Journal of Algebraic Combinatorics **11** (2000), 227–240

Hanlon and Stanley's Conjecture and the Milnor Fibre of a Braid Arrangement

G. DENHAM denham@math.lsa.umich.edu
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109–1109

Received August 26, 1997; Revised November 3, 1998; Accepted December 7, 1998

Abstract. Let $\mathcal{A}$ be a real arrangement of hyperplanes. Let $B = B(q)$ be Varchenko's quantum bilinear form of $\mathcal{A}$, introduced [15], specialized so that all hyperplanes have weight q. $B(q)$ is nonsingular for all complex q except certain roots of unity. Here, we examine the kernel of B at roots of unity in relation to the topology of the hyperplane singularity.

We use Varchenko's work [16] to relate $B(q)$ to a Salvetti complex for the Milnor fibration of $\mathcal{A}$. This paper's main result is specific to the arrangement of reflecting hyperplanes associated with the A_{n-1} root system. We use a geometric property of the Milnor fibre to resolve a conjecture due to Hanlon and Stanley regarding the $\mathfrak{S}_n$-module structure of the kernel of $B(q)$ at certain roots of unity.

Keywords: hyperplane arrangement, Milnor fibre, quantum bilinear form, braid arrangement

1. Introduction

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbf{R}^n$. In what follows, all hyperplanes are assumed to contain the origin. Let $L(\mathcal{A})$ denote their lattice of intersections. For each $H \in \mathcal{A}$, let ℓ_H be a linear functional whose kernel is H. Let $Q = \prod_{H\in\mathcal{A}} \ell_H$. Q is said to be a defining polynomial of the hyperplane arrangement $\mathcal{A}$. Also let $\mathcal{C}$ be the set of chambers of $\mathcal{A}$; that is, the set of connected components of $\mathbf{R}^n \setminus \bigcup_{H\in\mathcal{A}} H$.

In [15], Varchenko defines a matrix $B = B(\mathcal{A})$ whose rows and columns are indexed by the chambers $\mathcal{C}$. The entries of the matrix are $B(C, C') = \prod_H a_H$, where $\{a_H : H \in \mathcal{A}\}$ is a set of indeterminates, and where the product is taken over hyperplanes $H \in \mathcal{A}$ that separate chambers C and C'. In this paper, we restrict our attention to the case where each hyperplane has the same weight q; here $B(C, C') = q^{n(C,C')}$, where $n(C, C')$ is the number of hyperplanes that lie between chambers C and C'.

Let V be the complex vector space with basis $\mathcal{C}$. Regard B as an endomorphism of V by specializing q to a complex number. Varchenko gives a formula for the determinant of B in [15] that shows B is singular if and only if $q^{2k} = 1$, where

$$k = |\{H : H \supseteq X\}|$$

for some subspace $X \in L(\mathcal{A})$ for which Crapo's beta invariant [4] is nonzero. See [15] or [7] for a complete statement of this result.

When $\mathcal{A}$ is the set of reflecting hyperplanes given by a root system, the Weyl group G permutes the chambers $\mathcal{C}$, which gives V the structure of a G-module. For any $\sigma \in G$, one can see that $n(\sigma C, \sigma C') = n(C, C')$; therefore multiplication by B is a G-module endomorphism of V. Using the traditional labelling of the chambers of $\mathcal{A}$ with the elements of G, it turns out that B acts as multiplication by an element of the group algebra $\mathbf{C}G$.

The focus of this note shall be arrangement given by the root system A_{n-1}, also known as the braid arrangement. Here, the defining polynomial is

$$Q = \prod_{1 \leq i < j \leq n} (x_i - x_j), \tag{1.1}$$

and the Weyl group is the symmetric group $\mathfrak{S}_n$. Varchenko's determinant formula specializes to

$$\det B = \prod_{k=2}^{n} \left(1 - q^{k(k-1)}\right)^{\binom{n}{k}(k-2)!(n-k+1)!} \tag{1.2}$$

Thus B is singular if and only if $q = \zeta$ where ζ satisfies $\zeta^{k(k-1)} = 1$ for some k, $2 \leq k \leq n$.

Hanlon and Stanley [7] have shown that the kernel of B has an interesting structure as an $\mathfrak{S}_n$ module for some values of q. More precisely, let ξ be a kth root of unity. In context, we shall use ξ also to denote the one-dimensional representation of the cyclic subgroup of $\mathfrak{S}_n$ generated by the k-cycle $(12 \cdots k)$, whose value on the generator is ξ. They prove:

Theorem 1.3 (*Theorem* 3.3 *in* [7]**)** *Let $q = \zeta$ be a $n(n-1)$th root of unity, for which $\zeta^{j(j-1)} \neq 1$ for $1 \leq j < n$. Then, as an $\mathfrak{S}_n$-module,*

$$\ker B \cong \mathrm{Ind}_{C_{n-1}}^{\mathfrak{S}_n} \zeta^n - \mathrm{Ind}_{C_n}^{\mathfrak{S}_n} \zeta^{n-1}.$$

If ρ is a faithful, one-dimensional representation of C_n, then $\mathrm{Ind}_{C_n}^{\mathfrak{S}_n} \rho = \mathrm{Lie}_n$, where Lie_n is the representation of $\mathfrak{S}_n$ afforded on the multilinear part of the free Lie algebra with n generators.

Corollary 1.4 ([7]) *Let $q = \zeta$ be a primitive, $n(n-1)$th root of unity. Then*

$$\ker B \cong \mathrm{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \mathrm{Lie}_{n-1} - \mathrm{Lie}_n.$$

This representation has appeared in other contexts, such as the theory of noncommutative symmetric functions, and a version of graph cohomology [6, 13]. However, computational evidence given in Hanlon and Stanley's paper suggests that $\ker B$ has no comparably simple description for general values of q. In particular, without the condition that $\zeta^{j(j-1)} \neq 1$ for $j < n$, it is not always true that the virtual representation $\mathrm{Ind}_{C_{n-1}}^{\mathfrak{S}_n} \zeta^n - \mathrm{Ind}_{C_n}^{\mathfrak{S}_n} \zeta^{n-1}$ is even an actual representation. They conjecture that a generalization holds for certain values:

Conjecture 1.5 ([7]) For any k satisfying $2 \le k \le n$, let $q = \zeta$, where $\zeta^{j(j-1)} = 1$ for $2 \le j \le n$ if and only if $j = k$. Then, as a $\mathfrak{S}_n$-module,

$$\ker B \cong (n-k+1)\big(\mathrm{Ind}_{C_{k-1}}^{\mathfrak{S}_n}\zeta^k - \mathrm{Ind}_{C_k}^{\mathfrak{S}_n}\zeta^{k-1}\big). \tag{1.6}$$

The main objective of this paper is to show that the conjecture above is true, subject to the additional restriction that $k > n/2$. The proof occupies Section 3. However, it turns out that the conjectured result does not hold for any values of $k \le n/2$. The smallest value of n for which some k and ζ meet the other hypotheses is $n = 8$, with $k = 4$ and ζ a primitive, twelfth root of unity. Then the dimension of the right-hand side of (1.6) is strictly greater than that of $\ker B$, for reasons which will appear below.

The proof relies on identifying the role of the matrix B in a calculation of the singular homology of a topological space associated with the hyperplane arrangement. This is the subject of Section 2. For an arbitrary arrangement of m hyperplanes, consider the defining polynomial as a map $Q : \mathbf{C}^n \to \mathbf{C}$. Let $N = Q^{-1}(0)$, and $M = \mathbf{C}^n \backslash N$, respectively the variety and the complement of the arrangement. The restriction of $Q : M \to \mathbf{C}^*$ is a fibration (see [9]), and $F = Q^{-1}(1)$ is known as the Milnor fibre of the arrangement. Its topology is the subject of ongoing investigation: see, for example, [3] or [2].

We give an explicit chain complex that computes the homology of F for any real arrangement. A geometric property of F, its monodromy action, provides information about the algebraic properties of the matrix B.

Conversely, Theorem 1.3 applies to describe the representation of the alternating group afforded by certain monodromy eigenspaces in the homology of F; see Section 4.

2. A complex for the Milnor fibre

Here we describe the connection between Varchenko's matrix B and the homology of the Milnor fibre. Section 2.1 uses traditional algebraic topology to express the Milnor fibre as an infinite cyclic cover of the complement space, M. In [14], Salvetti gives a CW-complex that is homotopic to M, whose structure is determined by combinatorial data from the hyperplane arrangement (the face lattice.) With methods of Varchenko [16], one can use this to build a chain complex (Section 2.2) that computes the homology of F, where B appears as a chain map.

2.1. An infinite cyclic cover

Choose 1 as the base point of $\mathbf{C}^*$, and choose an arbitrary point $x_0 \in F$ as the base point of M. A standard device of homotopy theory [17] makes it possible to extend the fibration sequence

$$F \hookrightarrow M \xrightarrow{Q} \mathbf{C}^*$$

to the left, to obtain a new fibration, up to homotopy:

$$\Omega\mathbf{C}^* \to F \to M.$$

Here, $\Omega\mathbf{C}^*$ is the (based) space of homotopy classes of maps from S^1 to $\mathbf{C}^*$, homotopically equivalent to the integers $\mathbf{Z}$. The inclusion $F \hookrightarrow M$ is not itself a fibration, but one can replace F by a homotopically equivalent space $F(Q)$ to get an actual fibration over M:

$$\mathbf{Z} \to F(Q) \xrightarrow{\pi} M \tag{2.1}$$

The space $F(Q)$ is the *homotopy fibre* of Q:

$$F(Q) = \{(x, \omega) : x \in M, \omega \in [I, \mathbf{C}^*], \omega(0) = 1, \omega(1) = Q(x)\},$$

where $[I, X]$ is meant to denote the continuous maps between $I = [0, 1]$ and a space X, modulo homotopies that preserve endpoints. To see how (2.1) works, observe that $F(Q)$ consists of points in M paired with homotopy classes of paths in $\mathbf{C}^*$ leading from 1 (the base point) to the image $Q(x)$ of x under Q. The map π in (2.1) is given by projection onto the first coordinate.

The fibre $\pi^{-1}(x_0)$ consists of homotopy classes of loops in $\mathbf{C}^*$, since $Q(x_0) = 1$, and these are indexed naturally by the integers. Fix an explicit homotopy equivalence as follows. Define $\phi : F \to F(Q)$ by $\phi(x) = (x, \mathbf{1})$, where $\mathbf{1}$ is the constant path. Let $m = |\mathcal{A}|$, and define a path-lifting function $\Psi : F(Q) \to [I, M]$ by

$$\Psi(x, \omega)(t) = [\omega(1 - t)]^{-\frac{1}{m}} x$$

for points $(x, \omega) \in F(Q)$. Finally, define $\psi : F(Q) \to F$ by $\psi(x, \omega) = \Psi(x, \omega)(0)$. It is not hard to verify the following:

Proposition 2.2 *The maps ϕ and ψ establish a homotopy equivalence between F and $F(Q)$. Furthermore, suppose a group G acts on M in such a way that Q is constant on orbits. Then ϕ and ψ are equivariant with respect to the group action that G induces on F and $F(Q)$.*

In other words, the Milnor fibre F is homotopically equivalent to an infinite cyclic cover of M, in which the sheets of the cover are counted by the winding numbers of paths in $\mathbf{C}^*$. Milnor's article [8] describes this situation in generality. Our next observation will be that the deck transformations coincide with the geometric monodromy action on F.

The fundamental group $\pi_1(M)$ has a presentation with one generator for each hyperplane, given by a loop α_H around that hyperplane (Randell, [12]). Since the image under Q of such a loop is a loop around the origin in $\mathbf{C}^*$, each α_H has the same action on points $(x, \omega) \in F(Q)$: namely, it adds another loop around the origin to the path ω. Denote this map by $h' : F(Q) \to F(Q)$. One can check that a generator of $\pi_1(\mathbf{C}^*, 1)$ acts on $F(Q)$ as the self-map h'.

By way of comparison, let $h : F \to F$ be the monodromy action, given by $h(x) = e^{-2\pi i/m}x$. Then the following diagram commutes:

$$\begin{array}{ccc} F(Q) & \xrightarrow{h'} & F(Q) \\ \downarrow \psi & & \downarrow \psi \\ F & \xrightarrow{h} & F \end{array}$$

Since ψ is an isomorphism in homology, we can identify $H_*(F(Q), \mathbf{C})$ with $H_*(F, \mathbf{C})$. Then $h : F \to F$ and $h' : F(Q) \to F(Q)$ induce the same endomorphism h_* of $H_*(F, \mathbf{C})$. Since $h^m = \mathrm{Id}$, the action of $\mathbf{Z} = \pi_1(\mathbf{C}^*, 1)$ on $H_*(F, \mathbf{C})$ factors through $\mathbf{Z}/m\mathbf{Z}$, and h_* represents a generator of the group.

The Leray-Serre spectral sequence applied to the fibration (2.1) states that

$$E^2_{pq} = H_p(M, H_q(\mathbf{Z})) \Rightarrow H_{p+q}(F, \mathbf{C}).$$

This stage of the spectral sequence has only one nonzero row, so

$$H_*(F, \mathbf{C}) \cong H_*(M, \mathbf{C}[t, t^{-1}]). \tag{2.3}$$

It follows from the discussion above that the local coefficient system $\mathbf{C}[t, t^{-1}]$ is a $\pi_1(M)$-module in which each generator of $\pi_1(M)$ acts by multiplication by t. At the same time, $H_*(F, \mathbf{C})$ is a $\mathbf{C}[t, t^{-1}]$-module by identifying multiplication by t with the action of h_*.

2.2. Cellular homology of M

We continue by restating Varchenko's method of calculating the homology of a local coefficient system on M. The reader should refer to Chapter 2 of [16] for complete details. We begin with a brief description of Salvetti's complex; for the details of the construction, see [14], or the concise presentation in [11].

Three CW-complexes X, Y, A are required, where Y is a subcomplex of A. X is, by construction, homotopically equivalent to M. Complexes $A \subset Y$ are contractible, and the pair (A, Y) is homotopically equivalent to the pair $(\mathbf{C}^n, N)$.

The data for the construction comes from the face lattice of the real arrangement $\mathcal{A}$, which we shall denote $\mathcal{L} = \mathcal{L}(\mathcal{A})$. Let $\leq$ denote its partial ordering by reverse inclusion, and write $Q \prec P$ if P covers Q. The rank $\rho(P)$ of a face P is given by its codimension. Let $\mathcal{L}_p$ denote the faces of rank p; then $\mathcal{C} = \mathcal{L}_0$, the set of chambers. Recall the definition of the vector product $PC \in \mathcal{C}$ of any $P \in \mathcal{L}$ and $C \in \mathcal{C}$. PC is the chamber determined by "pushing" a point in the interior of P in the direction of chamber C.

Cells in the CW-complexes are labelled with symbols $E(P, Q)$, where P and Q are faces with $Q \leq P$. For $0 \leq r \leq n$ and $0 \leq s \leq n - r$, let

$$\mathcal{E}_{rs} = \{E(P, Q) : P \in \mathcal{L}_{r+s},\ Q \in \mathcal{L}_s,\ Q \leq P\}.$$

The cells of complex A in dimension r are indexed by the set $\cup_s \mathcal{E}_{rs}$. The cells of the subcomplex Y are indexed by $\cup_{s>0} \mathcal{E}_{rs}$. Last, the r-cells of X are indexed by $\mathcal{E}_{r0}$ alone.

In A, the boundary of a r-cell $E(P, Q)$ consists of all cells $E(P', Q)$, where $Q \leq P' \prec P$. In X, the boundary of a r-cell $E(P, Q)$ consists of all $r-1$-cells $E(Q, QC)$, where $Q \prec P$.

The cellular chain complexes corresponding to the complex X and the pair (A, Y) have the same bases, but differing boundary maps. $C_p(X)$ and $C_p(A, Y)$ are both isomorphic to the complex vector space generated by

$$\{E(P, C) : P \in \mathcal{L}_p, C \in \mathcal{C}, C \leq P\}.$$

In order to describe boundary maps that compute the homology of X and of the pair (A, Y), let Hyps $= \{H \in \mathcal{A} : P \subseteq H, Q \not\subseteq H\}$, and let $b(C, Q; P)$ be the number of $H \in$ Hyps that separate C and Q, minus the number that do not. Coorient the faces of the arrangement $\mathcal{A}$, and for faces $Q \prec P$ let $\varepsilon(P, Q)$ be $+1$ or -1 according to whether or not the coorientations of P and Q agree. See (2.4.2) in [16]. The following proposition is a specialization of Lemmas 2.5.13 and 2.5.15 of [16].

Proposition 2.4 ([16]) *Let $\mathcal{W}$ be a local coefficient system and let $s^4 = t$ be automorphisms of $\mathcal{W}$ so that each $\alpha_H \in \pi_1(X)$ acts by t. Then*

1. *$H_*(X, \mathcal{W})$ is computed by the chain complex $C_*(X) \otimes \mathcal{W}$, with boundary map*

$$\partial E(P, C) = \sum_{Q \prec P} \varepsilon(P, Q) s^{b(C,Q;P)} E(Q, QC).$$

2. *$H_*(A, Y, \mathcal{W})$ is computed by $C_*(A, Y) \otimes \mathcal{W}$, with boundary map*

$$\partial' E(P, C) = \sum_{C \leq Q \prec P} \varepsilon(P, Q) E(Q, C).$$

The main objective of this section is an application of the result above. Let $\mathcal{W} = \mathbf{C}[t, t^{-1}]$ be the local coefficient system of (2.3). Let $\tilde{C}_* = C_*(X) \otimes \mathcal{W}$, and $\tilde{C}'_* = C_*(A, Y) \otimes \mathcal{W}$.

Corollary 2.5 *Regard s as an indeterminate satisfying $s^4 = t$. Then*

$$H_*(F, \mathbf{C}) = H_*(\tilde{C}_*, \partial(t))$$

as $\mathbf{C}[t, t^{-1}]$-modules.

Proof: Let $\mathcal{W} = \mathbf{C}[t, t^{-1}]$ as in (2.3), and apply the proposition. Since X is homotopically equivalent to M,

$$H_*(M, \mathbf{C}[t, t^{-1}]) = H_*(\tilde{C}_*, \partial(t)).$$

The proof is completed by recalling Eq. (2.3). $\square$

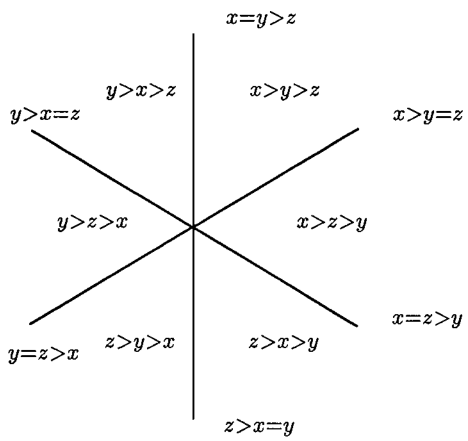

Figure 1. The A_2 arrangement.

Example 2.6 Consider the arrangement with defining polynomial $Q = (x - y)(x - z)(y - z)$ and faces labelled as in figure 1. The face lattice has maximum element $\hat{1} = \{(x, y, z) : x = y = z\}$. One can check that, for example,

- $\partial_2' E(\hat{1}, x>y>z) = -E(x = y>z, x>y>z) - E(x>y>z, x>y>z)$;
- $\partial_1 E(x = y>z, x>y>z) = s^{-1} E(x>y>z, x>y>z) - sE(y>x>z, y>x>z)$.
- $\partial_2 : \tilde{C}_2 \to \tilde{C}_1$ is a 6×12 matrix with invariant factors $1, 1-t, 1-t, 1-t, 1-t^3, 1-t^3$.

We shall also require the observation that, since the pair of spaces $(\mathbf{C}^n, N)$ is contractible,

Lemma 2.7 *The chain complex* $(\tilde{C}_*', \partial')$ *is exact.*

We introduce another important tool from [16]. For chambers $C, C' \in \mathcal{C}$, let $b(C, C')$ be the number of $H \in \mathcal{A}$ that separate C and C', minus the number that do not. Now define a bilinear form B' from the arrangement $\mathcal{A}$ whose matrix has C, C' entry equal to $s^{b(C,C')}$. We shall be interested in this bilinear form for various subarrangements of $\mathcal{A}$. For any $X \in L(\mathcal{A})$, let B_X' denote the corresponding bilinear form determined by the arrangement $\mathcal{A}_X$, the subarrangement of hyperplanes containing X. Note that if one sets $q = s^2$,

$$B = s^{|\mathcal{A}|} B'. \tag{2.8}$$

Define a map $S_* : \tilde{C}_* \to \tilde{C}_*'$ by

$$S_p E(P, C) = \sum_{C' \in \mathcal{C}} B_{|P|}'(C, C') E(P, C'),$$

for $C \in \mathcal{C}$, $P \in \mathcal{L}_p$, where $|P|$ denotes the subspace spanned by P. S_* is a block sum of matrices B'; via Corollary 2.5, then, S_* provides a relation between the homology of the Milnor fibre and Varchenko's matrices:

Proposition 2.9 ([16]) *$S_* : \tilde{C}_* \to \tilde{C}'_*$ is a homomorphism of chain complexes. S_* is an injection, and its cokernel is a torsion module over the ring $\mathbf{C}[t, t^{-1}]$.*

The last two lemmas have to do with the action of $\mathbf{Z}/m\mathbf{Z}$ on $H_*(M, \mathbf{C}[t, t^{-1}])$. Recall that the group's generator acts as multiplication by t. Thus t^m acts as the identity, and multiplication by $(1 - t^m)$ kills $H_*(M, \mathbf{C}[t, t^{-1}])$. Consequently:

Lemma 2.10 *When $\lambda \in \mathbf{C}$ satisfies $\lambda^m \neq 1$, the chain complex $(\tilde{C}_*, \partial(\lambda))$ is exact.*

To state the remaining lemma, let $\beta = \beta(\mathcal{A})$ denote Crapo's beta invariant of the matroid associated with $\mathcal{A}$. It is equal to the reduced Euler characteristic of the decone of $\mathcal{A}$.

Lemma 2.11 ([3]) *When $\lambda \in \mathbf{C}$ is an mth root of unity, the λ-eigenspace of h_*, acting on $H_{n-1}(F, \mathbf{C})$, has dimension at least $\beta(\mathcal{A})$.*

3. Generalizing Hanlon and Stanley's theorem

Our main theorem is based on Conjecture 1.5.

Theorem 3.1 *For any k satisfying $n/2 < k \le n$, let $q = \zeta$, where $\zeta^{j(j-1)} = 1$ for $2 \le j \le n$ if and only if $j = k$. Then, as a $\mathfrak{S}_n$-module,*

$$\ker B(n) \cong (n - k + 1)\left(\operatorname{Ind}_{C_{k-1}}^{\mathfrak{S}_n} \zeta^k - \operatorname{Ind}_{C_k}^{\mathfrak{S}_n} \zeta^{k-1}\right).$$

3.1. Preliminaries

For what follows, it will be convenient to restrict the A_{n-1} braid arrangement to $\mathbf{R}^{n-1}$ by eliminating the subspace contained in all hyperplanes, $x_1 + x_2 + \cdots + x_n = 0$. Call the arrangement $\mathcal{A}(\mathfrak{S}_n)$, and fix a defining polynomial for it by substituting $x_n = -\sum_{i=1}^{n-1} x_i$ in Eq. (1.1). Let $B(n) = B(\mathcal{A}(\mathfrak{S}_n))$, and note that the matrix is unaffected by this restriction. Take $B(1) = (1)$, corresponding to the empty arrangement. The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{A}(\mathfrak{S}_n)$ and M by permuting the coordinates.

The complexes defined in the previous section have nice descriptions for the braid arrangements. We begin with the face lattice: it is well known that the faces of the arrangement $\mathcal{A}(\mathfrak{S}_n)$ are determined by block-ordered partitions of the set of n elements. That is, to any face $P \in \mathcal{L}_{n-k}$ there corresponds a partition $[n] = \coprod_{r=1}^{k} X_r$, where each X_r is nonempty. The values $(|X_1|, \ldots, |X_k|)$ form a composition: an ordered sequence of positive integers whose sum is n. The points of P are those whose coordinates satisfy $x_i \le x_j$ when $x_i \in X_r$ and $x_j \in X_s$ with $r < s$, and $x_i = x_j$ exactly when $r = s$. In particular, chambers are indexed by permutations. Recall that the cells of the complexes $C_*(X)$ and $C_*(A, Y)$ are indexed by pairs (P, C) for which $P \in \mathcal{L}_k$, $C \in \mathcal{C}$, and $P \leq C$. Let $(a_1, \ldots, a_{n-k})$ be the composition of n corresponding to P, and $\sigma \in \mathfrak{S}_n$ the permutation given by C. We express the pair (P, C) by writing σ in one-line notation and delimiting the blocks of P's block-ordered partition with "/"'s:

$$\big(\sigma_1, \ldots, \sigma_{a_1}/\sigma_{a_1+1}, \ldots, \sigma_{a_1+a_2}/\cdots/\sigma_{a_1+\cdots+a_{n-k-1}+1}, \ldots, \sigma_n\big).$$

In [16], Varchenko shows that one can construct the complexes X, Y, and A so that $\mathfrak{S}_n$ acts cellularly.

Proposition 3.2 ([16]) *Let $E(P, C)$ be a cell of X or A indexed by the pair σ, $(a_1, \ldots, a_{n-k})$ as above. Then, each $\tau \in \mathfrak{S}_n$ induces the map $\tau E(P, C) = \varepsilon(\tau)E(P', C')$ in $C_*(X)$ and $C_*(A, Y)$, where ε is the sign character, and (P', C') is determined by the expression*

$$((\tau\sigma)_1, \ldots, (\tau\sigma)_{a_1}/\cdots/(\tau\sigma)_{a_1+\cdots+a_{n-k-1}+1}, \ldots, (\tau\sigma)_n).$$

Furthermore, ∂_* and ∂'_* are $\mathfrak{S}_n$-module homomorphisms.

We also require some notation to describe the invariant factors of $B(n)$ over the ring $\mathbf{C}[q, q^{-1}]$. For a reference on the Smith Normal Form of a matrix, see [10].

Definition 3.3 Let R be a Euclidean domain, A a matrix over R, and $u \in R$ a nonunit. Let $d_1, d_2, \ldots, d_k$ be the invariant factors of A, ordered as usual so that $d_i | d_{i+1}$. Let m_r be the number of invariant factors d_i that are divisible by u^r, but not by u^{r+1}. Let $\mu(A, u, x)$ be the generating function for the numbers m_r: that is, $\mu(A, u, x) = \sum_r m_r x^r$.

For example, over the integers,

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 6 & 0 & 4 \\ 1 & 0 & 7 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix},$$

so $\mu(A, 2, x) = 1 + x + x^2$, whereas $\mu(A, u, x) = 3$ for $u \neq 2$ or 4.

Note that Eq. (2.8) implies that $\mu(B, f, x) = \mu(B', f, x)$ for any polynomial $f \in \mathbf{C}[q]$, as long as $f(0) \neq 0$, subject to the usual identification $s^2 = q$.

We shall need to use the following property of the generating function.

Lemma 3.4 *Let $\mathcal{A} \oplus \mathcal{A}'$ denote the direct sum of two real hyperplane arrangements. Then for any nonunit $f \in \mathbf{C}[q]$,*

$$\mu(B(\mathcal{A} \oplus \mathcal{A}'), f, x) = \mu(B(\mathcal{A}), f, x) \cdot \mu(B(\mathcal{A}'), f, x).$$

Proof: From [16, Section 2.6], we have $B(\mathcal{A} \oplus \mathcal{A}') = B(\mathcal{A}) \otimes B(\mathcal{A}')$. Let $S(A)$ denote the Smith Normal Form of a matrix A. For any two square matrices A and A', it is known that $S(A \otimes A')$ equals $S(A) \otimes S(A')$, up to a reordering of the rows and columns. Then one only needs to verify the identity $\mu(A \otimes A', f, x) = \mu(A, f, x)\mu(A', f, x)$ when A and A' are diagonal. □

Definition 3.5 For any arrangement $\mathcal{A}$ of rank n and nonunit $f \in \mathbf{C}[q]$ satisfying $f(0) \neq 0$, let

$$\chi(\mathcal{A}, f, x) = \sum_{k=0}^{n} (-1)^k \mu(S_k, f, x).$$

Since μ is additive over direct sums of matrices, it follows from the definition of S_k in Section 2.2 that

$$\chi(\mathcal{A}, f, x) = \sum_{P \in \mathcal{L}} (-1)^{\rho(|P|)} \mu\big(B_{|P|}, f, x\big), \tag{3.6}$$

where $B_{|P|} = B(\mathcal{A}_{|P|})$.

The next lemma gives a more specific relation between functions χ and μ in the case of braid arrangements. For any nonunit $f \in \mathbf{C}[q, q^{-1}]$ with $f(0) \neq 0$, define a generating function

$$G(f, x, y) = 1 + \sum_{k \geq 1} \frac{\mu(B(k), f, x) y^k}{k!}.$$

Lemma 3.7 *$G(f, x, y)$ satisfies the identity*

$$G(f, x, y)^{-1} = 1 + \sum_{n \geq 1} \frac{(-1)^n \chi(\mathcal{A}(\mathfrak{S}_n)), f, x) y^n}{n!}.$$

Proof: Given a face $P \in \mathcal{L}_k$, let $(a_1, \ldots, a_{n-k})$ be the composition of n associated with it. It is not hard to verify that the arrangement $\mathcal{A}_{|P|}$ is the direct sum of the arrangements $\mathcal{A}(\mathfrak{S}_{a_r})$. By Lemma 3.4, then,

$$\mu\big(B_{|P|}, f, x\big) = \prod_{r=1}^{k} \mu(B(a_r), f, x). \tag{3.8}$$

Now let us determine the coefficient of $y^n/n!$ in $G(f, x, y)^{-1}$. Put $T = \sum_{k \geq 1} -\mu(B(k), f, x) y^k/k!$. Then

$$\begin{aligned} G(f, x, y)^{-1} &= \frac{1}{(1 - T)} \\ &= 1 + \sum_{n \geq 1} \sum_{(a_1, \ldots, a_k)} \binom{n}{a_1, \ldots, a_k} \prod_{r=1}^{k} \frac{(-1)^k \mu(B(a_r), f, x) y^n}{n!}, \end{aligned}$$

where the sum is taken over sequences of positive integers $(a_1, \ldots, a_k)$ whose sum is n. Using (3.8) to compare this with (3.6) yields the desired identity. □

Example 3.9 Let $f = q^2 - 1$, and let $P(\mathcal{A}, x)$ denote the Poincaré polynomial of the intersection lattice $L(\mathcal{A})$. With respect to the prime factors of $q^2 - 1$, the Smith Normal Form of Varchenko's matrix is known: we have

$$\begin{aligned} \mu(B(n),\, f, x) &= P(\mathcal{A}(\mathfrak{S}_n), x) \quad \text{By Theorem 3.1 of [5]},\\ &= \prod_{r=1}^{n-1}(1 + rx) \quad \text{by Arnold's Theorem [1].} \end{aligned}$$

Then $G(f, x, y) = (1-xy)^{-1/x}$, by the generalized binomial theorem. Clearly $G(f, x, y)^{-1} = G(f, -x, -y)$, from which

$$\chi(\mathcal{A}(\mathfrak{S}_n),\, f, x) = P(\mathcal{A}(\mathfrak{S}_n), -x).$$

(In fact, one can show that this last formula holds for any arrangement $\mathcal{A}$.)

Describing the invariant factors of Varchenko's matrices at primes other than $q \pm 1$ is closely related to describing the nontrivial monodromy eigenspaces of $H_*(F, \mathbf{C})$, however, and remains an open problem: see [5]. The alternating sum $\chi(\mathcal{A},\, f, x)$ introduced in Definition 3.5 is a weaker invariant of an arrangement. At the same time, one can regard it as a refinement of the Reidemeister torsion or zeta function of F that Milnor considers in [9].

3.2. Proof of Theorem 3.1

The proof of the theorem depends on considering the relation between the generating functions $\mu(B(n), q - \zeta, x)$ and $\chi(\mathcal{A}(\mathfrak{S}_n), q - \zeta, x)$ for appropriate roots of unity ζ.

Lemma 3.10 *Let $\mathcal{A}$ be an essential, n-dimensional arrangement of m hyperplanes, and let ζ be a nonzero complex number.*
1. *If $\zeta^{2m} \neq 1$, then $\chi(\mathcal{A}, q - \zeta, x) = 0$.*
2. *If $q = \zeta$ is a root of* $\det B(\mathcal{A})$, *but not of any* $\det B(\mathcal{A}_X)$ *for $X \in L(\mathcal{A})\backslash\{1\}$ satisfying $\beta(\mathcal{A}_X) \neq 0$, then $\chi(\mathcal{A}, q - \zeta, x) = (-1)^n(x - 1)\beta(\mathcal{A})$.*

Proof: In order to isolate the behaviour of $q - \zeta$, we shall localize $\mathbf{C}[t, t^{-1}]$ at the prime generated by $t - \zeta^2$. (Recall that $t = q^2$.) Let R denote the local ring, and assume this localization is in effect through this proof without further reference to it. Since S_* is an injection (Proposition 2.9), there is an exact sequence

$$0 \to \tilde{C}_*(t) \xrightarrow{S_*} \tilde{C}'_* \to \operatorname{coker} S_* \to 0.$$

To prove claim (1), decompose coker S_k for each k as a direct sum

$$\operatorname{coker} S_k = \bigoplus_{r\geq 1}\left(\frac{R}{(t-\zeta^2)^r}\right)^{a_{rk}}.$$

By the properties of the Smith Normal Form, a_{rk} is the coefficient of x^r in $\mu(S_k, t-\zeta^2, x)$. From Lemmas 2.10 and 2.7, respectively, the complexes $\tilde{C}_*(t)$ and $\tilde{C}'_*$ are exact. Using the long exact sequence in homology, we find that coker S_* is also exact. Since $\tilde{C}_*(t)$ and $\tilde{C}'_*$ are both exact sequences of free modules, they both split. This induces a splitting on cokerS_*, from which it follows that the alternating sum of the multiplicities a_{rk} is zero, for each r. That is, each coefficient of the polynomial $\chi(\mathcal{A}, q-\zeta, x)$ is zero.

Now we prove claim (2). Lemma 2.11 asserts that $\ker \partial_n(\zeta^2)$ has dimension at least β. From Lemma 2.7 and Proposition 2.9, ∂'_n and S_{n-1} are injections when $t=\zeta^2$. Since $\partial'_n S_n = S_{n-1}\partial_n(\zeta^2)$, the dimension of $\ker S_n$ is also at least β when $t=\zeta^2$. On the other hand, Varchenko's determinant formula shows that $t-\zeta^2$ divides $\det S_n$ exactly β times; it follows that $\mu(S_k, t-\zeta^2, x) = \beta x + c$ for a constant c.

To complete the argument, note that $\mu(S_k, f, 1) = \dim_{\mathbf{C}} C_k$ for each k. By exactness, then, $\chi(\mathcal{A}, t-\zeta^2, 1) = 0$, and we find that $\chi(\mathcal{A}, q-\zeta, x) = (-1)^n(x-1)\beta$. □

Lemma 3.11 *Let $k>1$. For any ζ satisfying $\zeta^{k(k-1)}=1$, let $n>1$ be the smallest integer satisfying $\zeta^{n(n-1)}=1$, excluding k. Suppose $n>k$. Then*
1. $\mu(B(r), q-\zeta, x) = r!$ *for* $1\leq r<k$;
2. $\mu(B(r), q-\zeta, x)$ *is a linear function of x for* $k\leq r<\min\{n, 2k\}$.

Proof: Apply the previous lemma to the arrangement $\mathcal{A}(\mathfrak{S}_r)$. For $2\leq r<n, r\neq k$, case (1) applies. For $r=k$, case (2) applies. Using the generating function identity (Lemma 3.7),

$$G(q-\zeta, x, y) = \left[1 - y + \frac{x-1}{k(k-1)}y^k + O(y^n)\right]^{-1}.$$

G is the exponential generating function for $\mu(B(r), q-\zeta, x)$. From the equation above, its coefficients are constant when $r<k$ and linear in x when both $r<2k$ and $r<n$. □

Lemma 3.12 *Let $k>1$ and $k\leq n<2k$. Suppose that ζ satisfies $\zeta^{r(r-1)}=1$ for $2\leq r\leq n$ only when $r=k$. For the arrangement $\mathcal{A}(\mathfrak{S}_n)$ and map $S_*: \tilde{C}_* \to \tilde{C}'_*$, let $q=\zeta$. Then*
1. $\ker S_r = 0$ *for* $0\leq r<k-1$.
2. *For $k-1\leq r\leq n-1$, there exists some $\alpha>0$ for which* $\ker S_r \cong \alpha(\operatorname{Ind}_{\mathfrak{S}_k}^{\mathfrak{S}_n}\ker B(k))$ *as $\mathfrak{S}_n$-modules.*

Proof: Assertion (1) follows from the determinant formula (1.2). To prove (2), suppose $k-1\leq r\leq n-1$, and let $\mathbf{a}=(a_1,\ldots,a_{n-r})$ be a composition of n. Let $B(\mathbf{a}) = \bigoplus_P B_{|P|}$, where the direct sum is taken over all faces $P\in\mathcal{L}_r$ whose block-ordered partition has block sizes $\mathbf{a}$. Using Proposition 3.2 and (2.8),

$$S_r = c \bigoplus_{\mathbf{a}} B(\mathbf{a}),$$

is a direct sum of $\mathfrak{S}_n$-homomorphisms, where c is some nonzero scalar, and the sum is taken over all compositions of n with $n - r$ parts. Then

$$\ker S_r \cong \bigoplus_{\mathbf{a}} \ker B(\mathbf{a}).$$

Since the hypotheses dictate that $n < 2k$, k appears at most once in each composition of n. Recall that $B_{|P|} \cong B(a_1) \otimes B(a_2) \otimes \cdots \otimes B(a_{n-r})$ as a $\mathfrak{S}_{a_1} \times \cdots \times \mathfrak{S}_{a_{n-r}}$-module homomorphism. At most one factor is $B(k)$, and the rest are isomorphisms. It follows that $\ker B(\mathbf{a}) = \mathrm{Ind}_{\mathfrak{S}_k}^{\mathfrak{S}_n} \ker B(k)$ if some $a_i = k$, and 0 if not. $\square$

Now we are prepared to prove Theorem 3.1:

Proof of 3.1: Suppose that n, k, and ζ satisfy the hypotheses of the theorem, and set $q = \zeta$. We shall use induction on $n - k$. If $n = k$, Theorem 1.3 applies. Otherwise, suppose further that, for all r satisfying $k \leq r < n$, $\ker B(r)$ is the direct sum of copies of the $\mathfrak{S}_r$-module $\mathrm{Ind}_{C_{k-1}}^{\mathfrak{S}_r} \zeta^k - \mathrm{Ind}_{C_k}^{\mathfrak{S}_r} \zeta^{k-1}$. We must show that the same is true when $r = n$.

By Lemma 3.12 and the induction hypothesis, $\ker S_{r-1} = b_r U$ for each $r < n$, for some numbers $b_r \geq 0$, where

$$U = \mathrm{Ind}_{C_{k-1}}^{\mathfrak{S}_n} \zeta^k - \mathrm{Ind}_{C_k}^{\mathfrak{S}_n} \zeta^{k-1}. \tag{3.13}$$

Consider the exact sequence of chain complexes over $\mathbf{C}\mathfrak{S}_n$

$$0 \to \ker S_* \to \tilde{C}_*(\zeta^2) \xrightarrow{S_*} \tilde{C}'_* \to 0. \tag{3.14}$$

From Lemmas 2.7 and 2.10, respectively, $\tilde{C}_*(\zeta^2)$ and $\tilde{C}'_*$ are exact. The long exact sequence in homology shows that $\ker S_*$ is also exact. It follows that $\ker S_{n-1} = b_n U$, where

$$b_n = - \sum_{1 \leq r < n} (-1)^{n-r} b_r.$$

Since $B(n) = q^{n(n-1)/4} S_n$, by (2.8), it remains only to determine b_n. From Lemma 3.11, the dimension of $\ker B(n)$ equals the multiplicity of $q - \zeta$ as a factor of $\det B(n)$. This equals $\binom{n}{k}(k-2)!(n-k+1)!$, by (1.2). Since $\dim U = n!/k(k-1)$, one finds $\ker B(n) = (n - k + 1)U$. $\square$

4. Remarks

The proof of Theorem 3.1 shows why Hanlon and Stanley's conjecture needs the restriction that $n < 2k$. When $n \geq 2k$, $\mathcal{A}(\mathfrak{S}_n)$ has edges that contain a direct sum of more than one braid sub-arrangement $\mathcal{A}(\mathfrak{S}_k)$. In this case, the methods used here describe the representation of $\mathfrak{S}_n$ on the kernel of $B(n)$ in terms of sums of tensor products of the representation (3.13) with itself.

At the same time, Theorem 1.3 (Theorem 3.3 of [7]) describes the representation of $\mathfrak{S}_n$ on the homology of $\tilde{C}_*(\zeta)$, where ζ is a root of unity satisfying the conditions of the theorem: one uses the exact sequence (3.14) as before. Equivalently, the theorem characterizes the representation of the alternating group $\mathfrak{A}_n$ on the ζ-eigenspace of $H_*(F, \mathbf{C})$. One might hope for an approach that simultaneously accounts for more of the structure of the (co)kernel of Varchenko's quantum bilinear form and of the homology of the arrangement's Milnor fibre.

References

1. V.I. Arnol'd, "The cohomology ring of the colored braid group," *Math Notes* **5** (1969), 138–140.
2. E. Artal-Bartolo, "Combinatorics and topology of line arrangements in the complex projective plane," *Proc. Amer. Math. Soc.* **121**(2) (1994), 385–390.
3. D.C. Cohen and A.I. Suciu, "On Milnor fibrations of arrangements," *J. London Math. Soc. (2)* **51**(1) (1995), 105–119.
4. H. Crapo, "A higher invariant for matroids," *J. Comb. Th.* **2** (1967), 406–417.
5. G. Denham and P. Hanlon, "On the Smith normal form of the Varchenko bilinear form of a hyperplane arrangement," *Pacific J. Math.* (Special Issue) (1997), 123–146. Olga Taussky-Todd: in memoriam.
6. G. Duchamp, A. Klyachko, D. Krob, and J.-Y. Thibon, "Noncommutative symmetric functions. III. Deformations of Cauchy and convolution algebras," *Discrete Math. Theor. Comput. Sci.* **1**(1) (1997), 159–216. Lie computations (Marseille, 1994).
7. P. Hanlon and R.P. Stanley, "A q-deformation of a trivial symmetric group action," *Trans. Amer. Math. Soc.* **350**(11) (1998), 4445–4459.
8. J.W. Milnor, "Infinite cyclic coverings," in *Conference on the Topology of Manifolds*, Michigan State Univ., E. Lansing, Mich., 1967, pp. 115–133, Prindle, Weber & Schmidt, Boston, Mass., 1968.
9. J.W. Milnor, *Singular Points of Complex Hypersurfaces*, Princeton University Press, 1968.
10. M. Newman, *Integral Matrices*, Academic Press, 1972.
11. P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer Verlag, 1992. Grundlehren der Mathematischen Wissenschaften, Vol. 300.
12. R. Randell, "The fundamental group of the complement of a union of complex hyperplanes," *Invent. Math.* **69**(1) (1982), 103–108.
13. A. Robinson and S. Whitehouse, "The tree representation of σ_{n+1}," *J. Pure Appl. Algebra* **111**(1/3) (1996), 245–253.
14. M. Salvetti, "Topology of the complement of real hyperplanes in $\mathbf{C}^n$," *Invent. Math.* **88** (1987), 603–618.
15. A.N. Varchenko, "Bilinear form of real configuration of hyperplanes," *Adv. Math.* **97**(1) (1993), 110–144.
16. A.N. Varchenko, *Multidimensional Hypergeometric Functions and Representation Theory of Lie Algebras and Quantum Groups*, World Scientific, 1995. Advanced Series in Mathematical Physics, Vol. 21.
17. G.W. Whitehead, *Elements of Homotopy Theory*, Springer Verlag, 1978. Graduate Texts in Mathematics, Vol. 61.