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A QUADRATIC FORM ON THE QUOTIENT OF A PERIODIC MAP

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Dedicated to Professor A. D. Wallace on his 68th birthday.

1. Introduction.

This note is concerned with the Witt equivalence class of the bilinear, symmetric and non-singular rational valued inner-product defined on $H^{2n}(M/T;Q)$ where (T,M^{4n}) is a diffeomorphism of prime period preserving the orientation of the manifold M^{4n} and M/T is the quotient space of the periodic map. Let us first discuss the background material. Associated to every closed oriented manifold, M^{4n}, of dimension divisible by four is the integral invariant signature, $\mathrm{sgn}(M)$. It is a fundamental principle of surgery theory that the quadratic form defined on $H^{2n}(M;Z)/\mathrm{Tor}$ is unimodular; that is, its determinant is ± 1. The value of this principle stems from the fact that the equivalence class in the rational Witt rings, $\mathscr{W}$, of any such quadratic form is uniquely determined by the signature, [7, lem. 8.9]. By contrast, however, for a compact oriented manifold with boundary, B^{4n}, the determinant of the quadratic form on the image of $j^*:H^{2n}(B,\partial B;Z) \to H^{2n}(B;Z)/\mathrm{Tor}$ may take on any non-zero integral value. Indeed this form is unimodular if and only if $\mathrm{im}(j^*) \subset H^{2n}(B;X)/\mathrm{Tor}$ is a direct summand. Thus in

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general the signature would reveal nothing about the Hasse symbols of the quadratic form of those finite primes which divide twice the determinant. For this reason the equivalence class of the quadratic form on $im(j^*)$ in the rational Witt ring is introduced as the invariant $w(B) \in \mathscr{W}$. A difference class, $w(B) - sgn(B) \cdot 1$, where 1 is the identity of the Witt ring, is also considered. This lies in the kernel of the signature homomorphism $sgn: \mathscr{W} \to Z$ and so has order 1, 2 or 4. This difference is a partial measure of the failure of $im(j^*)$ to be a summand. If W is a second compact oriented manifold with $\partial W = \partial B$ then a closed manifold $M = B \cup -W$ can be formed and from the usual additivity arguments, together with the fact that M is closed, it follows that $w(B) - w(W) = w(M) = sgn(M) \cdot 1 = sgn(B) \cdot 1 - sgn(W) \cdot 1$ and therefore the difference only depends on ∂B. From this there is then defined for every closed, oriented, bounding $(4n-1)$-manifold, V, a peripheral invariant in $\mathscr{W}$ by $per(V) = w(B) - sgn(B) \cdot 1$ where B is any compact oriented manifold with $\partial B = V$. For this peripheral invariant, $per(V_1 \cup -V_2) = per(V_1) - per(V_2)$ and if M^{4m} is a closed oriented manifold then $per(V \times M) = per(V)\, sgn(M)$. The peripheral invariant is quite well suited as we see in section 4 to applications to the study carried out in [1], which was of particular influence on us.

To return to the purpose of this note, it seemed to us that the use of the rational Witt ring might be applied in the study of orientation preserving periodic diffeomorphisms on closed manifolds to yield invariants which among other things, might fit in well with peripheral. For another usage of Witt rings in the study of PL-periodic maps we may refer the reader to [5]. Because we are concerned with the smooth case here our only candidate is the quadratic form on $H^{2n}(M/T;Z)/Tor$ since M/T in general has singularities and so the quadratic form may not be unimodular. Our results are incomplete, and therefore we shall only give a sketch of the ideas to outline questions needing further study.

Let (T,M^{4n}) denote an orientation preserving diffeomorphism of prime period, p, on a closed manifold. Let $\sigma \in H_{4n}(M;Z)$ be the orientation class and into $H^{2n}(M;Q)$ introduce the symmetric, bilinear and non-singular inner-product $(x,y) = p\langle xy,\sigma\rangle \in Q$ where $\langle xy,\sigma\rangle$ denotes the result of applying the augmentation homomorphism to the cap-product $xy \cap \sigma \in H_0(M;Q)$. The induced T^* is an isometry with respect to this inner-product and as a consequence the averaging operator
$\Sigma x = (x + Tx + \cdots + T^{p-1}x)/p$ is self-adjoint in the sense that $(\Sigma x,y) = (x,\Sigma y)$. As an immediate corollary this Σ exhibits the subspace of fixed vectors as an orthogonal summand of $H^{2n}(M;Q)$. Therefore, the restriction of the inner-product to the fixed vectors is still non-singular and the Witt class of this restriction is defined to be $w(T,M) \in \mathcal{W}$.

It is evident that $w(T,M)$ only depends on the oriented bordism class $[T,M^{4n}] \in \mathcal{O}_{4n}(Z_p)$, [3], and is an additive homomorphism of this group into $\mathcal{W}$. With respect to the $MSO_*(pt)$ module structure on $\mathcal{O}_*(Z_p)$ it is not difficult to see that for any closed, oriented manifold, K^{4m}, we have $w([T,M][K]) = w(T,M)\ \mathrm{sgn}(K)$. Since Stong has shown for an odd prime that $\mathcal{O}_*(Z_p)$ is a free $MSO_*(pt)$ · module with a homogeneous basis it will follow that if $2[T,M] = 0$ then $w(T,M) = 0 \in \mathcal{W}$, however, we have not been able to settle this point if $p = 2$.

If $\nu: M \to M/T$ is the quotient map then ν^* isomorphically identifies $H^{2n}(M/T;Q)$ with the subspace of T^*-fixed vectors in $H^{2n}(M;Q)$. For this reason we denote by $\mathrm{sgn}(M/T)$ the signature of $w(T,M)$. Let us show that $w(T,M) - \mathrm{sgn}(M/T) \cdot 1 = 0$ if (T,M) has no fixed points. In that case M/T is again a closed manifold with an

orientation $\sigma' \in H_{4n}(M/T;Z)$ for which $\nu_*(\sigma) = p\sigma'$.
But then for $x,y \in H^{2n}(M/T;Q)$

$$p^2\langle xy,\sigma'\rangle = p^2\langle \nu^*(x)\nu^*(y),\sigma/p\rangle = (\nu^*(x),\nu^*(y))$$

and thus in $\mathcal{W}$, $w(T,M) = \mathrm{sgn}(M/T) \cdot 1$.

We shall sum up what we do know about $w(T,M) - \mathrm{sgn}(M/T) \cdot 1$. We use $\langle p\rangle \in \mathcal{W}$ to denote the Witt class of the quadratic form pr^2.

<u>1.1 THEOREM</u>. <u>Let</u> (T,M^{2n}) <u>be an orientation preserving periodic diffeomorphism of prime period,</u> p, <u>on a closed manifold. Then</u>

(a) $w(T,M) - \mathrm{sgn}(M/T) \cdot 1$ <u>depends only on the oriented bordism class of the fixed point data in</u> (T,M);

(b) <u>if</u> $p = 2$ <u>then</u> $w(T,M) - \mathrm{sgn}(M/T) \cdot 1$ <u>takes on as values exactly the elements in the cyclic subgroup of order two generated by</u> $\langle 2\rangle - 1 \in \mathcal{W}$;

(c) <u>if</u> $p \equiv 3 \pmod 4$ <u>then</u> $w(T,M) - \mathrm{sgn}(M/T) \cdot 1$ <u>has as values exactly the elements in the cyclic subgroup of order four generated by</u> $\langle p\rangle - 1 \in \mathcal{W}$.

(d) <u>if</u> $p \equiv 5 \pmod 8$ <u>then the values of</u> $w(T,M) - \mathrm{sgn}(M/T) \cdot 1$ <u>lie in a subgroup of</u> $\mathcal{W}$ <u>isomorphic to</u> $Z_2 \oplus Z_2$ <u>which is generated by</u> $\langle p\rangle - 1$ <u>and the Witt class of</u> $(p-1/2)r_1^2 + (p+1)r_1r_2 + (p - 1/2)r_2^2$.

(e) <u>if</u> $p \equiv 1 \pmod 8$ <u>then the values of</u> $w(T,M) - \mathrm{sgn}(M/T) \cdot 1$ <u>lie in a subgroup of</u> $\mathcal{W}$ <u>isomorphic to</u> $Z_2 \oplus Z_2$, <u>one of whose generators is</u> $\langle p\rangle - 1$.

(f) <u>if</u> $p = 3$ <u>then</u> $w(T,M) - \mathrm{sgn}(M/T) \cdot 1 = \mathrm{sgn}(F)(\langle 3\rangle - 1)$ <u>where</u> $F \subset M$ <u>is the set of fixed points.</u>

(g) <u>if</u> $p = 2$ <u>and</u> (T,M) <u>is weakly complex then</u> $w(T,M) - \mathrm{sgn}(M/T) \cdot 1 = \mathrm{sgn}(F)(\langle 2\rangle - 1)$.

Statement (a) of course is an immediate corollary of the remarks preceding 1.1. The special formulas in (f) and (g) are postponed to the following section. For parts (b) - (e) there are three steps involved. The first is a lemma in which $Z(1/p)$ will denote the subring of rationals with denominator a power of p .

1.2 LEMMA. <u>If a quadratic form with coefficients in $Z(1/p)$ and determinant a unit in $Z(1/p)$ also has vanishing signature then either its rational Witt class is zero, or it contains an anisotropic form with integral coefficients, vanishing signature and determinant $\pm p$ or p^2.</u>

The proof of this lemma is also postponed until section 2. Now we have the problem of listing, up to rational congruence, all the anisotropic quadratic forms described in the conclusion of the lemma. Because the form is to have vanishing signature and be anisotropic (over Q) we need only consider binary and quaternary forms [4, 27d]. But then by [p. 40, 9b] the Hasse symbol $c_\infty(f)$ is 1, while the fact that f has integral coefficients and determinant $\pm p$ or p^2 tells us that for any finite prime $q \neq p$ or 2 we have $c_q(f) = 1$ also [4, sec. 12, Prop. 1]. Then of course from the product formula $c_p(f)c_2(f) = 1$. If $p = 2$ we are done as all Hasse symbols are $+1$ which precludes a quaternary anisotropic form and leaves us a single binary case, $r_1^2 - 2r_2^2$.

For $p \equiv 3 \pmod 4$ we write $g_1 = r_1^2 - pr_2^2$ and $g_2 = pr_1^2 - r_2^2$. Now from the discussion [4, sec. 13] of calculating c_2 we read off $c_2(g_1) = -1 = c_p(g_1)$ and $c_2(g_2) = +1 = c_p(g_2)$. Thus g_1 , g_2 are the only two binary forms we need. There is also the quaternary form $g_1 \oplus g_1 = r_1^2 - pr_2^2 + r_3^2 - pr_4^2$ with vanishing signature and determinant p^2 . Since all its coefficients are $+1 \pmod 4$ we again have $c_2(g_1 \oplus g_1) = -1 = c_p(g_1 \oplus g_1)$ and thus this form is also anisotropic [4, Th. 19]. It is evident $\langle g_1\rangle \in \mathcal{W}$ generates a cyclic subgroup of order 4 and $3\langle g_1\rangle = \langle g_2\rangle$.

For $p \equiv 5 \pmod 8$ we again have $g_1 = r_1^2 - pr_2^2$, but this time $c_2(g_1) = 1 = c_p(g_1)$. In addition, $pr_1^2 - r_2^2$ has exactly the same Hasse symbols and so $2\langle g_1\rangle = 0 \in \mathscr{W}$ in this case. Let us take instead $g_2 = (p - 1/2)r_1^2 + (p+1)r_1r_2 + (p - 1/2)r_2^2$. This is associated with the symmetric matrix $\begin{pmatrix} p - 1/2 & p + 1/2 \\ p + 1/2 & p - 1/2 \end{pmatrix}$ which has integral entires and determinant $-p$. Using $\begin{pmatrix} 0 & 1 \\ (p-1)/2 & -(p+1)/2 \end{pmatrix}$, and its transpose, we find g_2 is congruent to $(p - 1/2)g_1$ and hence $2\langle g_2\rangle = 0 \in \mathscr{W}$ also. Now from property 3 of the Hasse symbols we would have

$$c_p(g_2) = (p - 1/2, p)_p .$$

Next we apply property 5a, [4, p. 27], of the Hilbert symbols with $\alpha = \alpha_1 = p - 1/2$ and $\beta = p\beta_1$, $\beta_1 = 1$. Thus $a = 0$, $b = 1$ and $(p - 1/2, p)_p = (p - 1/2|p)$. This Legendre symbol is -1 since $p - 1/2$ cannot have a square root mod p when $p \equiv 5 \pmod 8$. Hence $c_2(g_2) = -1 = c_p(g_2)$ and g_2 is the other binary form we need. Next we set

$$g_1 \oplus g_2 = r_1^2 - pr_2^2 + (p - 1/2)r_3^2 + (p+1)r_3r_4 + (p - 1/2)r_4^2.$$

Then from property 4 of the Hasse symbols

$$c_p(g_1 \oplus g_2) = -(-1,-1)_p(-p,-p)_p .$$

Using again property 5a of Hilbert symbols we find $(-1,-1)_p = 1$, trivially, while $(-p,-p)_p = (-1,p)_p = 1$ because -1 has a square root mod p. Thus $g_1 \oplus g_2$ has determinant p^2, $c_2(g_1 \oplus g_2) = -1 = c_p(g_1 \oplus g_2)$ and therefore is anisotropic.

Finally we come to $p \equiv 1 \pmod 8$. We may still write $g_1 = r_1^2 - pr_2^2$ with $c_2(g_1) = c_p(g_1) = 1$ and $2\langle g_1\rangle = 0 \in \mathscr{W}$. But because p has a square root in the 2-adic numbers there is no binary form g_2 with determinant $-p$ and $c_2(g_2) = -1 = c_p(g_2)$ [4, Th. 29]. By the general existence theorem there is an integral quaternary form g_2 with determinant p , vanishing signature and $c_2(g_2) = -1 = c_p(g_2)$. This is anisotropic precisely because p is a 2-adic square. We must also appeal to the existence theorem for the quaternary form g_3 with integral coefficients, vanishing signature, determinant p^2 and $c_2(g_3) = -1 = c_p(g_3)$. To find that the subgroup generated by $\langle g_1\rangle, \langle g_2\rangle, \langle g_3\rangle$ is indeed $Z_2 \oplus Z_2$ we form $g_1 \oplus g_2$, a form in 6 variables with determinant $-p^2$ and vanishing signature. Then $c_\infty(g_1 \oplus g_2) = -1$ while $c_p(g_1 \oplus g_2) = -1,-1)_p(-p,p)_p = -1$. Hence by the product formula, $c_2(g_1 \oplus g_2) = +1$. Now add $r_5^2 - r_6^2$ to g_3 . This does not change $\langle g_3\rangle \in \mathscr{W}$. Call this sum g_4 . It has vanishing signature and determinant $-p^2$ so $c_\infty(g_4) = -1$ also. In addition

$$\begin{aligned} c_p(g_4) &= -(-1,-1)_p(p^2,-1)_p \\ &= -(-1,-1)_p(1,-1)_p = -1 . \end{aligned}$$

Hence g_4 is congruent to $g_1 \oplus g_2$ and thus $\langle g_3\rangle = \langle g_1\rangle + \langle g_2\rangle$. The reader may show $2\langle g_2\rangle = 0 \in \mathscr{W}$.

We remarked that three steps were involved in parts (b) - (e). The third of course consists in showing that the first two steps really do apply to $w(T,M) - \operatorname{sgn}(M/T) \cdot 1$. We shall do this during the discussion of (1.2). We point out that when $p \equiv 1 \pmod 4$ we do not know if $w(T,M) - \operatorname{sgn}(M/T) \cdot 1$ actually takes on any value other than 0 and $\langle p\rangle - 1$. It is clear that the fundamental problem is finding a general formula for $w(T,M) - \mathrm{sgn}(M/T) \cdot 1$ in terms of the fixed point data.

Let us examine now the significance of this difference class $w(T,M) - \mathrm{sgn}(M/T) \cdot 1$. Associated to (T,M) is the closed T-invariant normal tube, N, around the fixed point set. This is the union of small, non-intersecting, T-invariant normal tube, N, around the fixed point set. This is the union of small, non-intersecting, T-invariant normal cell bundles about each of the components of F. Then N is a compact oriented $4n$-manifold and we shall presently consider a technique for calculating $w(N)$. Now T acts freely on $\partial N/T$ is again a closed $(4n-1)$-manifold oriented so that the quotient map $\partial N \to \partial N/T$ has degree $+p$. Furthermore, $\partial N/T$ still bounds and so we may give a formula for $\mathrm{per}(\partial N/T)$.

<u>1.3 THEOREM</u>. <u>For any odd prime period</u>

$$\mathrm{per}(\partial N/T) = (\langle p\rangle w(N) - \mathrm{sgn}(N)\cdot 1) - (w(T,M) - \mathrm{sgn}(M/T)\cdot 1) .$$

Thus a formula for $w(T,M) - \mathrm{sgn}(M/T)\cdot 1$ would yield a computation of $\mathrm{per}(\partial N/T)$. From this for instance the peripheral invariant for every oriented $(4n-1)$-dimensional lens space L, with $\pi_1(L) \simeq Z_p$, would immediately follow.

We shall use the following invariant. Consider a pair (M^{4n+k},c) consisting of a closed oriented manifold together with a rational cohomology class $c \in H^k(M;Q)$. For each j, $0 \leq j \leq 4n+k$ there is the annihilator $A_j = \{x \mid x \in H^j(M;Q),\ xc = 0\}$. Since $A_j = H^j(M;Q)$ if $j > 4n$ we can write the graded, commutative quotient algebra with unit as $V^*(c) = \Sigma_0^{4n} H^j(M;Q)/A_j$. Into $H^*(M;Q)$ we introduce the inner product $(x,y) = \langle xyc,\sigma\rangle \in Q$. An elementary application of Poincaré duality then shows that the radical of this inner-product is precisely the annihilator ideal and so the induced inner-product on $V^*(c)$ is non-singular. Furthermore, it has the properties:

(a) $(xz,y) = (x,zy)$ for all $x,y,z \in V^*(c)$

(b) if $x \in V^j(c)$, $y \in V^i(c)$ then $(x,y) = 0$ if $i + j \neq 4n$, while $(x,y) = (-1)^{ij}(y,x)$ if $i+j = 4n$.

Clearly $V^*(c)$ is a rational 4n-dimensional Poincaré duality algebra. In particular $V^{2n}(c)$ receives a bilinear, symmetric and non-singular inner-product. The Witt class of the associated quadratic form is denoted by $w(M,c) \in \mathscr{W}$. Let us warn the reader that there is an oriented bordism theory for such pairs, roughly $MSO_{4n+k}(K(Q,k))$, however, if $k > 0$ then $w(M,c)$ is in no sense a bordism invariant.

Suppose $c \in H^k(M;Z)$ is an integral class, then $V_Z^{2n}(c)$ is the free abelian group obtained as the quotient of $H^{2n}(M;Z)$ by the subgroup of all elements for which xc has finite order.

1.4 LEMMA. *For an integral cohomology class the quadratic form on $V_Z^{2n}(c)$ is unimodular if and only if the image of*

$$\cup c : H^{2n}(M;Z) \to H^{2n+k}(M;Z)/\mathrm{Tor}$$

is a direct summand.

This is an elementary consequence of the fact that the pairing

$$(H^{2n}(M;Z)/\mathrm{Tor}) \otimes (H^{2n+k}(M;Z)/\mathrm{Tor}) \to Z$$

is unimodular. In case the image of $\cup c$ is a summand then of course $w(M,c) = \mathrm{sgn}(M,c) \cdot 1 \in \mathscr{W}$.

Our introduction of $V^*(c)$ may have seemed unnecessarily elaborate but it makes the product formula transparent. Let us sketch this point. Given (M^{4n+k},c) and (K^{4m+j},d), there is $(M \times K, c \otimes d)$. If we think of $V^*(c)$ as the image of $c : H^*(M;Q) \to H^*(M;Q)$, and do the same for d and $c \otimes d$, then we have an identification of $V^*(c \otimes d)$ with the graded tensor product $V^*(c) \otimes V^*(d)$. Merely as a consequence of the definition of graded tensor product the three inner-products are related by $(x_r \otimes y_s, z_{4n-r} \otimes w_{4m-s}) = (-1)^{rs}(x_r, z_{4n-r})(y_s, w_{4m-s})$. Now we may apply the familiar argument showing signature for closed manifolds is multiplicative in order to prove

1.5 LEMMA. *In* $\mathscr{W}$

$$w(M \times K, c \otimes d) = w(M,c)w(K,d) .$$

This $w(M,c)$ is related to 1.3 as follows. Suppose $\xi \to M^{4n+2j}$ is an oriented $SO(2j)$-bundle over a closed oriented manifold, then the total space of the associated closed 2j-disk bundle $E(\xi) \to M$ is a compact oriented $4(n+j)$-manifold.

1.6 LEMMA. *If* $e(\xi) \in H^{2j}(M;Z)$ *denotes the Euler class of the bundle then* $w(M, e(\xi)) = w(E(\xi))$.

For a map of odd prime period the normal bundle to each component of the fixed set receives a natural complex structure, [3]. Thus 1.6 may be applied to the top Chern class of each of these normal bundles and when the result is summed over all components of F we obtain $w(N)$.

We regard $\mathscr{W}$ both as the Witt classes of quadratic forms with non-zero determinant and as the ''cobordism'' ring of finite dimensional rational vector space equipped with bilinear, symmetric and non-singular inner-products. To illustrate this second viewpoint we mention

1.7 LEMMA. *Suppose* V *is a finite dimensional non-singular inner-product space. If* $L \subset V$ *is a subspace with orthogonal subspace* $L^{\perp}$ *and* $rad(L) = L \cap L^{\perp}$, *then* $rad(L^{\perp}) = rad(L)$ *and in* $\mathscr{W}$

$$\langle V\rangle = \langle L/\mathrm{rad}(L)\rangle + \langle L^{\perp}/\mathrm{rad}(L)\rangle \ .$$

2. Periodic Maps.

We shall first complete the argument for parts (b) - (e) of 1.1. The proof of 1.2 goes as follows. We shall employ only the fact that $Z(1/p)$ is a principal ideal domain. If V is a finitely generated free $Z(1/p)$ equipped with a bilinear, symmetric and $Z(1/p)$-valued inner-product then V is strongly by non-singular if and only if the naturally associated homomorphism $V \to \mathrm{Hom}(V,Z(1/p))$ is an isomorphism. Equivalently we may think of a quadratic form with coefficients in $Z(1/p)$ and determinant a unit in $Z(1/p)$. Suppose V is strongly non-singular and that $V \otimes_{Z(1/p)} Q$ contains an isotropy vector. By clearing denominators we may assume the isotropy vector lies in V . Using the fact that $Z(1/p)$ is a p.i.d. we may choose $x \in V$ with $(x,x) = 0$ and so that the submodule, $L \subset V$, generated by x is a direct summand. Let $L^{\perp} = \{y \mid y \in V, (x,y) = 0\}$. Surely $V/L^{\perp}$ has no torsion so $L^{\perp}$ is also a summand of V . There is induced on the free module $V' = L^{\perp}/L$ a $Z(1/p)$-valued inner-product which we must show is still strongly non-singular by proving that $V' \approx \mathrm{Hom}(V',Z(1/p))$. An element in $\mathrm{Hom}(V',Z(1/p))$ is a homomorphism $h:L^{\perp} \to Z(1/p)$ for which $h(x) = 0$. Using that $L^{\perp}$ is a summand of V and that V is strongly non-singular there is a $z \in V$ such that $(z,y) = h(y)$ for all $y \in L^{\perp}$. In particular, $(z,x) = h(x) = 0$ so that $z \in L^{\perp}$ also. It is trivial to see that $\mathrm{rank}(V') = \mathrm{rank}(V) - 2$. If 1.7 is applied to $L \otimes Q \subset V \otimes Q$ it follows that $\langle V\otimes Q\rangle = \langle V' \otimes Q\rangle \in \mathcal{W}$.

This shows that if f is a quadratic form with coefficients in $Z(1/p)$, determinant a unit in $Z(1/p)$ and $\mathrm{sgn}(f) = 0$ then either $\langle f\rangle = 0 \in \mathcal{W}$ or we may as well assume f is anisotropic over Q . In effect we know then that 2 or 4 variables are involved; that the (classical) index is 1 or 2 respectively; that

$c_\infty(f) = 1$ and $c_q(f) = 1$ for all finite primes different from p and 2. This leaves us with only $c_p(f)$, $c_2(f)$ and of course $c_p(f)c_2(f) = 1$. We are only concerned with rational congruence classes and in our discussion we exhibited representatives of these special anisotropic congruence classes which have integral coefficients and determinant $\pm p$ or p^2. Then knowing rational equivalence is determined by the number of variables, the index, the Hasse symbols and the determinant modulo non-zero rational squares, [4, Th. 1.5,28] the proof of 1.2 is complete.

We must show that our discussion does apply to $w(T,M) - \mathrm{sgn}(M/T) \cdot 1$. In the first place the quadratic form on $H^{2n}(M;Z)/\mathrm{Tor}$ given by $p\langle x^2,\sigma\rangle$ surely has integral coefficients and determinant a unit in $Z(1/p)$. In the second place we did not need to go all the way to Q to introduce the averaging operator. We may use $H^{2n}(M;Z(1/p)/\mathrm{Tor}$ and Σ will still exhibit the submodule of fixed elements as an orthogonal direct summand. Thus this submodule, V, is strongly $Z(1/p)$ non-singular. Furthermore, if $\mathrm{sgn}(V) = n \neq 0$ let $\mathcal{W}$ be the free $Z(1/p)$-module with generators $e_j, \cdots, e_{|n|}$ and put $(e_i,e_j) = -\delta_{i,j}$ if $n > 0$ or $\delta_{i,j}$ if $n < 0$. In either case $V \oplus W$ is still strongly $Z(1/p)$ non-singular and $\langle (V \oplus W) \otimes Q\rangle = w(T,M) - \mathrm{sgn}(M/T) \cdot 1$. This completes the proof of parts (b) - (e) in 1.1.

Next we would like to consider 1.3. We shall restrict our attention to a non-trivial periodic map (T,M^{4n}) on a closed connected manifold with orientation class $\sigma(M) \in H_{4n}(M;Z)$. We can write $M = B \cup N$ where B is a compact T-invariant regular submanifold, N is the closed T-invariant normal tube around the fixed point set F and $B \cap N = \partial B = \partial N$. It is important to note that T acts freely on B. The orientation of

$\sigma(B) \in H_{4n}(B,\partial B;Z)$ is the image of $\sigma(M)$ under $H_{4n}(M;Z) \to H_{4n}(M,N;Z) \simeq H_{4n}(B,\partial B;Z)$. The orientation $\sigma(N) \in H_{4n}(N,\partial N;Z)$ is similarly defined. Let us introduce the commutative quotient diagram

$$\begin{array}{ccccc} H_{4n}(M;Z) & \to & H_{4n}(M,N;Z) & \simeq & H_{4n}(B,\partial B;Z) \\ \downarrow & & \downarrow & & \downarrow \\ H_{4n}(M/T;Z) & \to & H_{4n}(M/T,N/T;Z) & \simeq & H_{4n}(B/T,\partial B/T;Z) \end{array} .$$

Since (T,B) is free there is a unique orientation $\sigma(B/T) \in H_{4n}(B/T,\partial B/T;Z)$ with $\nu_*(\sigma(B)) = p\sigma(B/T)$. Now $N/T \to F$ is a fibre bundle over each component of F with fibre the cone over a certain lens space. Since p is odd, every component of F has codimension at least two and therefore $H_{4n}(N/T;Z) \simeq H_{4n-1}(N/T;Z) = 0$. Now we have

$$\begin{array}{ccccccc} H_{4n}(M;Z) & \simeq & H_{4n}(M,N;Z) & \simeq & H_{4n}(B,\partial B;Z) & \simeq & Z \\ \downarrow & & \downarrow & & \downarrow & & \\ H_{4n}(M/T;Z) & \simeq & H_{4n}(M/T,N/T;Z) & \simeq & H_{4n}(B/T,\partial B/T;Z) & \simeq & Z \end{array} .$$

From this we find that there exists uniquely $\sigma(M/T) \in H_{4n}(M/T;Z) \simeq Z$ with $\nu_*(\sigma(M)) = p\sigma(M/T)$; that is, M/T has a uniquely defined ''orientation'' class. Let $\sigma(N/T) \in H_{4n}(N/T,\partial N/T;Z)$ be the image of $\sigma(M/T)$ under $H_{4n}(M/T;Z) \to H_{4n}(M/T,B/T;Z) \simeq H_{4n}(N/T,\partial N/T;Z)$ so that by naturality, $\nu_*(\sigma(N)) = p\sigma(N/T)$. Using rational coefficients and the orientation classes $\sigma(M/T)$, $\sigma(B/T)$ and $\sigma(N/T)$ we may define $w(M/T) = w(T,M)$, $w(B/T)$ and $w(N/T)$ just as we would for compact oriented manifolds. Since B/T is a compact oriented manifold with $\partial B/T = \partial N/T$ we may say that $\operatorname{per}(\partial N/T) = -(w(B/T) - \operatorname{sgn}(B/T) \cdot 1)$. The invariant $w(\cdot)$ here has an additivity property analogous to that of signature for compact manifolds, therefore

$$\begin{aligned} w(T,M) - \operatorname{sgn}(M/T)\cdot 1 &= (w(B/T) - \operatorname{sgn}(B/T)\cdot 1) \\ &\quad + (w(N/T) - \operatorname{sgn}(N/T)\cdot 1) \\ &= -\operatorname{per}(\partial N/T) + (w(N/T) - \operatorname{sgn}(N/T)\cdot 1) \ . \end{aligned}$$

Clearly we only need to settle $w(N/T)$. Using the canonical complex structure on the normal bundle to each component of F, the action (T,N) can be extended to a fibre preserving action of $U(1)$ on N. Therefore surely we may write

$$\begin{tikzcd} & H^{2n}(N,\partial N;Q) \arrow[r] & H^{2n}(N;Q) \\ H^{2n}(N/T,\partial N/T;Q) \arrow[ur, "\simeq"'] \arrow[rr] & & H^{2n}(N/T;Q) \arrow[u, "\simeq"'] \end{tikzcd} \ .$$

This, together with the relation $\nu_*(\sigma(N)) = p\sigma(N/T)$, leads us to conclude that $w(N/T) = \langle 1/p\rangle w(N) = \langle p\rangle w(N)$ in $\mathscr{W}$. Of course $\operatorname{sgn}(N) = \operatorname{sgn}(N/T)$ so we have completed the proof that $\operatorname{per}(\partial N/T) = (\langle p\rangle w(N) - \operatorname{sgn}(N)\cdot 1) - (w(T,M) - \operatorname{sgn}(M/T)\cdot 1)$. In the next section we shall establish 1.6 for $w(N)$.

We shall only give an argument for part (f) in 1.1 as (g) is entirely similar. First we need a general lemma.

<u>2.1 LEMMA</u>. <u>If the periodic map</u> (T,M^{4n}) <u>can be extended to a semi-free action of</u> $U(1)$, <u>then</u> $w(T,M) - \operatorname{sgn}(M/T)\cdot 1 = \operatorname{sgn}(F)(\langle p\rangle - 1)$. <u>If every non-empty component of</u> F <u>has codimension exactly</u> 2, <u>then</u> $w(T,M) - \operatorname{sgn}(M/T)\cdot 1 = 0$.

The two statements in the lemma are independent and both are trivial. If (T,M) extends to a semi-free $U(1)$-action then F must be exactly the set of stationary points under all of $U(1)$. For a semi-free action on a closed oriented manifold, it is known that $\operatorname{sgn}(F) = \operatorname{sgn}(M)$, [6]. On the other hand, T is homotopic to the identity so we find from the definition that $w(T,M) = \langle p\rangle w(M) = \langle p\rangle\operatorname{sgn}(M) = \operatorname{sgn}(F)\langle p\rangle$.

For the other part of the lemma the hypothesis guarantees that M/T is still locally Euclidean and hence the inner-product on $H^{2n}(M/T;Z)/\mathrm{Tor}$ given by $\langle xy, \sigma(M/T)\rangle = 1/p \langle \nu^*(x)\nu^*(y), \sigma(M)\rangle$ is unimodular and thus $w(M/T) = \mathrm{sgn}(M/T)\cdot 1$.

Let us turn now to $p = 3$, and suppose first that F is a connected proper submanifold. With $\lambda = \exp 2\pi i/3$ there is a complex structure on the normal bundle $\eta \to F$ in which T acts on each fibre as multiplication by λ . On the total space of the Whitney sum $\eta \oplus C \to F$ with a trivial line bundle a fibre preserving action of $U(1)$ is given by $t(\vec{v}, z) = (t\vec{v}, z)$. This will induce a fibre-wise semi-free action $(U(1), CP(\eta\oplus C))$ on the associated complex projective space bundle. The set of stationary points is the disjoint union of F with $CP(\eta)$. Let $(T, CP(\eta\oplus C))$ be the map of period 3 contained in the $U(1)$-action. Since $CP(\eta)$ has dimension $4n-2$ we obtain from 2.1 that $w(T, CP(\eta\oplus C)) - \mathrm{sgn}(CP(\eta\oplus C)/T)\cdot 1 = \mathrm{sgn}(F)(\langle 3\rangle - 1)$.

Now we observe that the normal bundle to $F \subset CP(\eta\oplus C)$ is still $\eta \to F$. Thus by removing the interior of a normal tube from around F in each of the manifolds, a new (T', K) can be formed by identifying along the resulting boundaries so that the fixed set of (T', K) is exactly the codimension 2 submanifold $CP(\eta)$ and $[T', K] = [T, M] - [T, CP(\eta\oplus C)]$ in $\mathcal{O}_{4n}(Z_3)$. Upon applying the second part of 2.1 to (T', K) we finally show

$$w(T, M) - \mathrm{sgn}(M/T) \cdot 1 = \mathrm{sgn}(F)(\langle 3\rangle - 1) .$$

It is also clear how to proceed when F is not connected. There is a definite obstacle to extending this argument to larger primes; namely, the occurrence of several distinct eigenvalues expressing the action of T in the normal bundle to the fixed set. We must admit the possibility that a general formula for $w(T, M) - \mathrm{sgn}(M/T)\cdot 1$ might involve this representation of Z_p in the normal fibres to the fixed set in some essential manner.

3. $w(M,c)$.

Obviously we may extend the definition of this invariant to a pair (B,b) consisting of a compact oriented manifold of dimension $4n+k$ and an absolute cohomology class $b \in H^k(B;Q)$. On the image of $j^*:H^{2n}(B,\partial B;Q) \to H^{2n}(B;Q)$ the inner-product is taken to be $(j^*(x),j^*(y)) = \langle xyb,\sigma\rangle$. From the Lefschetz duality theorem the radical consists of those $j^*(y)$ which annihilate b . The quotient of $\mathrm{im}(j^*)$ by this annihilator then represents $w(B,b) \in \mathcal{W}$. We note that if $0 \neq r \in Q$ then $w(B,r^2b) = w(B,b)$ and furthermore, $w(B,-b) = w-B,b) = -w(B,b)$. If $k = 0$ and $b = 1 \in H^0(B;Z)$ we simply obtain $w(B)$.

Consider now an oriented $SO(2_j)$-bundle $\xi \to B^{4n+2j}$ over a compact oriented manifold and let $E(\xi) \to B, S(\xi) \to B$ respectively denote the associated 2j-disk and (2j-1)-sphere bundles. If $\eta \to \partial B$ denotes the bundle induced over the boundary, then $E(\xi)$ is a compact oriented $4(n+j)$-manifold with $\partial E(\xi) = S(\xi) \cup E(\eta)$ and $S(\xi) \cap E(\eta) = S(\eta) \to \partial B$. There is the Thom class $U_\xi \in H^{2j}(E(\xi),S(\xi);Z)$ whose image under $H^{2j}(E(\xi),S(\xi);Z) \to H^{2j}(E(\xi);Z) \simeq H^{2j}(B;Z)$ is the Euler class $e(\xi)$. Identifying $H^*(B,\partial B;Q)$ with $H^*(E(\xi),E(\eta);Q)$ and using the cup-product pairing of $H^*(E(\xi);Q)$ with $H^*(E(\xi),E(\eta);Q)$ into $H^*(E(\xi),S(\xi) \cup E(\eta),Q)$ we find that there is a

$$U_\xi \cdot H^i(B,\partial B;Q) \to H^{i+2j}(E(\xi),\partial E(\xi);Q) .$$

From the exact sequence of the triple $(E(\xi),\partial E(\xi),S(\xi))$ it is clear that this is a relative Thom isomorphism for under the composition

$H^{2j}(E(\xi),S(\xi);Z) \to H^{2j}(S(\xi) \cup E(\eta),S(\xi);Z) \simeq$ $H^{2j}(E(\eta),S(\eta),Z)$ we see that $U_\xi \to U_\eta$ and so the five lemma will apply.

Next from the diagram

$$\begin{tikzcd}
 & H^{2(n+j)}(B;Q) \\
 & H^{2(n+j)}(E(\xi);Q) \arrow[u, "\simeq"'] \\
H^{2(n+j)}(E(\xi),\partial E(\xi);Q) \arrow[r] \arrow[uur] & H^{2(n+j)}(E(\xi),S(\xi);Q) \arrow[u] \\
H^{2n}(B,\partial B;Q) \arrow[u, "\simeq"'] \arrow[r] & H^{2n}(B,Q) \arrow[u, "\simeq"']
\end{tikzcd}$$

we find that we may identify the image of $H^{2(n+j)}(E(\xi),\partial E(\xi);Q) \to H^{2(n+j)}(E(\xi);Q)$ with the image of the composite homomorphism

$$H^{2n}(B,\partial B;Q) \xrightarrow{j^*} H^{2n}(B;Q) \xrightarrow{\cup e(\xi)} H^{2(n+j)}(B;Q) \ .$$

For $xy \in H^{2n}(B,\partial B;Q)$ we note that $(U_\xi x)(U_\xi y) = U_\xi^2(xy) =$ $U_\xi(e(\xi)xy) \in H^{4(n+j)}(E(\xi),\partial E(\xi);Q)$. Since the inner-product on $\operatorname{im}(j^*)$ used in defining $w(B,e(\xi))$ is $(j^*(x),j^*(x)) = \langle xye(\xi),\sigma\rangle$ we have shown

3.1 LEMMA. *In* $\mathscr{W}, w(E(\xi)) = w(B,e(\xi))$.

This is just the relative form of 1.6.

4. Directions of Application.

In this note we have tried to show how some current ideas about quadratic forms in topology; that is, the peripheral invariant and the invariant $w(M,c)$, can be fitted into the study of periodic diffeomorphisms. Consider (f) in 1.1. The formula is surely straightforward, indeed simple, and even if T^* is the identity on

$H^{2n}(M;Q)$ still yields the affirmative conclusion that $\mathrm{sgn}(M) \equiv \mathrm{sgn}(F) \pmod 4$. In view of (b) - (e) it seems reasonable to hope for a rather clean formula in general. We would be especially interested in seeing how the grouping of the odd primes by their mod 8 congruence classes works into the general formula.

In plumbing, [2, Ch. V], the peripheral invariant is recognized whenever the determinant of the plumbing matrix differs from ± 1 . Let us indicate the expected application of peripheral to questions raised in [1]. The authors have introduced a cobordism group $\Theta_k^{(2)}$ of closed oriented Z_2-homology spheres wherein addition is by connected sum and the 0 element is represented by those Z_2-homology spheres which bound Z_2-homology disks. Now every Z_2-homology sphere bounds some compact oriented manifold so for $n \geq 1$ there is an additive homomorphism per: $\Theta_{4n-1}^{(2)} \to \mathcal{W}$. The real question, however, are the Z_2-homology $(4n-1)$-spheres which bound odd framed manifolds. These make up a subgroup $bP_{4n}^{(2)} \subset \Theta_{4n-1}^{(2)}$ which is analogous to bP_{4n} . Indeed, $bP_{4n} \subset bP_n^{(2)}$ and obviously $\mathrm{per}|bP_{4n}$ is trivial so that there is an induced homomorphism per: $bP_{4n}^{(2)}/bP_{4n} \to \mathcal{W}$. In commenting on [1, 2.6] the authors noted that a problem obstructing the precise determination of $bP_{4n}^{(2)}$ was the need for the incorporation of the Hasse symbols into some homomorphism. It appears to us that peripheral plays this role.

4.1 THEOREM. *If* $n \geq 2$ *then*

$$\mathrm{per}: bP_{4n}^{(2)}/bP_{4n} \to \mathcal{W}$$

is an isomorphic embedding. Furthermore, an element in the kernel of $\mathrm{sgn}: \mathcal{W} \to Z$ *lies in this image if and only if it is represented by an integral quadratic form with odd de-*

terminant and vanishing signature.

In view of the importance of even quadratic forms in topology it is not unreasonable that we review them briefly as they relate the proof of 4.1. We follow [4] in denoting by $F(2)$ the 2-adic number field and by $R(2)$ the subring of 2-adic integers. We shall also use $\mathscr{W}(2)$ to denote the Witt ring of the field $F(2)$. Suppose V is a finitely generated free $R(2)$-module equipped with a bilinear symmetric inner-product with values in $R(2)$. We say V is even if and only if

(a) $V \simeq \mathrm{Hom}(V,R(2))$

(b) $(x,x) \in 2R(2)$ for all $x \in V$.

Equivalently we may think of a quadratic form with coefficients in $R(2)$, determinant a unit in $R(2)$ and which is improperly primitive. From [4, 33a] it now follows that $\operatorname{rank} V = 2m$ and V admits an orthogonal direct sum decomposition into rank 2 submodules $V = \Sigma_1^m V_j$ where for $1 < j \leq m$, V_j, is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and V_1 is either $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. As a consequence we see that if V is even then in $\mathscr{W}(2)$, $\langle V\rangle = 0$ or $\langle 2r_2^2 + 2r_1r_2 + r_3^2\rangle$, so $2\langle V\rangle = 0$ always.

Next suppose V_Z is a finitely generated free abelian group equipped with a Z-valued inner-product. We say V_Z is even if and only if $V_Z \otimes R(2)$ is even. The symmetric integral matrix representing the inner-product with respect to a basis of V_Z has odd determinant and all diagonal entries divisible by 2.

If $W_{\mathbf{z}}$ is a free abelian group with Z-valued inner-product, then associate to $W_{\mathbf{z}}$ the ordered triple.

$$(\operatorname{rank}(W_{\mathbf{z}}) (\mathrm{mod}\ 8),\ \det(W_{\mathbf{z}}) (\mathrm{mod}\ 8), c_2(W_{\mathbf{z}}))\ .$$

Now we can state a representation result.

4.2 LEMMA. *If the triple associated to* W_z *appears in the sequence* $(0,1,-1),(0,5,1),(2,7,1),(2,3,-1),(4,1,1),(4,5,-1),(6,7,-1),(6,3,1)$ *then there is an even* V_z *with* $\operatorname{rank} W_z = \operatorname{rank} V_z$, $\det W_z = \det V_z$ *and* W_z *rationally congruent to* V_z.

The sequence is found as follows. For each even rank $2n$ evaluate the triple associated to the n-fold sum of the hyperbolic rank 2 module $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with itself; then evaluate the triple associated to the sum of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ with (n-1)-copies of the hyperbolic module. This produces a sequence of triples which upon inspection has period 8 in the rank.

We wish to apply [4, Th. 46]. We need a new V_z whose field invariants are all exactly those of W_z and whose ring invariants at all odd primes are still exactly those of W_z. However, at the prime 2 we wish the ring invariant of V_z to be one of the two improperly primitive $R(2)$-forms. This fixes g_{02} and g_{i2} is trivial if $i > 0$. This will impose the five compatibility conditions on the rank (W_z), det (W_z) and $c_2(W_z)$. Any triple listed in the sequence, however, will fulfill the compatibility at 2. As an immediate corollary then.

4.3 LEMMA. *Suppose* W_z *has odd determinant and* $\operatorname{sgn}(W_z) = 0$ *then there is an even* V_z *with* $\langle W_z \otimes Q\rangle = \langle V_z \otimes Q\rangle - \operatorname{sgn}(V_z) \cdot 1$ *in* $\mathscr{W}$.

Obviously rank W_z is even. Consider the associated triple $(\mathrm{rk}(W_z)(\mathrm{mod}\ 8), \det(W_z)(\mathrm{mod}\ 8), c_2)$. It may be necessary to adjust the rank of W_z, without changing the other two invariants in order to get a term in the sequence of 4.2. Since det (W_z) is odd this may be done by adding copies of $\begin{pmatrix}1&0\\0&1\end{pmatrix}$. The even V_z is formed then and by construction

$$\langle W_z \otimes Q\rangle = \langle V_z \otimes Q\rangle - \operatorname{sgn}(V_z)\cdot 1 .$$

We need a version of Milnor's result on even integral froms which are unimodular [2, III.1A].

4.4 LEMMA. *Let* V_z *be an even inner-product module. If* N *is the order of* $\langle V_z \otimes Q\rangle - \operatorname{sgn}(V_z)\cdot 1$ *then*

$$N \operatorname{sgn}(V_z) \equiv 0 \pmod 8 .$$

We shall use in $\mathscr{W}$ the identity $\langle 2r_1^2 + 2r_1r_2 + 2r_2^2\rangle - 2\cdot 1 = \langle r_1^2 - 3r_2^2\rangle$. It is necessary to show $2r_1^2 + 2r_1r_2 + 2r_2^2 + r_4^2$ is rationally congruent to $r_1^2 - 3r_2^2 + r_3^2 + r_4^2$. Note both forms are quaternary and have determinant -3 so it is necessary only to observe that the Hasse symbols at 2 are both -1 .

Now suppose $\langle V_z \otimes Q\rangle = \operatorname{sgn}(V_z)\cdot 1 \in \mathscr{W}$. Since $\operatorname{sgn}(V_z) = 2m$ we can consider

$$\begin{aligned} &m\langle 2r_1^2 + 2r_1r_2 + 2r_2^2\rangle - \langle V_z \otimes Q\rangle \\ &= m(\langle 2r_1^2 + 2r_1r_2 + 2r_2^2\rangle - 2\cdot 1) \\ &= m(1 - \langle 3\rangle) . \end{aligned}$$

Since the direct sum of two even forms is still even this tells us that in $\mathscr{W}(2)$ the element $m(1 - \langle 3\rangle)$ contains a representative which is even over $R(2)$. Thus either $m(1 - \langle 3\rangle) = 0$ or $m(1 - \langle 3\rangle) = \langle 2r_1^2 + 2r_1r_2 + 2r_2^2\rangle$. Since $1 - \langle 3\rangle$ still has order 4 in $\mathscr{W}(2)$ it might appear that $2(1 - \langle 3\rangle) = \langle 2r_1^2 + 2r_1r_2 + 2r_2^2\rangle$ is still possible, but $r_1^2 - 3r_2^2 + r_3^2 - 3r_4^2$ is an anisotropic quaternary form while $2r_1^2 + 2r_1r_2 + 2r_2^2$ is an anisotropic binary form. Thus $m \equiv 0 \pmod 4$ is the only possibility which means $\mathrm{sgn}(V_z) \equiv 0 \pmod 8$. In general, if $N = 1,2$ or 4 is the order of $\langle V_z \otimes Q\rangle - \mathrm{sgn}(V_z) \cdot 1$ then $N\langle V_z \otimes Q\rangle = N\,\mathrm{sgn}(V_z) \cdot 1$ and hence $N \cdot \mathrm{sgn}(V_z) = 0 \pmod 8$.

Finally, then, we can return to 4.1. As noted in [1], since $n \geq 2$, an element in $bP_{4n}^{(2)}$ can be represented by plumbing together a family of 2n-spheres according to a symmetric integral matrix with odd determinant and all diagonal entries divisible by 2 and then killing the fundamental group of the boundary. Thus if K^{4n-1} is the resulting Z_2-homology sphere and V_z is the even inner-product module associated to the matrix then

(a) $[K] = 0 \in bP_{4n}^{(2)}$ if and only if $\langle V_z \otimes Q\rangle = 0 \in \mathscr{W}$;

(b) $\mathrm{per}\,[K] = \langle V \otimes Q\rangle - \mathrm{sgn}(V_z) \cdot 1$.

If $\mathrm{per}\,[K] = 0$ then by 4.4, $\mathrm{sgn}(V_z) \equiv 0 \pmod 8$. We use this fact to plumb together a homotopy (4n-1)-sphere in bP_{4n} which represents $[K]$. It follows then that

$$\mathrm{per}\colon bP_{4n}^{(2)}/bP_{4n} \to \mathscr{W}$$

is an isomorphic embedding. The image is found by 4.3. Using the peripheral invariant it is easy to see that $bP_{4n}^{(2)}/bP_{4n}$ is a direct summand of $\Theta_{4n-1}^{(2)}/bP_{4n}$.

We shall close by mentioning an invariant which we feel may be of value in future studies of diffeomorphisms of odd prime period on homotopy spheres. We shall show how the formula in (f) of 1.1 may be used to extend the definition of the peripheral invariant to diffeomorphisms of period 3.

Let (T,B^{4n}) be an orientation preserving diffeomorphism of period 3 on a compact manifold. On the image of

$H^{2n}(B,\partial B;Q) \to H^{2n}(B;Q)$ we introduce as before the inner-product $(j^*(x),j^*(y)) = 3\langle xy,\sigma\rangle$. Using Σ we exhibit the subspace of fixed vectors in $\mathrm{im}(j^*)$ as an orthogonal direct summand. The Witt class of the restriction of this inner-product to the fixed vectors is taken as $w(T,B)$. The signature of $w(T,B)$ is $\mathrm{sgn}(B/T)$. Every component of the fixed set $F \subset B$ has a canonical orientation. Next $w(F) \in \mathcal{W}$ is obtained by assigning to each component of F, whose dimension is divisible by 4 its Witt ring invariant and summing over all such components. If a component is closed; that is, lies in the interior of B then it contributes its signature multiplied by $1 \in \mathcal{W}$ to the sum. The difference

$$(w(T,B) - \mathrm{sgn}(B/T) \cdot 1) - w(F)(\langle 3\rangle - 1)$$

only depends on $(T,\partial B)$ and we may denote it by $\mathrm{per}(T,\partial B)$. We can also write

$$\mathrm{per}(T,\partial B) = (w(T,B) - \mathrm{sgn}(B/T) \cdot 1) - \mathrm{per}(\partial F)(\langle 3\rangle - 1) - \mathrm{sgn}(F)(\langle 3\rangle - 1) .$$

If (T,B) is fixed point free then $\mathrm{per}(T,\partial B) = \mathrm{per}(\partial B/T)$ while if T is the identity then $\mathrm{per}(T,\partial B) = \mathrm{per}(\partial B)$. This should provide additional motivation toward finding the generalization of the $p = 3$ formula.

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