Math. Z. (2006)
DOI 10.1007/s00209-005-0931-5

Mathematische Zeitschrift

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Asymptotic upper curvature bounds in coarse geometry

Received: 13 September 2005 / Accepted: 25 September 2005 / Published online: 28 March 2006

Abstract We define a notion of an asymptotic upper curvature bound for Gromov hyperbolic metric spaces that is invariant under rough-isometries and examine the basic properties of this concept.

1 Introduction

One of the fundamental problems of Riemannian geometry is the investigation of how restrictions on the curvature of a space affect its geometry. The impact of upper and lower bounds for the sectional curvature is rather well understood. For example, one knows that the sectional curvature of a Riemannian manifold is pointwise bounded by a constant κ from above precisely when small geodesic triangles are in a suitable sense "thinner" than comparison triangles in a model space of constant sectional curvature κ. This fact can be used as a basis of a definition and has led to the theory of metric spaces of curvature bounded from above in the sense of Alexandrov.

In the present paper we address the problem whether an appropriate theory of spaces with upper curvature bounds can also be developed in the context of coarse geometry. While in differential or metric geometry two spaces are considered indistinguishable if they are isometric, one relaxes this notion of equivalence between metric spaces in coarse geometry and is only concerned with the geometric features

M.B. was supported by NSF grants DMS 0200566 and DMS 0244421. Th.F. was supported by the Deutsche Forschungsgemeinschaft (FO 353/1-1).

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of the spaces on large scales. Here one often considers two spaces X and Y as equivalent if they are quasi-isometric. This means that there exists a mapping between X and Y that changes sufficiently large distances by at most a fixed multiplicative constant and satisfies an additional surjectivity condition (see Section 2). Since a rescaling of the metric of a smooth Riemannian manifold changes its curvature by a multiplicative factor, it is clear that the class of quasi-isometries is too large if one wants to define a reasonable concept of upper curvature bounds in coarse geometry that is invariant under this class.

In this paper we succeed in establishing a notion of upper curvature bounds for a large class of negatively curved spaces that is invariant under *rough-isometries*. In contrast to quasi-isometries, rough-isometries can change distances by at most a fixed additive amount. The class of spaces we consider are Gromov hyperbolic metric spaces.

Now in a geodesic Gromov hyperbolic space all geodesic triangles are rough-isometric to tripods. In particular, all triangles with given side-lengths look the same, namely like geodesic triangles in a metric tree. This means that if we want to introduce a notion of an upper curvature bound for Gromov hyperbolic spaces that is invariant under rough-isometries, then the definition cannot be based on triangle comparison statements similar in spirit to Alexandrov's definition of an upper curvature bound. More generally, all configurations of finitely many points and geodesics in a Gromov hyperbolic space are rough-isometric to configurations in a metric tree. This excludes any reasonable definition of an upper curvature bound with the desired invariance property based on finitely many points or configurations of geodesics. So the definition necessarily has to involve configurations of infinitely many points and geodesics, or a requirement on some asymptotic behavior as the number of elements in the configuration becomes arbitrarily large. Our definition of an upper curvature bound uses an asymptotic condition of this type.

To state this definition recall that the Gromov product $(x \cdot y)_p$ of points x and y in a metric space (X, d) with respect to a basepoint $p \in X$ is defined as

$$(x \cdot y)_p := \frac{1}{2}\big(d(p, x) + d(p, y) - d(x, y)\big). \tag{1}$$

The space X is called δ-hyperbolic, where $\delta \geq 0$, if there exists a basepoint $p \in X$ such that for all $x, y, z \in X$ we have

$$(x \cdot z)_p \geq \min\{(x \cdot y)_p, (y \cdot z)_p\} - \delta. \tag{2}$$

The dependence on the basepoint is not very serious in this definition. If there exists $p \in X$ such that inequality (2) is valid for all $x, y, z \in X$, then inequality (2) is true for all $p, x, y, z \in X$ if we replace δ by 2δ.

The space X is called Gromov hyperbolic if it is δ-hyperbolic for some $\delta \geq 0$. By applying (2) repeatedly one can show that in a δ-hyperbolic space there exists a constant $a \geq 0$ only depending on δ such that

$$(z \cdot z')_p \geq \min_{i=1,\ldots,n} (x_{i-1} \cdot x_i)_p - a \log n, \tag{3}$$

whenever $p, z, z' \in X$ and $x_0 = z, x_1, \ldots, x_n = z'$ is a chain (i.e., a finite sequence) of points in X with first element z and last element z'.

Our definition of an upper curvature bound is based on the constant a in (3).

Definition 1.1 Let X be a metric space, and $\kappa \in [-\infty, 0)$. We say that X has an asymptotic upper curvature bound κ if there exist $p \in X$ and a constant $c \geq 0$ such that for all $z, z' \in X$ and all chains $x_0 = z, x_1, \ldots, x_n = z'$ in X,

$$(z \cdot z')_p \geq \min_{i=1,\ldots,n} (x_{i-1} \cdot x_i)_p - \frac{1}{\sqrt{-\kappa}} \log n - c. \tag{4}$$

Here we interpret $1/\sqrt{\infty} = 0$. For convenience we say that X is an $\mathrm{AC}_u(\kappa)$-space if X has an asymptotic upper curvature bound $\kappa < 0$. If X is an $\mathrm{AC}_u(\kappa)$-space and $\kappa \leq \kappa' < 0$, then X is also an $\mathrm{AC}_u(\kappa')$-space.

Note that every $\mathrm{AC}_u(\kappa)$-space is Gromov hyperbolic. Conversely, inequality (3) implies that every Gromov hyperbolic space is an $\mathrm{AC}_u(\kappa)$-space for some $\kappa < 0$.

Definition 1.2 Let X be a Gromov hyperbolic metric space. Then

$$K_u(X) := \inf\{\kappa : X \text{ is an } \mathrm{AC}_u(\kappa)\text{-space}\} \in [-\infty, 0)$$

is called the asymptotic upper curvature of X.

Under rough-isometries, Gromov products only change by a fixed additive amount at most. Since this additive ambiguity can be absorbed in the constant c in (4), it is clear that if two metric spaces are rough-isometric, then one of them is an $\mathrm{AC}_u(\kappa)$-space if and only if the other one is as well. In particular, rough-isometric Gromov hyperbolic spaces have the same asymptotic upper curvature. In Proposition 3.4 we derive a sharp statement about the change of asymptotic upper curvature bounds under quasi-isometries.

Geodesic $\mathrm{AC}_u(\kappa)$-spaces can be characterized by an asymptotic "slimness" condition for geodesic polygons (see Section 3 for precise definitions).

Theorem 1.3 *Let X be a geodesic metric space, and $\kappa \in [-\infty, 0)$. Then the following conditions are equivalent:*

(i) *X is an $\mathrm{AC}_u(\kappa)$-space.*
(ii) *There exists $c' \geq 0$ such that every geodesic $(n+1)$-gon in X, $n \in \mathbb{N}$, $n \geq 2$, is Δ-slim with* $\Delta = \dfrac{1}{\sqrt{-\kappa}} \log n + c'$.

The methods for the proof of this theorem can also be used to study the behavior of inequality (4) under a change of the basepoint p. It is easy to see that (4) implies that a corresponding inequality holds for all basepoints p' with an additive constant c depending on p'. It turns out—and this is more difficult to establish—that in an $\mathrm{AC}_u(\kappa)$-space, inequality (4) holds for all basepoints and all chains with a constant c *independent* of the basepoint (cf. Proposition 3.3).

For $\kappa < 0$ and $n \in \mathbb{N}$ we denote n-dimensional real hyperbolic space of constant curvature κ by $\mathbb{H}^n_\kappa$. We let $\mathbb{H}^n = \mathbb{H}^n_{-1}$. The following proposition shows that our notion of curvature has some of the desired properties.

Proposition 1.4 (i) *Suppose X is a geodesic metric space. If X is a* $\mathrm{CAT}(\kappa)$-*space, $\kappa < 0$, then X is an $\mathrm{AC}_u(\kappa)$-space.*

(ii) *Let M be a Cartan-Hadamard manifold of pinched sectional curvature k, $-b^2 \leq k \leq -a^2 < 0$. Then the metric space (M, d), where d denotes the distance function on M induced by the Riemannian structure, is an* $\mathrm{AC}_u(\kappa)$*-space for $\kappa = -a^2$, but not for any $\kappa < -b^2$. In particular,*

$$-b^2 \leq K_u(M, d) \leq -a^2.$$

(iii) *If $k \in (-\infty, 0)$ and $n \in \mathbb{N}$, $n \geq 2$, then $\mathbb{H}^n_k$ is an* $\mathrm{AC}_u(\kappa)$*-space for $\kappa = k$, but not for any $\kappa < k$. In particular,*

$$K_u(\mathbb{H}^n_k) = k.$$

One can relate our notion of asymptotic upper curvature of a space X to its geometry at infinity, more precisely with the class of visual metrics on the boundary at infinity of X. We need the additional assumption that X is visual (see Section 2). Roughly speaking, a Gromov hyperbolic geodesic metric space X is visual if there exists a basepoint $p \in X$ such that each $x \in X$ lies uniformly close to a geodesic ray connecting p with the boundary at infinity $\partial_\infty X$ of X. This condition ensures that the boundary at infinity determines the geometry of the space up to rough-isometry.

For visual Gromov hyperbolic spaces the asymptotic upper curvature admits an interpretation in terms of a critical exponent.

Theorem 1.5 *Let X be a visual Gromov hyperbolic metric space.*

If there exists a visual metric on $\partial_\infty X$ with parameter $\epsilon > 0$, then X is an $\mathrm{AC}_u(\kappa)$*-space with $\kappa = -\epsilon^2$. Conversely, if X is an* $\mathrm{AC}_u(\kappa)$*-space, then for every $0 < \epsilon < \sqrt{-\kappa}$ there exists a visual metric on $\partial_\infty X$ with parameter ϵ.*

In particular,

$$K_u(X) = -a^2,$$

where

$$a := \sup\{\epsilon > 0 : \text{ there exists a visual metric on } \partial_\infty X \text{ with parameter } \epsilon\}.$$

For the precise definition of visual metrics and their visual parameters see Section 2.

Combined with some results from [BS] the previous theorem leads to the following embedding result for $\mathrm{AC}_u(\kappa)$-spaces if one imposes an additional "finite dimensionality" condition on the space (see Section 4 for the terminology used in the next theorem and related discussions).

Theorem 1.6 *Let $\kappa \in [-\infty, 0)$, and X be a geodesic* $\mathrm{AC}_u(\kappa)$*-space of bounded growth at some scale. Then for every κ' with $\kappa < \kappa' < 0$ there exists $n \in \mathbb{N}$ such that X is rough-isometric to a convex subset of hyperbolic space $\mathbb{H}^n_{\kappa'}$.*

In particular, X admits a rough-isometric embedding into $\mathbb{H}^n_{\kappa'}$. This statement is optimal in the sense that there are spaces X satisfying the conditions of the theorem that do not admit a rough-isometric embedding into a hyperbolic space of constant curvature κ (see Example 5.3).

In the other direction, every space admitting a rough-isometric embedding into $\mathbb{H}^n_\kappa$ or any $\mathrm{CAT}(\kappa)$-space is an $\mathrm{AC}_u(\kappa)$-space by Proposition 1.4.

Spaces X for which the extreme case $K_u(X) = -\infty$ occurs are characterized by the following rigidity result.

Theorem 1.7 *Let X be a Gromov hyperbolic geodesic metric space of bounded growth at some scale. If $K_u(X) = -\infty$, then X is rough-isometric to a convex subset of a regular tree T_l, $l \geq 2$.*

In the converse direction a stronger statement is true. Every simplicial tree is 0-hyperbolic, and hence an $\mathrm{AC}_u(-\infty)$-space. Therefore, every metric space X that is rough-isometric to a simplicial tree has the same property. In particular, $K_u(X) = -\infty$.

The concept of asymptotic upper curvature has applications in geometric group theory. One of the basic objects of investigation in this theory is the Cayley graph $C(\Gamma, S)$ associated with a finitely generated group Γ and a symmetric set S of generators of Γ (see Section 7). The following theorem is a simple consequence of our results.

Theorem 1.8 *Let Γ be a finitely generated group.*

If there exists a finite symmetric set S generating Γ with $K_u(C(\Gamma, S)) = -\infty$, then Γ is virtually free.

Conversely, if Γ is virtually free and S is a finite symmetric set generating Γ, then $C(\Gamma, S)$ is an $\mathrm{AC}_u(-\infty)$-space.

There are various other questions that can be studied in relation with the asymptotic upper curvature, for example, its behavior under constructions such as gluings along quasi-convex sets or hyperbolic products. A natural problem is also whether one can establish a corresponding concept of asymptotic lower curvature bounds. We hope to explore some of these issues in the future.

Outline of the paper: We start in Section 2 by setting up notation and recalling a number of basic definitions and facts. The relation between slimness conditions for geodesic polygons and asymptotic upper curvature bounds is discussed in Section 3 which includes a proof of Theorem 1.3.

In Section 4 we prove Proposition 1.4 and Theorem 1.5. Embedding theorems for Gromov hyperbolic spaces are treated in Section 5, where a proof of Theorem 1.6 can be found. The rigidity result Theorem 1.7 is the subject of Section 6, while the final Section 7 is devoted to hyperbolic groups including a proof of Theorem 1.8.

2 Preliminaries

Suppose (X, d_X) and (Y, d_Y) are metric spaces. A map $f : X \to Y$ is called a quasi-isometric embedding of X into Y if there exist constants $\lambda \geq 1$ and $k \geq 0$ such that

$$\frac{1}{\lambda} d_X(x, x') - k \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + k \tag{5}$$

for all $x, x' \in X$. If in addition, for each $y \in Y$ there exists $x \in X$ such that

$$d_Y(f(x), y) \leq k, \tag{6}$$

then f is called a quasi-isometry.

The notion of a rough-isometric embedding or a rough-isometry is defined similarly if we let $\lambda = 1$. Finally, the map f is called a bi-Lipschitz embedding if (5) holds with $k = 0$, and a bi-Lipschitz homeomorphism if it is surjective in addition. If we want to emphasize the parameters in these and similar definitions, then we speak of (λ, k)-quasi-isometries, etc.

These classes of maps lead to natural notions of equivalence for metric spaces. For example, we say that two metric spaces are rough-isometric if there exists a rough-isometry between them. To indicate that the metric spaces X and Y are rough-isometric, we use the notation $X \cong Y$.

A subset M of a metric space (X, d) is called cobounded if there exists a constant $k \geq 0$ such that for each point $x \in X$ there is a point $m \in M$ with $d(x, m) \leq k$. By (6) the image set of a quasi-isometry is cobounded in its target.

As a general source for the theory of Gromov hyperbolic spaces we refer to [GH]; most of the basic facts about these spaces that we need in the following can be found in this reference.

To each Gromov hyperbolic metric space (X, d) one can associate a boundary at infinity $\partial_\infty X$. A Gromov product $(\xi \cdot \xi')_p \in [0, \infty]$ can also be defined for points $\xi, \xi' \in \partial_\infty X$, $p \in X$. Here $(\xi \cdot \xi')_p = \infty$ if and only if $\xi = \xi'$.

A geodesic segment in a metric space X is the image of an isometric embedding of a compact interval in $\mathbb{R}$ into X. We denote any geodesic segment in X with endpoints x and y by $[x, y]$. If any two points in X can be connected by a geodesic segment, then X is called a geodesic metric space. A geodesic ray in X is the image of an isometric embedding of $[0, \infty) \subseteq \mathbb{R}$ into X. Based on k-rough-isometric embeddings, one can similarly define k-rough geodesic segments, k-rough geodesic rays, and the concept of a k-rough geodesic metric space. We use the notation $[x, y]_k$ for a k-rough geodesic segment in X with endpoints $x, y \in X$. If a metric space is k-rough geodesic for some $k \geq 0$, then it is called a rough geodesic metric space.

If X is Gromov hyperbolic, every k-rough geodesic ray converges to a unique point $\xi \in \partial_\infty X$. We write $[x, \xi]_k$ for such a ray if it starts at $x \in X$. Similarly, we denote by $[x, \xi]$ any geodesic ray starting at $x \in X$ and converging to $\xi \in \partial_\infty X$.

Conversely, if X is δ-hyperbolic and k-rough geodesic, then for every $x \in X$ and $\xi \in \partial_\infty X$ there exists a ray $[x, \xi]_{k'}$, where k' only depends on δ and k [BS, Prop. 5.2].

By definition a (possibly degenerate) tripod is a union $T = [0, q_1] \cup [0, q_2] \cup [0, q_3]$, $q_1, q_2, q_3 \in \mathbb{R}^2$, of three segments in $\mathbb{R}^2$ that have only the origin in common. The point 0 is called the center of the tripod, and the points q_1, q_2, q_3 its vertices. We think of the tripod T as being equipped with the natural path metric that agrees with the Euclidean metric on the segments $[0, q_i]$, $i = 1, 2, 3$.

Suppose $\Delta = [p, x] \cup [p, y] \cup [x, y]$ is a geodesic triangle in a geodesic metric space (X, d). Then there exist an essentially unique (up to isometry) tripod T and a map $f : \Delta \to T$ that sends vertices to vertices and is an isometry if restricted to any of the sides of Δ. We call such a map f a tripod map. It is useful to know that the points $u \in [p, x]$ and $v \in [p, y]$ with

$$d(p, u) = d(p, v) = (x \cdot y)_p$$

are mapped to the center of the tripod by f.

Geodesic triangles in Gromov hyperbolic spaces look like tripods. A quantitative version of this fact can be formulated as follows. Suppose (X, d) is a δ-hyperbolic geodesic metric space, $\Delta \subseteq X$ a geodesic triangle, and $f\colon \Delta \to T$ a tripod map. Then $d(u, v) \leq 8\delta$, whenever $u, v \in \Delta$ and $f(u) = f(v)$. (See [GH, Ch. 2, §3], in particular Prop. 21. Notice that our definition of a δ-hyperbolic space disagrees with the one given in [GH] in that we require (2) only for some basepoint p.) We will refer to this property as the thinness property of geodesic triangles in Gromov hyperbolic spaces.

If (X, d) is an arbitrary metric space, then the triangle inequality implies that

$$(x \cdot y)_p \leq \min\{d(p, x), d(p, y)\}$$

for all $p, x, y \in X$.

If X is a δ-hyperbolic geodesic metric space, then there exists a constant C only depending on δ such that

$$\big|(x \cdot y)_p - \mathrm{dist}(p, [x, y])\big| \leq C,$$

for all $p, x, y \in X$ and all geodesic segments $[x, y]$.

A metric space X is said to be visual if there exist $k \geq 0$ and a basepoint $p \in X$ such that each point in X lies on a k-rough geodesic ray emanating from p. Note that by increasing k if necessary, one can assume that the basepoint here is any given point in the space. Every visual Gromor hyperbolic space is tough geodesic [BS, Prop. 5.6].

A metric ρ on the boundary $\partial_\infty X$ of a Gromov hyperbolic space X is called visual if there exist $p \in X$, $\lambda \geq 1$, and $\epsilon > 0$ such that

$$\frac{1}{\lambda} \exp(-\epsilon(\xi \cdot \xi')_p) \leq \rho(\xi, \xi') \leq \lambda \exp(-\epsilon(\xi \cdot \xi')_p) \tag{7}$$

for all $\xi, \xi' \in \partial_\infty X$. We use the convention $\exp(-\infty) = 0$. If an inequality of this type is valid, then ϵ is called a (visual) parameter of the metric ρ.

If X is δ-hyperbolic, then there exists a visual metric ρ with parameter ϵ if $\epsilon > 0$ is small enough depending on δ. Two visual metrics ρ_1 and ρ_2 are "snowflake" equivalent; more precisely, if ϵ_i is a parameter for ρ_i, $i = 1, 2$, then there exists a constant $C \geq 1$ such that

$$\frac{1}{C}\rho_1^\alpha \leq \rho_2 \leq C\rho_1^\alpha, \tag{8}$$

where $\alpha = \epsilon_2/\epsilon_1$. This follows from (7) and the fact that if one shifts the basepoint p in the Gromov product to another point $q \in X$, then the Gromov product changes by at most a fixed additive amount.

3 Asymptotic upper curvature and geodesic polygons

Let X be a geodesic metric space, and $n \in \mathbb{N}$, $n \geq 2$. A geodesic n-gon in X is a set $P \subseteq X$ with distinguished points $x_1, \ldots, x_n, x_{n+1} = x_1$, called the vertices of P, and distinguished geodesic segments $[x_i, x_{i+1}]$, $i = 1, \ldots n$, called the sides or edges of P, such that

$$P = [x_1, x_2] \cup \cdots \cup [x_n, x_{n+1}].$$

The n-gon P is called Δ-slim for $\Delta \geq 0$ if each side of P is contained in the closed Δ-neighborhood of the union of the other $n-1$ sides of P, i.e., if

$$\operatorname{dist}\big(u, [x_1, x_2] \cup \cdots \cup \widehat{[x_i, x_{i+1}]} \cup \cdots \cup [x_n, x_{n+1}]\big) \leq \Delta$$

for all $i = 1, \ldots, n$, and all $u \in [x_i, x_{i+1}]$. Here $\widehat{[x_i, x_{i+1}]}$ indicates that the side $[x_i, x_{i+1}]$ of P is omitted from the union.

To relate the $\mathrm{AC}_u(\kappa)$-property with slimness properties of geodesic polygons, we start with a geometric fact for Gromov hyperbolic spaces.

Lemma 3.1 *Let (X, d) be a δ-hyperbolic geodesic metric space. Suppose that $p, z, y_1, y_2 \in X$, $u \in [p, z]$, and*

$$(z \cdot y_1)_p \geq t, \quad (y_1 \cdot y_2)_p \leq t + a,$$

where $t = d(u, p)$ and $a \geq 0$. Then

$$\operatorname{dist}(u, [y_1, y_2]) \leq a + b,$$

where $b \geq 0$ only depends on δ.

Proof All Gromov products will be with respect to the basepoint p which we drop from the notation. Pick $u' \in [p, y_1]$ such that

$$d(p, u') = d(p, u) = t \leq (z \cdot y_1) \leq d(p, y_1).$$

Then $d(u, u') \leq c := 8\delta$ by thinness of the geodesic triangle with vertices p, z, y_1.

If $u'' \in [p, y_1]$ is the point with

$$d(p, u'') = \min\{d(p, y_1), t + a\} \geq (y_1 \cdot y_2),$$

then $\operatorname{dist}(u'', [y_1, y_2]) \leq c$ by thinness of the geodesic triangle with vertices p, y_1, y_2.

Moreover,

$$d(u', u'') = |d(p, u'') - d(p, u')| = d(p, u'') - t \leq a.$$

It follows that

$$\operatorname{dist}(u, [y_1, y_2]) \leq d(u, u') + d(u', u'') + c \leq a + 2c.$$

So the desired inequality holds with $b = 2c = 16\delta$. □

The following lemma provides the crucial ingredient in the proof of Theorem 1.3.

Lemma 3.2 *Let (Y, d) be a δ-hyperbolic geodesic metric space. Suppose that $\kappa \in [-\infty, 0)$, $c \geq 0$, $p \in Y$, and $M \subseteq Y$ is a set such that*

$$(z \cdot z')_p \geq \min_{i=1,\ldots,n} (x_{i-1} \cdot x_i)_p - \frac{1}{\sqrt{-\kappa}} \log n - c \tag{9}$$

for all chains $x_0 = z, x_1, \ldots, x_n = z'$ in M.

Then there exists a constant $\tilde{c}$ only depending on δ and c such that every geodesic $(n+1)$-gon with vertices in M is Δ-slim with $\Delta = \dfrac{1}{\sqrt{-\kappa}} \log n + \tilde{c}$.

Proof All Gromov products will be with respect to the basepoint p.

Let

$$P = [x_0, x_1] \cup \cdots \cup [x_n, x_{n+1}]$$

be an arbitrary geodesic $(n+1)$-gon in Y with vertices $x_0, \ldots, x_n, x_{n+1} = x_0$ in M. To establish the desired slimness property of P let u' be an arbitrary point on one of the sides of P. By cyclically relabeling the vertices of P if necessary, we may assume without loss of generality that $u' \in [x_0, x_n]$. Let $v \in [x_0, x_n]$ be the unique point which is mapped to the center of the tripod under a tripod map of the geodesic triangle in Y with vertices p, x_0, x_n. We may assume that u' lies "to the left" of v, i.e., $u' \in [x_0, v] \subseteq [x_0, x_n]$. Thinness of the geodesic triangle with vertices p, x_0, x_n implies that there exists a point $u \in [p, x_0]$ with

$$t := d(p, u) \geq (x_0 \cdot x_n) \quad \text{and} \quad d(u, u') \leq b_1,$$

where $b_1 \geq 0$ is a constant only depending on δ. Then

$$(x_0 \cdot x_0) = d(p, x_0) \geq d(p, u) = t \geq (x_0 \cdot x_n) = (z \cdot z'), \tag{10}$$

where $z := x_0$ and $z' := x_n$. So by our assumptions there exists a number $i \in \{1, \ldots n\}$ such that

$$(x_{i-1} \cdot x_i) \leq (z \cdot z') + a \leq t + a,$$

where

$$a = \frac{1}{\sqrt{-\kappa}} \log n + c.$$

We may assume that i is the smallest number in $\{1, \ldots, n\}$ such that

$$(x_{i-1} \cdot x_i) \leq t + a. \tag{11}$$

Then

$$t \leq (z \cdot x_{i-1}). \tag{12}$$

Indeed, this is obviously true if $i = 1$, because then $x_{i-1} = x_0 = z$, and so by (10),

$$t \leq (x_0 \cdot x_0) = (z \cdot x_{i-1}).$$

To reach a contradiction suppose $i > 1$ and $t > (z \cdot x_{i-1})$. Consider the chain $x_0 = z, \ldots, x_{i-1}$. This is a chain in M with at most n elements. Hence our hypotheses show that there exists $j \in \{1, \ldots, i-1\}$ such that

$$(x_{j-1} \cdot x_j) \leq (z \cdot x_{i-1}) + a \leq t + a.$$

This contradicts the choice of i, and so (12) is true.

By inequalities (11) and (12) we can apply Lemma 3.1 with $y_1 = x_{i-1}$ and $y_2 = x_i$. It follows that

$$\operatorname{dist}(u, [x_{i-1}, x_i]) \leq a + b_2,$$

where $b_2 \geq 0$ is a constant only depending on δ. Hence

$$d(u', [x_{i-1}, x_i]) \leq a + b_1 + b_2.$$

This implies the desired slimness property with $\tilde{c} = c + b_1 + b_2$ which is a number only depending on c and δ. □

As an immediate consequence of this lemma we get a proof of Theorem 1.3.

Proof of Theorem 1.3 Let X be a geodesic metric space with metric d.

(i) $\Rightarrow$ (ii): Suppose that X is an $AC_u(\kappa)$-space, $\kappa \in [-\infty, 0)$. Then there exist a basepoint $p \in X$ and a constant $c \geq 0$ such that inequality (4) is true for all chains in X. In particular, X is δ-hyperbolic for some $\delta \geq 0$. The claim then follows from Lemma 3.2 applied to $M = Y = X$.

(ii) $\Rightarrow$ (i): Conversely, suppose that the slimness condition for geodesic polygons in X holds as stated. Then in particular, geodesic triangles are δ'-slim for some $\delta' \geq 0$ independent of the triangle. Since X is geodesic, this implies that X is δ-hyperbolic for some $\delta \geq 0$. Pick a basepoint $p \in X$. All Gromov products will be with respect to p.

Now let $z = x_0, x_1, \ldots, z' = x_n$ be an arbitrary chain in X, $n \in \mathbb{N}$, $n \geq 2$ (for $n = 1$ there is nothing to prove). Let $u \in [z, z']$ be a point such that

$$d(p, u) = \operatorname{dist}(p, [z, z']).$$

Since X is Gromov hyperbolic it follows that

$$|(z \cdot z') - d(p, u)| \leq b_1,$$

where $b_1 \geq 0$ is a constant only depending on δ. By the slimness property applied to the $(n + 1)$-gon

$$P = [z, z'] \cup [x_0, x_1] \cup \cdots \cup [x_{n-1}, x_n]$$

there exists $i \in \{1, \ldots, n\}$ such that

$$\operatorname{dist}(u, [x_{i-1}, x_i]) \leq \Delta := \frac{1}{\sqrt{-\kappa}} \log n + c'.$$

Since

$$|(x_{i-1} \cdot x_i) - \operatorname{dist}(p, [x_{i-1}, x_i])| \leq b_2,$$

where $b_2 \geq 0$ is a constant only depending on δ, it follows that

$$\begin{aligned}(x_{i-1} \cdot x_i) &\leq \operatorname{dist}(p, [x_{i-1}, x_i]) + b_2 \leq d(p, u) + \operatorname{dist}(u, [x_{i-1}, x_i]) + b_2 \\ &\leq (z \cdot z') + \Delta + b_1 + b_2.\end{aligned}$$

Hence (4) is true for all chains if we choose $c = c' + b_1 + b_2$. □

The following proposition shows that if inequality (4) is true for some basepoint p, then a similar inequality holds for all basepoints with a constant $c = c'$ independent of the basepoint.

Proposition 3.3 *Let (X, d) be an $\mathrm{AC}_u(\kappa)$-space, $\kappa \in [-\infty, 0)$. Then there exists $c' \geq 0$ such that for all $q \in X$ and all chains $x_0 = z, x_1, \dots, x_n = z'$ in X,*

$$(z \cdot z')_q \geq \min_{i=1,\dots,n} (x_{i-1} \cdot x_i)_q - \frac{1}{\sqrt{-\kappa}} \log n - c'.$$

Proof Let $p \in X$ be a point such that inequality (4) holds for all chains in X. Then X is δ-hyperbolic for some $\delta \geq 0$. We can isometrically embed X into a δ-hyperbolic geodesic metric space (Z, d) [BS, Thm. 4.1]. By this isometric embedding we can consider X as a subset of Z.

Now let $q \in X$, and consider an arbitrary chain $x_0 = z, \dots, x_n = z'$ in X. Pick a point $u \in [z, z']$ such that

$$d(q, u) = \mathrm{dist}(q, [z, z']).$$

Then

$$|d(q, u) - (z \cdot z')_q| \leq b_1,$$

where $b_1 \geq 0$ is a constant only depending on δ. By Lemma 3.2 applied to $M = X$ and $Y = Z$ there exists $i \in \{1, \dots, n\}$ such that

$$\mathrm{dist}(u, [x_{i-1}, x_i]) \leq \Delta = \frac{1}{\sqrt{-\kappa}} \log n + \tilde{c},$$

where $\tilde{c}$ only depends on c and δ. Hence

$$\begin{aligned}(x_{i-1} \cdot x_i)_q &\leq \mathrm{dist}(q, [x_{i-1}, x_i]) + b_2 \\ &\leq d(q, u) + \mathrm{dist}(u, [x_{i-1}, x_i]) + b_2 \\ &\leq (z \cdot z')_q + \Delta + b_1 + b_2,\end{aligned}$$

where $b_2 \geq 0$ is a constant only depending on δ. The claim follows with $c' = \tilde{c} + b_1 + b_2$. □

Proposition 3.4 *Let (X, d_X) be a geodesic metric space, Y an $\mathrm{AC}_u(\kappa)$-space with $\kappa \in [-\infty, 0)$, and $f\colon X \to Y$ a (λ, k)-quasi-isometric embedding. Then X is an $\mathrm{AC}_u(\kappa')$-space with $\kappa' = \kappa/\lambda^2$.*

Here for $\kappa = -\infty$ we interpret $\kappa' = -\infty$.

Proof Since Y is an $\mathrm{AC}_u(\kappa)$-space, it is δ-hyperbolic for some $\delta \geq 0$. We can isometrically embed Y into a δ-hyperbolic geodesic metric space (Z, d_Z) [BS, Thm. 4.1]. By this isometric embedding we can consider Y as a subset of Z. Since X is geodesic, it is enough to show that condition (ii) in Theorem 1.3 is satisfied (with κ replaced by κ').

So let $n \in \mathbb{N}$, $n \geq 2$, and

$$P = [x_0, x_1] \cup \dots \cup [x_n, x_{n+1}]$$

be an arbitrary geodesic $(n+1)$-gon in X with vertices $x_0, \dots, x_n, x_{n+1} = x_0$. Let u be a point on one of the sides of P, say $u \in [x_0, x_n]$. Define $v := f(u)$, and $y_i := f(x_i)$ for $i = 0, \dots, n+1$. Consider a geodesic $(n+1)$-gon

$$P' = [y_0, y_1] \cup \dots \cup [y_n, y_{n+1}]$$

in Z with the vertices $y_0, \dots, y_n, y_{n+1} = y_0$ that are points in Y.

Since Gromov hyperbolic spaces satisfy a geodesic stability property, the quasi-geodesic segment $f([x_0, x_n])$ and the geodesic segment $[y_0, y_n]$ have Hausdorff distance bounded by a constant $b_1 \geq 0$ only depending on λ, k, and δ (see, for example, [BS, p. 273]). In particular, there exists a point $v' \in [y_0, y_n]$ such that $d_Z(v, v') \leq b_1$. Since Y is an $\mathrm{AC}_u(\kappa)$-space, chains in Y satisfy an inequality as in (4). Lemma 3.2 (with $M = Y$ and $Y = Z$) shows that P' is Δ-slim with

$$\Delta = \frac{1}{\sqrt{-\kappa}} \log n + \tilde{c},$$

where $\tilde{c}$ only depends on δ and the constant c in (4). Hence there exist $i \in \{1, \ldots, n\}$ and a point $w' \in [y_{i-1}, y_i]$ such that $d_Z(v', w') \leq \Delta$. Again by geodesic stability there exists a point $w \in f([x_{i-1}, x_i])$ such that

$$d_Z(w, w') \leq b_2,$$

where b_2 is a constant only depending on λ, k, and δ.

If we write $w = f(z)$ with $z \in [x_{i-1}, x_i]$, then

$$\begin{aligned}(1/\lambda) d_X(u, z) - k &\leq d_Z(f(u), f(z)) = d_Z(v, w)\\ &\leq d_Z(v', w') + b_1 + b_2 \leq \Delta + b_1 + b_2,\end{aligned}$$

and so

$$\operatorname{dist}(u, [x_{i-1}, x_i]) \leq d_X(u, z) \leq \lambda\Delta + b_3 = \frac{1}{\sqrt{-\kappa'}} \log n + c',$$

where b_3 only depends on λ, k, and δ, and c' in addition to these parameters also on c. Hence P is Δ'-slim with

$$\Delta' = \frac{1}{\sqrt{-\kappa'}} \log n + c'.$$

Since c' does not depend on the polygon P, the claim follows from Theorem 1.3. $\square$

4 Asymptotic upper curvature and visual metrics

In this section we prove Proposition 1.4 and Theorem 1.5. We first need some preparation.

Lemma 4.1 *Let (X, d) be a visual Gromov hyperbolic space, and $p \in X$.*

Suppose there exist constants $a, b \geq 0$ such that for all $\xi, \xi' \in \partial_\infty X$ and all chains $\xi_0 = \xi, \xi_1, \ldots, \xi_n = \xi'$ in $\partial_\infty X$,

$$(\xi \cdot \xi')_p \geq \min_{i=1,\ldots,n} (\xi_{i-1} \cdot \xi_i)_p - a \log n - b. \tag{13}$$

Then there exists $b' \geq 0$ such that for all $z, z' \in X$ and all chains $x_0 = z, x_1, \ldots, x_n = z'$ in X,

$$(z \cdot z')_p \geq \min_{i=1,\ldots,n} (x_{i-1} \cdot x_i)_p - a \log n - b'. \tag{14}$$

Conversely, if inequality (14) *is valid for all chains in X, then there exists $b \geq 0$ such that* (13) *holds for all chains in $\partial_\infty X$.*

Proof By assumption, the space X is δ-hyperbolic for some $\delta \geq 0$, and k-visual for some $k \geq 0$ with respect to the basepoint p. By increasing k if necessary, we may also assume that for all $\xi \in \partial_\infty X$ there exists a ray $[p, \xi]_k$ [BS, Prop. 5.6 and Prop. 5.2].

Suppose inequality (13) is valid for all boundary chains, and let $z, z' \in X$ be arbitrary. Then there exist points $\xi, \xi' \in \partial_\infty X$, and k-rough geodesic rays $[p, \xi]_k$ and $[p, \xi']_k$ with $z \in [p, \xi]_k$ and $z' \in [p, \xi']_k$. We claim that there exists a constant C_1 only depending on δ and k such that

$$(\xi \cdot \xi')_p \geq (z \cdot z')_p - C_1. \tag{15}$$

The proof of this inequality is standard, and we will only give an outline. It relies on the fact that every configuration of n points and k-rough geodesic rays in a δ-hyperbolic space can be embedded into a metric tree by a c-rough-isometry with c only depending on n, k, and δ [GH, Ch. 2, §2]. Accordingly, up to a bounded error term only depending on δ and k the essentially different configurations that have to be considered for verifying inequality (15) are represented by Figure 1. Here we have to switch the roles of z and z' if necessary to reduce to these cases.

In a tree the Gromov product of two points is equal to the distance of the basepoint to the geodesic connecting these points. Hence in a tree inequality (15) is true with $C_1 = 0$. The general case of inequality (15) follows from this.

Now assume that $x_0 = z, x_1, \ldots, x_n = z'$ is a chain in X. We distinguish two cases according to Figure 1.

(a) Suppose first that the situation is as in Figure 1(a), which corresponds to the inequality

$$d(p, z) \leq (z \cdot z')_p + C_2$$

with $C_2 \geq 0$ only depending on δ and k. Since for all $y \in X$ we have $d(p, z) \geq (z \cdot y)_p$, we find

$$\begin{aligned}(z \cdot z')_p &\geq d(p, z) - C_2 \geq (z \cdot x_1)_p - C_2 \\ &\geq \min_{i=1,\ldots,n} (x_{i-1} \cdot x_i)_p - C_2.\end{aligned} \tag{16}$$

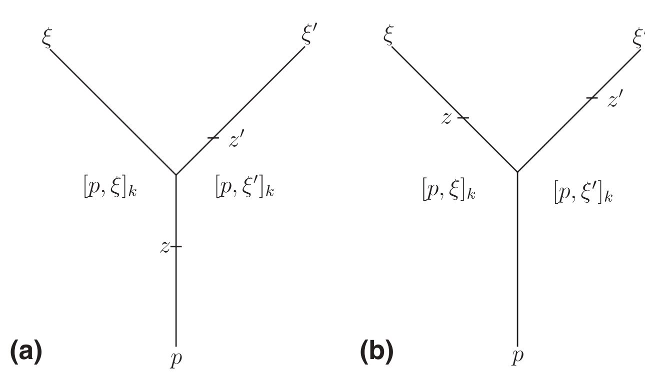

Fig. 1 This figure visualizes the different configurations that have to be considered in the proof of Lemma 4.1

(b) In the other case, represented by Figure 1(b), we have

$$(z \cdot z')_p \geq (\xi \cdot \xi')_p - C_3, \tag{17}$$

where C_3 is a constant only depending on δ and k. For $i = 1, \dots, n$ we can find points $\xi_i \in \partial_\infty X$ and k-rough geodesic rays $[p, \xi_i]_k$ through x_i. Using inequality (15) for the points $z = x_{i-1}$, $z' = x_i$, $\xi = \xi_{i-1}$, $\xi' = \xi_i$, and inequality (13), we obtain

$$\begin{aligned}(z \cdot z')_p &\geq (\xi \cdot \xi')_p - C_3 \\ &\geq \min_{i=1,\dots,n} (\xi_{i-1} \cdot \xi_i)_p - a \log n - (b + C_3) \\ &\geq \min_{i=1,\dots,n} (x_{i-1} \cdot x_i)_p - a \log n - (b + C_1 + C_3).\end{aligned} \tag{18}$$

From inequalities (16) and (18) the first part of the lemma follows.

To prove the other direction, suppose inequality (14) always holds, and let $\xi_0 = \xi, \dots, \xi_n = \xi' \in \partial_\infty X$ be an arbitrary chain in $\partial_\infty X$. To establish inequality (13), we may in addition assume that $\xi \neq \xi'$, and $\xi_{i-1} \neq \xi_i$ for $i = 1, \dots, n$. The general case of (13) follows from its validity under this additional restriction. For $i = 1, \dots, n$ we can find k-rough geodesic rays $[p, \xi_i]_k$. If we choose points $x_i \in [p, \xi_i]_k$ sufficiently far away from p, then we get a configuration as in Figure 1(b) for $x_0, x_n, \xi_0 = \xi, \xi_n = \xi'$ and for $x_{i-1}, x_i, \xi_{i-1}, \xi_i$, $i = 1, \dots, n$, corresponding to z, z', ξ, ξ', respectively. In other words, we will have

$$(x_0 \cdot x_n)_p - C_4 \leq (\xi \cdot \xi')_p \leq (x_0 \cdot x_n)_p + C_4,$$

and

$$(x_{i-1} \cdot x_i)_p - C_4 \leq (\xi_{i-1} \cdot \xi_i)_p \leq (x_{i-1} \cdot x_i)_p + C_4 \quad \text{for} \quad i = 1, \dots, n,$$

where C_4 is a constant only depending on δ and k. Inequality (13) with $b = b' + 2C_4$ follows from these inequalities and (14). $\square$

Suppose (X, d) is a geodesic metric space, let $\kappa < 0$, and consider the hyperbolic plane $\mathbb{H}^2_\kappa$ of constant curvature κ. We denote the metric on $\mathbb{H}^2_\kappa$ by $\tilde{d}$. If Δ is a geodesic triangle in X, then by definition a comparison triangle in $\mathbb{H}^2_\kappa$ is a geodesic triangle $\tilde{\Delta}$ in $\mathbb{H}^2_\kappa$ whose side-lengths are the same as those of Δ. We say that (X, d) is a CAT(κ)-space if every geodesic triangle Δ in X is thinner than a comparison triangle $\tilde{\Delta}$ in $\mathbb{H}^2_\kappa$ in the following sense. Suppose x, y, z are the vertices of Δ, and $\tilde{x}, \tilde{y}, \tilde{z}$ the corresponding vertices of $\tilde{\Delta}$. If u is an arbitrary point on the side $[y, z]$ of Δ and $\tilde{u}$ is the corresponding point on the side $[\tilde{y}, \tilde{z}]$ of $\tilde{\Delta}$ such that

$$d(y, u) = \tilde{d}(\tilde{y}, \tilde{u}) \text{ and } d(u, z) = \tilde{d}(\tilde{u}, \tilde{z}),$$

then we require that

$$d(x, u) \leq \tilde{d}(\tilde{x}, \tilde{u}).$$

This is one of various equivalent ways to define CAT(κ)-spaces [GH, Ch. 3, §1].

Let α and β be non-degenerate geodesic segments or rays in a CAT(κ)-space X with a common initial point $p \in X$. Then one can define an angle $\angle(\alpha, \beta)$ between

α and β (see [BBI, 3.6.5] for the precise definition and basic properties). In case $\alpha = [p, x]$ and $\beta = [p, y]$, where $x, y \in (X \cup \partial_\infty X) \setminus \{p\}$, we also use the notation $\angle(xpy) := \angle(\alpha, \beta)$.

If γ is a third geodesic segment or ray with initial point p, then

$$\angle(\alpha, \beta) \le \angle(\alpha, \gamma) + \angle(\gamma, \beta).$$

If α and β are non-overlapping subarcs of a geodesic segment containing p in its interior, then $\angle(\alpha, \beta) = \pi$.

If Δ is a geodesic triangle in a CAT(κ)-space, and $\tilde{\Delta}$ is a comparison triangle in $\mathbb{H}^2_\kappa$, then the angles at the vertices of Δ are not larger than the angles at the corresponding vertices of $\tilde{\Delta}$ [GH, Ch. 3, §1].

Lemma 4.2 *Suppose $\alpha = [p, u]$ and $\beta = [p, v]$ are non-degenerate geodesic segments or rays in the hyperbolic plane $\mathbb{H}^2 = \mathbb{H}^2_{-1}$, where $p \in \mathbb{H}^2$, $u, v \in (\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2) \setminus \{p\}$. Let $s = \angle(\alpha, \beta) \in [0, \pi]$ be the angle at p. Then*

$$\operatorname{dist}(p, [u, v]) \le -\log(\sin(s/2)) + C,$$

where $C \ge 0$ is an absolute constant.

In the formulation of this statement one uses the obvious interpretation $-\log(0) = \infty$ in case $s = 0$.

Proof There are unique points $\zeta, \xi \in \partial_\infty \mathbb{H}^2$ such that $u \in [p, \zeta]$ and $v \in [p, \xi]$ (so ζ and ξ are the projections of u and v from p to $\partial_\infty \mathbb{H}^2$, respectively). Let $\alpha' = [p, \zeta]$ and $\beta' = [p, \xi]$. Then for the angle at p we have

$$\angle(\alpha', \beta') = \angle(\alpha, \beta) = s.$$

If $\zeta = \xi$, then $s = 0$ and the claim is obvious. So we may assume $\zeta \ne \xi$. Then all quantities appearing below are finite.

One can show that

$$(\zeta \cdot \xi)_p = -\log(\sin(s/2)). \tag{19}$$

(This follows from the cosine theorem in the hyperbolic plane and a limiting argument, for example; cf. [Bo, pp. 85–87]. See also the related equation (25) below.)

There exists an absolute constant $C \ge 0$ such that

$$|\operatorname{dist}(p, [\zeta, \xi]) - (\zeta \cdot \xi)_p| \le C.$$

Hence

$$\begin{aligned} \operatorname{dist}(p, [u, v]) &\le \operatorname{dist}(p, [\zeta, \xi]) \le (\zeta \cdot \xi)_p + C \\ &= -\log(\sin(s/2)) + C. \end{aligned}$$

□

Proof of Proposition 1.4 In this proof we denote by $C_1, C_2, \ldots$ non-negative absolute constants.

(i) Suppose X is a CAT(κ)-space, $\kappa < 0$. In order to show that X is an $\mathrm{AC}_u(\kappa)$-space, we may assume that $\kappa = -1$ (by scaling the metric by a constant factor if necessary).
Let $p \in X$ be arbitrary, and assume that $x_0 = z, x_1, \dots, x_n = z'$ is an arbitrary chain in X. An inequality as in (4) will be true if we can show it under the additional assumption that $z \neq z'$. In this case we can pick a point $q \in [z, z']$ different from all the points in the chain such that

$$d(p, q) \leq \mathrm{dist}(p, [z, z']) + 1.$$

Then the angles $\angle(zqz')$ and $\angle(x_{i-1}qx_i)$, $i = 1, \dots, n$, are well-defined. We have $\angle(zqz') = \pi$, and so by the triangle inequality for angles,

$$\pi = \angle(zqz') \leq \sum_{i=1}^{n} \angle(x_{i-1}qx_i).$$

Therefore, there exists $j \in \{1, \dots, n\}$ such that for $u = x_{j-1}$, $v = x_j$, we have

$$\angle(uqv) \geq \pi/n.$$

Let Δ be the geodesic triangle in X with vertices q, u, v, and Δ' a comparison triangle for Δ in $\mathbb{H}^2$ with vertices q', u', v'. Since X is a CAT(-1)-space, we have

$$\pi/n \leq \angle(uqv) \leq \angle(u'q'v')$$

and

$$\mathrm{dist}(q, [u, v]) \leq \mathrm{dist}(q', [u', v']).$$

Thus, by Lemma 4.2 we see that

$$\begin{aligned} \mathrm{dist}(q, [u, v]) &\leq \mathrm{dist}(q', [u', v']) \\ &\leq -\log(\sin(\pi/(2n)) + C_1 \leq \log n + C_2. \end{aligned}$$

This and the choice of q imply that

$$\begin{aligned} (z \cdot z')_p &\geq \mathrm{dist}(p, [z, z']) - C_3 \geq d(p, q) - C_4 \\ &\geq d(p, q) + \mathrm{dist}(q, [x_{j-1}, x_j]) - \log n - C_5 \\ &\geq \mathrm{dist}(p, [x_{j-1}, x_j]) - \log n - C_5 \\ &\geq (x_{j-1} \cdot x_j)_p - \log n - C_6 \\ &\geq \min_{i=1,\dots,n} (x_{i-1} \cdot x_i)_p - \log n - C_6. \end{aligned}$$

The claim follows.

(ii) If M is a Cartan-Hadamard manifold as in the statement, then M is a $\mathrm{CAT}(\kappa)$-space with $\kappa = -a^2$, and so an $\mathrm{AC}_u(\kappa)$-space with the same κ by (i). Suppose that M is an $\mathrm{AC}_u(\kappa)$-space with $\kappa < -b^2$. To derive a contradiction, we may assume that $b = 1$. Pick a point $p \in M$, a geodesic $[\xi, \xi']$, $\xi, \xi' \in \partial_\infty M$, containing p, and a 2-dimensional subspace H of the tangent space of M at p that contains the tangent direction given by the geodesic $[\xi, \xi']$ at p. For arbitrary $n \in \mathbb{N}$ there are geodesic rays $[p, \xi_i]$, $\xi_i \in \partial_\infty M$, $i = 0, \ldots, n$, with tangent directions at p contained in H such that $\xi_0 = \xi$, $\xi_n = \xi'$ and such that the angle between successive rays $[p, \xi_{i-1}]$ and $[p, \xi_i]$, $i = 1, \ldots, n$, is equal to π/n. Let $p' \in \mathbb{H}^2$, and $u, v \in \partial_\infty \mathbb{H}^2$ be points such that the rays $[p', u]$ and $[p', v]$ in $\mathbb{H}^2$ form the angle π/n at p'. Then (cf. (19))

$$(u \cdot v)_{p'} = -\log \sin(\pi/(2n)).$$

By a standard comparison theorem in Riemannian geometry (cf. [Sa, Ch. IV, Thm. 4.2 (2)]) and a limiting argument,

$$(\xi_{i-1} \cdot \xi_i)_p \geq (u \cdot v)_{p'} \quad \text{for} \quad i = 1, \ldots, n.$$

Hence

$$\min_{i=1,\ldots,n} (\xi_{i-1} \cdot \xi_i)_p \geq (u \cdot v)_{p'} = -\log \sin(\pi/(2n)) \geq \log n - C_7.$$

On the other hand, by our assumption and Lemma 4.1 there exists a constant $b' \geq 0$ independent of n such that

$$\min_{i=1,\ldots,n} (\xi_{i-1} \cdot \xi_i)_p \leq (\xi \cdot \xi')_p + \frac{1}{\sqrt{-\kappa}} \log n + b'.$$

Since $\kappa < -1$, this and the previous inequality cannot both hold for large n. This is the desired contradiction.

(iii) This claim follows from (ii). □

Lemma 4.3 *Let Z be a set, and $\rho\colon Z \times Z \to [0, \infty)$ a function satisfying $\rho(\eta, \eta) = 0$ for all $\eta \in Z$. Suppose $0 < \alpha < 1$ and $C > 0$ are constants such that for all $\xi, \xi' \in Z$ and all chains $\xi_0 = \xi, \xi_1, \ldots, \xi_n = \xi'$ in Z,*

$$\rho(\xi, \xi') \leq C n^\alpha \max_{i=1,\ldots,n} \rho(\xi_{i-1}, \xi_i).$$

Then there exists a constant $K \geq 1$ such that

$$\rho(\xi, \xi') \leq K \sum_{i=1}^{n} \rho(\xi_{i-1}, \xi_i) \tag{20}$$

for all such chains.

Proof Let $K \geq 1$ be the smallest integer with

$$K \geq \bigl(4C\bigr)^{1/(1-\alpha)}.$$

We will prove (20) by induction on the length n of the chain. The inequality is obvious for $n = 1$. Suppose that $n \geq 2$ and that the inequality is true for all chains of length $< n$. Let $\xi_0, \xi_1, \dots, \xi_n$ be an arbitrary chain in Z of length n, and denote the sum on the right hand side in (20) by T. If $T = 0$, then each of the terms in T vanishes. Then our hypothesis implies that $\rho(\xi, \xi') = 0$ and so the desired inequality holds. Therefore we may assume $T > 0$.

We define integers $t_0 = 0 < t_1 < t_2 < \cdots < t_s = n$ recursively as follows. The number t_1 is the smallest integer such that

$$\sum_{i=1}^{t_1} \rho(\xi_{i-1}, \xi_i) \geq T/K.$$

Then $1 \leq t_1 \leq n$. If $t_1 = n$ the recursion terminates and $s = 1$. If $t_1 < n$ define t_2 as the smallest integer larger than t_1 such that

$$\sum_{i=t_1+1}^{t_2} \rho(\xi_{i-1}, \xi_i) \geq T/K$$

if such t_2 exists, otherwise let $t_2 = n$. We terminate if $t_2 = n$, and continue in a similar way to define t_3 if $t_2 < n$, etc. Corresponding to the sums used to define the numbers t_r we can split up the sum T into s terms. Among these terms $s - 1$ are $\geq T/K$. It follows that $(s-1)T/K \leq T$ and so $s \leq K + 1$. Moreover, by definition of $t_0, t_1, \dots, t_s$,

$$\sum_{i=t_{r-1}+1}^{t_r-1} \rho(\xi_{i-1}, \xi_i) < T/K \quad \text{for} \quad r = 1, \dots, s.$$

Hence

$$\rho(\xi_{t_{r-1}}, \xi_{t_r-1}) \leq T \quad \text{for} \quad r = 1, \dots, s$$

as follows from the induction hypothesis in case $t_{r-1} < t_r - 1$.

$$\rho(\xi_{t_{r-1}}, \xi_{t_r-1}) = 0 \leq T$$

in case $t_{r-1} = t_r - 1$.
Obviously,

$$\rho(\xi_{t_r-1}, \xi_{t_r}) \leq T \quad \text{for} \quad r = 1, \dots, s.$$

Thus

$$M := \max\{\rho(\xi_{t_{r-1}}, \xi_{t_r-1}), \rho(\xi_{t_r-1}, \xi_{t_r}) : r = 1, \dots, s\} \leq T.$$

There are $2s \le 2K + 2 \le 4K$ terms in the maximum defining M. The points appearing in the terms form a chain with initial point ξ_0 and endpoint ξ_n. Hence our assumptions imply

$$\rho(\xi_0, \xi_n) \le C(4K)^\alpha T \le KT,$$

where the second inequality follows from the definition of K. This provides the induction step. The claim follows. $\square$

Proof of Theorem 1.5 Suppose there exists a visual metric ρ on $\partial_\infty X$ with parameter $\epsilon > 0$. Then there exist $p \in X$ and $\lambda \ge 1$ such that

$$\frac{1}{\lambda}\exp(-\epsilon(\xi \cdot \xi')_p) \le \rho(\xi, \xi') \le \lambda \exp(-\epsilon(\xi \cdot \xi')_p) \tag{21}$$

for all $\xi, \xi' \in \partial_\infty X$.

If $\xi_0 = \xi, \xi_1, \ldots, \xi_n = \xi' \in \partial_\infty X$ is an arbitrary chain, then by the triangle inequality

$$\rho(\xi, \xi') \le \sum_{i=1}^{n} \rho(\xi_{i-1}, \xi_i),$$

and so there exists $j \in \{1, \ldots, n\}$ such that

$$\rho(\xi_{j-1}, \xi_j) \ge \frac{1}{n}\rho(\xi, \xi').$$

By (21) this implies

$$\begin{aligned}(\xi \cdot \xi')_p &\ge (\xi_{j-1} \cdot \xi_j)_p - \frac{1}{\epsilon}(\log n + 2\log\lambda)\\ &\ge \min_{i=1,\ldots,n}(\xi_{i-1} \cdot \xi_i)_p - \frac{1}{\epsilon}(\log n + 2\log\lambda).\end{aligned}$$

This shows that inequality (13) in Lemma 4.1 holds with $a = 1/\epsilon$ and $b = (2/\epsilon)\log\lambda$. So we get inequality (14) with the same constant a and some fixed $b' \ge 0$ independent of the chain. This means that X is an $\mathrm{AC}_u(\kappa)$-space with $\kappa = -\epsilon^2$ as desired.

Conversely, suppose X is an $\mathrm{AC}_u(\kappa)$-space with $\kappa < 0$, and let $0 < \epsilon < \sqrt{-\kappa}$ be arbitrary. Then there exist $p \in X$ and a constant $c \ge 0$ such that inequality (4) is valid for all chains in X. By Lemma 4.1 this means that inequality (13) is valid for all boundary chains, where $a = 1/\sqrt{-\kappa}$ and $b \ge 0$ is some constant independent of the chain. If we define

$$\rho(\xi, \xi') = \exp(-\epsilon(\xi \cdot \xi')_p) \quad \text{for} \quad \xi, \xi' \in \partial_\infty X,$$

this translates into the inequality

$$\rho(\xi, \xi') \le Cn^\alpha \max_{i=1,\ldots,n} \rho(\xi_{i-1}, \xi_i)$$

for all chains $\xi_0 = \xi, \xi_1, \ldots, \xi_n = \xi'$ in $\partial_\infty X$, where $\alpha = \epsilon/\sqrt{-\kappa} \in (0, 1)$ and $C = \exp(\epsilon b)$. By Lemma 4.3 inequality (20) holds for all chains, where $K \geq 1$ is a constant independent of the chain. Now define

$$\tilde{\rho}(\xi, \xi') = \inf\left\{\sum_{i=1}^{n} \rho(\xi_{i-1}, \xi_i)\right\}$$

for $\xi, \xi' \in X$, where the infimum is taken over all $n \in \mathbb{N}$ and all chains $\xi_0 = \xi, \xi_1, \ldots, \xi_n = \xi'$ in $\partial_\infty X$. Inequality (20) implies that $\tilde{\rho}$ is a metric on $\partial_\infty X$ satisfying $\rho/K \leq \tilde{\rho} \leq \rho$. So $\tilde{\rho}$ is a visual metric on $\partial_\infty X$ with parameter ϵ giving the second part of the theorem.

Finally, the assertion about $K_u(X)$ follows from the first two parts of the theorem. □

5 Embeddings of spaces with asymptotic upper curvature bounds

For the proof of Theorem 1.6 we first discuss some concepts and results that are relevant for embeddings of Gromov hyperbolic spaces. This is based on [BS].

Let (Z, ρ) be a metric space. We say that Z has finite Assouad dimension if there exists a number $D > 0$ satisfying the following condition: For $\alpha, \beta > 0$ let $S(\alpha, \beta)$ be the maximal cardinality of a set $V \subseteq Z$ such that $\alpha \leq \rho(x, y) \leq \beta$ for all $x, y \in V$, $x \neq y$. Then

$$S(\alpha, \beta) \leq K(\beta/\alpha)^D, \tag{22}$$

where $K \geq 0$ is a constant independent of α and β. The infimal D for which this condition holds is called the Assouad dimension of Z.

Assouad's Embedding Theorem [Ad] states that if (Z, ρ) is a metric space of finite Assouad dimension, and $p \in (0, 1)$, then the metric space (Z, ρ^p) admits a bi-Lipschitz embedding into some Euclidean space $\mathbb{R}^n$. Note that $(x, y) \in Z \times Z \mapsto \rho(x, y)^p$ defines a metric on Z whenever $p \in (0, 1]$.

A metric space X is said to be of bounded growth at some scale, if there exist constants r and R with $R > r > 0$, and $N \in \mathbb{N}$ such that every open ball of radius R in X can be covered by N open balls of radius r. According to [BS, Thm. 9.2] every Gromov hyperbolic geodesic metric space of bounded growth at some scale has a boundary $\partial_\infty X$ of finite Assouad dimension. Here $\partial_\infty X$ is equipped with any visual metric.

In [BS] it was proved that every Gromov hyperbolic geodesic metric space of bounded growth at some scale is rough-isometric to a convex subset of some real hyperbolic space $\mathbb{H}^n_\kappa$. We need a more precise statement of this type whose proof easily follows from the considerations in [BS]. In the proof we use the abstract "convex hull" $\mathrm{Con}(Y)$ of a metric space Y as defined in [BS, Section 7]. If (Y, d) is a bounded metric space, then as a set

$$\mathrm{Con}(Y) := Y \times (0, D(Y)],$$

where $D(Y) = \mathrm{diam}(Y)$ if $\mathrm{diam}(Y) > 0$ and $D(Y) = 1$ if $\mathrm{diam}(Y) = 0$, i.e., if Y consists of a single point. Moreover, $\mathrm{Con}(Y)$ is equipped with a metric σ defined by

$$\sigma\big((y,h),(y',h')\big) = 2\log\left(\frac{d(y,y')+\max\{h,h'\}}{\sqrt{hh'}}\right)$$

for $(y,h),(y',h') \in \mathrm{Con}(Y)$. If Y consists of a single point, then $\mathrm{Con}(Y)$ is isometric to the ray $[0,\infty)$. If X is a visual Gromov hyperbolic metric space and d is a visual metric on $\partial_\infty X$ with parameter $\epsilon = 1$, then $X \cong \mathrm{Con}(\partial_\infty X, d)$ (cf. the remark after Thm. 8.2 in [BS]). (Recall that the notation $M \cong N$ for two metric spaces M and N indicates that they are rough-isometric).

Proposition 5.1 *Suppose on the boundary at infinity $\partial_\infty X$ of a visual Gromov hyperbolic metric space X there exists a visual metric ρ with parameter* 1. *If $(\partial_\infty X, \rho)$ admits a bi-Lipschitz embedding into some Euclidean space $\mathbb{R}^{n-1}$, $n \in \mathbb{N}$, then X is rough-isometric to a convex subset of $\mathbb{H}^n$.*

Proof Since X is visual, $\partial_\infty X \neq \emptyset$, and so $\mathrm{Con}(\partial_\infty X, \rho)$ is defined. Under our assumptions, $X \cong \mathrm{Con}(\partial_\infty X, \rho)$. If $\partial_\infty X$ consists of only one point, $X \cong \mathrm{Con}(\partial_\infty X, \rho)$ is rough-isometric to the ray $[0,\infty)$. The result follows in this case.

If $\partial_\infty X$ consists of more than one point, let $Z \subseteq \mathbb{R}^{n-1}$ be the image of $(\partial_\infty X, \rho)$ under some bi-Lipschitz embedding of this space into $\mathbb{R}^{n-1}$. Then $\mathrm{Con}(\partial_\infty X, \rho) \cong \mathrm{Con}(Z)$ [BS, Thm. 7.4]. Since $(\partial_\infty X, \rho)$ is bounded and complete [BS, Prop. 6.2], the set Z has the same properties and is hence compact. We view $\mathbb{H}^n$ in the upper half-space model and consider $\mathbb{R}^{n-1}$ and hence Z as a subset of $\partial_\infty \mathbb{H}^n$. Since Z contains more than one point, $\mathrm{Con}(Z) \cong \mathrm{hull}(Z)$, where $\mathrm{hull}(Z) \subseteq \mathbb{H}^n$ denotes the hyperbolic convex hull of $Z \subseteq \partial_\infty \mathbb{H}^n$ [BS, Prop. 10.1]. The result also follows in this case. □

Suppose X is a geodesic metric space of bounded growth at some scale. Let $0 < r < R$ be parameters as in the definition of this concept. We can find a maximal set of points X_0 in X with the property that the distance between any two points in X_0 is at least $5R$. By definition, the visualization $\hat{X}$ of X is obtained by gluing an isometric copy of the ray $[0,\infty)$ to each point $x_0 \in X_0$ and identifying x_0 with 0, the initial point of the ray. The new space $\hat{X}$ carries a unique metric $\hat{d}$ that agrees with the metric on X and the metrics on the rays glued to X, and makes $\hat{X}$ a geodesic space (cf. [BS, p. 298]). We will always assume that $\hat{X}$ is equipped with this metric $\hat{d}$.

Lemma 5.2 *Suppose X is a geodesic $\mathrm{AC}_u(\kappa)$-space of bounded growth at some scale. Then $\hat{X}$ has the same properties, and is visual in addition.*

Proof The stated properties of the metric space $(\hat{X}, \hat{d})$ are obvious except for the fact that $\hat{X}$ is an $\mathrm{AC}_u(\kappa)$-space.

To verify this statement fix a basepoint $p \in \hat{X}$. We may assume that $p \in X \subseteq \hat{X}$. All Gromov products will be with respect to this basepoint. For an arbitrary point $x \in \hat{X}$ we define its "projection" $\hat{x}$ to X as follows. If $x \in X$ let $\hat{x} := x$. If $x \in \hat{X}\backslash X$, then x is contained in a unique ray that was glued to X in the construction of $\hat{X}$. In this case let $\hat{x}$ be the unique point in X to which this ray was glued. The definition of the metric on $\hat{X}$ implies that

$$(x \cdot y) \geq (\hat{x} \cdot \hat{y}) \tag{23}$$

for all $x, y \in \hat{X}$. Here we actually have equality unless one of the rays that was glued to X in the construction of $\hat{X}$ contains both x and y.

Now let $x_0 = z, x_1, \ldots, x_n = z'$ be an arbitrary chain in $\hat{X}$. If one of the rays glued to X contains both z and z', then

$$\begin{aligned}(z \cdot z') &= \min\{\hat{d}(z, p), \hat{d}(z', p)\}\\ &\geq \min_{i=0,\ldots,n} \hat{d}(x_i, p)\\ &\geq \min_{i=1,\ldots,n} (x_{i-1} \cdot x_i).\end{aligned}$$

Suppose no ray glued to X contains both z and z'. Then $(z \cdot z') = (\hat{z} \cdot \hat{z}')$. Moreover, in this case we will show that

$$\min_{i=1,\ldots,n} (x_{i-1} \cdot x_i) = \min_{i=1,\ldots,n} (\hat{x}_{i-1} \cdot \hat{x}_i). \tag{24}$$

An inequality as in (4) now follows from the corresponding inequality in X for the chain $\hat{x}_0 = \hat{z}, \hat{x}_1, \ldots, \hat{x}_n = \hat{z}'$. This together with the inequality in the previous case implies that $\hat{X}$ is an $\mathrm{AC}_u(\kappa)$-space.

To verify (24), it is enough by (23) to show that the left hand side is bounded from above by the right hand side. Pick $j \in \{1, \ldots, n\}$ such that

$$(\hat{x}_{j-1} \cdot \hat{x}_j) = \min_{i=1,\ldots,n} (\hat{x}_{i-1} \cdot \hat{x}_i).$$

If $(x_{j-1} \cdot x_j) = (\hat{x}_{j-1} \cdot \hat{x}_j)$, then (24) follows. Otherwise, $(x_{j-1} \cdot x_j) > (\hat{x}_{j-1} \cdot \hat{x}_j)$ and so there exists a ray S that was glued to X containing x_{j-1} and x_j. Note that S is glued to X at the point $y := \hat{x}_{j-1} = \hat{x}_j$. Since not both points z and z' are contained in S, there exists $k \in \{1, \ldots, n\}$ such that one of the points x_{k-1} and x_k is contained in S and the other is not. Then

$$\begin{aligned}\min_{i=1,\ldots,n} (x_{i-1} \cdot x_i) \leq (x_{k-1} \cdot x_k) &\leq d(p, y)\\ &= (\hat{x}_{j-1} \cdot \hat{x}_j) = \min_{i=1,\ldots,n} (\hat{x}_{i-1} \cdot \hat{x}_i),\end{aligned}$$

and again (24) follows. □

Proof of Theorem 1.6 It is enough to show that every $\mathrm{AC}_u(\kappa)$-space X with $\kappa < -1$ is rough-isometric to a convex subset of some hyperbolic space $\mathbb{H}^n$, $n \in \mathbb{N}$, if it is geodesic and of bounded growth at some scale. The general case follows from this by rescaling the metric on X.

First assume that X is visual in addition. By Theorem 1.5 there exists a visual metric ρ on $\partial_\infty X$ with parameter $\epsilon > 1$. Then $\rho^{1/\epsilon}$ is a visual metric on $\partial_\infty X$ with parameter 1. Since X has bounded growth at some scale, $(\partial_\infty X, \rho)$ has finite Assouad dimension. According to Assouad's embedding theorem, $(\partial_\infty X, \rho^{1/\epsilon})$

admits a bi-Lipschitz embedding into some $\mathbb{R}^{n-1}$, $n \in \mathbb{N}$. By Proposition 5.1, the space X is rough-isometric to a convex subset of $\mathbb{H}^n$.

In the general case, consider the visualization $\hat{X}$ of X. By Lemma 5.2 we can apply the considerations of the first part of the proof to $\hat{X}$. Hence $\hat{X}$ admits a rough-isometric embedding into some hyperbolic space $\mathbb{H}^n$. Let $U \subseteq \mathbb{H}^n$ be the image of $X \subseteq \hat{X}$ under this rough-isometric embedding. Since X is geodesic, U is rough geodesic. The geodesic stability of $\mathbb{H}^n$ then implies that U is cobounded in the set

$$V = \bigcup_{x,y \in U} [x, y]$$

consisting of all geodesic segments with endpoints in U. In particular, $U \cong V$ (a rough-isometry is given by the inclusion of U in V). Let $W = \mathrm{hull}(U)$ be the convex hull of U in $\mathbb{H}^n$. Then $V \subseteq W$, and V is cobounded in W [BS, Prop. 10.1 (1)]. So $W \cong V \cong U \cong X$ which shows that X is rough-isometric to the convex subset W of $\mathbb{H}^n$. □

Example 5.3 We now give an example showing that in Theorem 1.6 the space $\mathbb{H}^n_{\kappa'}$ cannot be replaced by $\mathbb{H}^n_\kappa$ in general. In our discussion we will leave some of the details to the reader.

Let

$$B = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$$

be the unit ball in $\mathbb{C}^2$ equipped with the metric d defined by

$$\cosh^2 d(z, w) = \frac{(1 - \langle z, w\rangle)(1 - \langle w, z\rangle)}{(1 - \langle z, z\rangle)(1 - \langle w, w\rangle)}$$

for $z, w \in B$. Here

$$\langle x, y\rangle = x_1\bar{y}_1 + x_2\bar{y}_2$$

denotes the standard Hermitian product of

$$x = (x_1, x_2),\ \ y = (y_1, y_2) \in \mathbb{C}^2.$$

Then B equipped with d is isometric to complex hyperbolic space of real dimension 4 [BH, p. 310]. The space B is a Cartan-Hadamard manifold normalized so that B has sectional curvatures between -4 and -1. In particular, B is geodesic and a CAT(-1)-space, which implies by Proposition 1.4 that B is an $\mathrm{AC}_u(-1)$-space. Since its sectional curvature is pinched, B has bounded growth at some scale. So B satisfies all the hypotheses of Theorem 1.6 with $\kappa = -1$.

As a set its Gromov boundary $\partial_\infty B$ can be identified with the unit sphere

$$S = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$$

in $\mathbb{C}^2$. A direct computation involving the explicit expression for the metric gives the formula

$$(u \cdot v)_o = \lim_{z \to u,\, w \to v} (z \cdot w)_o = -\frac{1}{2}\log(|1 - \langle u, v\rangle|/2) \tag{25}$$

for the Gromov product of two points $u, v \in S$ with respect to the origin $o = (0, 0)$ in $\mathbb{C}^2$.

One can show that ρ defined by

$$\rho(u, v) = e^{-(u \cdot v)_o} = \sqrt{|1 - \langle u, v\rangle|/2}$$

for $u, v \in S$ is a metric on S [KR, p. 321]. So ρ is a visual metric with parameter 1. The space S equipped with the metric ρ is closely related to the Heisenberg group equipped with its Carnot metric. Indeed, the Heisenberg group H is obtained from S by stereographic projection from any point of S [KR, Sec. 1.F]. In particular, every bounded open set in H is bi-Lipschitz equivalent to an open set in S. It is well-known that no nonempty open subset of the Heisenberg group admits a bi-Lipschitz embedding into a Euclidean space $\mathbb{R}^n$, $n \in \mathbb{N}$ [Se, Sect. 7]. Hence (S, ρ) cannot be embedded into any Euclidean space by a bi-Lipschitz map.

Now consider real hyperbolic space $\mathbb{H}^n$, $n \in \mathbb{N}$ (as viewed in the unit ball model). The boundary at infinity $\partial_\infty \mathbb{H}^n$ can be identified with the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. The restriction of the Euclidean metric to $\mathbb{S}^{n-1}$ is a visual metric with parameter 1.

The space B does not admit any rough-isometric embedding into $\mathbb{H}^n_{-1} = \mathbb{H}^n$. Indeed, any such embedding induces a bi-Lipschitz embedding of $\partial_\infty B = S$ into $\partial_\infty \mathbb{H}^n = \mathbb{S}^{n-1}$ if we equip S and $\mathbb{S}^{n-1}$ with visual metrics of parameter 1. In particular, there would be a bi-Lipschitz embedding of (S, ρ) into Euclidean space $\mathbb{R}^n$. This is impossible as we have seen.

6 Spaces with asymptotic upper curvature $-\infty$

Recall that a metric ρ on a space Z is called an ultrametric if it satisfies $\rho(x, y) \leq \max\{\rho(x, z), \rho(z, y)\}$ for all $x, y, z \in Z$.

We call a chain $z_0 = z, \dots, z_n = z'$ in a metric space (Z, ρ) a λ-chain for $\lambda \geq 0$, if $\rho(z_{i-1}, z_i) \leq \lambda$ for all $i = 1, \dots, n$. A metric space (Z, ρ) is said to be uniformly disconnected if there exists $\epsilon > 0$ such that no distinct points $z, z' \in Z$ can be connected by a $(\epsilon\rho(z, z'))$-chain.

The following lemma is proved in [DS, Proposition 15.7]. For the convenience of the reader we include the proof.

Lemma 6.1 *A metric space (Z, ρ) is uniformly disconnected if and only if there exist a constant $C \geq 1$ and an ultrametric $\tilde{\rho}$ on Z such that*

$$\frac{1}{C}\rho(z, z') \leq \tilde{\rho}(z, z') \leq C\rho(z, z') \quad \textit{for all} \quad z, z' \in Z.$$

Proof Suppose Z is uniformly disconnected. For $z, z' \in Z$ define

$$\tilde{\rho}(z, z') := \inf\{\lambda \geq 0 : \text{there exists a } \lambda\text{-chain connecting } z \text{ and } z'\}.$$

Then $\tilde{\rho}$ is a symmetric distance function on Z satisfying

$$\epsilon\rho \leq \tilde{\rho} \leq \rho,$$

where $\epsilon > 0$ is the parameter given by the uniform disconnectedness of Z. It follows from the definition of $\tilde{\rho}$ that $\tilde{\rho}(z, z') \leq \max\{\tilde{\rho}(z, z''), \tilde{\rho}(z'', z')\}$ for all $z, z', z'' \in Z$. Hence $\tilde{\rho}$ is an ultrametric with the desired properties.

For the converse we may assume that ρ itself is an ultrametric. Then Z is uniformly disconnected with $\epsilon = 1/2$ as parameter. Indeed, if $z, z' \in Z$, $z \neq z'$, are arbitrary, then there is no $(\rho(z, z')/2)$-chain $z_0 = z, z_1, \dots, z_n = z'$; for otherwise,

$$\rho(z, z') \leq \max_{i=1,\dots,n} \rho(z_{i-1}, z_i) \leq \rho(z, z')/2,$$

which is impossible. □

A crucial ingredient in the proof of Theorem 1.7 is the following lemma.

Lemma 6.2 *Let (X, d) be a visual Gromov hyperbolic geodesic metric space of bounded growth at some scale. Assume $K_u(X) = -\infty$. Then $\partial_\infty X$ equipped with any visual metric is uniformly disconnected.*

Proof Since two visual metrics on $\partial_\infty X$ are related by an inequality as in (8), it is enough to show that $(\partial_\infty X, \rho)$ is uniformly disconnected, where ρ is some suitably chosen visual metric.

Since $K_u(X) = -\infty$, there exist visual metrics on $\partial_\infty X$ for all positive parameters according to Theorem 1.5. Fix such a metric ρ with parameter 1. Since X has bounded growth at some scale, the space $(\partial_\infty X, \rho)$ has finite Assouad dimension (see Section 5). So we can fix numbers $D > 0$ and $K > 0$ such that an inequality as in (22) is valid for $(\partial_\infty X, \rho)$. Let $\sigma = D + 1$. Then there also exists a visual metric $\tilde{\rho}$ on $\partial_\infty X$ with parameter σ. For some constant $C_1 \geq 1$ we have

$$(1/C_1)\rho^\sigma \leq \tilde{\rho} \leq C_1 \rho^\sigma. \tag{26}$$

Let $\epsilon > 0$, and $\xi, \xi' \in \partial_\infty X$ with $\xi \neq \xi'$ be arbitrary. We want to show that if $\epsilon > 0$ is small enough independently of ξ and ξ', then there is no $(\epsilon\rho(\xi, \xi'))$-chain connecting these points.

Suppose there is such a chain. Then among all $(\epsilon\rho(\xi, \xi'))$-chains starting at ξ and leaving the open ball B centered at ξ with radius $r = \rho(\xi, \xi')$ there is one with minimal cardinality $N_\epsilon + 1$, where $N_\epsilon \in \mathbb{N}$. Let $\xi = \xi_0, \dots, \tilde{\xi} := \xi_{N_\epsilon}$ be such a chain. Then $\tilde{\xi} \notin B$, while $\xi_i \in B$ for $i = 0, \dots, N_\epsilon - 1$. In particular, $\rho(\xi, \tilde{\xi}) \geq \rho(\xi, \xi')$. Therefore, inequality (26) gives

$$\begin{aligned} N_\epsilon \cdot (\epsilon\rho(\xi, \tilde{\xi}))^\sigma \geq N_\epsilon \cdot (\epsilon\rho(\xi, \xi'))^\sigma &\geq \sum_{i=1}^{N_\epsilon} \rho(\xi_{i-1}, \xi_i)^\sigma \\ &\geq \frac{1}{C_1} \sum_{i=1}^{N_\epsilon} \tilde{\rho}(\xi_{i-1}, \xi_i) \geq \frac{1}{C_1} \tilde{\rho}(\xi, \tilde{\xi}) \\ &\geq \frac{1}{C_2} \rho(\xi, \tilde{\xi})^\sigma, \end{aligned}$$

where $C_2 = C_1^2$. Thus,

$$N_\epsilon \geq \frac{1}{C_2}\left(\frac{1}{\epsilon}\right)^\sigma. \tag{27}$$

Due to the minimality of N_ϵ, we have $\rho(\xi_i, \xi_j) > \epsilon\rho(\xi, \xi')$ whenever $|i - j| \geq 2$. This implies that there exists a subset $I \subseteq \{0, \ldots, N_\epsilon - 1\}$ of cardinality $\#I \geq N_\epsilon/2$ such that $\rho(\xi_i, \xi_j) \geq \epsilon\rho(\xi, \xi')$ for all $i, j \in I$ with $i \neq j$. Since the set $\{\xi_i : i \in I\}$ is contained in B, we can use inequality (22) for $\alpha = \epsilon\rho(\xi, \xi')$ and $\beta = 2\rho(\xi, \xi')$, and obtain

$$\frac{N_\epsilon}{2} \leq K\left(\frac{2}{\epsilon}\right)^D. \tag{28}$$

Since $\sigma = D + 1$ we get the inequality

$$\frac{1}{C_2}\left(\frac{1}{\epsilon}\right)^{D+1} \leq 2K\left(\frac{2}{\epsilon}\right)^D.$$

This results in a positive lower bound for ϵ independent of ξ and ξ'. It follows that if $\epsilon > 0$ is small enough, then it is impossible to connect any points $\xi, \xi' \in \partial_\infty X$ with $\xi \neq \xi'$ by a $(\epsilon\rho(\xi, \xi'))$-chain in $(\partial_\infty X, \rho)$. So $(\partial_\infty X, \rho)$ is uniformly disconnected, and the claim follows. □

Let $N \in \mathbb{N}$, $N \geq 2$, and consider the space

$$C_N = \{1, \ldots, N\}^{\mathbb{N}_0}$$

consisting of all sequences (x_n) such that $x_n \in \{1, \ldots, N\}$ for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Equipped with the product topology each of these spaces C_N is homeomorphic to a Cantor set. We define a metric σ on C_N as follows. For two sequences $\alpha = (x_n)$ and $\beta = (y_n)$ in C_N set $\sigma(\alpha, \beta) = 0$ if $\alpha = \beta$. Otherwise, there exists a smallest number $k \in \mathbb{N}_0$ such that $x_k \neq y_k$. Define

$$\sigma(\alpha, \beta) = e^{-k}.$$

Then σ is an ultrametric on C_N. We will always think of C_N as being equipped with this metric σ.

The spaces C_N serve as model spaces for compact ultrametric spaces of finite Assouad dimension as the following embedding theorem shows.

Proposition 6.3 *Suppose (Z, ρ) is a compact ultrametric space of finite Assouad dimension. Then (Z, ρ) admits a bi-Lipschitz embedding into C_N for sufficiently large N.*

Proof We may assume that the diameter of Z is equal to 1. Since Z has finite Assouad dimension there exists a number $N \in \mathbb{N}$, $N \geq 2$, with the following property. If $R > 0$ and $V \subseteq Z$ is a set of diameter $\leq 2R$ such that the mutual distance of distinct elements in V is $\geq R/e$, then the cardinality of V is bounded by N.

We will show that Z admits a bi-Lipschitz embedding into C_N. For this purpose we first define open balls $B_{i_0 \ldots i_k}$ in Z inductively. Here $k \in \mathbb{N}_0$ and

$$i_0, \ldots, i_k \in \{1, \ldots, N\}.$$

It is convenient to let $B = Z$ corresponding to $k = -1$ and an empty sequence of indices $i_0, \ldots, i_k$. Choose a maximal 1-separated set $V \subseteq B = Z$. So distinct elements in V have mutual distance ≥ 1 and there is no proper superset of V with the same property. It follows from the definition of N (applied with $R = e$) that V has at most N elements, say $x_1, \dots, x_n$, where $n \leq N$. Let B_i, $i = 1, \dots, n$, be the open ball in Z which has radius 1 and is centered at x_i. By the definition of V the balls $B_1, \dots, B_n$ form a cover of B. Moreover, from the fact that ρ is an ultrametric it follows that each B_i is a closed and hence compact subset of Z and that

$$\mathrm{dist}(B_i, B_j) \geq 1 \text{ for } i \neq j.$$

Now each ball B_i generates "children" B_{ij} in a similar way as in the first step of the construction. Fix $i \in \{1, \dots, n\}$. Let V_i be a maximal $(1/e)$-separated subset of B_i. It follows from the definition of N (applied with $R = 1$) that V_i contains at most N elements, say $x_{i1}, \dots, x_{in_i}$, where $n_i \leq N$. Let B_{ij}, $j = 1, \dots, n_i$, be the open ball with radius $1/e$ centered at x_{ij}. From the fact that ρ is an ultrametric it follows again that each B_{ij} is a compact subset of Z and that

$$\mathrm{dist}(B_{ij}, B_{ik}) \geq 1/e \quad \text{whenever} \quad j \neq k.$$

Moreover, we have that

$$B_{ij} \subseteq B_i \quad \text{for all} \quad i, j.$$

We continue in this manner decreasing the radii of the balls by the factor $1/e$ in each step. In this way we obtain balls $B_{i_0 \dots i_k}$, where $k \in \mathbb{N}_0$ and $1 \leq i_0, \dots, i_k \leq N$, with the following properties:

(i) each ball $B_{i_0 \dots i_k}$ is an open ball of radius e^{-k} and is a compact subset of Z,
(ii) $\mathrm{dist}(B_{i_0 \dots i_{k-1} i}, B_{i_0 \dots i_{k-1} j}) \geq e^{-k}$, whenever $i \neq j$,
(iii) $B_{i_0 \dots i_k} \subseteq B_{i_0 \dots i_{k-1}}$ for all indices,
(iv) $\displaystyle\bigcup_i B_{i_0 \dots i_{k-1} i} = B_{i_0 \dots i_{k-1}}$ for all indices.

Note that if $k = 0$ we interpret $i_0 \dots i_{k-1}$ as an empty sequence and so $B_{i_0 \dots i_{k-1}} = B = Z$.

The properties (i)–(iv) of the balls imply that for each $x \in Z$ there exists a unique sequence $(i_n) \in C_N$ such that

$$\{x\} = \bigcap_{n \in \mathbb{N}_0} B_{i_0 \dots i_n}.$$

The correspondence $x \in Z \mapsto (i_n) \in C_N$ defines a map $\phi \colon Z \to C_N$. We claim that ϕ is the desired bi-Lipschitz embedding. To see this we have to show that for arbitrary $x, y \in Z$, $x \neq y$, with associated sequences $\phi(x) = (i_n)$ and $\phi(y) = (j_n)$, the distance $\rho(x, y)$ is comparable to $\sigma(\phi(x), \phi(y))$.

Since $x \neq y$, there exists a smallest integer $k \in \mathbb{N}_0$ such that $i_k \neq j_k$. Then by definition of σ we have

$$\sigma(\phi(x), \phi(y)) = e^{-k}. \tag{29}$$

On the other hand, $i_0 = j_0, \dots, i_{k-1} = j_{k-1}$, and so $x, y \in B_{i_0 \dots i_{k-1}}$. Hence by property (i) of the balls we have

$$\rho(x, y) \leq 2e^{-(k-1)}. \tag{30}$$

Since $i_k \neq j_k$, the definition of ϕ and property (ii) of the balls imply

$$\rho(x, y) \geq e^{-k}. \tag{31}$$

From the relations (29)–(31) we infer that

$$\frac{1}{2e}\rho(x, y) \leq \sigma(\phi(x), \phi(y)) \leq \rho(x, y) \quad \text{for all} \quad x, y \in Z.$$

The claim follows. □

We now summarize some facts about trees. By definition a (simplicial) tree is a locally finite connected graph without cycles. If we assign to each edge e in a tree T a positive number $\ell(e) > 0$, then there exists a unique path metric on T such that each edge e of T is a geodesic segment of length $\ell(e)$. In the following we will assume that a tree carries a metric induced by such a length assignment for its edges. Trees are 0-hyperbolic geodesic metric spaces.

The regular tree T_l, $l \in \mathbb{N}$, $l \geq 2$, is the unique tree such that each vertex has valence l and each edge has length 1. If we fix a vertex $p \in T_l$ as a basepoint, then the geodesic rays in T_l emanating from p admit a natural coding by sequences $\alpha = (x_n)$ such that $x_0 \in \{1, \dots, l\}$ and $x_n \in \{1, \dots, l-1\}$ for $n \in \mathbb{N}$. This coding is obtained as follows: If we label the edges emanating from a vertex v of T and "leading away from p" by the numbers $1, \dots, l-1$ (for $v = p$ we label by the numbers $1, \dots, l$), then each geodesic ray in T_l with initial point p gives rise to a sequence by recording the labels of edges that we traverse by traveling along the ray starting from p.

This coding leads to the identification

$$\partial_\infty T_l = \{1, \dots, l\} \times \{1, \dots, l-1\}^{\mathbb{N}}.$$

In analogy with the metric σ on the Cantor set C_N as discussed above, we can define a metric σ_p on $\partial_\infty T_l$ as follows. If ξ and ζ are two distinct points in $\partial_\infty T_l$, let $\alpha = (x_n)$ and $\beta = (y_n)$ be the sequences representing the geodesic rays $[p, \xi]$ and $[p, \zeta]$, respectively. Since $\xi \neq \zeta$, there exists a smallest $k \in \mathbb{N}_0$ such that $x_k \neq y_k$, and we have $(\xi \cdot \zeta)_p = k$. Set

$$\sigma_p(\xi, \zeta) = e^{-k} = e^{-(\xi\cdot\zeta)_p}.$$

Then σ_p is a visual ultrametric on $\partial_\infty T_l$ with parameter 1.

If U is an arbitrary subset of a tree T, we denote by hull(U) the convex hull of U in T, i.e., the smallest convex subset of T containing U. Note that hull(U) is the union of all geodesic segments $[p, q]$, where $p, q \in U$. This easily follows from the following fact valid in every tree: if $p_1, p_2, p_3, p_4 \in T$, and $x \in [p_1, p_2]$, $y \in [p_3, p_4]$, then

$$[x, y] \subseteq \bigcup_{1 \leq i < j \leq 4} [p_i, p_j]. \tag{32}$$

Suppose a subset U of a tree T has the property that every two points $x, y \in U$ can be connected by a k-rough geodesic segment $[x, y]_k \subseteq U$ with k independent of x and y. By geodesic stability (valid for all Gromov hyperbolic spaces), the Hausdorff distance of $[x, y]_k$ and the geodesic segment $[x, y]$ is bounded by a constant only depending on k (and hence independent of x and y). Since the convex hull of U is the union of all geodesic segments with endpoints in U, this implies that U is cobounded in $\mathrm{hull}(U)$. So in this case $U \cong \mathrm{hull}(U)$.

Suppose Z is a subset of the boundary $\partial_\infty T$ of a tree T. If Z has at least two points, its convex hull, $\mathrm{hull}(Z) \subseteq T$, is defined to be the smallest convex subset of T whose boundary at infinity contains Z. Based on the inclusion (32), which is also true if $p_1, \dots, p_4$ are points in $\partial_\infty T$, one can show that $\mathrm{hull}(Z)$ is the union of all geodesics $[\xi, \zeta]$ with $\xi, \zeta \in \partial_\infty T$, $\xi \neq \zeta$. In particular, $\mathrm{hull}(Z)$ is a visual Gromov hyperbolic geodesic metric space and we have the identification $\partial_\infty\big(\mathrm{hull}(Z)\big) = Z$.

If the tree T is a regular tree T_l, then we can equip Z with the restriction σ of the metric σ_p defined above, where p is a vertex in $Y = \mathrm{hull}(Z)$. Then σ is a visual metric on $Z = \partial_\infty Y$ with parameter 1. Based on [BS, Remark after Thm. 8.2] we conclude that (cf. the discussion in Section 5)

$$\mathrm{Con}(Z, \sigma) \cong Y = \mathrm{hull}(Z).$$

Proof of Theorem 1.7 Based on the previous embedding theorem, the reasoning is similar as in the proofs of Proposition 5.1 and Theorem 1.6.

First assume that X is visual in addition. Then $\partial_\infty X \neq \emptyset$.

If $\partial_\infty X$ consists of only one point, then X contains a rough geodesic ray that is cobounded in X. Then X is rough-isometric to the ray $[0, \infty)$, and the statement is obviously true.

Assume that $\partial_\infty X$ consists of more than one point. By Theorem 1.5 there exists a visual metric ρ on $\partial_\infty X$ with parameter 1. By Lemma 6.2 and Lemma 6.1 we may assume that ρ is an ultrametric. Moreover, the space $(\partial_\infty X, \rho)$ has finite Assouad dimension. Hence by Proposition 6.3 the space $(\partial_\infty X, \rho)$ is bi-Lipschitz equivalent to a subset Z of some Cantor set C_N, $N \geq 2$ (equipped with the metric σ). As in the proof of Proposition 5.1 we see that

$$X \cong \mathrm{Con}(\partial_\infty X, \rho) \cong \mathrm{Con}(Z).$$

By the discussion above, we can consider C_N equipped with its metric σ as a subset of the boundary $\partial_\infty T_l$ of the regular tree T_l, $l = N + 1$, equipped with a visual metric with parameter 1. Since $\partial_\infty X$ and hence also Z consist of more than one point,

$$\mathrm{Con}(Z) \cong \mathrm{hull}(Z) \subseteq T_l.$$

It follows that X is rough-isometric to the convex subset $\mathrm{hull}(Z)$ of T_l. The claim follows in this case.

If X is not visual, consider the visualization $\hat{X}$ of X. By Lemma 5.2 the space $\hat{X}$ satisfies the hypotheses of the theorem and is visual in addition. So by the considerations of the first part of the proof, $\hat{X}$ admits a rough-isometric embedding into some regular tree T_l. Let $U \subseteq T_l$ denote the image of $X \subseteq \hat{X}$ under this rough-isometric embedding. Since X is geodesic, U is rough-geodesic. Since every rough-geodesic subset of a tree is cobounded in its convex hull, it follows that $X \cong U \cong \mathrm{hull}(U)$. The statement follows. □

7 0-hyperbolic spaces and virtually free groups

In this section we will prove Theorem 1.8. We will first establish some facts about 0-hyperbolic spaces that are of independent interest and are related to the discussion in the previous section.

The following lemma shows that the existence of visual ultrametrics on the boundary of a Gromov hyperbolic space is equivalent to a condition on its inner geometry.

Lemma 7.1 *Let (X, d) be a visual Gromov hyperbolic metric space. Suppose that there exists a visual metric on $\partial_\infty X$ that is also an ultrametric. Then for all $p \in X$ there exists $\Delta \geq 0$ such that for all $z, z' \in X$ and all chains $x_0 = z, x_1, \ldots, x_n = z'$ in X,*

$$(z \cdot z')_p \geq \min_{i=1,\ldots,n} (x_{i-1} \cdot x_i)_p - \Delta. \tag{33}$$

Conversely, suppose that there exists $\Delta \geq 0$ such that (33) holds for all chains in X. Then for each $\epsilon > 0$ the boundary $\partial_\infty X$ carries a visual ultrametric with parameter ϵ.

Note that an inequality as in (33) holds if and only if X is an $\mathrm{AC}_u(-\infty)$-space.

Proof Suppose there exists a visual ultrametric ρ on $\partial_\infty X$. According to Lemma 4.1 it is enough to establish an inequality as in (33) for chains in the boundary. So let $\xi_0 = \xi, \xi_1, \ldots, \xi_n = \xi'$ be an arbitrary chain in $\partial_\infty X$. Since ρ is a visual metric, it satisfies an inequality as in (7). Together with the fact that ρ is an ultrametric this leads to

$$\begin{aligned}\exp(-\epsilon(\xi \cdot \xi')_p) \leq \lambda\rho(\xi, \xi') &\leq \lambda \max_{i=1,\ldots,n} \rho(\xi_{i-1}, \xi_i) \\ &\leq \lambda^2 \max_{i=1,\ldots,n} \exp(-\epsilon(\xi_{i-1} \cdot \xi_i)_p),\end{aligned}$$

and therefore

$$(\xi \cdot \xi')_p \geq \min_{i=1,\ldots,n} (\xi_{i-1} \cdot \xi_i)_p - \frac{2}{\epsilon} \log \lambda. \tag{34}$$

The first part of the claim follows.

Conversely, suppose an inequality as in (33) holds for fixed $p \in X$ and all chains in X. Then by Lemma 4.1 there exists $\Delta' \geq 0$ such that

$$(\xi \cdot \xi')_p \geq \min_{i=1,\ldots,n} (\xi_{i-1} \cdot \xi_i)_p - \Delta' \tag{35}$$

for all chains $\xi_0 = \xi, \xi_1, \ldots, \xi_n = \xi'$ in $\partial_\infty X$. For arbitrary $\xi, \xi' \in \partial_\infty X$ define

$$[\xi \cdot \xi']_p := \sup \min_{i=1,\ldots,n} (\xi_{i-1} \cdot \xi_i)_p,$$

where the supremum is taken over all $n \in \mathbb{N}$ and all chains $\xi_0 = \xi, \xi_1, \ldots, \xi_n = \xi'$ in $\partial_\infty X$. Then

$$[\zeta \cdot \xi]_p \geq \min\{[\zeta \cdot \eta]_p, [\eta \cdot \xi]_p\} \quad \text{for all} \quad \zeta, \eta, \xi \in \partial_\infty X, \tag{36}$$

and (35) implies that

$$(\xi \cdot \xi')_p \leq [\xi \cdot \xi']_p \leq (\xi \cdot \xi')_p + \Delta' \quad \text{for all} \quad \xi, \xi' \in \partial_\infty X. \tag{37}$$

Let $\epsilon > 0$ be arbitrary and define

$$\rho(\xi, \xi') = \exp(-\epsilon[\xi \cdot \xi']_p) \quad \text{for} \quad \xi, \xi' \in \partial_\infty X.$$

Then inequalities (36) and (37) imply that ρ is a visual ultrametric with parameter $\epsilon > 0$. The second part of the claim follows. □

The following lemma shows that inequality (33) is related to the characterization of 0-hyperbolic spaces up to rough-isometry. This lemma is due to Gromov [Gr, 6.1.A]. For the convenience of the reader, we will sketch its proof (see [GH, Ch. 2] for related considerations).

Lemma 7.2 *Let (X, d) be a Gromov hyperbolic metric space. Then X is rough-isometric to a 0-hyperbolic space if and only if there exist $p \in X$ and a constant $\Delta \geq 0$ such that for all $z, z' \in X$ and all chains $x_0 = z, x_1, \ldots, x_n = z'$ in X,*

$$(z \cdot z')_p \geq \min_{i=1,\ldots,n} (x_{i-1} \cdot x_i)_p - \Delta. \tag{38}$$

In other words, a Gromov hyperbolic metric space is rough-isometric to a 0-hyperbolic space if and only if it is an $\mathrm{AC}_u(-\infty)$-space.

Proof Every 0-hyperbolic space is an $\mathrm{AC}_u(-\infty)$-space. Hence every metric space rough-isometric to a 0-hyperbolic space has the same property. This implies the “only if”-part of the statement.

For the converse direction suppose that inequality (38) holds for all chains. Similarly as in the proof of Lemma 7.1, for $z, z' \in X$ we define

$$[z \cdot z']_p := \sup \min_{i=1,\ldots,n} (x_{i-1} \cdot x_i)_p,$$

where the supremum is taken over all $n \in \mathbb{N}$ and all chains $x_0 = z, x_1, \ldots, x_n = z'$ in X. Then we have

$$[x \cdot z]_p \geq \min\{[x \cdot y]_p, [y \cdot z]_p\} \quad \text{for all} \quad x, y, z \in X. \tag{39}$$

Moreover, our hypothesis implies that

$$(z \cdot z')_p \leq [z \cdot z']_p \leq (z \cdot z')_p + \Delta \quad \text{for all} \quad z, z' \in X. \tag{40}$$

We define a distance function d' on X by

$$d'(z, z') := d(z, p) + d(z', p) - 2[z \cdot z']_p \quad \text{for} \quad z, z' \in X.$$

Then by (40),

$$d(z, z') - 2\Delta \leq d'(z, z') \leq d(z, z') \quad \text{for all} \quad z, z' \in X. \tag{41}$$

One can check that d' is a pseudometric, i.e., it satisfies all the requirements for a metric, but it may happen that $d'(z, z') = 0$ for points $z \neq z'$. We let $\tilde{X}$ be the quotient space of X obtained by identifying all points z and z' in X with $d'(z, z') = 0$, and equip $\tilde{X}$ with the metric $\tilde{d}$ induced by d'. As follows from (41), the natural projection map $X \to \tilde{X}$ is a rough-isometry. Moreover, (39) implies that $(\tilde{X}, \tilde{d})$ is 0-hyperbolic. The claim follows. □

Proposition 7.3 *Let X be a geodesic metric space which admits a quasi-isometric embedding into an* $\mathrm{AC}_u(-\infty)$*-space. Then X is rough-isometric to a* 0*-hyperbolic metric space.*

Proof By Proposition 3.4, X is an $\mathrm{AC}_u(-\infty)$-space. Hence X is rough-isometric to a 0-hyperbolic space by Lemma 7.2. □

Note that the assumptions of the proposition are satisfied in particular if X is a geodesic metric space that admits a quasi-isometric embedding into a 0-hyperbolic space.

Before we now prove Theorem 1.8, we record some relevant facts about groups. Suppose Γ is a finitely generated group, and S is a finite and symmetric set of generators. Here the symmetry of S means that if $s \in S$, then $s^{-1} \in S$. The Cayley graph $C(\Gamma, S)$ associated with Γ and S is the graph whose vertices are the elements in Γ. Moreover, two vertices $x, y \in \Gamma$ are connected by an edge if there exists $s \in S$ such that $y = xs$. The graph $C(\Gamma, S)$ carries a natural path metric so that its edges are geodesic segments of length 1. Equipped with this metric, $C(\Gamma, S)$ is a geodesic metric space.

A finitely generated group Γ is called hyperbolic, if its Cayley graph $C(\Gamma, S)$ associated with a finite and symmetric set S of generators is Gromov hyperbolic. The Gromov hyperbolicity of the Cayley graph does not depend on the choice of the generating set, because two such choices yield Cayley graphs that are bi-Lipschitz equivalent.

As is common in geometric group theory, we say that a group Γ has *virtually* a certain property, if Γ has a subgroup of finite index with the property in question. In particular, Γ is called virtually free, if there exists a free subgroup in Γ of finite index.

Proof of Theorem 1.8 Suppose first that there exists a finite and symmetric set S of generators of Γ such that $K_u(X) = -\infty$, where $X = C(\Gamma, S)$. Then X is a Gromov hyperbolic geodesic metric space, and it is of bounded growth at some scale, because the valence of each vertex in the graph $C(\Gamma, S)$ is uniformly bounded by the number of elements in S.

By Theorem 1.7 the space X is rough-isometric to a convex subset C of a regular tree. Then $\partial_\infty X$ is homeomorphic to $\partial_\infty C$. Since $\partial_\infty C$ is a subset of a Cantor set, $\partial_\infty X$ is totally disconnected. It is well-known that this implies that Γ is virtually free (see, for example, [KB, Thm. 8.1]).

For the converse let S be an arbitrary finite and symmetric set of generators of Γ. Note that if Γ' is a subgroup of finite index in Γ, then Γ' is also finitely generated. Moreover, if S' is a finite and symmetric set of generators of Γ', then the Cayley graphs $C(\Gamma, S)$ and $C(\Gamma', S')$ are quasi-isometric (see [GH, Ch. 2, §3]). If Γ is virtually free, then we can choose Γ' and S' so that $C(\Gamma', S')$ is isometric to a regular tree, and hence 0-hyperbolic. Since $C(\Gamma, S)$ is geodesic, we can apply Proposition 7.3 and conclude that $C(\Gamma, S)$ is an $\mathrm{AC}_u(-\infty)$-space. □

Theorem 1.8 shows that if Γ is a virtually free group, then $K_u(C(\Gamma, S)) = -\infty$ independently of the choice of the generating set S. For general hyperbolic groups Γ, the asymptotic upper curvature of $C(\Gamma, S)$ will depend on S. One can show that if Γ is such a group, then there are generating sets S of Γ such that $K_u(C(\Gamma, S))$

is arbitrarily small. It is an interesting question whether $K_u(C(\Gamma, S))$ is always bounded away from 0 independently of S.

Acknowledgement It is a pleasure to thank Bruce Kleiner and Viktor Schroeder for useful discussions.

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