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Technical Report No. 17

THE THREE-DIMENSIONAL ELASTIC CHARACTERISTICS OF CORD-RUBBER LAMINATES

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NOMENCLATURE

English Letters:

a_{ij}, c_{ij} Constants associated with generalized Hooke's law, using properties based on cord tension.

a'_{ij} Constants associated with generalized Hooke's law, using properties based on cord compression.

E,F,G Elastic constants for orthotropic laminates with cords in tension.

E',F',G' Elastic constants for orthotropic laminates with cords in compression.

ℓ Direction cosine.

u,v,w Displacements in the x, y, z directions, respectively.

W Elastic strain energy.

x,y,z Orthogonal coordinates aligned along and normal to the cord direction.

Greek Letters:

α One-half the included angle between cords in adjoining plies in a two-ply laminate.

ϵ Strain.

μ Poisson's ratio.

ξ,η,ζ Orthogonal coordinates aligned along and normal to the principal axes of elasticity, or orthotropic axes, in an orthotropic laminate.

σ Stress.

σ' Interply stress.

I. FOREWORD

It has been known for some time that the stress-strain relationships perpendicular to the plane of a tire carcass could be of some importance in cases where the carcass becomes thick such as in aircraft or industrial tires. Such an analysis will require knowledge of the elastic properties of cord-rubber combinations perpendicular to the thickness of a number of plies, but will yield information concerning the possible interaction between forces in the plane of the carcass and ground pressure forces perpendicular to the plane of the carcass.

Previous work on the elastic characteristics of cord rubber laminates has ignored stresses in this thickness direction. Thus, this report will serve two purposes: one is to give some indication of the size of the elastic constants perpendicular to the carcass of a tire, the second is to investigate what effects an inclusion of stresses in this direction might have on the previous development of plane elastic characteristics.

This report is an extension of the development in Ref. 1; hence, much of the detail of Ref. 1 will be utilized in this development.

II. SUMMARY

A three-dimensional analysis of the elastic characteristics of cord-rubber laminates is presented in this report. The analysis is a continuation of Ref. 1, which was based on the principals of orthotropic materials. The investigation is confined to those structures which have the cords of all plies in tension.

This report shows that the extension moduli and cross-moduli, relating stress and strain in the plane of the cords, are not affected by the stress and strain in the thickness direction; hence, there are no coupling effects between the normal strains in the plane of the cords and the stresses in the direction of the thickness. The shear modulus is affected, but the magnitude of this change is very small for the numerical calculations presented in this report.

In addition, the extension moduli and cross-moduli representing the relationships between normal stresses and strains in the thickness direction are presented, as well as the additional shearing effects, which are shown to be completely independent of the stresses in the plane of the cords.

Interply shear stresses again enter into the analysis. They are used only insofar as they influence the elastic characteristics considered in this report. However, these stresses are shown to be different from those previously obtained by ignoring effects in the thickness direction.

It may generally be concluded from the work reported here that the inclusion of the third direction is entirely possible when investigating the

elastic characteristics of orthotropic cord-rubber laminates. The only additional information needed is that relating stress and strain in the thickness direction of a single sheet of the constituent material.

III. THEORETICAL ANALYSIS

A two-ply laminated structure is considered in which the cords are separated angularly in the two plies by an included angle 2α, as shown in Fig. 1. In that illustration, the heavy diagonal lines representing typical textile cords are shown being bisected by coordinate axes ξ and η, two arms of the orthogonal ξ, η, ζ system. The elastic properties of this type of structure will be obtained and expressed in terms of the elastic properties of a single sheet of parallel textile fibers embedded in rubber and forming one lamina.

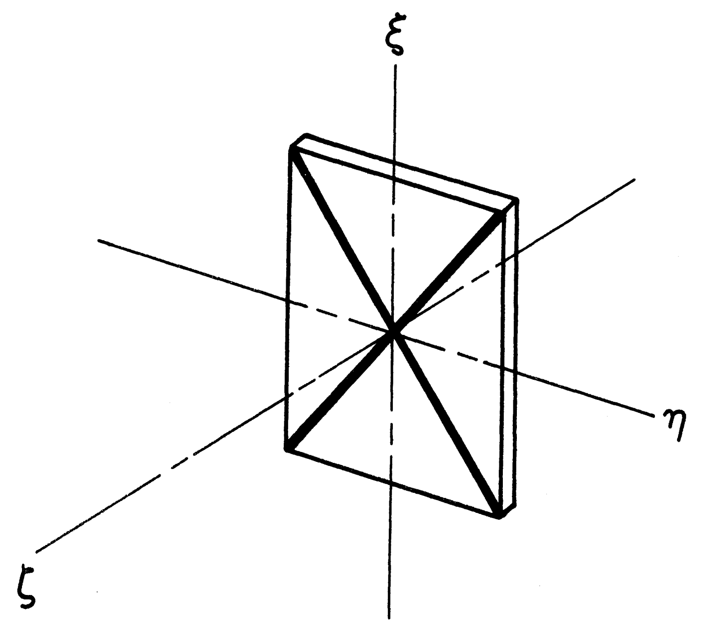

Fig. 1. Schematic view of a small section of a two-ply laminate.

This report will extend Ref. 1 by including the effects of stress in the ζ direction. The details of Ref. 1 will be referred to often in this analysis.

Although discussion in this report is limited to the characteristics of a two-ply system, the ideas and techniques developed allow fairly easy extension to any number of plies, provided that the structure remains orthotropic, that is, that it maintains three planes of elastic symmetry.

Observation of the physical nature of the structure under investigation, shows that the usual form of Hooke's law can be applied directly, so long as the relationships between stress and strain are written in the ζ, ξ, and η directions, since these are the only directions in which coupling between normal stresses and shearing strains vanishes. In other words, these are the only directions in which we can apply normal stresses and have no shearing strain. The equations relating stress and strain for this condition are:

$$\epsilon_\xi = \frac{\sigma_\zeta}{E_\zeta} - \frac{\mu_{\eta\xi}}{E_\eta}\sigma_\eta - \frac{\mu_{\zeta\xi}}{E_\zeta}\sigma_\zeta$$

$$\epsilon_\eta = -\frac{\mu_{\xi\eta}}{E_\xi}\sigma_\xi + \frac{\sigma_\eta}{E_\eta} - \frac{\mu_{\zeta\eta}}{E_\zeta}\sigma_\zeta$$

$$\sigma_\zeta = -\frac{\mu_{\xi\zeta}}{E_\xi}\sigma_\xi - \frac{\mu_{\eta\zeta}}{E_\eta}\sigma_\eta + \frac{\sigma_\zeta}{E_\zeta}$$

$$\epsilon_{\xi\eta} = \frac{\sigma_{\xi\eta}}{G_{\xi\eta}} \qquad \epsilon_{\eta\xi} = \frac{\sigma_{\eta\xi}}{G_{\eta\xi}}$$

$$\epsilon_{\xi\zeta} = \frac{\sigma_{\xi\zeta}}{G_{\xi\zeta}} \qquad \epsilon_{\zeta\xi} = \frac{\sigma_{\zeta\xi}}{G_{\zeta\xi}}$$

$$\epsilon_{\eta\zeta} = \frac{\sigma_{\eta\zeta}}{G_{\eta\zeta}} \qquad \epsilon_{\zeta\eta} = \frac{\sigma_{\zeta\eta}}{G_{\eta\zeta}}$$

Some simplification of Eq. (1) results when one recalls (Ref. 2) that the stresses and strains are components of symmetric tensors and, hence, $\sigma_{ij} = \sigma_{ji}$ and $\epsilon_{ij} = \epsilon_{ji}$. With this in mind, it can be seen that $G_{\xi\eta} = G_{\eta\xi}$, $G_{\xi\zeta} = G_{\zeta\xi}$, $G_{\eta\zeta} = G_{\zeta\eta}$. Note however, that $\mu_{\xi\eta}$ and $\mu_{\eta\xi}$, $\mu_{\xi\zeta}$ and $\mu_{\zeta\xi}$, etc. are not necessarily equal. Therefore, from this viewpoint one has twelve elastic constants: E_ξ, E_η, E_ζ, $\mu_{\xi\eta}$, $\mu_{\eta\xi}$, $\mu_{\xi\zeta}$, $\mu_{\zeta\xi}$, $\mu_{\eta\zeta}$, $\mu_{\zeta\eta}$, $G_{\xi\eta}$, $G_{\xi\zeta}$, $G_{\eta\zeta}$.

Equations (1) can be obtained by a somewhat more general approach, which will also yield additional information. Several authors, eg., Ref. 2, have shown that for an orthotropic material the relationship between stress and strain can be expressed in the following manner:

$$
\begin{aligned}
\sigma_\xi &= C_{11}\,\epsilon_\xi + C_{12}\,\epsilon_\eta + C_{13}\,\epsilon_\zeta + 0 + 0 + 0 \\
\sigma_\eta &= C_{21}\,\epsilon_\xi + C_{22}\,\epsilon_\eta + C_{23}\,\epsilon_\zeta + 0 + 0 + 0 \\
\epsilon_\zeta &= C_{31}\,\epsilon_\xi + C_{32}\,\epsilon_\eta + C_{33}\,\epsilon_\zeta + 0 + 0 + 0 \\
\sigma_{\eta\zeta} &= 0 + 0 + 0 + 2C_{44}\,\epsilon_{\eta\zeta} + 0 + 0 \\
\sigma_{\xi\zeta} &= 0 + 0 + 0 + 0 + 2C_{55}\,\epsilon_{\xi\zeta} + 0 \\
\sigma_{\xi\eta} &= 0 + 0 + 0 + 0 + 0 + 2C_{66}\,\epsilon_{\xi\eta} \quad ,
\end{aligned}
\tag{2}
$$

when C_{ij} represent elastic constants of the material. In addition, there is a function which represents the strain energy per unit volume stored in the material; this function is of the form

$$W = \Big[C_{11}\,\epsilon_\xi^2 + C_{22}\,\epsilon_\eta^2 + C_{33}\,\epsilon_\zeta^2 + (C_{13} + C_{31})\,\epsilon_\xi\,\epsilon_\zeta + 4C_{44}\,\epsilon_{\eta\zeta} + 4C_{55}\,\epsilon_{\xi\zeta} + 4C_{66}\,\epsilon_{\xi\eta}\Big] \quad . \tag{3}$$

Since this function is quadratic in form, interchanging the subscripts on the C's will not alter its value; hence, $C_{ij} = C_{ji}$. A physical consequence of the unchanging nature of W is discussed in Ref. 1.

With this in mind, one can rewrite Eq. (2) in this form:

$$\begin{aligned}
\sigma_\xi &= C_{11}\,\epsilon_\xi + C_{12}\,\epsilon_\eta + C_{13}\,\epsilon_\zeta \\
\sigma_\eta &= C_{12}\,\epsilon_\xi + C_{22}\,\epsilon_\eta + C_{23}\,\epsilon_\zeta \\
\sigma_\zeta &= C_{13}\,\epsilon_\xi + C_{23}\,\epsilon_\eta + C_{33}\,\epsilon_\zeta \\
\sigma_{\eta\zeta} &= 2C_{44}\,\epsilon_{\eta\zeta} \\
\sigma_{\xi\zeta} &= 2C_{55}\,\epsilon_{\xi\zeta} \\
\sigma_{\xi\eta} &= 2C_{66}\,\epsilon_{\xi\eta} \quad .
\end{aligned} \tag{4}$$

Rewriting Eq. (4) for strain in terms of stress, we have

$$\begin{aligned}
\epsilon_\xi &= \frac{(C_{22}\,C_{33} - C_{23}^2)}{C_{22}}\,\sigma_\xi + \frac{(C_{13}\,C_{23} - C_{12}\,C_{33})}{C_{22}}\,\sigma_\eta + \frac{(C_{12}\,C_{23} - C_{13}\,C_{22})}{C_{22}}\,\sigma_\zeta \\
\epsilon_\eta &= \frac{(C_{13}\,C_{23} - C_{12}\,C_{33})}{C_{22}}\,\sigma_\xi + \frac{(C_{11}\,C_{33} - C_{13}^2)}{C_{22}}\,\sigma_\eta + \frac{(C_{12}\,C_{13} - C_{11}\,C_{23})}{C_{22}}\,\sigma_\zeta
\end{aligned} \tag{5}$$

$$\epsilon_{\zeta} = \frac{(C_{12}\,C_{23} - C_{13}\,C_{22})}{C_{22}}\,\sigma_{\xi} + \frac{(C_{12}\,C_{13} - C_{11}\,C_{23})}{C_{22}}\,\sigma_{\eta} + \frac{(C_{11}\,C_{22} - C_{12}^2)}{C_{22}}\,\sigma_{\zeta}$$

$$\epsilon_{\eta\zeta} = \frac{\sigma_{\eta\zeta}}{2C_{44}}$$

$$\epsilon_{\zeta\xi} = \frac{\sigma_{\xi\zeta}}{2C_{55}} \qquad (5)$$

$$\epsilon_{\xi\eta} = \frac{\sigma_{\xi\eta}}{2C_{66}} \quad .$$

Comparing Eqs. (1) and Eqs. (5), observe that

$$\frac{(C_{13}\,C_{23} - C_{12}\,C_{33})}{C_{22}} = \frac{-\mu_{\eta\xi}}{E_{\eta}} = \frac{-\mu_{\xi\eta}}{E_{\xi}} \equiv \frac{1}{F_{\xi\eta}}$$

$$\frac{(C_{12}\,C_{23} - C_{13}\,C_{22})}{C_{22}} = \frac{-\mu_{\zeta\xi}}{E_{\zeta}} = \frac{-\mu_{\xi\zeta}}{E_{\xi}} \equiv \frac{1}{F_{\xi\zeta}}$$

$$\frac{(C_{12}\,C_{13} - C_{11}\,C_{23})}{C_{22}} = \frac{-\mu_{\zeta\eta}}{E_{\zeta}} = \frac{-\mu_{\eta\zeta}}{E_{\eta}} \equiv \frac{1}{E_{\eta\zeta}} \qquad (6)$$

$$\frac{(C_{22}\,C_{33} - C_{23}^2)}{C_{22}} = \frac{1}{E_{\xi}} \; ; \quad \frac{(C_{11}\,C_{33} - C_{13}^2)}{C_{22}} = \frac{1}{E_{\eta}} \; ; \quad \frac{(C_{11}\,C_{22} - C_{12}^2)}{C_{22}} = \frac{1}{E_{\zeta}}$$

$$2C_{66} = G_{\xi\eta} \; ; \quad 2C_{55} = G_{\xi\zeta} \; ; \quad 2C_{44} = G_{\eta\zeta} \quad .$$

The substitution of Eqs. (6) into Eqs. (5) yields the stress-strain relations for any orthotropic body in the form

$$\epsilon_\xi = \frac{\sigma_\xi}{E_\xi} + \frac{\sigma_\eta}{F_{\xi\eta}} + \frac{\sigma_\zeta}{F_{\xi\zeta}}$$

$$\epsilon_\eta = \frac{\sigma_\xi}{F_{\xi\eta}} + \frac{\sigma_\eta}{E_\eta} + \frac{\sigma_\zeta}{F_{\eta\zeta}}$$

$$\epsilon_{\xi\eta} = \frac{\sigma_{\xi\eta}}{G_{\xi\eta}}$$

$$\epsilon_{\xi\zeta} = \frac{\sigma_{\xi\zeta}}{G_{\xi\zeta}}$$

$$\epsilon_{\eta\zeta} = \frac{\sigma_{\eta\zeta}}{G_{\eta\zeta}}$$

The form and terminology of Eqs. (7) will be used throughout the following discussion involving stress-stress relations. Note that Eqs. (7) contain nine independent elastic constants: the extension moduli E_ξ, E_η, and E_ζ; the cross-moduli $F_{\xi\eta}$, $F_{\xi\zeta}$, and $F_{\eta\zeta}$; and the shear moduli $G_{\xi\eta}$, $G_{\xi\zeta}$, and $G_{\eta\zeta}$.

Leaving the general two-ply structure, we will now investigate a single ply which is considered to be constructed by imbedding a series of parallel, straight cords lying in a plane into a sheet-like matrix of elastic material. This construction is illustrated in Fig. 2. This sheet of material is also an orthotropic body if the axes of elastic symmetry are taken to be the x, y, and z axes. Since it is orthotropic it will have elastic constants similar to those expressed in Eqs. (7) except that the subscripts will now be x, y,

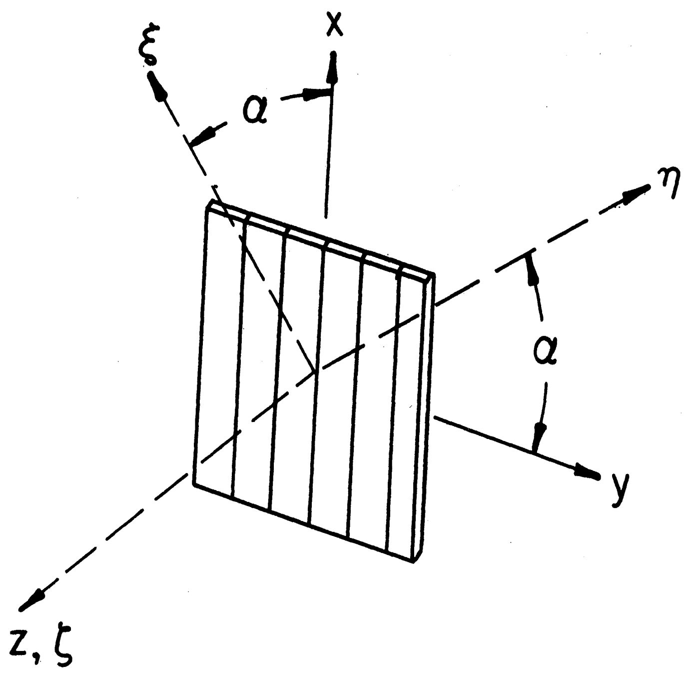

Fig. 2. Schematic view of a single ply of cord imbedded in rubber.

and z instead of ξ, η and ζ.

It will be assumed that E_x, E_y, E_z, F_{xz}, F_{yz}, F_{xy}, G_{xy}, G_{xz}, and G_{yz} are known quantities. Some previous work has been done in obtaining these properties.[5] Again, note that these are the properties of a single ply in the orthotropic directions only when all of the cords are in tension.

Again referring to Fig. 2, consider a known stress σ_x, σ_y, σ_z, σ_{xy}, σ_{xz}, and σ_{yz} with a corresponding strain state ϵ_x, ϵ_y, ϵ_z, ϵ_{xy}, ϵ_{xz}, and ϵ_{yz}. It is desired to determine the corresponding stress and strain state referred to the orthogonal axes ξ, η, and ζ, obtained by rotating the x, y, z system about the z arm through an angle of α. The direction cosines of the different axes with respect to each other due to this rotation are shown in Table I.

TABLE I

DIRECTION COSINES

	x	y	z
ξ	$\cos\alpha$	$-\sin\alpha$	0
η	$\sin\alpha$	$\cos\alpha$	0
ζ	0	0	1

It has been shown[3] that the stress components referred to the ξ, η, ζ system, written in terms of the known components of the x, y, z system, take the form

$$\begin{aligned}
\sigma_\xi &= \cos^2\alpha\ \sigma_x + \sin^2\alpha\ \sigma_y - 2\sin\alpha\cos\alpha\ \sigma_{xy} \\
\sigma_\eta &= \sin^2\alpha\ \sigma_x + \cos^2\alpha\ \sigma_y + 2\sin\alpha\cos\alpha\ \sigma_{xy} \\
\sigma_\zeta &= \sigma_z \\
\sigma_{\xi\eta} &= \sin\alpha\cos\alpha\ \sigma_x - \sin\alpha\cos\alpha\ \sigma_y + (\cos^2\alpha - \sin^2\alpha)\sigma_{xy} \\
\sigma_{\eta\zeta} &= \cos\alpha\ \sigma_{yz} + \sin\alpha\ \sigma_{xz} \\
\sigma_{\xi\zeta} &= -\sin\alpha\ \sigma_{yz} + \cos\alpha\ \sigma_{xz}
\end{aligned} \tag{8}$$

Similarly, for the strain components referred to the ξ, η, ζ system written in terms of the known components of the x, y, z system,

$$\epsilon_\xi = \cos^2\alpha \ \epsilon_x + \sin^2\alpha \ \epsilon_y - \sin\alpha \cos\alpha \ \epsilon_{xy}$$

$$\epsilon_\eta = \sin^2\alpha \ \epsilon_x + \cos^2\alpha \ \epsilon_y + \sin\alpha \cos\alpha \ \epsilon_{xy}$$

$$\epsilon_\zeta = \epsilon_z$$

$$\epsilon_{\xi\eta} = 2 \sin\alpha \cos\alpha \ \epsilon_x - \alpha \sin\alpha \cos\alpha \ \epsilon_y + (\cos^2\alpha - \sin^2\alpha)\epsilon_{xy} \tag{9}$$

$$\epsilon_{\eta\zeta} = \cos\alpha \ \epsilon_{yz} + \sin\alpha \ \epsilon_{xz}$$

$$\epsilon_{\xi\zeta} = - \sin\alpha \ \epsilon_{yz} + \cos\alpha \ \epsilon_{xz} \quad .$$

In like manner, a similar set of equations could be written for the components of stress and strain in the x, y, z system in terms of the components in the ξ, η, ζ system. These will take the following form:

$$\sigma_x = \cos^2\alpha \ \sigma_\xi + \sin^2\alpha \ \sigma_\eta + 2 \sin\alpha \cos\alpha \ \sigma_{\xi\eta}$$

$$\sigma_y = \sin^2\alpha \ \sigma_\xi + \cos^2\alpha \ \sigma_\eta - 2 \sin\alpha \cos\alpha \ \sigma_{\xi\eta}$$

$$\sigma_z = \sigma_\zeta$$

$$\sigma_{xy} = - \sin\alpha \cos\alpha \ \sigma_\xi + \sin\alpha \cos\alpha \ \sigma_\eta + (\cos^2\alpha - \sin^2\alpha)\sigma_{\xi\eta} \tag{8}^1$$

$$\sigma_{yz} = \cos\alpha \ \sigma_{\eta\zeta} - \sin\alpha \ \sigma_{\xi\zeta}$$

$$\sigma_{xz} = \sin\alpha \ \sigma_{\eta\zeta} + \cos\alpha \ \sigma_{\xi\zeta}$$

Similarly, for the strain components referred to the x, y, z system written in terms of the known components of the ξ, η, ζ system,

$$\epsilon_x = \cos^2\alpha\,\epsilon_\xi + \sin^2\alpha\,\epsilon_\eta + \sin\alpha\cos\alpha\,\epsilon_{\xi\eta}$$

$$\epsilon_y = \sin^2\alpha\,\epsilon_\xi + \cos^2\alpha\,\epsilon_\eta - \sin\alpha\cos\alpha\,\epsilon_{\xi\eta}$$

$$\epsilon_z = \epsilon_\zeta$$

$$(9)^{1}$$

$$\epsilon_{xy} = -2\sin\alpha\cos\alpha\,\epsilon_\xi + 2\sin\alpha\cos\alpha\,\epsilon_\eta + (\cos^2\alpha - \sin^2\alpha)\epsilon_{\xi\eta}$$

$$\epsilon_{yz} = \cos\alpha\,\epsilon_\zeta - \sin\alpha\,\epsilon_{\xi\zeta}$$

$$\epsilon_{xz} = \sin\alpha\,\epsilon_{\eta\zeta} + \cos\alpha\,\epsilon_{\xi\zeta} \quad .$$

The generalized form of Hooke's law may be written using the stresses as independent variables with respect to any desired coordinate axes; for example, for the ξ, η, ζ system

$$\begin{aligned}
\epsilon_\xi &= a_{11}\sigma_\xi + a_{12}\sigma_\eta + a_{13}\sigma_\zeta + a_{14}\sigma_{\xi\eta} + a_{15}\sigma_{\xi\zeta} + a_{16}\sigma_{\eta\zeta} \\
\epsilon_\eta &= a_{21}\sigma_\xi + a_{22}\sigma_\eta + a_{23}\sigma_\zeta + a_{24}\sigma_{\xi\eta} + a_{25}\sigma_{\xi\zeta} + a_{26}\sigma_{\eta\zeta} \\
\epsilon_\zeta &= a_{31}\sigma_\xi + a_{32}\sigma_\eta + a_{33}\sigma_\zeta + a_{34}\sigma_{\xi\eta} + a_{35}\sigma_{\xi\zeta} + a_{36}\sigma_{\eta\zeta} \\
\epsilon_{\xi\eta} &= a_{41}\sigma_\xi + a_{42}\sigma_\eta + a_{43}\sigma_\zeta + a_{44}\sigma_{\xi\eta} + a_{45}\sigma_{\xi\zeta} + a_{46}\sigma_{\eta\zeta} \qquad (10) \\
\epsilon_{\xi\zeta} &= a_{51}\sigma_\xi + a_{52}\sigma_\eta + a_{53}\sigma_\zeta + a_{54}\sigma_{\xi\eta} + a_{55}\sigma_{\xi\zeta} + a_{56}\sigma_{\eta\zeta} \\
\epsilon_{\eta\zeta} &= a_{61}\sigma_\xi + a_{62}\sigma_\eta + a_{63}\sigma_\zeta + a_{64}\sigma_{\xi\eta} + a_{65}\sigma_{\xi\zeta} + a_{66}\sigma_{\eta\zeta} \quad .
\end{aligned}$$

The values of the various a_{ij} coefficients of Eqs. (10) will be determined by using the procedure outlined in Ref. 1. The coefficient a_{14} will be determined to illustrate the procedure.

The stress $\sigma_{\xi\eta}$ may be considered known, and σ_ξ, σ_η, σ_ζ, $\sigma_{\zeta\xi}$, and $\sigma_{\eta\zeta}$ are set equal to zero. Then it follows from Eqs. (10) that

$$a_{14} = \frac{\epsilon_\xi}{\sigma_{\xi\eta}} \quad ; \tag{11}$$

however, from Eq. (9)

$$\epsilon_\xi = \cos^2\alpha\ \epsilon_x + \sin^2\alpha\ \epsilon_y - \sin\alpha\cos\alpha\ \epsilon_{xy} \quad . \tag{12}$$

Next, recalling that the single ply under consideration is orthotropic with the x, y, z axes as the axes of orthotropy, the equations relating strain and stress can be written by referring to Eqs. (7).

$$\begin{aligned}
\epsilon_x &= \frac{\sigma_x}{E_x} + \frac{\sigma_y}{F_{xy}} + \frac{\sigma_z}{F_{xz}} \\
\epsilon_y &= \frac{\sigma_x}{F_{xy}} + \frac{\sigma_y}{E_y} + \frac{\sigma_z}{F_{yz}} \\
\epsilon_z &= \frac{\sigma_x}{F_{xz}} + \frac{\sigma_y}{F_{yz}} + \frac{\sigma_z}{E_z} \\
\epsilon_{xy} &= \frac{\sigma_{xy}}{G_{xy}}\ , \quad \epsilon_{xz} = \frac{\sigma_{xz}}{G_{xz}}\ , \quad \epsilon_{yz} = \frac{\sigma_{yz}}{G_{yz}}
\end{aligned} \tag{13}$$

Now, referring to Eqs. (8)[1], σ_x, σ_y, σ_z, σ_{xy}, σ_{xz}, and σ_{yz} can be expressed in terms of the non-zero $\sigma_{\xi\eta}$:

$$\sigma_x = 2 \sin \alpha \cos \alpha \; \sigma_{\xi\eta} \; , \quad \sigma_y = -2 \sin \alpha \cos \alpha \; \sigma_{\xi\eta}$$

$$\sigma_{xy} = (\cos^2\alpha - \sin^2\alpha)\sigma_{\xi\eta} \; , \quad \sigma_z = \sigma_{yz} = \sigma_{xz} = 0 \quad . \tag{14}$$

Substituting Eqs. (14) into Eqs. (13) gives

$$\epsilon_x = 2 \sin \alpha \cos \alpha \left(\frac{1}{E_x} - \frac{1}{F_{xy}}\right)\sigma_{\xi\eta}$$

$$\epsilon_y = 2 \sin \alpha \cos \alpha \left(\frac{-1}{E_y} + \frac{1}{F_{xy}}\right)\sigma_{\xi\eta}$$

$$\epsilon_z = 2 \sin \alpha \cos \alpha \left(\frac{1}{F_{xz}} - \frac{1}{F_{yz}}\right)\sigma_{\xi\eta} \tag{15}$$

$$\epsilon_{xy} = \frac{(\cos^2\alpha - \sin^2\alpha)}{G_{xy}} \sigma_{\xi\eta}$$

$$\epsilon_{xz} = \epsilon_{yz} = 0 \quad .$$

Now, substituting Eqs. (15) into Eqs. (12),

$$\epsilon_\xi = \sigma_{\xi\eta} \left[\frac{2 \cos^3\alpha \sin \alpha}{E_x} - \frac{2 \sin^3\alpha \cos \alpha}{E_y} - \sin \alpha \cos \alpha \left(\frac{\cos^2\alpha - \sin^2\alpha}{G_{xy}}\right) + \frac{2}{F_{xy}}(- \cos^3\alpha \sin \alpha + \sin^3\alpha \cos \alpha)\right] \quad .$$

Therefore, a_{14} becomes

$$a_{14} = \frac{\epsilon_\xi}{\sigma_{\xi\eta}} = \frac{2\cos^3\alpha\sin\alpha}{E_x} - \frac{2\sin^3\alpha\cos\alpha}{E_y}$$

$$- \cos\alpha\cos\alpha\,(\cos^2\alpha - \sin^2\alpha)\left(\frac{1}{G_{xy}} + \frac{2}{F_{xy}}\right) .$$

By identical procedures one may obtain

$$a_{15} = a_{16} = a_{26} = a_{35} = a_{36} = a_{45} = a_{46} = a_{51} = a_{52} = a_{53}$$

$$a_{25} = a_{54} = a_{61} = a_{62} = a_{63} = a_{64} = 0$$

$$a_{41} = a_{14}$$

$$a_{11} = \left[\frac{\cos^4\alpha}{E_x} + \frac{\sin^4\alpha}{E_y} + \cos^2\alpha\sin^2\alpha\left(\frac{1}{G_{xy}} + \frac{2}{F_{xy}}\right)\right]$$

$$a_{22} = \left[\frac{\sin^4\alpha}{E_x} + \frac{\cos^4\alpha}{E_y} + \cos^2\alpha\sin^2\alpha\left(\frac{1}{G_{xy}} + \frac{2}{F_{xy}}\right)\right]$$

$$a_{33} = \frac{1}{E_z}$$

$$a_{44} = 4\cos^2\alpha\sin^2\alpha\left[\frac{1}{E_x} + \frac{1}{E_y} - \frac{2}{F_{xy}}\right] + \frac{(\cos^2\alpha - \sin^2\alpha)^2}{G_{xy}}$$

$$a_{55} = \frac{\cos^2\alpha}{G_{xz}} + \frac{\sin^2\alpha}{G_{yz}} \qquad (16)$$

$$a_{66} = \frac{\cos^2\alpha}{G_{yz}} + \frac{\sin^2\alpha}{G_{xz}}$$

$$a_{12} = a_{21} = \sin^2\alpha \cos^2\alpha \left[\frac{1}{E_x} + \frac{1}{E_y} - \frac{1}{G_{xy}}\right] + \frac{(\sin^4\alpha + \cos^4\alpha)}{F_{xy}}$$

$$a_{13} = a_{31} = \frac{\cos^2\alpha}{F_{xz}} + \frac{\sin^2\alpha}{F_{yz}}$$

$$a_{23} = a_{32} = \frac{\sin^2\alpha}{F_{xz}} + \frac{\cos^2\alpha}{F_{yz}}$$

$$a_{24} = a_{42} = \frac{2 \sin^3\alpha \cos \alpha}{E_x} - \frac{2 \cos^3\alpha \sin \alpha}{E_y} + \sin \alpha \cos \alpha(\cos^2\alpha - \sin^2\alpha)\left(\frac{1}{G_{xy}} + \frac{2}{F_{xy}}\right)$$

$$a_{34} = a_{43} = 2 \cos \alpha \sin \alpha \left[\frac{1}{F_{xz}} - \frac{1}{F_{yz}}\right]$$

$$a_{56} = a_{65} = \cos \alpha \sin \alpha \left[\frac{1}{G_{xz}} - \frac{1}{G_{yz}}\right] \quad . \qquad (16)$$

Equations (10) and Eqs. (16) now allow the properties of an orthotropic sheet to be predicted in any direction ξ, η, ζ at an angle α with the cord direction. If the results of Eqs. (16) are used, Eqs. (10) take the form

$$\epsilon_\xi = a_{11} \sigma_\xi + a_{12} \sigma_\eta + a_{13} \sigma_\zeta + a_{14} \sigma_{\xi\eta} \qquad (a)$$

$$(17)$$

$$\epsilon_\eta = a_{12} \sigma_\xi + a_{22} \sigma_\eta + a_{23} \sigma_\zeta + a_{24} \sigma_{\xi\eta} \qquad (b)$$

$$\epsilon_\zeta = a_{13}\,\sigma_\xi + a_{23}\,\sigma_\eta + a_{33}\,\sigma_\zeta + a_{34}\,\sigma_{\xi\eta} \quad (c)$$

$$\epsilon_{\xi\eta} = a_{14}\,\sigma_\xi + a_{24}\,\sigma_\eta + a_{34}\,\sigma_\zeta + a_{44}\,\sigma_{\xi\eta} \quad (d)$$

$$\epsilon_{\xi\zeta} = a_{55}\,\sigma_{\xi\zeta} + a_{56}\,\sigma_{\eta\zeta} \quad (e) \qquad (17)$$

$$\epsilon_{\eta\zeta} = a_{65}\,\sigma_{\xi\zeta} + a_{66}\,\sigma_{\eta\zeta} \quad (f)$$

Note that Eqs. (17e,f) are independent of Eqs. (17a-d). One concludes from this that there are no coupling effects between the stresses σ_ξ, σ_η, σ_ζ, and $\sigma_{\xi\eta}$ and the shearing strains in the ξ - ζ plane and the η - ζ plane. In like manner, there are no coupling effects between the shearing stresses $\sigma_{\xi\zeta}$ and $\sigma_{\eta\zeta}$ and the strains ϵ_ξ, ϵ_η, ϵ_ζ, and $\epsilon_{\xi\eta}$.

To determine the relationship between the stresses and strains in Eqs. (17a-d) consider now a sheet of the type shown in Fig. 2 but inclined at some angle α to the orthotropic sheet axes x and y. This is illustrated in Fig. 3. Imagine that is is desired to extend this sheet in the ξ, η, and ζ directions by means of normal stresses only. Because the orthotropic axes x and y do not coincide with the ξ, η axes, this is not possible; distortions $\epsilon_{\xi\eta}$ will inevitably accompany the application of any set of normal stresses σ_ξ, σ_η, and σ_ζ. However, because the ζ axis coincides with the third orthotropic axis, the z axis, there are no accompanying shear strains $\epsilon_{\xi\zeta}$ or $\epsilon_{\eta\zeta}$. This is also seen in (17e,f). Because there are shearing distortions in the ξ - η plane, one must admit the existence of shearing stresses $\sigma_{\zeta\eta}$ as necessary for the distortionless extension of an element such as that represented by Fig. 3,

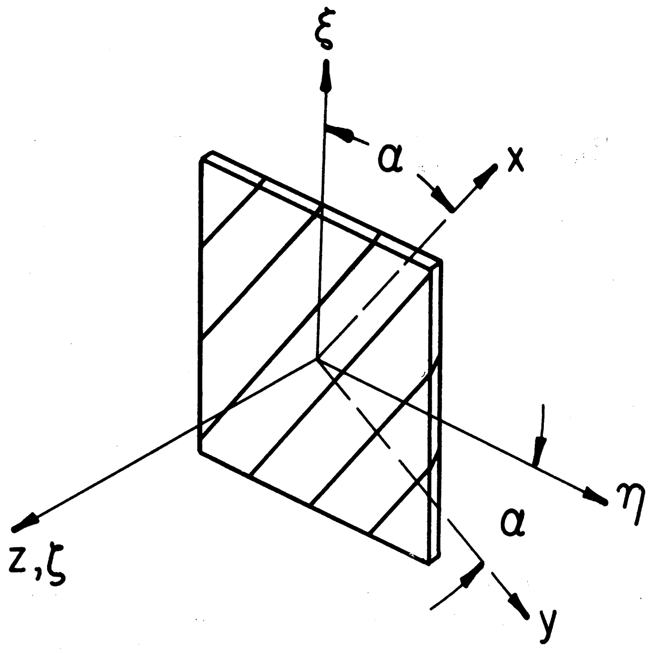

Fig. 3. Single ply of cord imbedded in rubber at angle α to the vertical ξ - axis.

where the reference axes do not coincide with the orthotropic axes. With this provision in mind, one may go directly to (17a-d) and presume no shear strain $\epsilon_{\xi\eta}$, that is, a $\sigma_{\xi\eta}$ will exist to prevent the shear distortion when the normal stresses are applied to the structure. This will result in having only normal strains ϵ_ξ, ϵ_η, and ϵ_ζ. For this condition Eqs. (17) become

$$\begin{aligned} \epsilon_\xi &= a_{11}\,\sigma_\xi + a_{12}\,\sigma_\eta + a_{13}\,\sigma_\zeta + a_{14}\,\sigma_{\xi\eta} \\ \epsilon_\eta &= a_{12}\,\sigma_\xi + a_{22}\,\sigma_\eta + a_{23}\,\sigma_\zeta + a_{24}\,\sigma_{\xi\eta} \\ \epsilon_\zeta &= a_{13}\,\sigma_\xi + a_{23}\,\sigma_\eta + a_{33}\,\sigma_\zeta + a_{34}\,\sigma_{\xi\eta} \\ 0 &= a_{14}\,\sigma_{\xi\eta} + a_{24}\,\sigma_\eta + a_{34}\,\sigma_\zeta + a_{44}\,\sigma_{\xi\eta} \quad . \end{aligned} \tag{18}$$

We now have four equations and four unknowns (ϵ_ξ, ϵ_η, ϵ_ζ, $\sigma_{\xi\eta}$). Solving the last of Eqs. (18) for $\sigma_{\xi\eta}$, one obtains

$$\sigma_{\xi\eta} = -\frac{a_{41}}{a_{44}}\sigma_\xi - \frac{a_{42}}{a_{44}}\sigma_\eta - \frac{a_{43}}{a_{44}}\sigma_\zeta \quad . \tag{19}$$

This is the stress that must be supplied from some external source to obtain distortionless extension. The use of this stress in the first three of Eqs. (18) yields

$$\begin{aligned}
\epsilon_\xi &= \sigma_\xi\left(a_{11} - \frac{a_{14}\,a_{14}}{a_{44}}\right) + \sigma_\eta\left(a_{12} - \frac{a_{14}\,a_{12}}{a_{44}}\right) + \sigma_\zeta\left(a_{13} - \frac{a_{14}\,a_{34}}{a_{44}}\right) \\
\epsilon_\eta &= \sigma_\xi\left(a_{12} - \frac{a_{24}\,a_{14}}{a_{44}}\right) + \sigma_\eta\left(a_{22} - \frac{a_{24}\,a_{24}}{a_{44}}\right) + \sigma_\zeta\left(a_{23} - \frac{a_{24}\,a_{34}}{a_{44}}\right) \qquad (20) \\
\epsilon_\zeta &= \sigma_\xi\left(a_{13} - \frac{a_{14}\,a_{34}}{a_{44}}\right) + \sigma_\eta\left(a_{23} - \frac{a_{34}\,a_{24}}{a_{44}}\right) + \sigma_\zeta\left(a_{33} - \frac{a_{34}\,a_{34}}{a_{44}}\right) \quad .
\end{aligned}$$

Comparison of Eqs. (20) with Eqs. (7) shows that

$$\begin{aligned}
\frac{1}{E_\xi} &= \left(a_{11} - \frac{a_{14}^2}{a_{44}}\right); \quad \frac{1}{E_\eta} = \left(a_{22} - \frac{a_{24}^2}{a_{44}}\right); \quad \frac{1}{E_\zeta} = \left(a_{33} - \frac{a_{34}^2}{a_{44}}\right) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (21) \\
\frac{1}{F_{\xi\eta}} &= \left(a_{12} - \frac{a_{14}\,a_{24}}{a_{44}}\right); \quad \frac{1}{F_{\xi\zeta}} = \left(a_{13} - \frac{a_{14}\,a_{34}}{a_{44}}\right); \quad \frac{1}{F_{\eta\zeta}} = \left(a_{23} - \frac{a_{24}\,a_{34}}{a_{44}}\right) \quad .
\end{aligned}$$

Now consider the shear moduli $G_{\xi\eta}$, $G_{\xi\zeta}$, and $G_{\eta\zeta}$, which relate the shearing strains and stresses. Note again that this entire development is based on the assumption that all cords are in a state of tension. This is a necessary assumption to assure that the sheet properties E_x, E_y, etc., which are assumed to be known, are constants for all plies involved.

To obtain $G_{\xi\eta}$ associated with all cords in tension, one must postulate an extension-free distortion or a pure strain $\epsilon_{\xi\eta}$, which occurs as the result of the proper application of the stresses σ_ξ, σ_η, σ_ζ, and $\sigma_{\xi\eta}$. It has already been mentioned that the application of either or both of the stresses $\sigma_{\xi\zeta}$ and $\sigma_{\eta\zeta}$ does not affect the strain $\epsilon_{\xi\eta}$. Equations (17) become, when applied to this case,

$$\begin{aligned}
0 &= a_{11}\,\sigma_\xi + a_{12}\,\sigma_\eta + a_{13}\,\sigma_\zeta + a_{14}\,\sigma_{\xi\eta} \\
0 &= a_{12}\,\sigma_\xi + a_{22}\,\sigma_\eta + a_{23}\,\sigma_\zeta + a_{24}\,\sigma_{\xi\eta} \\
0 &= a_{13}\,\sigma_\xi + a_{23}\,\sigma_\eta + a_{33}\,\sigma_\zeta + a_{34}\,\sigma_{\xi\eta} \\
\epsilon_{\xi\eta} &= a_{14}\,\sigma_\xi + a_{24}\,\sigma_\eta + a_{34}\,\sigma_\zeta + a_{44}\,\sigma_{\xi\eta}\quad .
\end{aligned} \tag{22}$$

Solving the first three of Eqs. (22) for σ_ξ, σ_η, and σ_ζ gives

$$\begin{aligned}
\sigma_\xi &= \sigma_{\xi\eta}\left[-a_{14}(a_{22}a_{33}-a_{23}^2) - a_{12}(a_{34}a_{23}-a_{24}a_{33}) + a_{13}(a_{22}a_{34}-a_{24}a_{23})\right]\Big/D \\
\sigma_\eta &= \sigma_{\xi\eta}\left[a_{11}(a_{23}a_{34}-a_{24}a_{33}) + a_{14}(a_{21}a_{33}-a_{23}a_{31}) + a_{13}(a_{24}a_{31}-a_{21}a_{34})\right]\Big/D \\
\sigma_\zeta &= \sigma_{\xi\eta}\left[a_{11}(a_{23}a_{24}-a_{22}a_{34}) - a_{12}(a_{13}a_{24}-a_{12}a_{34}) + a_{14}(a_{12}a_{23}-a_{22}a_{13})\right]\Big/D,
\end{aligned}$$

$$\text{where } D = \left[a_{11}(a_{22}a_{33}-a_{23}^2) - a_{12}(a_{12}a_{33}-a_{23}a_{13}) + a_{13}(a_{12}a_{23}-a_{22}a_{13})\right]. \tag{23}$$

Equations (23) represent the normal stress necessary for an extensionless distortion. Substituting these into the last of Eqs. (22) gives

$$\frac{1}{G_{\xi\eta}} = \frac{\epsilon_{\xi\eta}}{\sigma_{\xi\eta}} = \Bigg\{\Bigg[a_{14}^2(a_{23}^2-a_{22}a_{33}) + a_{24}^2(a_{13}^2-a_{11}a_{33}) + a_{34}^2(a_{12}^2-a_{11}a_{22})$$

$$+ 2a_{14}a_{24}(a_{12}a_{33}-a_{13}a_{24}) + 2a_{24}a_{34}(a_{11}a_{23}-a_{12}a_{13})\Bigg]\Bigg/\Bigg[a_{33}(a_{11}a_{22}-a_{12}^2)$$

$$+ a_{23}(a_{12}a_{13}-a_{11}a_{23}) + a_{13}(a_{12}a_{23}-a_{13}a_{22})\Bigg]\Bigg\} + a_{44} \quad . \tag{24}$$

The last two of Eqs. (17) will now be considered in an effort to obtain $G_{\xi\zeta}$ and $G_{\eta\zeta}$. In the first case, a known $\sigma_{\xi\zeta}$ will be applied and it will be assumed that the stresses will so distribute themselves that $\epsilon_{\eta\zeta}$ will be zero. For this case Eqs. (17) become

$$\begin{aligned} \epsilon_{\xi\zeta} &= a_{55}\,\sigma_{\xi\zeta} + a_{56}\,\sigma_{\eta\zeta} \\ 0 &= a_{56}\,\sigma_{\xi\zeta} + a_{66}\,\sigma_{\eta\zeta} \quad . \end{aligned} \tag{25}$$

Solving the second of Eqs. (25) for $\sigma_{\eta\zeta}$, one obtains the shear stress $\sigma_{\eta\zeta}$, which must be supplied from some external source in order to have a pure shear strain $\epsilon_{\xi\zeta}$ resulting from the application of only the sheer stress $\sigma_{\xi\zeta}$:

$$\sigma_{\eta\zeta} = -\frac{a_{65}}{a_{66}}\,\sigma_{\xi\zeta} \quad . \tag{26}$$

Substituting Eq. (26) into the first of Eqs. (25) gives

$$\frac{1}{G_{\xi\zeta}} = \frac{\epsilon_{\xi\zeta}}{\sigma_{\xi\zeta}} = \left(a_{55} - \frac{a_{56}^2}{a_{66}}\right) \quad . \tag{27}$$

For a second case, a known $\sigma_{\eta\xi}$ is applied and it is assumed that the stresses will be so distributed that $\epsilon_{\xi\zeta}$ is zero. For this case Eqs. (17) become

$$\begin{aligned} 0 &= a_{55}\, \sigma_{\xi\zeta} + a_{56}\, \sigma_{\eta\zeta} \\ \epsilon_{\eta\zeta} &= a_{56}\, \sigma_{\xi\zeta} + a_{66}\, \sigma_{\eta\zeta} \quad . \end{aligned} \tag{28}$$

Solving the second of Eqs. (28) for $\sigma_{\xi\zeta}$, one obtains the shear stress $\sigma_{\xi\zeta}$ which must be supplied from some external source in order to have a pure shear strain $\epsilon_{\eta\zeta}$ resulting from the application of the shear stress $\sigma_{\eta\zeta}$:

$$\sigma_{\xi\zeta} = -\frac{a_{56}}{a_{55}}\, \sigma_{\eta\zeta} \quad . \tag{29}$$

Substituting Eq. (29) into the second of Eqs. (28) gives

$$\frac{1}{G_{\eta\zeta}} = \frac{\epsilon_{\eta\zeta}}{\sigma_{\eta\zeta}} = \left(a_{66} - \frac{a_{56}^2}{a_{55}}\right) \quad . \tag{30}$$

Equations (21), (24), (27), and (30) now represent a set of equations from which one can calculate the elastic properties of an orthotropic sheet in any direction ξ, η, ζ at an angle α with the cord direction x, y. It should be made clear at this point that the development of these equations has been obtained directly from the analysis of a single ply of material and, it has been observed by comparing these with Eqs. (7), that they are the same elastic properties that relate the stress and strain of the two-ply composite body. Hence, the moduli of the two-ply combination of Fig. 2 are the same as those of the single sheet of Fig. 3 when referred to the ξ, η, ζ axis. Of course, these are the same only so long as the two plies are exactly the same in physical make-up and so long as the single sheet has been provided with the "extra" stresses expressed by Eqs. (19), (23), (26), and (29).

It is appropriate at this point to compare the form of the elastic properties as developed in this report to those of Ref. 1. First of all, E_ξ, E_η, and $F_{\xi\eta}$ are exactly the same for the two developments. This implies that the relationships between normal stresses and strains in the ξ - η plane are unaffected by stresses and strains in the ζ direction. However, note that the "extra" stresses $\sigma_{\xi\eta}$ (Eq. (19) of this report and Eqs. (14) of Ref. 1) are not the same. Hence, the magnitude of the additional stress necessary for distortionless extension is different for the two developments. The magnitude of this difference is discussed in a subsequent portion of this report.

The next comparison to be made is that of the shear modulus, $G_{\xi\eta}$. A comparison of Eq. (17) of Ref. 1 to Eqs. (24) of this development shows that there is a considerable difference in the two expressions for $G_{\xi\eta}$. The magnitude of this difference is discussed in a later part of this report. Note, also, in conjunction with $G_{\xi\eta}$ that the "extra" stress σ_ξ and σ_η necessary for extensionless distortion are different in the two developments. These differences imply that the "extra" stresses necessary for pure distortion are affected by the stresses and strains in the third direction. These "extra" stresses are the so-called "interply shear stresses" that have been discussed by this group in previous reports (Refs. 1, 4, etc.).

In addition to the changes noted in the relationships of stress and strain in the ξ - η plane, one can now predict the relationships of stress and strain in the n - ζ and ξ - ζ planes. Also, note that additional interply stresses are present when one includes the stresses and strains in the ζ direction.

IV. CALCULATION OF ELASTIC PROPERTIES

Two topics are discussed in this section, supported by evidence from actual calculations. First a comparison is made between those elastic properties which are common to this report and to Ref. 1. Second, typical calculations are presented for those additional properties that have been developed in this report.

Equations (21), (24), etc., show that in order to calculate the elastic properties of the orthotropic sheet in a non-orthotropic direction, one must first calculate the a_{ij} coefficients. However, in order to calculate the a_{ij}'s one must have some knowledge of the basic sheet properties E_x, E_y, F_{xy}, G_{xy}, E_z, F_{xz}, F_{yz}, G_{xz}, and G_{yz}. Previous work has been done in obtaining the first four of these quantities, and some typical values from Refs. 2 and 6 are used in the calculations which follow. Because the remaining five sheet properties have not been measured previously, there is no experimental information on their magnitudes. However, the approximate value of these quantities can be estimated from the geometry of construction of a single ply.

By referring to Fig. 2, one can reasonably assume that E_z will be similar in magnitude to E_y, because, like E_y, E_z is a ratio of stress to strain in a direction perpendicular to the cords. In a like manner, one can reasonably assume that F_{xz} will be similar in magnitude to F_{xy}, because, like F_{xy}, F_{xz} is a ratio of either stress in the cord direction and strain in the direction perpendicular to the cords or vice-versa.

On the other hand, F_{yz} is assumed to be similar in magnitude to $2E_y$. This assumption is based on the fact that F_{yz} is a ratio of stress in a direction perpendicular to the cords to strain in a direction perpendicular to the cords but also perpendicular to the stress. Hence it will behave like the quantity E_1/μ_1, where E_1 refers to the extension modulus in a direction perpendicular to the cords, while μ_1 refers to the Poisson's ratio in a direction perpendicular to the cords. This in turn implies that F_{yz} is similar in magnitude to the ratio of $E_Y/\mu_R \simeq 2E_Y$.

It is also reasonable to assume that G_{xz} and G_{yz} are similar in magnitude to G_{xy}, since they are all ratios of shear stress and shear strain taking place primarily in the material in which the cords are imbedded.

These sheet properties are only estimates, with no experimental confirmation. However, they should serve adequately in the following calculations, since the calculations are only intended to illustrate some of the differences in the development of this report as compared to that in Ref. 1. With this in mind, the following typical values of sheet properties used in the calculations of this section are listed below:

$$E_x = 223500 \text{ psi} \qquad F_{xy} = -405000 \text{ psi}$$

$$E_y = 970 \text{ psi} \qquad F_{xz} = -400000 \text{ psi}$$

$$E_z = 1000 \text{ psi} \qquad F_{yz} = -2000 \text{ psi}$$

$$G_{xy} = 308 \text{ psi}$$

$$G_{xz} = 300 \text{ psi}$$

$$G_{yz} = 350 \text{ psi}$$

$$\alpha = 0°, 15°, 30°, 45°, 60°, 75°, 90°.$$

As mentioned previously, there is no difference between the E_ξ, E_η, and $F_{\xi\eta}$ of this report and those of Ref. 1. Therefore no discussion of these properties is necessary except to compare the interply shear stresses required for the distortionless extension necessary in these three properties. To do this a comparison must be made between the $\sigma_{\xi\eta}$ obtained from Eq. (19) of this report and that obtained from Eq. (14) of Ref. 1. The results of this comparison are shown in Table II and illustrated in Fig. 4. The difference in the two formulations can be attributed to the significant effect of σ_ζ, represented by the third term on the right-hand side of Eq. (19).

In comparing the $G_{\xi\eta}$ of this report (Eq. (24)) and that of Ref. 1 (Eq. (17)) observe that there is a considerable difference in the two expressions. This is because the interply stresses necessary for extensionless distortion are different in the two developments. However, Table I shows that the numerical value of $G_{\xi\eta}$ is altered very little by the inclusion of the effects of the ζ direction. This is further borne out by observing that the normal interply stresses are unaltered numerically. The additional normal interply stress σ_ζ, illustrated in Fig. 5, is small and contributes very little to the value of $G_{\xi\eta}$.

Figure 5 is a graphical representation of $F_{\xi\zeta}$ and $F_{\eta\zeta}$, and it is interesting to note that these values change sign over the range of cord angles used in the calculations. This implies that it is possible, with the correct combination of cord angle and sheet properties, to apply an extensional stress

TABLE II

NUMERICAL VALUES OF ELASTIC PROPERTIES AND INTERPLY STRESSES

Property	α°						
	0	15	30	45	60	75	90
$\sigma_{\xi\eta}$—Eq. (19); this report*	0	9.81	28.8	50.8	28.8	9.81	0
$\sigma_{\xi\eta}$—Eqs. (14); Ref. 1*	0	19.0	55.8	98.6	55.8	19.0	0
$\sigma_{\xi}/\sigma_{\xi\eta}$—Eqs. (23); this report	0	3.66	1.72	1.00	.58	.27	0
$\sigma_{\xi}/\sigma_{\xi\eta}$—Eqs. (16); Ref. 1	0	3.65	1.72	1.00	.58	.26	0
$\sigma_{\eta}/\sigma_{\xi\eta}$—Eqs. (23); this report	0	.27	.58	1.00	1.72	3.66	0
$\sigma_{\eta}/\sigma_{\xi\eta}$—Eqs. (16); Ref. 1	0	.26	.58	1.00	1.72	3.65	0
$\sigma_{\zeta}/\sigma_{\xi\eta}$—Eqs. (23); this report	0	.0079	.0047	.0040	.0046	.0079	0
$\sigma_{\eta\zeta}/\sigma_{\zeta\xi}$—Eq. (26); this report	0	-.0412	-.0693	-.0769	-.0642	-.0361	0
$\sigma_{\xi\zeta}/\sigma_{\eta\zeta}$—Eq. (29); this report	0	-.0361	-.0642	-.0769	-.0693	-.0412	0
E_{ξ}—Eqs. (21); this report and Eq. (15) of Ref. 1	223500	90380	7112	1225	842	942	970
E_{η}—Eqs. (21); this report and Eq. (15) of Ref. 1	970	924	842	1225	7112	90380	223500
E_{ζ}—Eqs. (21); this report	1000	1023	1131	1312	1131	1023	1000
$F_{\xi\eta}$—Eqs. (21); this report and Eq. (15) of Ref. 1	-405000	-12480	-2519	- 1239	-2519	-12480	-405000
$F_{\xi\zeta}$—Eqs. (21); this report	-400000	+31300	+8149	-172416	-2600	- 2053	- 2000
$F_{\eta\zeta}$—Eqs. (21); this report	- 2000	- 2053	-2600	-172416	+8149	+31300	-400000
$G_{\xi\eta}$—Eq. (24); this report	308	14221	42048	55962	42048	14221	308
$G_{\xi\eta}$—Eq. (17); Ref. 1	308	14210	42020	55920	42020	14210	308
$G_{\xi\zeta}$—Eq. (27); this report	300	303	312	325	337	347	350
$G_{\eta\zeta}$—Eq. (30); this report	350	347	337	325	312	303	300

*Calculated with normal stresses of 100.

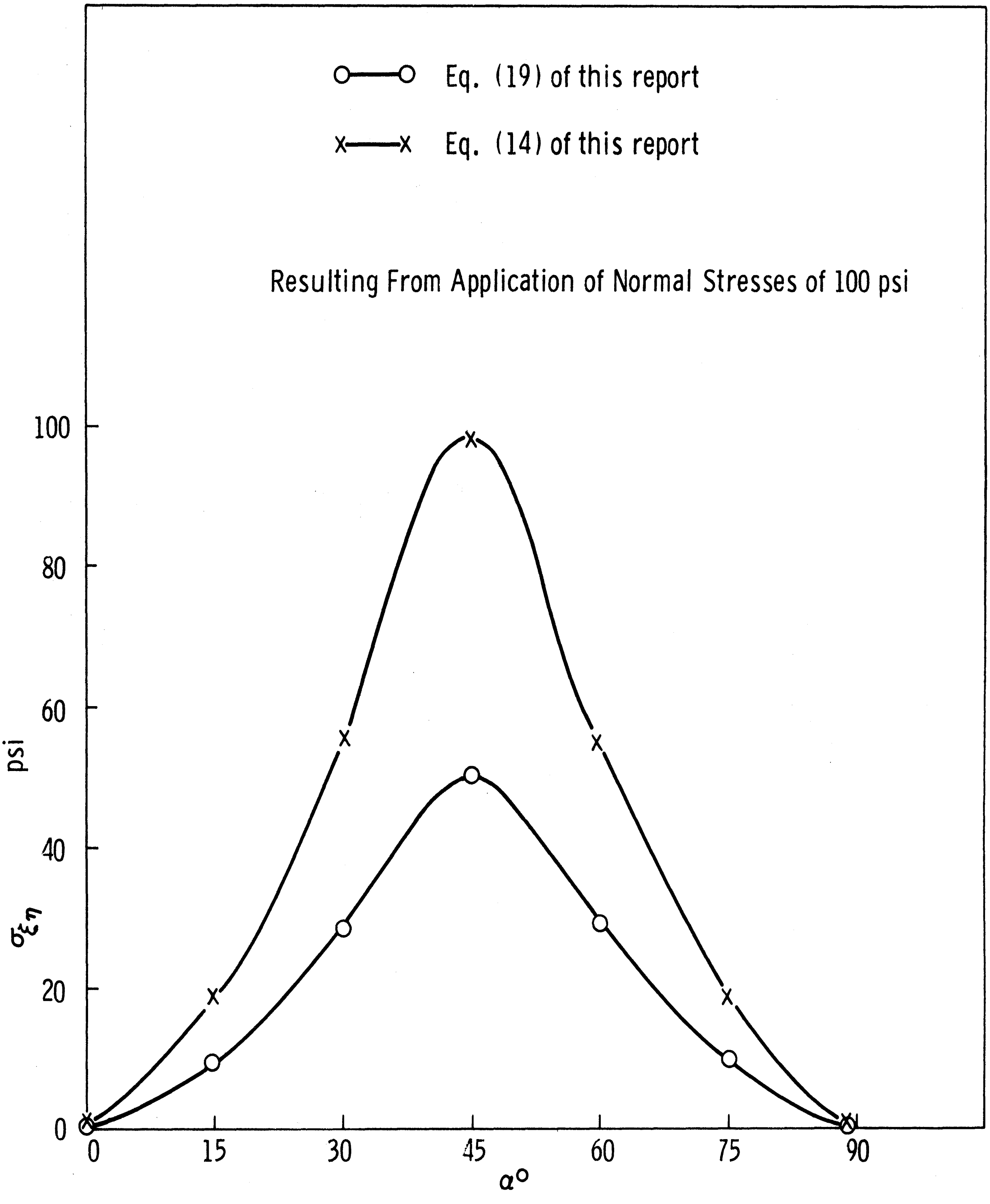

Fig. 4. Comparison of interply shear stress $\sigma'_{\xi\eta}$ of this report and that of Ref. 1.

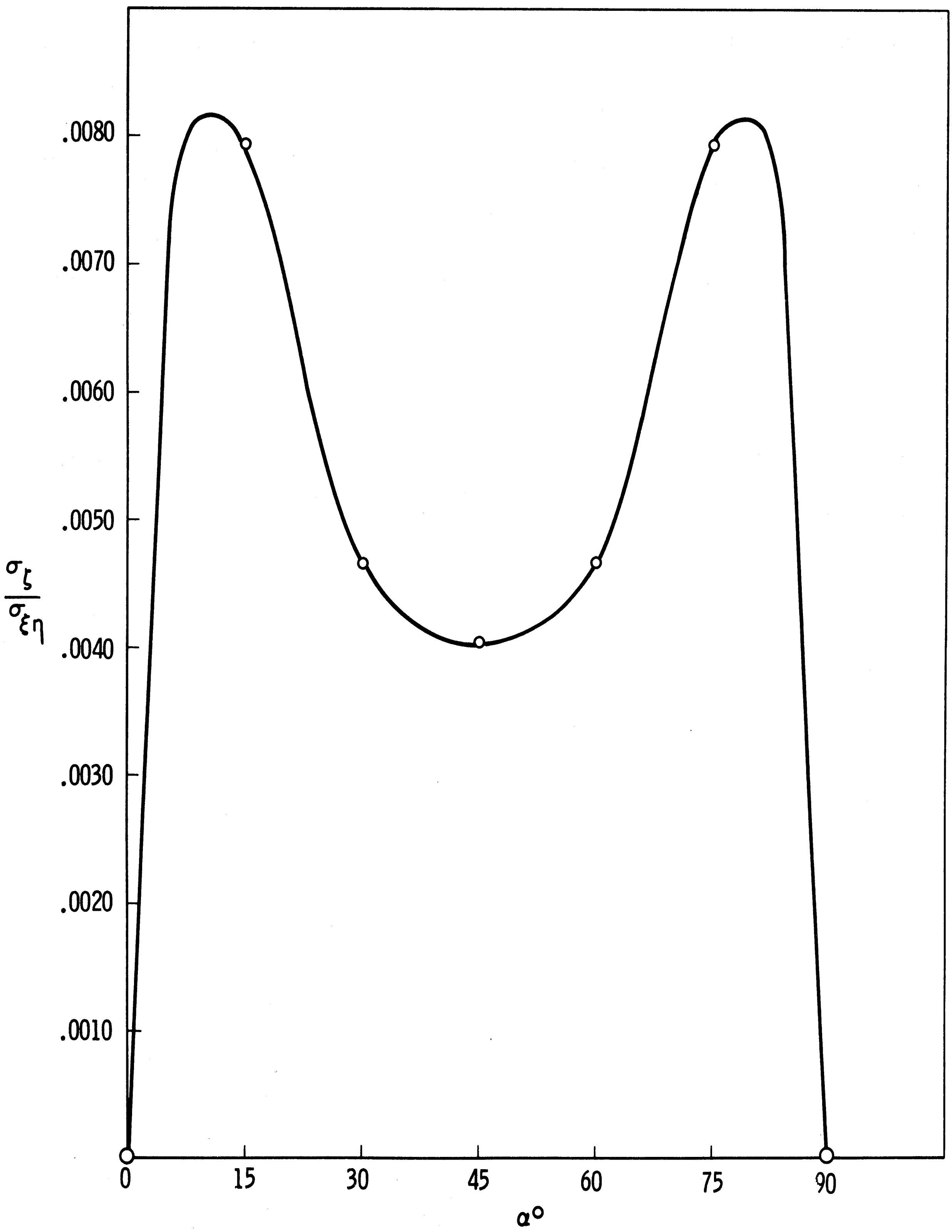

Fig. 5. Interply stress σ_ζ' as calculated from Eq. (23).

in the ξ or η direction and have the thickness increase. In a similar manner, one may apply an extensional stress in the ζ direction and have an increase in length in either the ξ or η direction.

Figures 6, 7, and 8 illustrate the remaining additional elastic characteristics obtainable from the analysis presented in this report.

It is emphasized again that the results presented here are those of a typical example. There are many special combinations of geometry and basic sheet properties that would probably result in slightly different conclusions being drawn concerning the influence of various parameters on the elastic characteristics.

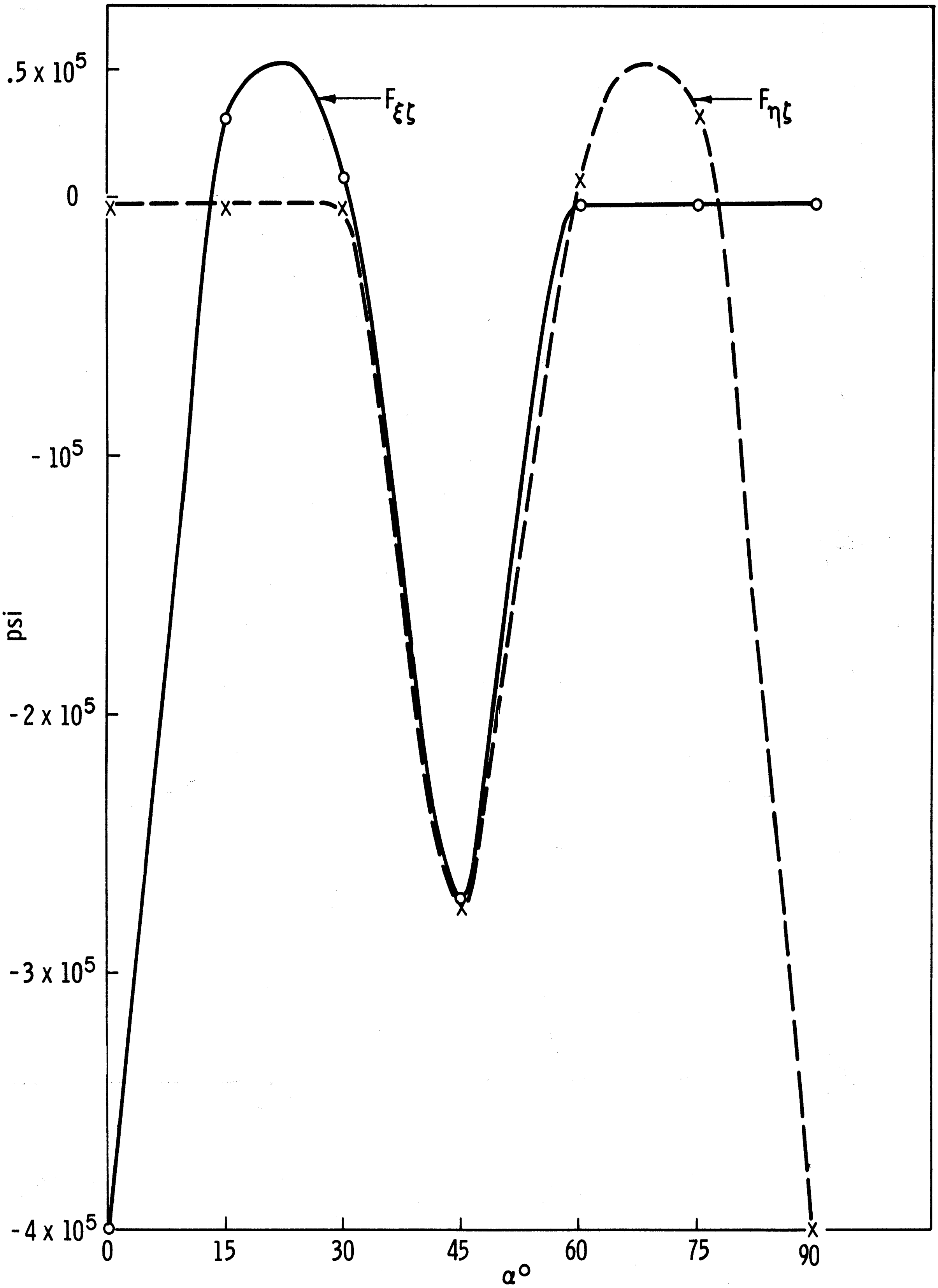

Fig. 6. Cross-moduli $F_{\xi\zeta}$ and $F_{\eta\zeta}$ as calculated from Eq. (21).

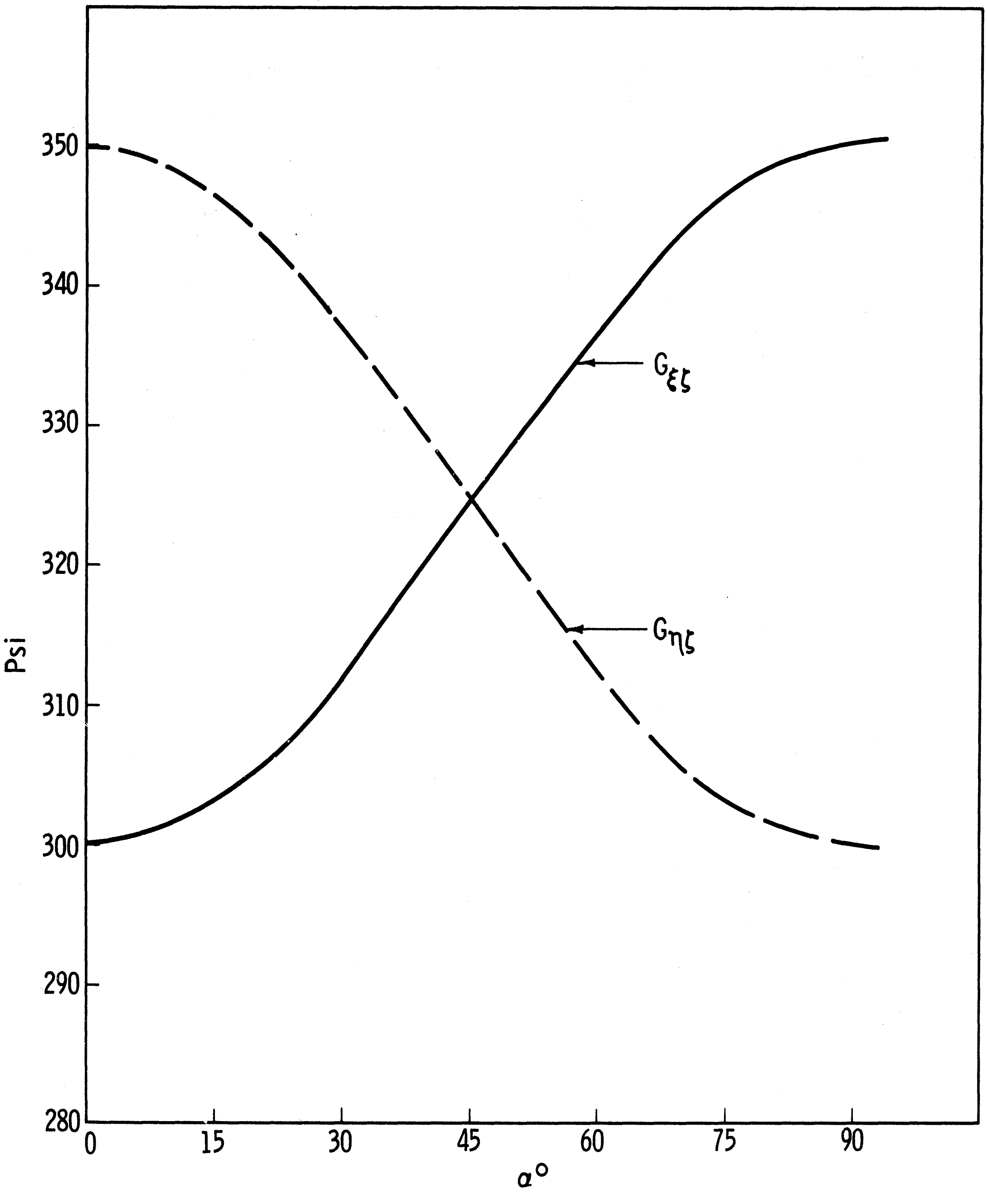

Fig. 7. Shear moduli $G_{\xi\zeta}$ and $G_{\eta\zeta}$ as calculated from Eqs. (27) and (30).

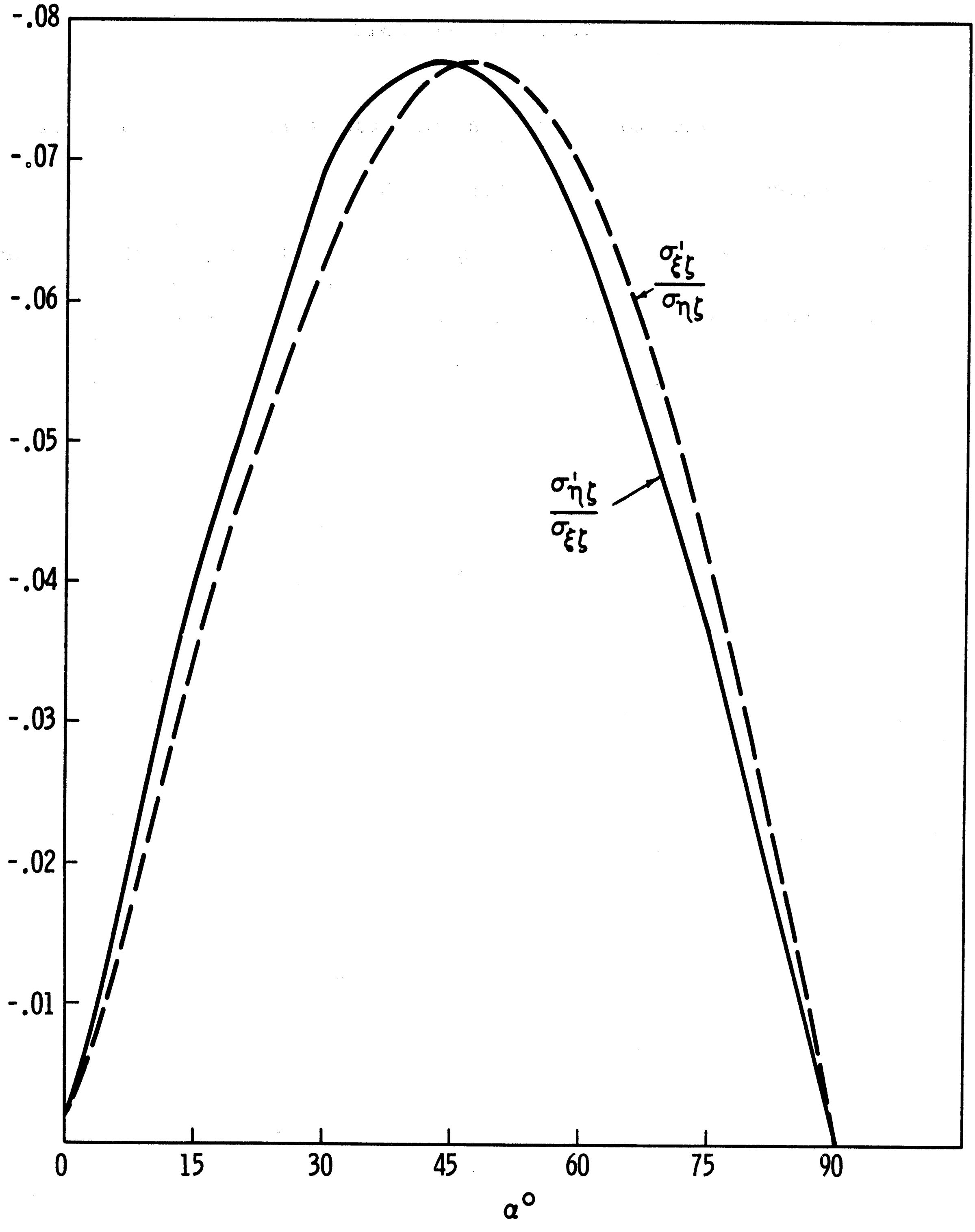

Fig. 8. Interply shear stresses $\sigma'_{\xi\zeta}$ and $\sigma'_{\eta\zeta}$ calculated from Eqs. (26) and (29).

V. ACKNOWLEDGMENTS

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