\infty \int_0^w \mu^2 \frac{e^{-\theta_1 x} e^{-\theta_2 w} a(x + y)}{1 - a^*(\mu)} .
$$

$$
\left\{ \int_0^\infty e^{-\mu(y+t)} a(w - y + t)\,dt + \int_0^{w-y} \int_0^{w-y} e^{-\mu(\xi + y + t)} a(w - y - \xi + t) \right.
$$

$$
\left. \sum_{j=1}^\infty a_j(\xi)\,d\xi\,dt \right\} dy\,dx\,dw
$$

which upon simplification reduces to

$$
\frac{\mu^2 [a^*(\theta_2) - a^*(\mu)][a^*(\theta_2 + \mu) - a^*(\theta_1)]}{(\mu - \theta_2)(1 - a^*(\mu))[1 - a^*(\mu + \theta_2)][\theta_1 - (\theta_2 + \mu)]} .
$$

Then

$$
\begin{aligned}
E[I_{n+1} X_n] &= \frac{\partial^2}{\partial \theta_1 \partial \theta_2} E[e^{-\theta_1 I_{n+1}} e^{-\theta_2 X_n}] \Big|_{\theta_1 = \theta_2 = 0} \\
&= \frac{-\dfrac{\mu^2}{\lambda} a^{*\prime}(\mu) - [1 - a^*(\mu)]^2 + \dfrac{\mu}{\lambda}\left[\dfrac{\mu}{\lambda} - (1 - a^*(\mu))\right]}{\mu^2 [1 - a^*(\mu)]^2} .
\end{aligned}
\tag{5}
$$

Equating (5) with (1) and simplifying results in the Ricatti type differential equation

$$
\frac{\mu^2}{\lambda} a^{*\prime}(\mu) + [1 - a^*(\mu)] = 0
$$

whose solution is $a^*(\mu) = \dfrac{\lambda}{\lambda + \mu}$ which implies arrival intervals have a negative exponential distribution with parameter λ.

Bibliography

SCHMIDT, G.: Über die in einem einfachen Verlustsystem induzierten stochastischen Prozesse, Unternehmensforschung, 11, 1967, pp. 95–110.