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One-Dimensional Stability of Viscous Strong Detonation Waves

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Abstract

Building on Evans-function techniques developed to study the stability of viscous shocks, we examine the stability of strong-detonation-wave solutions of the Navier-Stokes equations for reacting gas. The primary result, following [1, 17], is the calculation of a *stability index* whose sign determines a necessary condition for spectral stability. We show that for an ideal gas this index can be evaluated in the Zeldovich-von Neumann-Döring limit of vanishing dissipative effects. Moreover, when the heat of reaction is sufficiently small, we prove that strong detonations are spectrally stable provided that the underlying shock is stable. Finally, for completeness, we include the calculation of the stability index for a viscous shock solution of the Navier-Stokes equations for a nonreacting gas.

1. Introduction and Preliminaries

1.1. Introduction

Laboratory and numerical experiments indicate that detonations have quite sensitive stability properties. Indeed, steady planar detonations subjected to one-dimensional longitudinal perturbations may change form to "galloping" detonations in which the velocity fluctuates periodically in time. Such detonations have been been predicted numerically [15] and observed experimentally in various settings by GORDON, MOORADIAN & HARPER [22], MANSON *et al*. [45], and MUNDY *et al.* [50]. Another instability, this one with 3-dimensional structure, is the "spinning detonation" long-known in laboratory experiments [4, 5] and more recently captured numerically for the Zeldovich-von Neumann-Döring (ZND)[1] model in [3, 28]. A three-dimensional perturbation of a steady detonation wave propagating down a

[1] The ZND model is an inviscid combustion model. See the discussion in Section 1.3.

tube with a circular cross-section may cause the wave to bifurcate to a wave with a complex rotating structure which traces a helical path along the boundary of the tube. This structure is typically followed by localized regions of extremely high pressure.

Due to this sensitivity and the complexity of models for reacting fluids, stability analyses of detonation waves are largely numerical studies of the ZND model (which neglects dissipative effects) as in [3, 31, 55–57] or are restricted to various incarnations of the Majda or Majda-Rosales[2] models (nonphysical analogues of Burgers' equation) as in [32, 38, 39, 33–37, 51, 58]. Our approach, utilizing the Evans function, allows the treatment of the Navier-Stokes equations for reacting fluids and yields an explicitly computable quantity known as the *stability index*.

In Eulerian coordinates, the Navier-Stokes equations for reacting fluids modeling the simplest possible one-step chemical reaction can be written as

$$\rho_t + (\rho u)_x = 0, \tag{1.1}$$

$$(\rho u)_t + (\rho u^2 + p)_x = (\nu u_x)_x, \tag{1.2}$$

$$\tilde{\mathcal{E}}_t + [\rho u \tilde{\mathcal{E}} + up]_x = (\theta T_x)_x + (q\rho d Y_x)_x + (\nu u u_x)_x, \tag{1.3}$$

$$(\rho Y)_t + (\rho u Y)_x = (\rho d Y_x)_x - k\rho Y \varphi(T). \tag{1.4}$$

In (1.1)–(1.4) and below, we use, unless stated otherwise, the notation: ρ, u, p, $\mathcal{E}$, T, and Y represent respectively density, velocity, pressure, total energy, temperature, and mass fraction of reactant. The use of the tilde denotes that the energy $\tilde{\mathcal{E}} = \rho(u^2/2+\tilde{e})$ is modified from the standard gas-dynamical energy $\mathcal{E} = \rho(u^2/2+e) = \rho E$ due to heat produced in the chemical reaction by $\tilde{e} = e + qY$. The positive constants ν, θ, and d are the coefficients of viscosity, heat conductivity, and species diffusion. The positive constants k and q measure the rate of reaction and the heat released in reaction, respectively, and the form of the so-called ignition function φ is discussed in detail below. The system is closed by specifying equations of state, $p = p(\rho, e, Y)$ and $T = T(\rho, e, Y)$. We begin by assuming only that p and T are independent of Y, but for some portions of the analysis we shall assume further an ideal, polytropic gas, that is,

$$p(\rho, e) = \Gamma \rho e, \quad T(\rho, e) = c_v^{-1} e,$$

where c_v, the specific heat at constant volume, and Γ, known as the Gruneisen coefficient, are constants. Equations (1.1)–(1.4) are standard; a derivation can be found in [62].

Often φ is assumed to satisfy the Arrhenius law, that is,

$$\varphi(T) = \exp\left(-\frac{E_A}{RT}\right),$$

where E_A is the activation energy and R is the gas constant (assuming the ideal gas law, $R = c_v \Gamma$). However, nonvanishing of the exponential creates a problem

[2] For the remainder of the paper, we refer to all models with the simplifying feature of scalar kinetics as the Majda model.

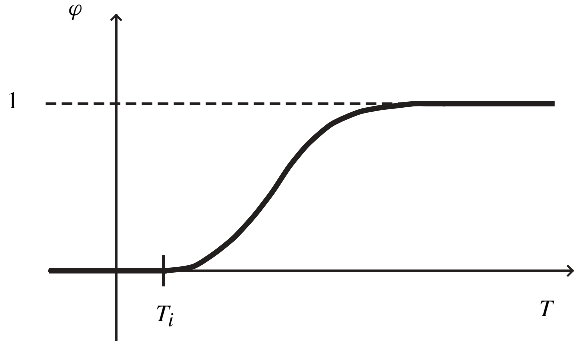

Fig. 1.1. The ignition function φ

known as the “cold-boundary difficulty.” Essentially, nonvanishing of φ precludes the unburned state from being a rest point of the traveling-wave equation. In place of the Arrhenius kinetics, we make the standard assumption that the smooth function φ vanishes for temperatures below some ignition temperature, T_i, and is identically 1 for some larger value of T. This circumvents the cold-boundary difficulty; see Fig. 1.1. The model (1.1)–(1.4) includes dissipative effects neglected by the ZND model and allows for complete gas-dynamical effects unlike the Majda model. We remark that an artificially strictly parabolic multi-dimensional version is considered in the appendix of [65]; here, we include the additional difficulty of partial parabolicity.

1.2. Plan of the paper

We begin in Section 1 by gathering the relevant background material for our analysis. We first discuss the standard ZND model as a prelude to a discussion of the analysis of [18]: ZND solutions are singular solutions in the context of geometric singular perturbation theory used therein. The structure of these singular solutions allows us to evaluate the stability index; we perform the evaluation in Section 3. The backgound material in Section 1 concludes with a description of the Evans-function theory for the stability of viscous shock waves. Section 2 contains the calculation and evaluation of the stability index for a viscous shock. This proves useful in Section 3, where we examine the system (1.1)–(1.4) which models a reacting fluid. Appendix A contains a revised version of an appendix of [65] which is used in the calculations of the stability indexes in both Section 2 and Section 3.

1.3. The Zeldovich-von Neumann-Döring model

Setting the constants $\nu, \theta,$ and d equal to zero in (1.1)–(1.4) yields a model introduced independently by Zeldovich, von Neumann, and Döring [8, 14]. In their formulation, known as the ZND model, dissipative effects are neglected and the reaction rate is assumed to be finite. This is a refinement of the earlier Chapman-Jouguet model in which the reaction was assumed to take place instantaneously. The ZND model, in Eulerian coordinates, then has the form

$$\begin{aligned}
\rho_t + (\rho u)_x &= 0,\\
(\rho u)_t + (\rho u^2 + p)_x &= 0,\\
\tilde{\mathcal{E}}_t + ((\tilde{\mathcal{E}} + p)u)_x &= 0,\\
(\rho Y)_t + (\rho u Y)_x &= -k\rho Y\varphi(T).
\end{aligned}$$

The constant k is the reaction rate, and φ is the ignition function. Strong detonations in the ZND model are initiated by a (purely) gas-dynamical shock, called the Neumann shock, which heats the gas by compressing it. The increase in the temperature to a sufficiently high level "turns on" φ and starts the reaction. Thus, these waves have the structure of a gas-dynamical shock followed by a reaction zone resolving to the final burned state. This is seen in the characteristic "detonation spikes" in the temperature and pressure profiles in strong agreement with observed features in laboratory experiments. We shall see this structure in our discussion of [18] below.

1.4. Existence of strong detonations

Existence of traveling-wave solutions of (1.1)–(1.4) is studied in [18] using the techniques of geometric singular perturbation theory (GSPT). As the orientations of the singular manifolds constructed in that argument will play a role in our analysis, we recap the argument here. We remark that an interesting feature of the GSPT analysis of (1.1)–(1.4) is the recovery of the shock-layer analysis of [21].

1.4.1. The Hugoniot curve. Traveling-wave solutions of (1.1)–(1.4) are those which depend only on $\xi = x - st$. This ansatz reduces the system (1.1)–(1.4) to a system of ordinary differential equations. By Galilean invariance we may, without loss of generality, set $s = 0$, so the system is

$$(\rho u)' = 0, \tag{1.5}$$

$$(\rho u^2 + p)' = (\nu u')', \tag{1.6}$$

$$\left(\rho u(\frac{u^2}{2} + \tilde{e}) + up\right)' = (\theta T')' + (q\rho d Y')' + (\nu u u')', \tag{1.7}$$

$$(\rho u Y)' = (\rho d Y')' - k\rho Y\varphi(T), \tag{1.8}$$

where $'$ denotes differentiation with respect to x. From equation (1.5), it follows that the mass flux $m = \rho u$ has a constant value. Moreover, each of (1.6) and (1.7) can be integrated once. We suppose that an unburned state $(\rho_+, u_+, p_+, Y_+ = 1)$ has been fixed at $+\infty$. Then, the momentum equation integrates to

$$\rho u^2 + p - (\rho_+ u_+^2 + p_+) = \nu u'.$$

For a possible connection to a completely burned state $(\rho, u, p, Y = 0)$ at $x = -\infty$, it is necessary that the state be a rest point of the differential equation; more precisely, (ρ, u, p, Y) must satisfy

$$\rho u^2 + p - (\rho_+ u_+^2 + p_+) = 0.$$

Searching for all such states leads to the expression

$$p - p_+ = -m^2 \left(\frac{1}{\rho} - \frac{1}{\rho_+} \right).$$

This equation describes a line in the specific volume-pressure plane with slope $-m^2$. It is referred to as the *Rayleigh Line*. Similarly, integrating the energy equation and searching for rest points again, leads to the equation for the *Hugoniot Curve*

$$\tilde{e} - \tilde{e}_+ = -\frac{1}{2}(p + p_+) \left(\frac{1}{\rho} - \frac{1}{\rho_+} \right).$$

The intersection of the Rayleigh line and the Hugoniot curve in the specific volume-pressure plane determines the possible burned states corresponding to the fixed unburned state (ρ_+, u_+, p_+, Y_+). An important distinction from the nonreacting case is the fact that there may be one, two, or no possible burned states. See Fig. 1.2. Another interesting feature of Fig. 1.2 is the fact that the Hugoniot curve splits into two branches. This indicates that the conservation relations are compatible with two distinct types of processes, just as observed by early experimentalists [8]. The compressive solutions are called *detonations* while the expansive solutions are referred to as *deflagrations*. Accordingly, the two branches of the Hugoniot curve

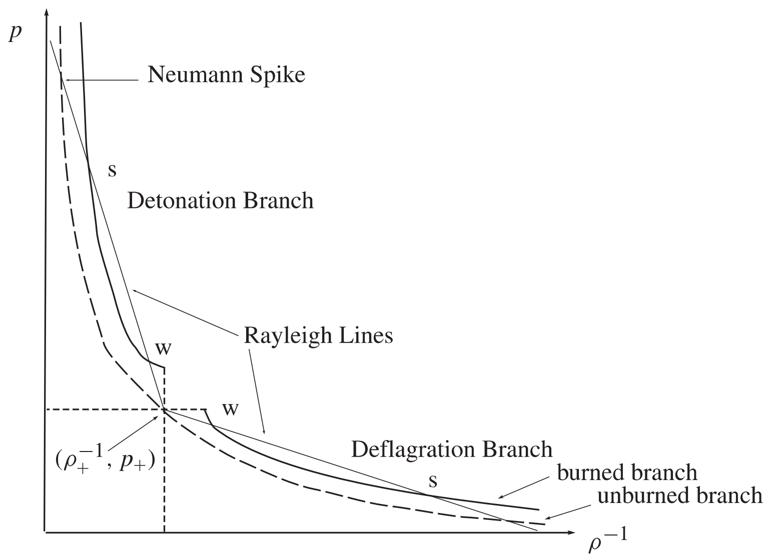

Fig. 1.2. The Hugoniot curve

are called the detonation branch and deflagration branch. Each of these branches is further subdivided into two sections. Here, we focus on the detonation branch, but similar characterizations apply to the deflagration branch. From the diagram, it is clear that there is a unique value of $-m^2$ so that the Rayleigh line is tangent to the Hugoniot curve. The point of tangency is called the Chapman-Jouguet point, and a detonation connecting to the burned state identified by that point is called a Chapman-Jouguet detonation. For values of $-m^2$ which are smaller than this unique value, the Rayleigh line intersects the Hugoniot curve twice. The larger of these two intersections is the burned state corresponding to a *strong detonation* while the smaller corresponds to the burned state of a *weak detonation*. One distinction between these waves is the following: strong detonations satisfy the Lax characteristic condition while weak detonations are undercompressive. In this context, compressivity refers to the number of incoming characteristics. Finally, if $-m^2$ is too large, the Rayleigh line and the Hugoniot curve do not intersect and there are no possible burned states which are compatible with the unburned state.

1.4.2. GSPT analysis of detonation waves. We begin with the briefest of introductions to GSPT. Consider a system of singularly perturbed ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f(x, y), \\ \epsilon \frac{\mathrm{d}y}{\mathrm{d}t} &= g(x, y), \end{aligned} \tag{Slow$_\epsilon$}$$

where ϵ is small. We call such a system the *slow* system. By rescaling the independent variable by $\tau = t/\epsilon$, we obtain the equivalent (when $\epsilon \neq 0$) *fast* system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\tau} &= \epsilon f(x, y), \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &= g(x, y). \end{aligned} \tag{Fast$_\epsilon$}$$

Looking at (Slow$_\epsilon$) and (Fast$_\epsilon$), it is clear that there are then two distinguished limiting systems when $\epsilon = 0$. They are the *reduced* problem

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f(x, y), \\ 0 &= g(x, y), \end{aligned} \tag{Slow$_0$}$$

and the *layer* problem

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\tau} &= 0, \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &= g(x, y). \end{aligned} \tag{Fast$_0$}$$

The basic idea, then, is to construct solutions of the original system as smooth perturbations of the composite orbits of the decoupled limiting systems (Slow$_0$) and (Fast$_0$). For more details consult [13, 59].

With this framework in mind, we take up the analysis of [18]. We note that here, following [18], we are assuming an ideal, polytropic gas, so that $p = R\rho T$, $e = c_v T$, and $\Gamma = \gamma - 1 = R/c_v$. We remark that existence for more general equations of state has been shown by different methods in [24], but the singular manifolds constructed in the GSPT analysis [18] play a key role in the evaluation of our stability condition. Namely, they contain the necessary geometric information about the profile. Using m and the integrated versions of (1.6) and (1.7), we find by some further manipulation that

$$\nu u_x = m(u - u_\pm) + mR\left(\frac{T}{u} - \frac{T_\pm}{u_\pm}\right),$$

$$\theta T_x + \nu u u_x + q\rho d Y_x = m\left((R + c_v)(T - T_\pm) + q(Y - Y_\pm) + \frac{1}{2}(u^2 - u_\pm^2)\right).$$

Next, we define the variable Z by the relationship

$$Z = Y - \rho d\frac{Y_x}{m} = Y - d\frac{Y_x}{u},$$

and note that Y_x vanishes at $\pm\infty$, to obtain

$$Z_- = Y_-, \qquad Z_+ = Y_+.$$

The equation for Y can thus be rewritten as

$$Z_x = -k\frac{Y}{u}\varphi(T), \quad \text{(note: } u \neq 0\text{)}.$$

Finally, rescaling to make the equations dimensionless, we arrive at the system

$$\nu u_x = u - 1 + \frac{1}{\gamma M^2}\left(\frac{T}{u} - 1\right), \tag{1.9}$$

$$\theta T_x = T - 1 - \frac{\gamma - 1}{\gamma}(T - u) + qZ - \frac{(\gamma - 1)M^2}{2}(u - 1)^2, \tag{1.10}$$

$$dY_x = u(Y - Z), \tag{1.11}$$

$$Z_x = -\frac{Y}{u}\varphi(T). \tag{1.12}$$

All quantities in (1.9)–(1.12) have been rescaled; M, defined by $M^2 = u_+^2/(\gamma R T_+)$, is the Mach number.

The values of the dissipative coefficients (ν, θ, and d) are typically quite small. Taking advantage of this smallness, the next step is to fix small values $\hat{\nu}$, $\hat{\theta}$, and $\hat{d}$, and then to set $\nu = \epsilon\hat{\nu}$, $\theta = \epsilon\hat{\theta}$ and $d = \epsilon\hat{d}$ so that the system (1.9)–(1.12) takes the form

$$\epsilon\hat{\nu} u_x = u - 1 + \frac{1}{\gamma M^2}\left(\frac{T}{u} - 1\right), \tag{1.13}$$

$$\epsilon\hat{\theta} T_x = T - 1 - \frac{\gamma - 1}{\gamma}(T - u) + qZ - \frac{(\gamma - 1)M^2}{2}(u - 1)^2, \tag{1.14}$$

$$\epsilon\hat{d} Y_x = u(Y - Z), \tag{1.15}$$

$$Z_x = -\frac{Y}{u}\varphi(T). \tag{1.16}$$

Here, ϵ is supposed to be small, and this system is singularly perturbed. Setting $\epsilon = 0$ in (1.13)–(1.16) yields the reduced (slow flow) system

$$0 = u - 1 + \frac{1}{\gamma M^2}\left(\frac{T}{u} - 1\right), \tag{1.17}$$

$$0 = T - 1 - \frac{\gamma - 1}{\gamma}(T - u) + qZ - \frac{(\gamma - 1)M^2}{2}(u - 1)^2, \tag{1.18}$$

$$0 = u(Y - Z), \tag{1.19}$$

$$Z_x = -\frac{Y}{u}\varphi(T). \tag{1.20}$$

Equations (1.17)–(1.19) define a one-dimensional manifold $\mathcal{C}$ upon which equation (1.20) describes a flow. Using the facts that (1.17) is independent of Y and Z, (1.18) is independent of Y, and (1.19) implies that $Y = Z$ (as long as $u \neq 0$), we discover that $\mathcal{C}$ can be visualized in three-dimensional (u, T, Z)-space.

The equation (1.17) describes a parabolic trough. Using (1.17) in (1.18) yields

$$0 = T\frac{\gamma + 1}{2\gamma} + qZ + u\frac{\gamma - 1}{2\gamma}(1 + \gamma M^2) - 1 - \frac{(\gamma - 1)M^2}{2},$$

which describes a plane $\mathcal{K}$ in (u, T, Z)-space. The manifold $\mathcal{C}$ is exactly the intersection of this plane and the parabolic trough. This intersection is pictured in Fig. 1.3. Note that $\mathcal{C}$ splits into two branches; the requirement that there be two burned end states corresponding to the fixed unburned state forces the vertex of the intersection to have a negative Z coordinate. In fact, the vertex is exactly at $Z = 0$ in the Chapman-Jouguet case. Note also that all the rest points of (1.13)–(1.16) are contained in $\mathcal{C}$. Rescaling the independent variable by $\xi = x/\epsilon$ in (1.13)–(1.16) yields the equivalent (when $\epsilon \neq 0$) system

$$\hat{\nu} u_\xi = u - 1 + \frac{1}{\gamma M^2}\left(\frac{T}{u} - 1\right), \tag{1.21}$$

$$\hat{\theta} T_\xi = T - 1 - \frac{\gamma - 1}{\gamma}(T - u) + qZ - \frac{(\gamma - 1)M^2}{2}(u - 1)^2, \tag{1.22}$$

$$\hat{d} Y_\xi = u(Y - Z), \tag{1.23}$$

$$Z_\xi = \epsilon\left(-\frac{Y}{u}\varphi(T)\right). \tag{1.24}$$

Setting $\epsilon = 0$ in (1.21)–(1.24) yields the layer (fast flow) system

$$\hat{\nu} u_\xi = u - 1 + \frac{1}{\gamma M^2}\left(\frac{T}{u} - 1\right), \tag{1.25}$$

$$\hat{\theta} T_\xi = T - 1 - \frac{\gamma - 1}{\gamma}(T - u) + qZ - \frac{(\gamma - 1)M^2}{2}(u - 1)^2, \tag{1.26}$$

$$\hat{d} Y_\xi = u(Y - Z), \tag{1.27}$$

$$Z_\xi = 0, \tag{1.28}$$

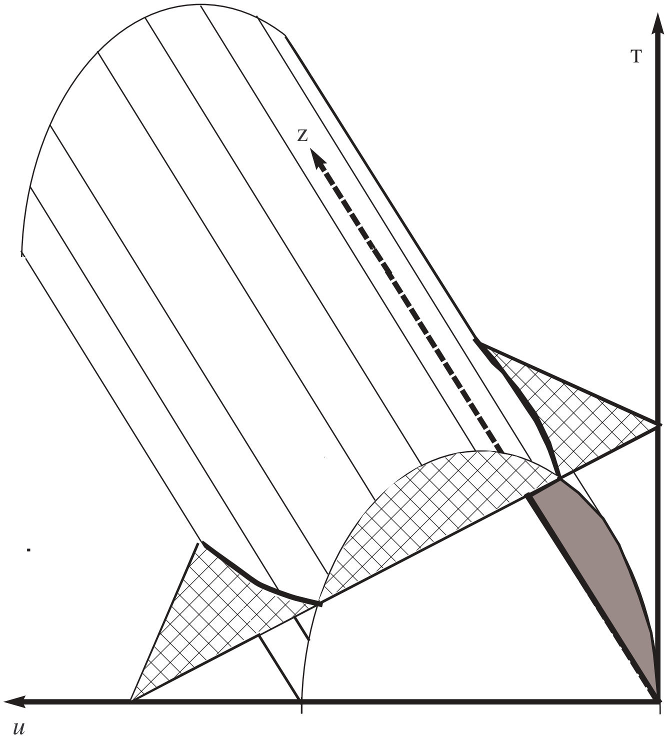

Fig. 1.3. The intersection of $\mathcal{K}$ and the trough

which is the shock-layer problem of GILBARG [21] coupled to (1.27). Thus, for each constant-Z slice, the fast flow is described by the shock-layer analysis of [21]. Figure 1.4 represents the structure of the singular ($\epsilon = 0$) flow looking down on $\mathcal{C}$ from a vantage point perpendicular to the plane $\mathcal{K}$. Dark arrows represent fast flow while single arrows represent slow flow. The fast flow in the plane $Z =$ constant corresponds to a nonreacting gas-dynamical shock. On the other hand, the slow flow proceeds on each branch of $\mathcal{C}$ representing the progress of the reaction. Combining the fast and slow flows, we see the perturbed composite orbit of a strong detonation in Fig. 1.4. Note the presence of the ZND structure. There is a gas-dynamical shock to the Neumann spike, which raises the temperature above ignition, followed by a reaction resolving to the final totally burned state. The same basic structure can be seen in the analysis of the scalar Majda model [42, 51, 41]. Having dealt with the question of existence (at least for small ν, θ, and d), we turn our attention to the tools of our stability analysis.

1.5. The Evans function and the gap and tracking lemmas

The proper notion of stability for traveling waves connecting constant end states, such as the detonations and shocks we consider, is that of *orbital stability*, that is, the convergence of the perturbed solution to the manifold of solutions which connect

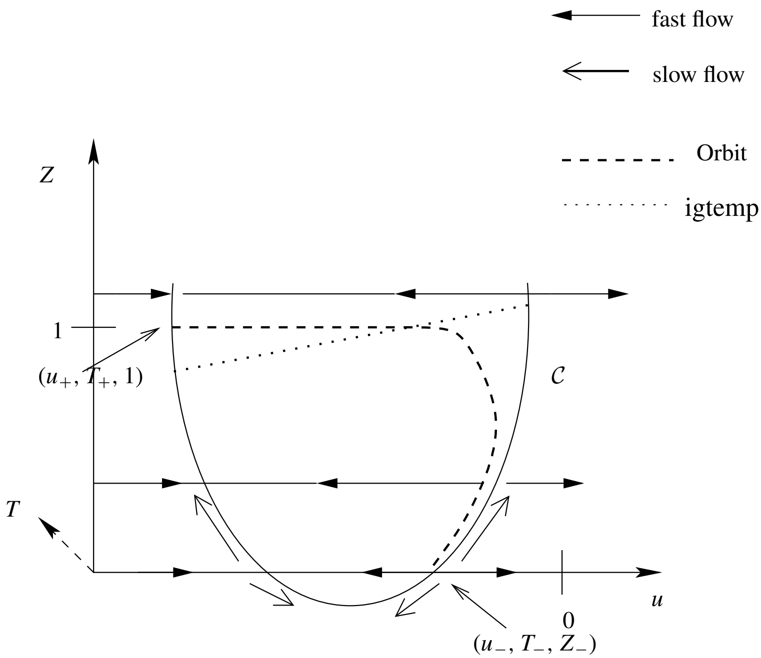

Fig. 1.4. Singular flow and composite orbit

the same two end states. We refer to this as nonlinear stability. A weaker notion is that of *linearized orbital stability* defined as convergence of solutions of the linearized equations about the wave to the tangent manifold to the manifold of solutions connecting the same two end states. Closely related to this concept of linear stability is *spectral stability*; a profile is spectrally stable if the linearized operator about the wave L has no spectrum in the set $\{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda \geqq 0\}$ except $\lambda = 0$. We note that translation invariance implies that 0 is an eigenvalue of the linearized operator, so necessarily $0 \in \sigma(L)$. Spectral stability is clearly necessary for linear stability, itself necessary for nonlinear stability. Recent work by MASCIA & ZUMBRUN [46, 48, 47] extending earlier work of ZUMBRUN & HOWARD [66] indicates that spectral stability implies nonlinear orbital stability in the settings of viscous conservation laws and relaxation systems which are closely related to the combustion systems we consider. In light of these results, the determination of spectral stability can be regarded as the essential initial step in determining the stability of detonation waves. We also note that [69] contains large-amplitude non-linear stability results which are relevant to the detonation problem.

1.5.1. Background. As noted above, a vital step in determining stability is locating the spectrum of a linear operator. The search for spectrum is facilitated by the

Evans function, $D(\lambda)$, a powerful tool for the investigation of stability of traveling waves. This function, analytic on the unstable halfplane, is an infinite-dimensional analogue of the characteristic polynomial. Zeros of $D(\cdot)$ correspond to eigenvalues of the linearized operator about the wave. The Evans function was introduced in [9–12] specifically to study nerve-axon equations and further developed in [1] to the case of semilinear parabolic systems. The use of the Evans function was extended in [17] to the case of 2×2 conservation laws with viscosity. The extension to $n \times n$ systems was completed in [2].

The construction of the Evans function is accomplished by analytically parametrizing the unstable/stable manifolds of the variable-coefficient eigenvalue equations. This is done by comparing these objects to the corresponding unstable/stable manifolds for the limiting constant-coefficient systems at $\pm\infty$. We now give an abbreviated description of the construction for the case of a system of conservation laws with viscosity,

$$U_t + f(U)_x = (B(U)U_x)_x,$$

where $U, f \in \mathbf{R}^n$ and B is an $n \times n$ matrix. As we shall be concerned with the important physical cases of gas dynamics and combustion, it will be the case that the matrix B is incompletely parabolic. However, the equations of compressible-gas dynamics satisfy the symmetrizability, dissipativity, and block structure conditions of Kawashima [30]:

symmetrizability

$$\begin{gathered}\text{There exists a symmetrizer } A^0(U),\\ \text{symmetric and positive definite,}\\ \text{such that } A^0(U)A(U) \text{ is symmetric and } A^0(U)B(U)\\ \text{is symmetric and positive semidefinite.}\end{gathered} \tag{1.29}$$

dissipativity

$$\text{There is no eigenvector of } A(U) \text{ lying in the kernel of } B(U). \tag{1.30}$$

block structure

$$\text{The right kernel of } B(U) \text{ is independent of } U. \tag{1.31}$$

In the above, $A(U)$ denotes the Jacobian matrix of the flux f. We discuss the construction in the case of combustion in more detail below. A viscous profile,

$$U(x,t) = \bar{U}(x - st), \qquad \bar{U}(\pm\infty) = U_\pm,$$

is a solution of the (integrated) traveling-wave equation

$$-s(U - U_-) + f(U) - f(U_-) = (B(U)U')'.$$

Taking, without loss of generality, $s = 0$ and linearizing about $\bar{U}(x)$, we obtain an equation modeling the approximate evolution of a small disturbance, v. This equation has the form

$$v_t = Lv := (Bv_x)_x - (Av)_x.$$

Here, $B(x) = B(\bar{U}(x))$ and $A(x)$ is determined by the relationship

$$Av = f'(\bar{U}(x))v - B'(\bar{U}(x))v\bar{U}_x.$$

The operator L is the linearized operator about the wave $\bar{U}$.

Definition 1. The profile $\bar{U}(\cdot)$ is *spectrally stable* if the linearized operator L about the wave has no spectrum in the closed unstable complex halfplane $\{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda \geqq 0\}$ except at $\lambda = 0$.

The next lemma allows us to narrow our search for the spectrum.

Lemma 1. *If the Kawashima conditions* (1.29)–(1.31) *hold, then the operator L has no essential spectrum in* $\{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda \geqq 0\}\backslash 0$.

The lemma follows by a standard argument of [23] provided that the constant solutions $U \equiv U_\pm$ are linearly stable. Such stability follows at once from the condition

$$\operatorname{Re}\sigma(i\xi A(U) - |\xi|^2 B(U)) \leqq \frac{-\theta|\xi|^2}{1+|\xi|^2}, \quad \theta > 0,$$

which is equivalent to (1.30) in the presence of (1.29) by an argument of [54]. Details can be found in any of [17, 63, 65–67]. Thus, for the systems of our interest, $\sigma_{\text{ess}}(L)$ is confined to the left complex half plane except the origin, and the only possible unstable spectrum consists of isolated eigenvalues of finite multiplicity. Determination of spectral stability is then reduced to checking that the operator L has no unstable point spectrum. The focus then is on the eigenvalue equation for this operator,

$$Lw = \lambda w,$$

and solutions $w \in L^2$ with corresponding eigenvalue λ such that $\operatorname{Re}\lambda > 0$.

The eigenvalue equation can be recast as a system of first-order ordinary differential equations

$$W' = \mathbb{A}(x,\lambda)W, \quad W \in \mathbf{C}^N. \tag{1.32}$$

Because the wave $\bar{U}$ connects constant states $U_\pm$, the matrix $\mathbb{A}$ has limits as $x \to \pm\infty$. Thus,

$$\mathbb{A}(x,\lambda) \longrightarrow \mathbb{A}_\pm(\lambda) \quad \text{as} \quad x \to \pm\infty.$$

The idea is to connect the stable (or unstable) manifolds of the system $W' = \mathbb{A}(x,\lambda)W$ at $\pm\infty$ to the stable (resp. unstable) subspaces $\mathcal{S}^\pm(\lambda)$ (resp. $\mathcal{U}^\pm(\lambda)$) of the constant-coefficient systems at each of $\pm\infty$. The procedure we are outlining requires that the system have *consistent splitting*.

Definition 2. We say that the system (1.32) has *consistent splitting on* $\Omega \subset \mathbf{C}$ if the matrices $\mathbb{A}_\pm(\lambda)$ are both hyperbolic for all λ in a region $\Omega \subset \mathbf{C}$, and there is an integer k such that the stable (or unstable) subspaces of $\mathbb{A}_+$ and $\mathbb{A}_-$ are both k-dimensional (resp. $(N-k)$-dimensional).

The set Ω is called the *region of consistent splitting*. It is shown in [17] that the linear stability of the constant solutions is equivalent to:

$$\text{Eigenvalues } \mu^{\pm}(\lambda) \text{ of } \mathbb{A}_{\pm}(\lambda) \text{ have nonvanishing real parts for all } \lambda \text{ with } \operatorname{Re}\lambda > 0. \tag{1.33}$$

It follows that Ω, the region of consistent splitting, contains at least the unstable complex half plane. For, using (1.33), we can count the number of stable/unstable eigenvalues of $\mathbb{A}_{\pm}(\lambda)$ as $\lambda \to +\infty$ along the real axis. Then, direct calculation yields that the numbers agree at both infinities and thus verifies consistent splitting. See the explicit computations below in the main body of the paper or the more general analysis of the Appendix. Then, given bases $\{\phi_1^+, \dots \phi_k^+\}$ and $\{\phi_{k+1}^-, \dots \phi_N^-\}$ of the stable manifold at $+\infty$ and the unstable manifold at $-\infty$, the idea is to define the Evans function as

$$D(\lambda) = \det(\phi_1^+, \dots \phi_k^+, \phi_{k+1}^-, \dots, \phi_N^-)|_{x=0}.$$

A natural way to attempt such a procedure is to choose as bases for $\mathcal{S}^{\pm}(\lambda)$ and $\mathcal{U}^{\pm}(\lambda)$ the purely exponential normal modes of $W' = \mathbb{A}_{\pm}(\lambda)W$. Unfortunately, it is not possible to make this choice analytically with respect to λ as some eigenvalues of $\mathbb{A}_{\pm}$ may coalesce as λ varies. The utility and power of the Evans function comes from the fact that this difficulty can be surmounted, and $D(\lambda)$ can be chosen to be analytic.

One solution, following the elegant construction of [1], is to track volume forms rather than individual solutions. We associate with any collection

$$V_1, V_2, \dots, V_p$$

of vectors the wedge product

$$V_1 \wedge V_2 \wedge \cdots \wedge V_p.$$

This determines an embedding of the manifold of p-dimensional bases into the manifold of p-forms. More precisely this determines an embedding into the submanifold of p-forms expressible as a single product, the *pure* p-forms. The benefit of this approach can be seen by taking a set of p solutions $W_1, W_2, \dots, W_p$ of the eigenvalue equation $W' = \mathbb{A}(x, \lambda)W$ and noticing that the corresponding p-form $\zeta = W_1 \wedge W_2 \wedge \cdots \wedge W_p$ solves the "lifted" linear ordinary differential equation

$$\zeta' = \mathcal{A}(x, \lambda)\zeta,$$

where the operator $\mathcal{A}$ is determined by

$$\mathcal{A} = \mathbb{A}W_1 \wedge W_2 \wedge \cdots \wedge W_p + \cdots + W_1 \wedge W_2 \wedge \cdots \wedge \mathbb{A}W_p.$$

If the collection $W_1, W_2, \dots, W_p$ consists of eigenvectors of $\mathbb{A}_{\pm}$ with corresponding eigenvalues $\mu_1, \mu_2, \dots, \mu_p$, then it immediately follows that the wedge $W_1 \wedge W_2 \wedge \cdots \wedge W_p$ is an eigenvector of $\mathcal{A}_{\pm}$ with corresponding eigenvalue $\mu_1 + \mu_2 + \cdots + \mu_p$. In the cases of interest, $\mu_1, \mu_2, \dots, \mu_p$ is either the set of stable eigenvalues of $\mathbb{A}$ ($p = N - k$) or the set of unstable eigenvalues ($p = k$), and $\mu_1 + \mu_2 + \cdots + \mu_p$

is *simple*. Thus, the volume form associated with any basis of $\mathcal{S}^{\pm}(\lambda)$ or $\mathcal{U}^{\pm}(\lambda)$ is a simple eigenvector of $\mathcal{A}_{\pm}$ corresponding to a purely exponential growth or decay mode. In this lifted setting, since the eigenvectors are simple, they depend on λ in an analytic fashion. This construction demonstrates the important fact that the eigenspaces vary analytically with respect to a parameter even when the individual eigenvectors do not.

For notational convenience, we follow the standard convention of associating the full N-volume forms with the complex numbers via the coordinate representation in the standard basis. That is, we write

$$V_1 \wedge \cdots \wedge V_N = \det(V_1, \dots, V_N).$$

1.5.2. The gap lemma. The gap lemma of [17] and [27] is the key technical result that allows Evans-function techniques for the stability analysis of traveling waves to be extended to the case of viscous conservation laws. This lemma extends the analytic framework of [1] to cases in which the essential spectrum of the linearized operator touches the imaginary axis, that is, there is no spectral gap between the essential spectrum and the unstable half plane $\{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda > 0\}$. In the presence of such a gap, a standard argument of [7] provides a relationship between the behavior of solutions near $\pm\infty$ of a system of asymptotically constant-coefficient eigenvalue equations and the corresponding solutions of the limiting, constant-coefficient equations.

More precisely, consider an ordinary differential equation with parameter (as obtained above by rewriting the eigenvalue equation as a first-order system)

$$W' = \mathbb{A}(x, \lambda) W, \quad W \in \mathbf{C}^N, \tag{1.34}$$

where the differentiation is with respect to x, and $\mathbb{A}$ is continuous in x and analytic with respect to λ. Moreover, suppose also that $\mathbb{A} \to \mathbb{A}_{\pm}$ as $x \to \pm\infty$. Provided that

$$\int_0^{\pm\infty} |\mathbb{A} - \mathbb{A}_{\pm}| \, \mathrm{d}x < +\infty, \tag{1.35}$$

then there is a one-to-one correspondence between the normal modes $V_j^{\pm} \mathrm{e}^{\mu_j^{\pm} x}$ of the constant-coefficient limiting system

$$W' = \mathbb{A}_{\pm}(\lambda) W,$$

where $\mu_j^{\pm}(\lambda)$, $V_j^{\pm}(\lambda)$ is an eigenvalue, eigenvector pair corresponding to $\mathbb{A}_{\pm}(\lambda)$, and solutions $W_j^{\pm}$ of (1.34) having the same limiting behavior. That is,

$$W_j^{\pm}(\lambda, x) = V_j^{\pm} \mathrm{e}^{\mu_j^{\pm} x} (1 + o(1)) \quad \text{as} \quad x \to \pm\infty.$$

The argument in [7] uses a fixed-point iteration scheme depending on the sign of differences of the real parts of the eigenvalues μ_j. In the case of strict separation of the eigenvalues, a *spectral gap*, the fixed point is the uniform limit of an analytic sequence of iterates, and thus analyticity in λ is preserved. In our case of interest, there is no spectral gap; the above argument breaks down. The key observation in [17] is that, in the absence of such a gap, analyticity can be preserved if we replace the hypothesis (1.35) with the stronger assumption

$$|\mathbb{A} - \mathbb{A}_{\pm}| = O(\mathrm{e}^{-\alpha|x|}) \quad \text{as} \quad x \to \pm\infty. \tag{1.36}$$

Theorem 1 (The gap lemma)**.** *Let $\mathbb{A}(x, \lambda)$ be continuous in x and analytic in λ with*

$$\mathbb{A}(x, \lambda) \to \mathbb{A}_{\pm}(\lambda) \quad as \quad x \to \pm\infty,$$

at an exponential rate $\mathrm{e}^{-\alpha|x|}$, $\alpha > 0$, and let $\zeta^-(\lambda)$ and $\eta^-(\lambda)$ be analytic N- and $(N-k)$-forms associated with the complementary invariant subspaces of $\mathbb{A}_-(\lambda)$, C^- and E^- with spectral gap β. Furthermore put τ_{C^-} equal to the trace of $\mathcal{A}$ restricted to C^-. Then there exists a solution $\mathcal{W}(x, \lambda)$ of the lifted ordinary differential equation $\zeta' = \mathcal{A}(x, \lambda)\zeta$ of the form

$$\mathcal{W}(x, \lambda) = \zeta(x, \lambda) \exp(\tau_{C^-} x),$$

where ζ (and thus $\mathcal{W}$) is C^1 in x and locally analytic in λ. Moreover $\zeta(x, \lambda)$ satisfies

$$(\partial_\lambda)^j \zeta(x, \lambda) = (\partial_\lambda)^j \zeta^-(\lambda) + O(\mathrm{e}^{-\bar{\alpha}|x|}|\zeta^-(\lambda)|),$$

when $x < 0$ for $j = 0, 1, \ldots$.

See [63, 65, 66, 17, 27] for further discussion and the proof. Appealing to the gap lemma, we thus obtain bases $\{\phi_1^+(x, \lambda), \ldots \phi_k^+(x, \lambda)\}$ of the stable manifold at $+\infty$ and $\{\phi_{k+1}^-(x, \lambda), \ldots \phi_N^-(x, \lambda)\}$ of the unstable manifold at $-\infty$. Therefore, we can indeed define $D(\lambda)$ by

$$D(\lambda) = \det(\phi_1^+, \ldots \phi_k^+, \phi_{k+1}^-, \ldots, \phi_N^-)|_{x=0}.$$

An important feature of the construction is that $D(\lambda)$ can be chosen to be real valued for real λ .

Theorem 2. *There exist bases $\phi_j^{\pm}$ such that $D(\lambda)$ satisfies*

$$D(\bar{\lambda}) = \overline{D(\lambda)}.$$

In particular, $D(\lambda)$ is real valued for $\lambda \in \mathbf{R}$.

See [63, 65] for details. A proof involves tracing through the various steps in the construction of the Evans function and verifying that complex symmetry is preserved at each stage.

1.5.3. The tracking lemma. In the calculation of the stability conditions (described below), it will be necessary to connect information about the sign of the Evans function $D(\lambda)$ as $\lambda \longrightarrow \infty$ along the real axis to the normalizations for the bases of stable and unstable manifolds chosen at $\lambda = 0$. This can be accomplished by using the "tracking lemma" appearing in various forms in [17, 63, 65, 69, 47].

As described in [17, 63, 65, 69, 47], the eigenvalue equation $W' = \mathbb{A}(x, \lambda)W$ for $|\lambda| \to \infty$ may often be transformed by rescaling/change of coordinates, to an equation of form

$$Z^{\epsilon\prime} = \mathbb{M}^\epsilon Z^\epsilon + \Theta^\epsilon Z^\epsilon, \tag{1.37}$$

$$\mathbb{M}^\epsilon := L\mathbb{A}R(x, \lambda) = \begin{pmatrix} M_1^\epsilon & 0 \\ 0 & M_2^\epsilon \end{pmatrix}(x), \tag{1.38}$$

where $\epsilon \to 0$ as $|\lambda| \to \infty$, M_j satisfy an approximate *uniform spectral gap*

$$\max \sigma(\operatorname{Re}(M_1^\epsilon)) - \alpha^\epsilon(x) \leqq -\eta(\epsilon) < 0 < \eta(\epsilon) \leqq \min \sigma(\operatorname{Re}(M_2^\epsilon)) + \alpha^\epsilon(x) \tag{1.39}$$

for all x, $\eta(\epsilon) > 0$, $\int |\alpha^\epsilon(x)|\, dx \leqq C$ independent of ϵ, and

$$|\Theta^\epsilon| \leqq \delta(\epsilon), \tag{1.40}$$

where $\delta(\epsilon)$ is a small constant relative to gap $\eta(\epsilon)$. In (1.39), we have used the notation $\operatorname{Re} M := (1/2)(M + M^*)$ to denote the real part of the matrix M. Thus, (1.39) is a "geometric" gap involving the numerical range of M_j^ϵ rather than an "algebraic" gap involving the spectrum alone. The condition (1.39) yields decay bounds via simple energy estimates even in cases (e.g., M_j^ϵ unbounded or rapidly varying in x) where the spectral condition does not. (In practice, (1.39) follows from the asymptotic relations $\max \sigma(\operatorname{Re}(M_1^\epsilon(\pm\infty))) \leqq -\eta(\epsilon) < 0 < \eta(\epsilon) \leqq \min \sigma(\operatorname{Re}(M_2^\epsilon(\pm\infty)))$; see [47].)

Then, denoting by $P_{\mathcal{S}}$ and $P_{\mathcal{U}}$ the projections onto the stable and unstable eigenspaces of $\mathbb{M}(x, \delta)$, we have (see [47, 69]):

Theorem 3 (The tracking lemma)**.** *Under assumption* (1.39)*, for δ/η sufficiently small, solutions Z^+ decaying as $x \to +\infty$ and Z^- decaying as $x \to -\infty$ of* (1.37) *lie, respectively, within cones*

$$\mathbb{K}_- = \left\{ Z \,\middle|\, \frac{|P_{\mathcal{S}} Z|}{|P_{\mathcal{U}} Z|} \leqq \frac{C\delta}{\eta} \right\}, \tag{1.41}$$

and

$$\mathbb{K}_+ = \left\{ Z \,\middle|\, \frac{|P_{\mathcal{U}} Z|}{|P_{\mathcal{S}} Z|} \leqq \frac{C\delta}{\eta} \right\}. \tag{1.42}$$

As a consequence, we may determine the sign of the Evans function for $|\lambda|$ sufficiently large by "tracking" the stable subspace of $\mathbb{M}$ from $x = +\infty$ to $x = 0$ and the unstable subspace of $\mathbb{M}$ from $x = -\infty$ to $x = 0$ and calculating the orientation of their intersection, *provided they are transverse*. Since $\mathbb{M}$ is similar to $\mathbb{A}$, we may equivalently carry out this computation in the original W coordinates, once the existence of transformation (1.37) has been established. For the models considered here, such a transformation always exists for $|\lambda|$ sufficiently large, with $\mathbb{M}^{\epsilon} = O(|\lambda|)$, $\eta = O(|\lambda|^{1/2})$, and $\delta = O(1)$. From now on, we shall take this fact (existence of a transformation) as given; for details, see [47] or [69].

1.6. Discussion

1.6.1. Stability conditions. Even though the Evans function is not typically evaluable, it is possible to obtain information about its zeros in the following way. Due to a translational eigenvalue at $\lambda = 0$, $D(0) = 0$. It is then possible to calculate $\operatorname{sgn} D'(0)$ and the sign of $D(\cdot)$ as $\lambda \longrightarrow \infty$ along the real axis. (Recall: $D(\lambda)$ can be chosen to be real for real λ.) Combining this information yields a parity for the number of unstable zeros of $D(\cdot)$, hence unstable eigenvalues for the linearized operator. We call the quantity $\operatorname{sgn} D'(0)D(+\infty)$ the *stability index*. When the signs agree, there must be an even number (possibly 0) of real unstable eigenvalues, and when they disagree, there must be an odd number of such eigenvalues. Recall that complex eigenvalues occur in conjugate pairs, hence they do not affect the parity. Clearly then,

$$\operatorname{sgn} D'(0)D(+\infty) \geqq 0$$

is necessary for spectral stability. On the other hand, when the index is negative, a positive growth rate is detected, and the wave under consideration is determined to be unstable. We remark that since the stability index only determines the parity of unstable eigenvalues, the condition $\operatorname{sgn} D'(0)D(+\infty) \geqq 0$ is not sufficient on its own to conclude spectral stability. The index yields only partial stability information; the possibilities of complex-conjugate unstable eigenvalues and/or even numbers of unstable real eigenvalues are not detected by this approach. Nonetheless, the stability index serves as a useful starting place in stability investigation.

1.6.2. Results. We now describe the two main results.

Theorem 4. *The stability index for a strong-detonation solution of* (1.1)–(1.4) *with Lax 3-shock structure has the form*

$$\tilde{\Gamma} = \operatorname{sgn} D'(0)D(+\infty) = \operatorname{sgn} \bar{\gamma}\Delta,$$

where $\bar{\gamma}$ is a constant measuring transversality of the stable/unstable manifolds of the traveling-wave equation and

$$\Delta = \det(r_1^-, r_2^-, [U] + \boldsymbol{q}). \tag{1.43}$$

Moreover, for an ideal gas, the sign of the stability index in the ZND limit is consistent with spectral stability.

Theorem 5. *Consider a sequence of strong detonations indexed by q converging as $q \to 0$ to a nonreacting shock. Then, for q sufficiently small, such a strong detonation is spectrally stable provided the underlying gas-dynamical shock (of arbitrary strength) is stable.*

In these theorems, $r_j^{\pm}$ are right eigenvectors of the flux Jacobian, $[U]$ is a vector of jumps in the gas-dynamical conserved quantities $\rho, m, \mathcal{E}$ (density, momentum, total energy), and $\boldsymbol{q}$ is the vector $\boldsymbol{q} = (0, 0, q)^{\mathrm{tr}}$. We use a superscript tr to denote the transpose of a vector. Finally, $q > 0$ represents the energy liberated during the exothermic chemical reaction. We note that the term Δ in (1.43), which appears in the stability index due to the low-frequency calculation of $D'(0)$, is related to the Lopatinski determinant, a "stability function" for inviscid shocks. See [26] and the references therein. In Section 3, we detail the reduction of the equation $\Delta = 0$ to

$$M^2[1/\rho]p_e - M - 1 = 0,$$

which has the form of the well-known instability condition of MAJDA [43] for inviscid shocks. We remark that there is no explicit dependence on q in this expression. We also remark that the finding $\tilde{\Gamma} > 0$ for an ideal gas in the ZND limit is consistent with an instability, if it occurs, of "galloping" type, i.e., corresponding to the crossing of a complex-conjugate pair of eigenvalues into the right half plane. This kind of instability has been observed in laboratory and numerical experiments.

The second theorem is related to results of TAN & TESEI [60] for the Navier-Stokes equations for reacting fluids, and LIU & YING [38] and LI, LIU & TAN [32] for various versions of the Majda model. In [60], the authors consider strong detonations with a one-step reaction scheme, a Heaviside ignition function, and no species diffusion. Their argument, valid for perturbations with zero mass, proceeds by a complicated energy estimate. Their small-strength assumption implies that the result is only valid for small q. In [38], again proceeding via energy estimates, full nonlinear stability is established for strong detonations in the Majda model when q is sufficiently small. In [32], the authors prove nonlinear stability for strong detonations in a version of the Majda model with species diffusion using techniques in the spirit of the Evans function. Finally, we remark again that, by the program of [66, 46, 48], it is expected that spectral stability should be equivalent to nonlinear stability.

1.6.3. Extensions. For the calculations in this paper, we have made the simplifying assumption that the equations of state are independent of the progress of the reaction. Though standard in the literature, this is clearly an idealization since the nature of the gas changes during the chemical reaction, as pointed out in [6]. One extension is to carry out the analysis in the more realistic setting of reaction-dependent equations of state, as discussed in the context of the Majda model in [41]. Also, we note that the $q \to 0$ argument in Section 3.7 fails in this more realistic setting since the $q = 0$ gas equation is still coupled to the reaction equation through the equation of state.

Another interesting direction of future study is a more detailed examination of the effect (if any) of multiple reactants on the stability index and its sign. In particular, the analysis of [19] provides the geometric information required to evaluate (in the ZND limit) the transversality coefficient of the stability index for the interesting two-species reaction they consider. In particular, while an exothermic-exothermic two-step reaction behaves much as the one-step exothermic reaction we consider, an exothermic-endothermic two-step reaction has a richer structure [14].

2. Nonreacting gas

In this section, we consider viscous-shock solutions of the Navier-Stokes equations in one space dimension; our main focus is the calculation of the stability index. These computations will prove useful when we shift our focus to detonations in the next section. The system takes the form

$$\rho_t + (\rho u)_x = 0, \tag{2.1}$$

$$(\rho u)_t + (\rho u^2 + p)_x = (\nu u_x)_x, \tag{2.2}$$

$$\mathcal{E}_t + (u\mathcal{E} + up)_x = (\theta T_x)_x + (\nu u u_x)_x. \tag{2.3}$$

The system (2.1)–(2.3) features five unknowns (ρ, u, e, p, T) and three equations. The system is completed by equations of state which incorporate the physical properties of the particular gas being modeled. We close the system by assuming that p and T are given functions of ρ and e. The three differential equations (2.1)–(2.3) and the equations of state give a set of five equations for the five variables ρ, u, e, T and p. The sound speed is

$$c = \sqrt{p_\rho + \rho^{-2} p p_e}.$$

For some portions of the analysis, we will further assume that the gas under consideration is ideal and polytropic so that the specific forms of the equations of state are

$$p(\rho, e) = \Gamma \rho e, \quad T(\rho, e) = c_v^{-1} e,$$

where the constants c_v and Γ are as in the previous section. We note that in this case the sound speed satisfies

$$c^2 = \Gamma e + \Gamma^2 e = (1 + \Gamma)\Gamma e = \gamma \Gamma e, \tag{2.4}$$

where $\gamma = 1 + \Gamma$.

We rewrite the system (2.1)–(2.3) in terms of the conserved quantities ρ, m, and $\mathcal{E}$ to obtain

$$\rho_t + m_x = 0, \tag{2.5}$$

$$m_t + \left(\frac{m^2}{\rho} + p\right)_x = \left(\nu \left(\frac{m}{\rho}\right)_x\right)_x, \tag{2.6}$$

$$\mathcal{E}_t + \left(\frac{m}{\rho}(\mathcal{E} + p)\right)_x = \left((\theta T)_x + \left(\nu \left(\frac{m}{\rho}\right)\left(\frac{m}{\rho}\right)_x\right)\right)_x. \tag{2.7}$$

Rewriting (2.1)–(2.3) in the form of a conservation law with viscosity, $U_t + f(U)_x = (B(U)U_x)_x$, we obtain

$$\begin{pmatrix}\rho_t\\ m_t\\ \mathcal{E}_t\end{pmatrix} + \partial_x\begin{pmatrix} m\\ m^2/\rho + p\\ m\rho^{-1}(\mathcal{E}+p)\end{pmatrix}$$
$$= \partial_x\left[\begin{pmatrix} 0 & 0 & 0\\ -\frac{\nu m}{\rho^2} & \frac{\nu}{\rho} & 0\\ \theta T_\rho + T_e e_\rho - \frac{\nu m^2}{\rho^3} & \theta T_e e_m + \frac{\nu m}{\rho^2} & \theta T_e e_{\mathcal{E}}\end{pmatrix}\begin{pmatrix}\rho_x\\ m_x\\ \mathcal{E}_x\end{pmatrix}\right],$$

from which the form of the viscosity matrix B becomes apparent. The Jacobian matrix $A(U)$ of the flux f has the form

$$\begin{pmatrix} 0 & 1 & 0\\ -u^2 + p_\rho + p_e e_\rho & 2u - u\frac{p_e}{\rho} & \frac{p_e}{\rho}\\ u(-E - \frac{p}{\rho} + p_\rho + p_e e_\rho) & E + \frac{p}{\rho} - u^2\frac{p_e}{\rho} & u + u\frac{p_e}{\rho}\end{pmatrix}.$$

To calculate the eigenvalues/eigenvectors of A, we use the device (see [26] and the isentropic-gas section of [64]) of conjugating by appropriate "shift" matrices so that the conjugated matrix has a particularly simple form. Following this procedure we obtain eigenvalues

$$\begin{aligned} a_1 &= u - c,\\ a_2 &= u,\\ a_3 &= u + c,\end{aligned}$$

and right eigenvectors

$$r_1 = \begin{pmatrix} 1\\ u - c\\ \frac{u^2}{2} - cu + \frac{p}{\rho} + e\end{pmatrix},$$
$$r_2 = \begin{pmatrix} \frac{p_e}{\rho}\\ \frac{p_e}{\rho}u\\ \frac{p_e}{\rho}\frac{u^2}{2} - \left(p_\rho - \frac{p_e}{\rho}e\right)\end{pmatrix},$$
$$r_3 = \begin{pmatrix} 1\\ u + c\\ \frac{u^2}{2} + cu + \frac{p}{\rho} + e\end{pmatrix}.$$

The left eigenvectors are

$$l_1 = \left(p_\rho - \frac{p_e}{\rho}e + cu + \frac{p_e u^2}{2\rho}, -c - \frac{p_e}{\rho}u, \frac{p_e}{\rho}\right),$$
$$l_2 = \left(-e - \frac{p}{\rho} + \frac{u^2}{2}, -u, 1\right),$$
$$l_3 = \left(p_\rho - \frac{p_e}{\rho}e - cu + \frac{p_e u^2}{2\rho}, c - \frac{p_e}{\rho}u, \frac{p_e}{\rho}\right).$$

We shall restrict our attention to a Lax 3-shock. That is, we suppose that the shock speed s satisfies the inequalities

$$a_2^- < s < a_3^-, \qquad a_3^+ < s. \tag{2.8}$$

The calculations for a 1-shock follow in a similar fashion.

2.1. Traveling-wave and linearized equations

The traveling-wave equation is

$$m' = 0, \tag{2.9}$$

$$\left(\frac{m^2}{\rho} + p\right)' = \left(\nu\left(\frac{m}{\rho}\right)'\right)', \tag{2.10}$$

$$\left(\frac{m}{\rho}(\mathcal{E} + p)\right)' = \left((\theta T)' + \left(\nu\left(\frac{m}{\rho}\right)\left(\frac{m}{\rho}\right)'\right)\right)', \tag{2.11}$$

where, without loss of generality, we have taken $s = 0$. Each of the equations (2.9)–(2.11) may be integrated up once. Thus, fixing a state at $-\infty$, we have

$$m - m_- = 0, \tag{2.12}$$

$$\left(\frac{m^2}{\rho} + p\right) - \left(\frac{m^2}{\rho} + p\right)_- = \nu\left(\frac{m}{\rho}\right)', \tag{2.13}$$

$$\left(\frac{m}{\rho}(\mathcal{E} + p)\right) - \left(\frac{m}{\rho}(\mathcal{E} + p)\right)_- = (\theta T)' + \nu\left(\frac{m}{\rho}\right)\left(\frac{m}{\rho}\right)'. \tag{2.14}$$

The requirement for a connection, that both end states be rest points of the system, leads from (2.12)–(2.14) to the well-known Rankine-Hugoniot conditions

$$[m] = 0,$$
$$m^2[1/\rho] = -[p],$$
$$\left[\frac{\mathcal{E}}{\rho}\right] = -\left[\frac{p}{\rho}\right].$$

We suppose that $\bar{U}(x) = (\bar{\rho}(x), \bar{m}(x), \bar{\mathcal{E}}(x))^{\mathrm{tr}}$ is a stationary profile connecting end states $U_\pm = (\rho_\pm, m_\pm, \mathcal{E}_\pm)^{\mathrm{tr}}$ which satisfy the Rankine-Hugoniot conditions. We note that such profiles, if they exist, are transverse. See [21, 49]. Indeed, global existence for connecting profiles has been shown in [21] for equations of state which satisfy the thermodynamic conditions of [61]. More precisely, existence is shown for equations of state such that the isentropes are convex and do not cross in the pressure-volume plane. We note that a polytropic, ideal gas satisfies this condition.

Linearizing (2.1)–(2.3) about this profile, we find equations for the evolution of small perturbations $(\rho, m, \mathcal{E})$. These equations can be written in the general form

$$w_t + (Aw)_x = (Bw_x)_x,$$

where $w = (\rho, m, \mathcal{E})^{\mathrm{tr}}$ and the matrices A and B depend only on x. More precisely, we find

$$\rho_t + m_x = 0, \tag{2.15}$$

$$m_t + (\alpha_{21}(x)\rho + \alpha_{22}(x)m + \alpha_{23}(x)\mathcal{E})_x = (b_{21}(x)\rho_x + b_{22}(x)m_x)_x, \tag{2.16}$$

$$\mathcal{E}_t + (\alpha_{31}(x)\rho + \alpha_{32}(x)m + \alpha_{33}(x)\mathcal{E})_x = (b_{31}(x)\rho_x + b_{32}(x)m_x + b_{33}(x)\mathcal{E}_x)_x, \tag{2.17}$$

where the coefficient functions $\alpha_{ij}(x)$ can be expressed in terms of the entries of the flux Jacobian and derivatives of the entries of the viscosity matrix. The terms $b_{ij}(x)$ correspond to the ij-entries of the viscosity matrix B. In both cases x-dependence arises from evaluation along the known profile $\bar{U}(x)$.

As our interest is in spectral stability of the profile, we focus on the eigenvalue equations corresponding to (2.15)–(2.17). They are

$$\lambda\rho + m' = 0, \tag{2.18}$$

$$\lambda m + (\alpha_{21}(x)\rho + \alpha_{22}(x)m + \alpha_{23}(x)\mathcal{E})' = (b_{21}(x)\rho' + b_{22}(x)m')', \tag{2.19}$$

$$\lambda\mathcal{E} + (\alpha_{31}(x)\rho + \alpha_{32}(x)m + \alpha_{33}(x)\mathcal{E})' = (b_{31}(x)\rho' + b_{32}(x)m' + b_{33}(x)\mathcal{E}')', \tag{2.20}$$

and the corresponding limiting system as $x \to \pm\infty$ takes the form

$$\begin{aligned}
\lambda\rho + m' &= 0,\\
\lambda m + (\alpha_{21}^{\pm}\rho + \alpha_{22}^{\pm}m + \alpha_{23}^{\pm}\mathcal{E})' &= (b_{21}^{\pm}\rho' + b_{22}^{\pm}m')',\\
\lambda\mathcal{E} + (\alpha_{31}^{\pm}\rho + \alpha_{32}^{\pm}m + \alpha_{33}^{\pm}\mathcal{E})' &= (b_{31}^{\pm}\rho' + b_{32}^{\pm}m' + b_{33}^{\pm}\mathcal{E}')'.
\end{aligned}$$

We make the invertible change of variables as in [65]:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{pmatrix}}_{C} \begin{pmatrix} \rho \\ m \\ \mathcal{E} \end{pmatrix}. \tag{2.21}$$

Thus,

$$\begin{aligned}
z_1 &= m,\\
z_2 &= b_{21}\rho + b_{22}m,\\
z_3 &= b_{31}\rho + b_{32}m + b_{33}\mathcal{E},
\end{aligned}$$

and

$$C^{-1} = \begin{pmatrix} -b_{21}^{-1}b_{22} & b_{21}^{-1} & 0 \\ 1 & 0 & 0 \\ b_{21}^{-1}b_{33}^{-1}b_{31}b_{22} - b_{33}^{-1}b_{32} & -b_{33}^{-1}b_{21}^{-1}b_{31} & b_{33}^{-1} \end{pmatrix}.$$

We can rewrite the eigenvalue equation in z-coordinates as

$$(B(C^{-1}z)')' = (AC^{-1}z)' + \lambda C^{-1}z,$$

or more explicitly as

$$0 = z_1' + \lambda(-b_{21}^{-1} b_{22} z_1 + b_{21}^{-1} z_2), \tag{2.22}$$

$$z_2'' = (\beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3)' + \lambda z_1, \tag{2.23}$$

$$z_3'' = (\eta_1 z_1 + \eta_2 z_2 + \eta_3 z_3)' + \lambda g(z_1, z_2, z_3), \tag{2.24}$$

where the linear function g is given by

$$g(z_1, z_2, z_3) = (b_{21}^{-1} b_{33}^{-1} b_{31} b_{22} - b_{33}^{-1} b_{32}) z_1 - (b_{33}^{-1} b_{21}^{-1} b_{31}) z_2 + b_{33}^{-1} z_3, \tag{2.25}$$

and the β and η coefficients can be calculated in terms of the α_{ij} and b_{ij}. The exact form of these coefficients is not used below, so we omit the calculation. From (2.22)–(2.24) it is a simple matter to recast the eigenvalue equation as a first-order system of the form

$$Z' = \mathbb{A}(x, \lambda) Z, \quad Z = (z_1, z_2, z_3, z_2', z_3')^{\mathrm{tr}}, \tag{2.26}$$

with a corresponding limiting system

$$Z' = \mathbb{A}_\pm(\lambda) Z.$$

The matrix $\mathbb{A}$ takes the form

$$\mathbb{A}(x, \lambda) = \begin{pmatrix} -\lambda b_{21}^{-1} b_{22} & \lambda b_{21}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \lambda + \beta_1' - \lambda b_{21}^{-1} b_{22} & \beta_2' + \lambda b_{21}^{-1} & \beta_3' & \beta_2 & \beta_3 \\ \mathbb{A}_{51} & \mathbb{A}_{52} & \eta_3' + \lambda b_{33}^{-1} & \eta_2 & \eta_3 \end{pmatrix},$$

where

$$\mathbb{A}_{51} = \eta_1' - \lambda b_{21}^{-1} b_{22} + \lambda(b_{21}^{-1} b_{33}^{-1} b_{31} b_{22} - b_{33}^{-1} b_{32}),$$

and

$$\mathbb{A}_{52} = \eta_2' + \lambda b_{21}^{-1} - \lambda b_{33}^{-1} b_{21}^{-1} b_{31}.$$

We verify the consistent-splitting hypotheses, without loss of generality, in the original w-coordinates. The characteristic equation has the form

$$(\lambda I + \mu A_\pm - \mu^2 B_\pm) v = 0.$$

We obtain a sequence of lemmas.

Lemma 2. *For* $\operatorname{Re} \lambda > 0$*, the matrix* $\mathbb{A}_\pm(\lambda)$ *has eigenvalues*

$$\mu_1^\pm(\lambda), \mu_2^\pm(\lambda) < 0 < \mu_3^\pm(\lambda), \mu_4^\pm(\lambda), \mu_5^\pm(\lambda),$$

(*with ordering referring to real parts*). *The eigenspaces* $\mathcal{S}^\pm(\lambda)$ *and* $\mathcal{U}^\pm(\lambda)$ *associated with the eigenvalues* $\mu_1^\pm(\lambda)$, $\mu_2^\pm(\lambda)$ *and* $\mu_3^\pm(\lambda)$, $\mu_4^\pm(\lambda)$, $\mu_5^\pm(\lambda)$ *depend analytically on* λ.

Proof. As established in the introduction, the number of positive/negative roots is constant for $\operatorname{Re}\lambda > 0$, so that roots can be counted as $\lambda \to +\infty$ along the real axis. There is one root with $\mu \sim \lambda$, and there are four roots with $\mu \sim \lambda^{1/2}$. See the Appendix for more details. □

Moreover, in a neighborhood of $\lambda = 0$, a bifurcation analysis yields:

Lemma 3. *For each j, there are analytic extensions of $\mu_j^\pm(\lambda)$ to a neighborhood N of $\lambda = 0$. Moreover, there are analytic choices of individual eigenvectors $V_j^\pm(\lambda)$ corresponding to $\mu_j^\pm(\lambda)$ in N.*

Proof. When $\lambda = 0$, the characteristic equation reduces to

$$(\mu A_\pm - \mu^2 B_\pm)v = 0,$$

which has a triple root at zero. Nonzero roots must satisfy $\mu^{-1} \in \sigma(A^{-1}B)$ which has two nonzero eigenvalues. One of them switches signs at $\pm\infty$. The zero roots bifurcate analytically from zero, for, linearizing about $(\lambda, \mu) = (0, 0)$, we obtain

$$(\lambda I + \mu A_\pm)v = 0.$$

We find that, on the $-\infty$ side,

$$\mu_{2,3,4}^-(\lambda) = -\frac{\lambda}{a_{1,2,3}^-} + O(\lambda^2)$$

and

$$v_{2,3,4}^-(\lambda) = r_{1,2,3}^- + O(\lambda),$$

while on the $+\infty$ side,

$$\mu_{3,4,5}^+(\lambda) = -\frac{\lambda}{a_{1,2,3}^+} + O(\lambda^2)$$

and

$$v_{3,4,5}^+(\lambda) = r_{1,2,3}^+ + O(\lambda).$$

Finally, the existence of analytic choices of eigenvectors $V_j^\pm$ follows from the fact that $V_j^\pm$ are polynomial functions of $\mu_j^\pm$ and $v_j^\pm$; specifically, $V_j = (z_1, z_2, z_3, \mu z_2, \mu z_3)^{\mathrm{tr}}$ by (2.26) for $z = Cv$, with C as in (2.21) □

Lemma 4. *There are choices of bases*

$$\mathcal{B}_{\mathcal{S}}^\pm(\lambda) = \{\phi_1^\pm(\lambda), \phi_2^\pm(\lambda)\}$$

and

$$\mathcal{B}_{\mathcal{U}}^\pm(\lambda) = \{\phi_3^\pm(\lambda), \phi_4^\pm(\lambda), \phi_5^\pm(\lambda)\}$$

of $\mathcal{S}^\pm(\lambda)$ and $\mathcal{U}^\pm(\lambda)$ which are analytic with respect to λ in $N \cup \{\operatorname{Re}\lambda > 0\}$. In the neighborhood N, they satisfy

$$\mathcal{B}_{\mathcal{S}}^+(\lambda) = k_+(\lambda) V_1^+(\lambda) \wedge V_2^+(\lambda),$$

and

$$\mathcal{B}_{\mathcal{U}}^-(\lambda) = k_-(\lambda) V_3^-(\lambda) \wedge V_4^-(\lambda) \wedge V_5^-(\lambda),$$

where $V_j^\pm$ are as in the previous lemma and $k_\pm(\lambda)$ are scalar functions such that $k_\pm(0) = 1$.

Proof. The proof follows from Lemmas 2 and 3 and a standard (nontrivial!) result of matrix perturbation theory [29] that analytic eigenprojections induce analytic eigenbases on simply connected domains. □

Finally, using the gap lemma (Theorem 1), we obtain:

Lemma 5. *There are bases $\mathcal{B}_{\mathcal{S}}(x,\lambda)$ and $\mathcal{B}_{\mathcal{U}}(x,\lambda)$ of the spaces of solutions of the eiegnvalue equations decaying at $x = \pm\infty$ which are tangent to $\mathcal{S}^+(\lambda)$ as $x \to +\infty$ and $\mathcal{U}^-(\lambda)$ as $x \to -\infty$. That is,*

$$\mathcal{S}^+(\lambda) = \lim_{x\to+\infty} \operatorname{span} \mathcal{B}_{\mathcal{S}}(x,\lambda),$$

and

$$\mathcal{U}^-(\lambda) = \lim_{x\to-\infty} \operatorname{span} \mathcal{B}_{\mathcal{U}}(x,\lambda).$$

A word on notation is in order. Working again in z-coordinates, we refer to the elements of the bases $\mathcal{B}_{\mathcal{S}}(x,\lambda)$ and $\mathcal{B}_{\mathcal{U}}(x,\lambda)$ by $Z_j^{\pm}(x,\lambda)$, so that

$$Z_j^{\pm} = (z_{1,j}^{\pm}, z_{2,j}^{\pm}, z_{3,j}^{\pm}, z_{2,j}^{\pm'}, z_{3,j}^{\pm'})^{\mathrm{tr}},$$

and we denote by $z_j^{\pm}$ with a solitary subscript the first three components of $Z_j^{\pm}$. Thus

$$z_j^{\pm} = (z_{1,j}^{\pm}, z_{2,j}^{\pm}, z_{3,j}^{\pm})^{\mathrm{tr}}.$$

2.2. The Evans function

Definition 3. The *Evans function* is

$$D(\lambda) = \det(Z_1^+, Z_2^+, Z_3^-, Z_4^-, Z_5^-)|_{x=0} = \det \begin{pmatrix} z_{1,1}^+ & z_{1,2}^+ & z_{1,3}^- & z_{1,4}^- & z_{1,5}^- \\ z_{2,1}^+ & z_{2,2}^+ & z_{2,3}^- & z_{2,4}^- & z_{2,5}^- \\ z_{3,1}^+ & z_{3,2}^+ & z_{3,3}^- & z_{3,4}^- & z_{3,5}^- \\ z_{2,1}^{+'} & z_{2,2}^{+'} & z_{2,3}^{-'} & z_{2,4}^{-'} & z_{2,5}^{-'} \\ z_{3,1}^{+'} & z_{3,2}^{+'} & z_{3,3}^{-'} & z_{3,4}^{-'} & z_{3,5}^{-'} \end{pmatrix}_{|x=0}.$$

As usual, at $\lambda = 0$ we are free to set

$$z_1^+ = z_5^- = C\bar{U}_x, \tag{2.27}$$

and at $\lambda = 0$ we also set

$$z_2^+(+\infty) = 0, \quad z_3^-(-\infty) = Cr_1^-, \quad z_4^-(-\infty) = Cr_2^-. \tag{2.28}$$

2.2.1. Calculation of $D'(0)$.

Proposition 1. *The Evans function* $D(\lambda)$ *satisfies* $D(0) = 0$ *and*

$$\operatorname{sgn} D'(0) = \operatorname{sgn} \gamma_{\mathrm{NS}} \det(r_1^-, r_2^-, [U]),$$

where

$$\gamma_{\mathrm{NS}} = \begin{pmatrix} z_{2,1}^+ & z_{2,2}^+ \\ z_{3,1}^+ & z_{3,2}^+ \end{pmatrix}$$

measures transversality of the intersection of stable/unstable manifolds in the traveling-wave equation ($\gamma_{\mathrm{NS}} \neq 0 \Leftrightarrow$ *transversality*).

Proof. That $D(0) = 0$ follows immediately from (2.27). Using the Leibniz rule to compute $D'(0)$, we find

$$D'(0) = \det(\partial_\lambda Z_1^+, Z_2^+, Z_3^-, Z_4^-, Z_5^-)|_{x=0} + \cdots + \det(Z_1^+, Z_2^+, Z_3^-, Z_4^-, \partial_\lambda Z_5^-)|_{x=0}.$$

We combine the two nonzero determinants above to obtain

$$D'(0) = \det(Z_1^+, Z_2^+, Z_3^-, Z_4^-, \tilde{Z})|_{x=0}, \tag{2.29}$$

where

$$\tilde{Z} = \partial_\lambda(Z_5^- - Z_1^+).$$

Differentiating the eigenvalue equation with respect to λ at $\lambda = 0$, we find that $\tilde{z} = \partial_\lambda(z_5^- - z_1^+)$ satisfies (after an integration)

$$B(C^{-1}\tilde{z})' = AC^{-1}\tilde{z} + [U]. \tag{2.30}$$

Also, at $\lambda = 0$, the eigenvalue equations simplify considerably for $j = 1, 2, 3, 4$ to (omitting $\pm$)

$$\begin{aligned} 0 &= z_{1,j}', \\ z_{2,j}'' &= (\beta_1 z_{1,j} + \beta_2 z_{2,j} + \beta_3 z_{3,j})', \\ z_{3,j}'' &= (\eta_1 z_{1,j} 1 + \eta_2 z_{2,j} + \eta_3 z_{3,j})', \end{aligned}$$

which can be integrated up using the boundary conditions supplied by (2.27) and (2.28). Thus, we have

$$B(C^{-1}z_j^+)' = AC^{-1}z_j^+ \quad j = 1, 2, \tag{2.31}$$

$$B(C^{-1}z_3^-)' = AC^{-1}z_3^- - a_1^- r_1^-, \tag{2.32}$$

$$B(C^{-1}z_4^-)' = AC^{-1}z_4^- - a_2^- r_2^-. \tag{2.33}$$

The first equation of each of (2.30)–(2.33) allows a simplification in the first row of the determinant (2.29). More precisely, we can replace four of the entries in the first row using

$$\begin{aligned}\tilde{z}_1 &= -[\rho],\\ z^+_{1,j} &= 0 \quad j = 1, 2,\\ z^-_{1,3} &= a^-_1 (r^-_1)_1,\\ z^-_{1,4} &= a^-_2 (r^-_2)_1.\end{aligned}$$

Similarly, the second equation of each of (2.30)–(2.33) allows a row operation to simplify the fourth row. Thus, we use

$$\begin{aligned}\tilde{z}'_2 &= \beta_1\tilde{z}_1 + \beta_2\tilde{z}_2 + \beta_3\tilde{z}_3 + [m],\\ z^{+'}_{2,j} &= \beta_1 z^+_{1,j} + \beta_2 z^+_{2,j} + \beta_3 z^+_{3,j}, \quad j = 1, 2,\\ z^{-'}_{2,3} &= \beta_1 z^-_{1,3} + \beta_2 z^-_{2,3} + \beta_3 z^-_{3,3} - a^-_1 (r^-_1)_2,\\ z^{-'}_{2,4} &= \beta_1 z^-_{1,4} + \beta_2 z^-_{2,4} + \beta_3 z^-_{3,4} - a^-_2 (r^-_2)_2\end{aligned}$$

to simplify the fourth row. Continuing in this vein, we see that the third equation of each of (2.30)–(2.33) indicates that a row operation will simplify the fifth row of (2.29). That is, we use

$$\begin{aligned}\tilde{z}'_3 &= \eta_1\tilde{z}_1 + \eta_2\tilde{z}_2 + \eta_3\tilde{z}_3 + [\mathcal{E}],\\ z^{+'}_{3,j} &= \eta_1 z^+_{1,j} + \eta_2 z^+_{2,j} + \eta_3 z^+_{3,j}, \quad j = 1, 2,\\ z^{-'}_{3,3} &= \eta_1 z^-_{1,3} + \eta_2 z^-_{2,3} + \eta_3 z^-_{3,3} - a^-_1 (r^-_1)_3,\\ z^{-'}_{3,4} &= \eta_1 z^-_{1,4} + \eta_2 z^-_{2,4} + \eta_3 z^-_{3,4} - a^-_2 (r^-_2)_3\end{aligned}$$

to simplify the fifth row. In the above, we have used $(r^-_j)_i$ to denote the ith component of r^-_j. Putting these operations together, we see that (2.29) simplifies to

$$\det \begin{pmatrix} 0 & 0 & a^-_1 & a^-_2 & -[\rho] \\ z^+_{2,1} & z^+_{2,2} & * & * & * \\ z^+_{3,1} & z^+_{3,2} & * & * & * \\ 0 & 0 & -a^-_1(r^-_1)_2 & -a^-_2(r^-_2)_2 & [m] \\ 0 & 0 & -a^-_1(r^-_1)_3 & -a^-_2(r^-_2)_3 & [\mathcal{E}] \end{pmatrix}_{|x=0}. \tag{2.34}$$

It follows from (2.34) that

$$D'(0) = a^-_1 a^-_2 \det\begin{pmatrix} z^+_{2,1} & z^+_{2,2} \\ z^+_{3,1} & z^+_{3,2}\end{pmatrix} \det(r^-_1, r^-_2, [U]).$$

From the shock inequalities (2.8), it follows that $a^-_1 a^-_2 > 0$, and the proposition is proved. □

2.2.2. Behavior for large λ. We appeal to the Appendix to determine sgn $D(\lambda)$ as $\lambda \to +\infty$ along the real axis and complete the calculation of the stability index. In the Appendix, this calculation is carried out for abstract "real viscosity" systems of the form:

$$U_t + F(U)_x = (B(U)U_x)_x,$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad F = \begin{pmatrix} f \\ g \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix},$$

and

$$u, f \in \mathbf{R}^{n-r}, \quad v, g \in \mathbf{R}^r, \quad b_1 \in \mathbf{R}^{r\times(n-r)}, \quad b_2 \in \mathbf{R}^{r\times r}.$$

With a slight abuse of notation, we adopt in this section the notation of the Appendix. Then, it follows from Lemma 9 that for λ real and sufficiently large,

$$D(\lambda) \neq 0.$$

Furthermore, we can relate sgn $D(\lambda)$ for λ large to our normalizations at $\lambda = 0$ by Lemma 10. We find for λ real and sufficiently large that

$$\operatorname{sgn} \tilde{D}(\lambda) = \operatorname{sgn} \mathbb{S}^+ \det(\pi\mathbb{W}^+, \varepsilon\mathbb{S}^+) \det(\pi\mathbb{W}^-)|_{\lambda=0}, \tag{2.35}$$

where $\tilde{D}$ is the Evans function computed in the original w-coordinates, so that D and $\tilde{D}$ differ by a nonvanishing real factor. In (2.35), $\mathbb{S}^+$ is a basis for the one-dimensional stable subspace of $\tilde{A} = A_{11} - A_{12}b_2^{-1}b_1$, where A_{ij} are the entries of the Jacobian matrix of F. In particular, we note that $(A_{11}, A_{12}) = \mathrm{d}f(\bar{U})$. In this case, $\tilde{A}$ is simply the particle velocity (the "original" u) and is thus negative since we consider a 3-shock. The symbols $\mathbb{W}^{\pm}$ denote bases for decaying solutions at each of $\pm\infty$, and π, ε denote projection and extension respectively. We work now in z-coordinates and follow the discussion of the Appendix to see that the form of (2.35) simplifies considerably. The function $D(\lambda)$ satisfies:

Lemma 6. *For λ real and sufficiently large,*

$$\operatorname{sgn} D(\lambda) = \operatorname{sgn} \gamma_{\mathrm{NS}}(\mathbb{S}^+)^2 \det(r_1^-, r_2^-, \bar{U}_x). \tag{2.36}$$

2.3. The stability index

We combine Lemma 6 and Proposition 1 to obtain

Proposition 2. *The stability index for a Lax 3-shock is*

$$\tilde{\Gamma}_{\mathrm{NS}} = \operatorname{sgn} D'(0)D(+\infty) = \operatorname{sgn} \det(r_1^-, r_2^-, [U]) \det(r_1^-, r_2^-, \bar{U}_x|_{-\infty}),$$

where $[U] = ([\rho], [m], [\mathcal{E}])^{\mathrm{tr}}$ is a vector of jumps and $\bar{U}_x = (\bar{\rho}_x, \bar{m}_x, \bar{\mathcal{E}}_x)^{\mathrm{tr}}$.

We remark that the stability index is unaffected by change in the viscosity matrix. That is, the index agrees with that of an artificially parabolic system, at least in the presence of (1.29)–(1.31). Further, we note that the calculation of $D'(0)$ captures low-frequency information. In this setting, this corresponds to "inviscid" behavior or the stability of shock solutions of the Euler Equations,

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= 0, \\ \mathcal{E}_t + [u\,(\mathcal{E} + p)]_x &= 0, \end{aligned}$$

and Δ takes the form of a Lopatinski determinant, familiar from the stability analysis of such shocks. See [26] for the calculation of this determinant for the multi-dimensional Euler equations. In the weak-shock limit we note that

$$[U] \sim \bar{U}_x \sim r_3^-,$$

which implies that the stability index satisfies

$$\operatorname{sgn} D'(0)D(+\infty) \sim \operatorname{sgn} \det(r_1^-, r_2^-, r_3^-)^2 = +1$$

consistent with stability in the weak-shock limit. Moreover, in the ideal-gas case, combining the nonvanishing of Δ which is well known for an ideal gas (see, e.g., [43, 52]), the global existence result of [21] which guarantees transversality of connections, and nonvanishing of $D(+\infty)$, we can then conclude consistency with stability for shocks of any strength in the ideal-gas case.

We remark that in the more general case some rudimentary knowledge about the connecting orbit is neccessary to evaluate

$$\operatorname{sgn} \det(r_1^-, r_2^-, \bar{U}_x),$$

namely the direction of $\bar{U}_x|_{-\infty}$.

2.4. Isentropic gas dynamics

Consider the case of isentropic gas dynamics:

$$\rho_t + (\rho u)_x = 0, \tag{2.37}$$

$$(\rho u)_t + (\rho u^2 + p)_x = (\nu u_x)_x, \tag{2.38}$$

where the pressure satisfies

$$p = p(\rho), \quad c^2 = p'(\rho) > 0.$$

A calculation analogous to but simpler than that of the previous section leads to the stability index

$$\begin{aligned} \tilde{\Gamma}_i = \operatorname{sgn} D'(0)D(+\infty) &= \operatorname{sgn} \gamma_i^2 \det(\mathbb{S}^+)^2 \det(r_1^-, [U]) \det(r_1^-, \bar{U}_x) \\ &= \operatorname{sgn} \det(r_1^-, [U]) \det(r_1^-, \bar{U}_x), \end{aligned}$$

Fig. 2.1. The phase portrait

for a 2-shock. Here, $[U] = ([\rho], [m])^{\mathrm{tr}}$ is a vector of jumps in the gas-dynamical conserved quantities, $\bar{U}_x = (\bar{\rho}_x, \bar{m}_x)^{\mathrm{tr}}$, $r_1^- = (1, u_- - c_-)^{\mathrm{tr}}$ is the outgoing right eigenvector, and the term γ_i is a transversality coefficient. In this case, evaluation of the index is straightforward since the traveling-wave equation has a one-dimensional phase space. The phase portrait is shown in Fig. 2.1. Then, $\operatorname{sgn}\det(r_1^-, [U])\det(r_1^-, \bar{U}_x)$ becomes simply $\operatorname{sgn}[\rho]\bar{\rho}_x(u_- - c_-)^2 = +1$, consistent with stability, since both $[\rho]$ and $\bar{\rho}_x$ are negative. Moreover, we note that for one-dimensional isentropic gas dynamics, HUMPHERYS [25] has shown that there are no unstable *real* eigenvalues for shocks of arbitrary strength.

3. Reacting Gas

In this section, we extend the analysis of the previous section to our main interest: strong detonation waves, which are particular traveling-wave solutions of equations (1.1)–(1.4). As described in the Introduction, we assume that the ignition function $\varphi(T)$ identically vanishes for temperatures below some ignition temperature T_i and is identically 1 for some larger value of T. We also assume that the pressure p and the temperature T are given functions of ρ and e; thus, p and T are independent of the progress of the reaction. At some points in the analysis, we further specify that the gas is ideal and polytropic, so that

$$p = \Gamma\rho e, \qquad T = c_v^{-1} e. \tag{3.1}$$

Here, $\tilde{e}$ is related to e, the specific internal energy not due to reaction, by

$$\tilde{e} = e + qY.$$

By virtue of this relation, the third and fourth equations can be combined, and the energy-balance equation can be rewritten as

$$\left(\rho\left(\frac{u^2}{2} + e\right)\right)_t + \left(\rho u(\frac{u^2}{2} + e) + up\right)_x = (\theta T_x)_x + qk\rho Y\varphi(T) + (\nu u u_x)_x.$$

This form of the energy-balance equation allows us to write the system (1.1)–(1.4) in the general framework (3.87)–(3.88) discussed below with $U = (\rho, m, \mathcal{E})^{\mathrm{tr}}$, $z = \rho Y$, $Q = \boldsymbol{q} = (0, 0, q)^{\mathrm{tr}}$, $\Phi(U) = k\varphi(T)$, $D^1(U) = d$, and $D^2(U, z) = (-\frac{dz}{\rho}, 0, 0)$. We remark at this point that, since the flux f in the kinematic variables U is as in the Navier-Stokes case, the left and right eigenvectors l_j,r_j of the flux

Jacobian $A(U)$ and their corresponding eigenvalues a_j are precisely as calculated in the previous section

In accordance with our previous analysis, we assume that the strong detonation has Lax 3-shock structure, so that

$$a_2^- < s < a_3^-, \quad a_3^+ < s.$$

Without loss of generality, we set $s = 0$, so the reaction front is stationary. The shock inequalities imply that $u_\pm < 0$, so that fluid particles cross the reaction front from right to left; alternatively, the front is "moving" to the right connecting an unburned state at $+\infty$ to a completely burned state at $-\infty$. Without loss of generality, we normalize $\rho_+ = 1$ so that the total reactant variable z satisfies

$$z_+ = 1, \quad z_- = 0. \tag{3.2}$$

We also assume that the end states are such that the temperature on the unburned $(+)$ side is below ignition so that $\varphi = 0$ and that the temperature on the burned $(-)$ side is sufficiently large so that $\varphi = 1$. The first assumption is necessary to ensure that the end state at $+\infty$ is a rest point of the traveling-wave equation. Thus, φ satisfies

$$\varphi_+ = 0, \quad \varphi_- = 1. \tag{3.3}$$

3.1. Traveling-wave and linearized equations

The $(s = 0)$ traveling-wave equation is

$$m_x = 0, \tag{3.4}$$

$$\left(\frac{m^2}{\rho} + p\right)_x = \left(\nu\left(\frac{m}{\rho}\right)_x\right)_x, \tag{3.5}$$

$$\left(\frac{m\tilde{\mathcal{E}}}{\rho} + \frac{mp}{\rho}\right)_x = (\theta T_x)_x + \left(q\rho d\left(\frac{z}{\rho}\right)_x\right)_x + \left(\nu\left(\frac{m}{\rho}\right)\left(\frac{m}{\rho}\right)_x\right)_x, \tag{3.6}$$

$$\left(\frac{mz}{\rho}\right)_x = (dz_x)_x + \left(-\frac{dz}{\rho}\rho_x\right)_x - k\varphi(T)z. \tag{3.7}$$

We fix a state at $-\infty$ and then integrate the first three equations above to find that

$$m - m_- = 0,$$

$$\left(\frac{m^2}{\rho} + p\right) - \left(\frac{m^2}{\rho} + p\right)_- = \nu\left(\frac{m}{\rho}\right)_x,$$

$$\left(\frac{m}{\rho}(\tilde{\mathcal{E}} + p)\right) - \left(\frac{m}{\rho}(\tilde{\mathcal{E}} + p)\right)_- = \theta T_x + q\rho d\left(\frac{z}{\rho}\right)_x + \nu\left(\frac{m}{\rho}\right)\left(\frac{m}{\rho}\right)_x.$$

Requiring the state at $+\infty$ to be a rest point of the system, we obtain from the first two equations above the familiar Rankine-Hugoniot conditions

$$[m] = 0, \tag{RH1}$$

$$\left[\frac{m^2}{\rho} + p\right] = 0, \tag{RH2}$$

where the brackets as usual indicate the difference between the state at $+\infty$ and that at $-\infty$. In the case of the third equation, we find (because $\tilde{\mathcal{E}} = \mathcal{E} + qz$) that

$$\left[\frac{m\tilde{\mathcal{E}}}{\rho} + \frac{mp}{\rho}\right] = \left[\frac{m\mathcal{E}}{\rho} + \frac{mp}{\rho}\right] + q\left[\frac{mz}{\rho}\right],$$

and from (3.2) the third jump condition becomes

$$0 = \left[\frac{m\mathcal{E}}{\rho} + \frac{mp}{\rho}\right] + qu_+. \tag{RH3}$$

Linearizing the system (1.1)–(1.4) about a profile $(\bar{\rho}, \bar{m}, \bar{\mathcal{E}}, \bar{z})$, we obtain

$$\rho_t + m_x = 0, \tag{3.8}$$

$$m_t + (\alpha_{21}\rho + \cdots + \alpha_{23}\mathcal{E})_x = (b_{21}\rho_x + b_{22}m_x)_x, \tag{3.9}$$

$$\mathcal{E}_t + (\alpha_{31}\rho + \cdots + \alpha_{33}\mathcal{E})_x = (b_{31}\rho_x + \cdots + b_{33}\mathcal{E}_x)_x + qkl, \tag{3.10}$$

$$z_t + (v_1\rho + v_2 m + v_4 z)_x = (dz_x)_x + (\tilde{d}\rho_x)_x - kl, \tag{3.11}$$

where α_{ij} and b_{ij} are as in the previous section, and

$$l = l_z z + l_\rho \rho + l_m m + l_{\mathcal{E}}\mathcal{E},$$

with

$$\begin{aligned} l_z(x) &= \varphi(\bar{T}), \\ l_\rho(x) &= \varphi'(\bar{T})(\bar{T}_\rho + \bar{T}_e\bar{e}_\rho), \\ l_m(x) &= -\varphi'(\bar{T})\bar{T}_e\frac{\bar{u}}{\bar{\rho}}, \\ l_{\mathcal{E}}(x) &= \varphi'(\bar{T})\bar{T}_e/\bar{\rho}, \end{aligned}$$

and

$$\begin{aligned} v_1(x) &= -\bar{u}\bar{Y} - \frac{d\bar{z}\bar{\rho}_x}{\bar{\rho}^2}, \\ v_2(x) &= \frac{\bar{z}}{\bar{\rho}}, \\ v_4(x) &= \bar{u} + \frac{d\bar{\rho}_x}{\bar{\rho}}. \end{aligned}$$

We note that, due to the structure of φ and (3.3), it follows that l_ρ, l_m, and $l_{\mathcal{E}}$ vanish at both $\pm\infty$ while $l_{z+} = 0$ and $l_{z-} = 1$. The equations (3.8)–(3.11) can also be written in the more compact form

$$(Bw_x)_x = (Aw)_x + w_t + \boldsymbol{q}kg(w, z),$$
$$(dz_x)_x + (\tilde{d}w_x)_x = (V_w w)_x + (V_z z)_x + z_t - kg(w, z).$$

In an abuse of notation, we have written $\tilde{d}$ to stand for both the 1×3 matrix $(-\frac{dz}{\rho}, 0, 0)$ and the $(1, 1)$-entry of that matrix. The meaning will be clear from the context.

Under the invertible change of coordinates

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ \tilde{d} & 0 & 0 & d \end{pmatrix}}_{C} \begin{pmatrix} \rho \\ m \\ \mathcal{E} \\ z \end{pmatrix}, \tag{3.12}$$

the eigenvalue equations corresponding to (3.8)–(3.11) take the simple form

$$0 = \zeta_1' + \lambda(-b_{21}^{-1}b_{22}\zeta_1 + b_{21}^{-1}\zeta_2), \tag{3.13}$$
$$\zeta_2'' = (\beta_1\zeta_1 + \cdots + \beta_3\zeta_3)' + \lambda\zeta_1, \tag{3.14}$$
$$\zeta_3'' = (\eta_1\zeta_1 + \cdots + \eta_3\zeta_3)' + \lambda g(\hat{\zeta}) + qkl \cdot \zeta, \tag{3.15}$$
$$\zeta_4'' = (\theta_1\zeta_1 + \cdots + \theta_4\zeta_4)' + \lambda h(\zeta) - kl \cdot \zeta. \tag{3.16}$$

Here, we have used the notation $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ and $\hat{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$. The linear functions g and h take the form

$$g(\hat{\zeta}) = (b_{21}^{-1}b_{33}^{-1}b_{31}b_{22} - b_{33}^{-1}b_{32})\zeta_1 - (b_{33}^{-1}b_{21}^{-1}b_{31})\zeta_2 + b_{33}^{-1}\zeta_3$$

and

$$h(\zeta) = d^{-1}b_{21}^{-1}b_{22}\tilde{d}\zeta_1 - d^{-1}b_{21}^{-1}\tilde{d}\zeta_2 + d^{-1}\zeta_4.$$

The coefficients β_j and η_j are as in the previous section, while the θ_j depend on $d, \tilde{d}, b_{ij}$ and v_j. Also, $l \cdot \zeta = (l_1\zeta_1 + \cdots + l_4\zeta_4)$ where l_j depends on $l_\rho, l_m, l_{\mathcal{E}}, l_z, b_{ij}$, d and $\tilde{d}$. The change of coordinates matrix C has block structure respecting the division of the variables into gas-dynamical variables $w = (\rho, m, \mathcal{E})^{\mathrm{tr}}$, and reaction variable z. Thus, we have

$$\zeta = \left(\begin{array}{c|c} C_{\mathrm{NS}} & 0 \\ \hline \tilde{d}\ 0\ 0 & d \end{array}\right) \begin{pmatrix} w \\ z \end{pmatrix},$$

where C_{NS} denotes the change of variables in the nonreacting case considered in the previous section. The inverse of the matrix C also respects this structure and is given by

$$C^{-1} = \left(\begin{array}{c|c} C_{\mathrm{NS}}^{-1} & 0 \\ \hline d^{-1}b_{21}^{-1}b_{22}\tilde{d}^{-1}\ \ -d^{-1}b_{21}^{-1}\tilde{d}\ \ 0 & d^{-1} \end{array}\right).$$

We note that when $\lambda = 0$, by a substitution of (3.16) into (3.15) through the term $l \cdot \zeta$ which appears in both equations, we recover a third equation in which every term is differentiated; this means that we can integrate up the first three equations subject to appropriate boundary conditions. This fact is what will allow for a simplification of the Evans-function determinant via row operations in the calculation of $D'(0)$. The eigenvalue equations (3.13)–(3.16) can be written as a first-order system of the form

$$Z' = \mathbb{A}(x, \lambda)Z, \quad Z = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_2', \zeta_3', \zeta_4')^{\mathrm{tr}}, \tag{3.17}$$

with corresponding constant-coefficient system, as $x \to \pm\infty$,

$$Z' = \mathbb{A}_\pm(\lambda)Z. \tag{3.18}$$

The characteristic equation in original (w, z)-coordinates is

$$M_\pm \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the characteristic matrix $M_\pm$ has block structure

$$\left(\begin{array}{c|c} M_\pm^1 & M_\pm^2 \\ \hline M_\pm^3 & M_\pm^4 \end{array}\right),$$

with blocks

$$M_\pm^1 = \begin{pmatrix} \lambda & \mu & 0 \\ \mu a_{21} - \mu^2 b_{21} & \lambda + \mu a_{22} - \mu^2 b_{22} & \mu a_{23} \\ \mu a_{31} - \mu^2 b_{31} & \mu a_{32} - \mu^2 b_{32} & \lambda + \mu a_{33} - \mu^2 b_{33} \end{pmatrix},$$

$$M_\pm^2 = \begin{pmatrix} 0 \\ 0 \\ -qk\varphi_\pm \end{pmatrix},$$

$$M_\pm^3 = \begin{pmatrix} \mu v_1 - \mu^2 \tilde{d} & \mu v_2 & 0 \end{pmatrix},$$

and

$$M_\pm^4 = \lambda - \mu^2 d + \mu v_4 + k\varphi_\pm.$$

We take advantage of the fact that that the coefficient matrices $\mathbb{A}_\pm$ have no center subspaces, (1.33), so that we may count the number of stable/unstable roots in the limit $\lambda \to +\infty$ along the real axis. Due to the incomplete parabolicity of the viscosity matrix B, we expect one root which scales as

$$\mu \sim \tilde{\mu}\lambda, \quad \tilde{\mu} = O(1),$$

hence the matrix $M_\pm$ takes the form

$$\left(\begin{array}{ccc|c} \lambda & \tilde\mu\lambda & 0 & 0 \\ \tilde\mu\lambda a_{21}-\tilde\mu^2\lambda^2 b_{21} & \lambda+\tilde\mu\lambda a_{22}-\tilde\mu^2\lambda^2 b_{22} & \tilde\mu\lambda a_{23} & 0 \\ \tilde\mu\lambda a_{31}-\tilde\mu^2\lambda^2 b_{31} & \tilde\mu\lambda a_{32}-\tilde\mu^2\lambda^2 b_{32} & \lambda+\tilde\mu a_{33}-\tilde\mu^2\lambda^2 b_{33} & -qk\varphi_\pm \\ \hline \tilde\mu\lambda u_\pm Y_\pm-\tilde\mu^2\lambda^2\tilde d & \tilde\mu\lambda Y_\pm & 0 & N \end{array}\right)_\pm,$$

where

$$N=\lambda-\tilde\mu^2\lambda^2 d+\tilde\mu\lambda u_\pm+k\varphi_\pm.$$

Upon dividing each row by the highest power of λ, we find that in the limit $\lambda\to+\infty$, the roots satisfy a block triangular system

$$\left(\begin{array}{ccc|c} 1 & \tilde\mu & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ \hline \tilde d & 0 & 0 & d \end{array}\right)_\pm \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.19}$$

where the upper left-hand block is easily recognized as the gas-dynamics block which appears in the Appendix. Thus there is one root

$$-\tilde\mu^{-1}\in\sigma(u_\pm),$$

and since we are considering detonations with 3-shock structure, there is one unstable root at each of $\pm\infty$. The remaining roots scale as

$$\mu\sim\tilde\mu\lambda^{1/2},\quad \tilde\mu=O(1).$$

Thus, upon substituting and dividing as before, we obtain a different block triangular system,

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -\tilde\mu^2 b_{21} & -\tilde\mu^2 b_{22}+1 & 0 & 0 \\ \tilde\mu^2 b_{31} & -\tilde\mu^2 b_{32} & -\tilde\mu^2 b_{33}+1 & 0 \\ \hline -\tilde\mu^2\tilde d & 0 & 0 & -\tilde\mu^2 d+1 \end{array}\right)_\pm \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.20}$$

It follows immediately from (3.20) that $w_1=0$; thus, roots must satisfy

$$\left[\left(\begin{array}{cc|c} b_{22} & 0 & 0 \\ b_{32} & b_{33} & 0 \\ \hline 0 & 0 & d \end{array}\right)-\tilde\mu^{-2}I\right]\begin{pmatrix} w_2 \\ w_3 \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

that is, $\tilde\mu^{-2}$ must be an eigenvalue of the block diagonal matrix

$$\left(\begin{array}{cc|c} b_{22} & 0 & 0 \\ b_{32} & b_{33} & 0 \\ \hline 0 & 0 & d \end{array}\right)_\pm. \tag{3.21}$$

This yields 3 stable and 3 unstable roots at each of $\pm\infty$. We note that, due to the block diagonal structure of (3.21), there is a 2-1 split at each infinity of kinematic versus reactive roots. Summarizing, we have found that there are 3 stable and 4 unstable roots at each of $\pm\infty$ as long as $\operatorname{Re}\lambda > 0$. Thus, we have now verified consistent splitting by direct calculation.

Consistent splitting breaks down at $\lambda = 0$. We look first at the $x = +\infty$ case. Due to (3.3), the characteristic equation has lower-block triangular structure when $\lambda = 0$. Thus, the characteristic matrix has the form (in which the $+$ subscript is dropped):

$$\left(\begin{array}{ccc|c} 0 & \mu & 0 & 0 \\ \mu a_{21} - \mu^2 b_{21} & \mu a_{22} - \mu^2 b_{22} & \mu a_{23} & 0 \\ \mu a_{31} - \mu^2 b_{31} & \mu a_{32} - \mu^2 b_{32} & \mu a_{33} - \mu^2 b_{33} & 0 \\ \hline \mu u_+ - \mu^2 \tilde{d} & \mu & 0 & -\mu^2 d + \mu u_+ \end{array}\right).$$

From our analysis of the nonreacting case, we know that the stable roots corresponding to the upper left-hand kinematic block do not vanish at $\lambda = 0$, thus they correspond to *fast* kinematic modes. On the other hand, looking at the reaction block,

$$-\mu^2 d + \mu u_+ = 0,$$

it is clear that there are solutions $\mu = 0$ corresponding to a slow unstable reactive mode and $\mu = \frac{u_+}{d}$, a fast stable reactive mode. A bifurcation analysis as in the previous section shows that the roots have analytic extensions in a neighborhood of λ=0.

In the $x = -\infty$ case, we find when $\lambda = 0$ that, due to (3.2), the characteristic matrix has the upper-block triangular form (in which the $-$ subscript is dropped):

$$\left(\begin{array}{ccc|c} 0 & \mu & 0 & 0 \\ \mu a_{21} - \mu^2 b_{21} & \mu a_{22} - \mu^2 b_{22} & \mu a_{23} & 0 \\ \mu a_{31} - \mu^2 b_{31} & \mu a_{32} - \mu^2 b_{32} & \mu a_{33} - \mu^2 b_{33} & -qk \\ \hline 0 & 0 & 0 & -\mu^2 d + \mu u + k \end{array}\right).$$

From our previous analysis of the nonreacting case, we know that two of the unstable roots from the gas block vanish at $\lambda = 0$. By block structure, the corresponding vectors have the expansion

$$\begin{pmatrix} r_j^- \\ 0 \end{pmatrix} + O(\lambda), \quad j = 1, 2,$$

where r_j^- is an eigenvector of the flux Jacobian A. Since $d, k > 0$, the roots coming from the reaction block satisfy

$$\mu = \frac{u_-}{2d} \mp \frac{\sqrt{u_-^2 + 4dk}}{2d} \neq 0.$$

This implies that all reactive modes are fast modes on the $x = -\infty$ side. We note that this discussion shows that all the reactive modes of our interest (stable at $+\infty$/unstable at $-\infty$) are fast modes, and thus asymptotically vanish in both kinematic and reaction components. We also note that the inclusion of the differing block structure at each of $\pm\infty$ corrects an omission in the abstract development considered in [65].

Finally, applying the gap lemma, we obtain bases

$$\{Z_1^+, Z_2^+, Z_3^+\} \text{ and } \{Z_4^-, Z_5^-, Z_6^-, Z_7^-\}$$

spanning the stable manifold at $+\infty$ and the unstable manifold at $-\infty$. We use the notation

$$Z_j^\pm = (\zeta_{1,j}^\pm, \zeta_{2,j}^\pm, \zeta_{3,j}^\pm, \zeta_{4,j}^\pm, \zeta_{2,j}^{\pm'}, \zeta_{3,j}^{\pm'}, \zeta_{4,j}^{\pm'})^{\mathrm{tr}} \tag{3.22}$$

when we need to indicate the components.

Following our standard convention, we identify Z_1^+/Z_7^- with the derivative of the profile; this is possible since the derivative of the profile satisfies the linearized system and decays at both $\pm\infty$. We let Z_2^+ and Z_3^+ correspond to fast kinematic and reactive modes respectively. On the $-\infty$ side, we let Z_4^- and Z_5^- correspond to slow kinematic modes while Z_6^- is a fast kinematic mode. Note that implicit in this assignment is the assumption that the profile approaches the burned end state parallel to the reactive mode. This assumption is generic in the case where the reaction is slower and actually occurs in the ZND limit as evidenced in our discussion of existence in Section 1.

3.2. The Evans function

Definition 4. The *Evans function* is

$$D(\lambda) = \det(Z_1^+, Z_2^+, Z_3^+, Z_4^-, Z_5^-, Z_6^-, Z_7^-)|_{x=0}. \tag{3.23}$$

We make the usual normalizations at $\lambda = 0$. Namely, we put

$$\begin{pmatrix} \zeta_{1,1}^+ \\ \zeta_{2,1}^+ \\ \zeta_{3,1}^+ \\ \zeta_{4,1}^+ \end{pmatrix} = \begin{pmatrix} \zeta_{1,7}^- \\ \zeta_{2,7}^- \\ \zeta_{3,7}^- \\ \zeta_{4,7}^- \end{pmatrix} = C \begin{pmatrix} \bar\rho_x \\ \bar m_x \\ \bar{\mathcal{E}}_x \\ \bar z_x \end{pmatrix}, \tag{3.24}$$

and the other fast modes satisfy

$$\begin{aligned} &Z_2^+(+\infty) = Z_3^+(+\infty) = 0, \\ &Z_6^-(-\infty) = 0, \end{aligned} \tag{3.25}$$

while the slow modes satisfy

$$\begin{pmatrix} \zeta_{1,4}^- \\ \zeta_{2,4}^- \\ \zeta_{3,4}^- \\ \zeta_{4,4}^- \end{pmatrix} = C \begin{pmatrix} r_1^- \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \zeta_{1,5}^- \\ \zeta_{2,5}^- \\ \zeta_{3,5}^- \\ \zeta_{4,5}^- \end{pmatrix} = C \begin{pmatrix} r_2^- \\ 0 \end{pmatrix}. \tag{3.26}$$

3.2.1. Calculation of $D'(0)$.

Proposition 3. *The Evans function* $D(\lambda)$ *satisfies* $D(0) = 0$ *and* $D'(0) = \gamma_d \Delta$ *where*

$$\Delta = \det(r_1^-, r_2^-, [U] + \boldsymbol{q}), \tag{3.27}$$

and

$$\gamma_d = \det \begin{pmatrix} \zeta_{2,1}^+ & \zeta_{2,2}^+ & \zeta_{2,3}^+ & \zeta_{2,6}^- \\ \zeta_{3,1}^+ & \zeta_{3,2}^+ & \zeta_{3,3}^+ & \zeta_{3,6}^- \\ \zeta_{4,1}^+ & \zeta_{4,2}^+ & \zeta_{4,3}^+ & \zeta_{4,6}^- \\ \zeta_{4,1}^{+'} & \zeta_{4,2}^{+'} & \zeta_{4,3}^{+'} & \zeta_{4,6}^{-'} \end{pmatrix} \tag{3.28}$$

measures transversality of the stable and unstable manifolds in the traveling-wave equation ($\gamma_d \neq 0 \Leftrightarrow$ *transversality*).

Before beginning the proof, we remark that in contrast to the case considered earlier, γ_d is not extreme, that is, it involves fast modes from both infinities.

Proof. As before, $D(0) = 0$ follows immediately from the normalization (3.24) chosen for the basis elements. Applying the Leibniz rule,

$$D'(0) = \det(\partial_\lambda Z_1^+, Z_2^+, \dots, Z_7^-)|_{x=0} + \cdots + \det(Z_1^+, \dots, Z_6^-, \partial_\lambda Z_7^-)|_{x=0}, \tag{3.29}$$

and combining the two nonzero determinants in the above equation, we obtain

$$D'(0) = \det(Z_1^+, Z_2^+, Z_3^+, Z_4^-, Z_5^-, Z_6^-, \tilde{Z})|_{x=0}, \tag{3.30}$$

where $\tilde{Z} = \tilde{Z}^- - \tilde{Z}^+ = \partial_\lambda Z_7^- - \partial_\lambda Z_1^+$. For clarity, we perform the necessary manipulations in the original (w, z)-coordinates and then translate the results to the ζ-coordinates. Thus, we write down the general form of the eigenvalue equation:

$$(Bw')' = (Aw)' + \lambda w + \boldsymbol{q} k\varphi z, \tag{3.31}$$

$$(dz')' + (\tilde{d}w')' = (V_w w)' + (V_z z)' + \lambda z - k\varphi z. \tag{3.32}$$

When $\lambda = 0$, we can make a substitution from (3.32) into (3.31) to obtain a modified equation

$$(Bw')' = (Aw)' + \boldsymbol{q}(-(dz')' - (\tilde{d}w')' + (V_w w)' + (V_z z)'), \tag{3.33}$$

in which every term is differentiated, and which therefore can be integrated to

$$Bw' - Bw_\pm = Aw - Aw_\pm + \boldsymbol{q}(-dz' + dz_\pm - \tilde{d}w' + \tilde{d}w_\pm + V_w w - V_w w_\pm + V_z z - V_z z_\pm),$$

with the $\pm$ subscripts indicating boundary conditions to be supplied by the normalizations (3.24)–(3.26). It follows that the fast modes satisfy

$$Bw' + \boldsymbol{q} dz' + \boldsymbol{q}\tilde{d}w' = (A + \boldsymbol{q} V_w)w + \boldsymbol{q} V_z z, \tag{3.34}$$

while the slow modes satisfy

$$Bw' + \boldsymbol{q}dz' + \boldsymbol{q}\tilde{d}w' = (A + \boldsymbol{q}V_w)w + \boldsymbol{q}V_z z - a_j^- r_j^-. \tag{3.35}$$

On the other hand, $(\tilde{w}^{\pm}, \tilde{z}^{\pm})$ satisfy the variational equations at $\lambda = 0$:

$$(B\tilde{w}^{\pm'})' = (A\tilde{w}^{\pm})' + \bar{U}_x + \boldsymbol{q}k\varphi\tilde{z}^{\pm}, \tag{3.36}$$

$$(d\tilde{z}^{\pm'})' + (\tilde{d}\tilde{w}^{\pm'})' = (V_w\tilde{w}^{\pm})' + (V_z\tilde{z}^{\pm})' + \bar{z}_x - k\varphi\tilde{z}^{\pm}. \tag{3.37}$$

We make the same substitution from (3.37) into (3.36) so that every term is a derivative. Then, we integrate $(\tilde{w}^{+}, \tilde{z}^{+})$ from x to $+\infty$ to obtain

$$B\tilde{w}^{+'} + \boldsymbol{q}d\tilde{z}^{+'} + \boldsymbol{q}\tilde{d}\tilde{w}^{+'} = (A + \boldsymbol{q}V_w)\tilde{w}^{+} + \boldsymbol{q}V_z\tilde{z}^{+} + \bar{U} - U_+ + \bar{z} - z_+, \tag{3.38}$$

and $(\tilde{w}^{-}, \tilde{z}^{-})$ from $-\infty$ to x to obtain

$$B\tilde{w}^{-'} + \boldsymbol{q}d\tilde{z}^{-'} + \boldsymbol{q}\tilde{d}\tilde{w}^{-'} = (A + \boldsymbol{q}V_w)\tilde{w}^{-} + \boldsymbol{q}V_z\tilde{z}^{-} + \bar{U} - U_- + \bar{z} - z_-. \tag{3.39}$$

It follows by subtracting (3.38) from (3.39) that $(\tilde{w}, \tilde{z})$ satisfy

$$B\tilde{w}' + \boldsymbol{q}d\tilde{z}' + \boldsymbol{q}\tilde{d}\tilde{w}' = (A + \boldsymbol{q}V_w)\tilde{w} + \boldsymbol{q}V_z\tilde{z} + [U] + \boldsymbol{q}. \tag{3.40}$$

Translating this information to ζ-coordinates, we have for fast modes (kinematic and reactive) $j = 1, 2, 3, 6$

$$\begin{aligned} 0 &= \zeta_{1,j}, \\ \zeta'_{2,j} &= \beta_1\zeta_{1,j} + \cdots + \beta_3\zeta_{3,j}, \\ \zeta'_{3,j} &= \eta_1\zeta_{1,j} + \cdots + \eta_3\zeta_{3,j}, +qk(-\zeta'_{4,j} + \theta_1\zeta_{1,j} + \cdots + \theta_4\zeta_{4,j}). \end{aligned}$$

Note that the equation for $\zeta''_{4,j}$ is unchanged. We do not get any simplification from that equation. Also, when $j = 4$,

$$\begin{aligned} 0 &= \zeta_{1,4} - a_1^-(r_1^-)_1, \\ \zeta'_{2,4} &= \beta_1\zeta_{1,4} + \cdots + \beta_3\zeta_{3,4} - a_1^-(r_1^-)_2, \\ \zeta'_{3,4} &= \eta_1\zeta_{1,4} + \cdots + \eta_3\zeta_{3,4} + qk(-\zeta'_{4,4} + \theta_1\zeta_{1,4} + \cdots + \theta_4\zeta_{4,4}) - a_1^-(r_1^-)_3, \end{aligned}$$

and similarly for $j = 5$,

$$\begin{aligned} 0 &= \zeta_{1,5} - a_2^-(r_2^-)_1, \\ \zeta'_{2,5} &= \beta_1\zeta_{1,5} + \cdots + \beta_3\zeta_{3,5} - a_2^-(r_2^-)_2, \\ \zeta'_{3,5} &= \eta_1\zeta_{1,5} + \cdots + \eta_3\zeta_{3,5} + qk(-\zeta'_{4,5} + \theta_1\zeta_{1,5} + \cdots + \theta_4\zeta_{4,5}) - a_2^-(r_2^-)_3. \end{aligned}$$

Finally, $\tilde{\zeta}$ satisfies

$$\begin{aligned} 0 &= \tilde{\zeta}_1 + [\rho], \\ \tilde{\zeta}'_2 &= \beta_1\tilde{\zeta}_1 + \cdots + \beta_3\tilde{\zeta}_3 + [m], \\ \tilde{\zeta}'_3 &= \eta_1\tilde{\zeta}_1 + \cdots + \eta_3\tilde{\zeta}_3 + qk(-\tilde{\zeta}'_4 + \theta_1\tilde{\zeta}_1 + \cdots + \theta_4\tilde{\zeta}_4) + [\mathcal{E}] + q. \end{aligned}$$

These equations allow us to perform row operations simplifying rows 1, 5, and 6. Then, we find by rearranging rows and columns that

$$D'(0) = \det \left(\begin{array}{cccc|c} 0 & \cdots & \cdots & 0 & \\ 0 & \cdots & \cdots & 0 & a_1^- r_1^- \; a_2^- r_2^- \; [U] + \boldsymbol{q} \\ 0 & \cdots & \cdots & 0 & \\ \hline \zeta_{2,1}^+ & \zeta_{2,2}^+ & \zeta_{2,3}^+ & \zeta_{2,6}^- & * \; * \; * \\ \zeta_{3,1}^+ & \zeta_{3,2}^+ & \zeta_{3,3}^+ & \zeta_{3,6}^- & \vdots \; \vdots \; \vdots \\ \zeta_{4,1}^+ & \zeta_{4,2}^+ & \zeta_{4,3}^+ & \zeta_{4,6}^- & \vdots \; \vdots \; \vdots \\ \zeta_{4,1}^{+'} & \zeta_{4,2}^{+'} & \zeta_{4,3}^{+'} & \zeta_{4,6}^{-'} & * \; * \; * \end{array} \right),$$

from which the result follows. □

3.2.2. Behavior for large λ. To finish the calculation of the stability index, it remains to determine

$$\operatorname{sgn} D(\lambda) \quad \text{as} \quad \lambda \to +\infty, \text{ real.}$$

Here, block triangular structure plays a key role. In particular, it allows for certain "diagonal" kinematic and reaction components of eigenvectors to be separately analytically specifiable. This means that we will be able to calculate the sign in two pieces. For the gas-dynamical piece, the analysis of the Appendix applies. On the other hand, the reaction piece can be treated directly.

For ease of comparison and consistency with our previous analyses, we shift now to the notation of the Appendix. Thus, (u, v) represent the gas-dynamical variables, $\tilde{A} = A_{11} - A_{12} b_2^{-1} b_1$ where A_{ij} is the ij-entry of the flux Jacobian, and b_1, b_2 are the nonzero blocks of the matrix B. Recall that in this case $\tilde{A}$ is simply the particle velocity (u in the usual notation). For 3-shocks, this quantity is uniformly negative. We also use z to represent the reaction variable.

Proposition 4. *For real λ sufficiently large,*

$$\operatorname{sgn} D(\lambda) = \operatorname{sgn} \underbrace{\det(\mathbb{S}^+) \det(\pi \mathbb{W}^+, \epsilon \mathbb{S}^+) \det(\pi \mathbb{W}^-)}_{\text{kinematic only } j=1,2,4,5,6} \times \underbrace{z_3^+ z_7^-}_{\text{reaction}} \neq 0, \tag{3.41}$$

where π denotes projection of the kinematic variable $W = (u, v, v')$ onto (u, v) components, and $\mathbb{S}(x)$ is a real basis of the stable subspace of $\tilde{A}$, and $\varepsilon u := (u, -b_2^{-1} b_1 u)$ denotes extension.

Proof. For real λ sufficiently large, from the block triangular structure in (3.19) and (3.20), it follows that at any (fixed) x the stable/unstable manifolds of the frozen eigenvalue equation are spanned by vectors of the forms

$$\underbrace{\begin{pmatrix} u \\ -b_2^{-1} b_1 u \\ * \\ * \\ * \end{pmatrix}, \begin{pmatrix} 0 \\ v \\ \mp \tilde{\mu} \lambda^{1/2} v \\ * \\ * \end{pmatrix}}_{\text{(kinematic)}}, \text{ and } \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \pm \lambda^{1/2} d^{-1/2} \end{pmatrix}}_{\text{(reaction)}}.$$

By the tracking lemma and the discussion of Section 1.5.3, the Evans function therefore satisfies

$$D(\lambda) \sim \det \left\{ V_1 \underbrace{\left(\begin{array}{c|c} \beta_1^+ & 0 \\ \hline * & \alpha_z^+ \end{array}\right)}_{\bar{\alpha}^+}, V_2 \underbrace{\left(\begin{array}{c|c} \beta_1^+ & 0 \\ \hline * & \alpha_z^+ \end{array}\right)}_{\bar{\alpha}^+} \right\}, \tag{3.42}$$

where

$$V_1 = \begin{pmatrix} 0 & 0 & 0 \\ v_1 & v_2 & 0 \\ -\tilde{\mu}_1 \lambda^{1/2} v_1 & -\tilde{\mu}_2 \lambda^{1/2} v_2 & 0 \\ * & * & 1 \\ * & * & -\lambda^{1/2} d^{-1/2} \end{pmatrix} \tag{3.43}$$

and

$$V_2 = \begin{pmatrix} u & 0 & 0 & 0 \\ -b_2^{-1} b_1 u & v_1 & v_2 & 0 \\ * & \tilde{\mu}_1 \lambda^{1/2} v_1 & \tilde{\mu}_2 \lambda^{1/2} v_2 & 0 \\ * & * & * & \lambda^{1/2} d^{-1/2} \end{pmatrix}, \tag{3.44}$$

so long as the right-hand side does not vanish. In (3.42), $\alpha_j^\pm$ and $\beta_j^\pm$ are as in the nonreacting case, so that

$$\det\left(\beta_1^+\right) \quad \text{and} \quad \det \begin{pmatrix} \alpha_1^- & \alpha_2^- \\ \beta_2^- & \beta_1^- \end{pmatrix} \tag{3.45}$$

are real nonzero quantities while $\alpha_z^\pm$ are real, nonvanishing scalar functions of x. Note that block triangular structure plays a key role here since it allows the matrices $\bar{\alpha}^\pm$ to be block triangular. The right-hand side of (3.42) is equal to the product of

$$\det \left(V_1 \middle| V_2 \right) \tag{3.46}$$

and

$$\det \begin{pmatrix} \bar{\alpha}^+ & 0 \\ 0 & \bar{\alpha}^- \end{pmatrix}. \tag{3.47}$$

Interchanging columns, we find that the the determinant (3.46) simplifies into the product

$$- \det \begin{pmatrix} 0 & 0 & u & 0 & 0 \\ v_1 & v_2 & -b_2^{-1} b_1 u & v_1 & v_2 \\ -\tilde{\mu}_1 \lambda^{1/2} v_1 & -\tilde{\mu}_2 \lambda^{1/2} v_2 & * & \tilde{\mu}_1 \lambda^{1/2} v_1 & \tilde{\mu}_2 \lambda^{1/2} v_2 \end{pmatrix} \times \det \begin{pmatrix} 1 & 1 \\ -\left(\frac{\lambda}{d}\right)^{1/2} & \left(\frac{\lambda}{d}\right)^{1/2} \end{pmatrix} \tag{3.48}$$

of kinematic and reaction parts. The first, kinematic factor then reduces as in Lemma 9 to

$$\det(\mathbb{S}^+)\det(\pi\mathbb{W}^+, \varepsilon\mathbb{S}^+)\det(\pi\mathbb{W}^-),$$

while the sign of the second, reaction factor is (where we have included the minus sign from (3.48))

$$-z_3^+ z_7^-,$$

whence the result follows. Nonvanishing follows from the corresponding result for nonreacting gas dynamics and the fact the $\alpha_z^\pm$ are nonvanishing for all x. □

Nonvanishing of $D(+\infty)$ implies that the stability index $\tilde{\Gamma}$ satisfies either

$$\tilde{\Gamma} = \operatorname{sgn}\gamma_d\Delta$$

or

$$\tilde{\Gamma} = -\operatorname{sgn}\gamma_d\Delta$$

as model parameters are varied smoothly. Thus, we find that the *relative stability index*, defined to be

$$\operatorname{sgn}\gamma_d\Delta,$$

gives a measure of *spectral flow*. That is, changes in the sign of the relative stability index indicate a change in stability.

In this case, however, we can do more and actually evaluate the (absolute) stability index,

$$\operatorname{sgn} D'(0)D(+\infty),$$

by relating our formula above for large λ to the normalizations we have chosen at $\lambda = 0$.

Proposition 5. *For λ real and sufficiently large,*

$$\operatorname{sgn} D(\lambda) = -\operatorname{sgn}\det(\mathbb{S}^+)\det(\pi\mathbb{W}^+, \varepsilon\mathbb{S}^+)\det(\pi\mathbb{W}^-)z_3^+ z_7^-|_{\lambda=0} \neq 0. \tag{3.49}$$

Proof. The form of (3.41) allows us to connect to $\lambda = 0$ separately in kinematic and reaction terms. Since the pair (A, B) satisfy the semidissipativity conditions, the result of the Appendix holds. As for the reaction term, the pull back to $\lambda = 0$ is evident since "reaction" vectors have the form

$$(*, *, *, *, *, \underbrace{1}_{z}, *),$$

hence projection onto the z component never vanishes and cannot change sign. □

3.3. The stability index

Combining Propositions 3 and 5, we obtain the desired result.

Theorem 6. *The stability index for a strong detonation with Lax 3-shock structure is*

$$\tilde{\Gamma} := -\operatorname{sgn} \gamma_d \Delta \gamma_{\mathrm{NS}} z_3^+ z_7^- \det(r_1^-, r_2^-, U_6^-),$$

where γ_{NS} involves only the gas-dynamical components.

Proof. The stability index has the form $\tilde{\Gamma} = \operatorname{sgn} D'(0) D(+\infty)$. From Proposition 3, we find that $D'(0) = \gamma_d \Delta$. On the other hand, from Proposition 5, the term $\det(\pi \mathbb{W}^-)$ simplifies to $\det(r_1^-, r_2^-, U_6^-)$ as in the Appendix, while the term $\det(\pi \mathbb{W}^+, \varepsilon \mathbb{S}^+)$ breaks into the product of

$$\gamma_{\mathrm{NS}} = \det \begin{pmatrix} \zeta_{2,1}^+ & \zeta_{2,2}^+ \\ \zeta_{3,1}^+ & \zeta_{3,2}^+ \end{pmatrix}$$

and $\det(\mathbb{S}^+)$, the latter of which cancels with the factor $\det(\mathbb{S}^+)$ appearing in (3.49).
□

3.4. Reduction of Δ and nonvanishing of Δ for an ideal gas

The term Δ in the stability index has the form of a modified Lopatinski determinant

$$\Delta = \det(r_1^-, r_2^-, [U] + \boldsymbol{q}).$$

In place of the determinant above, it is more convenient for calculations to rewrite the quantity as

$$\Delta = l_3^- \cdot ([U] + \boldsymbol{q}) = l_3^- \cdot [U] + l_3^- \cdot \boldsymbol{q},$$

where

$$l_3^- = \left(p_\rho - \frac{p_e}{\rho} e - cu + \frac{p_e u^2}{2\rho}, c - \frac{p_e}{\rho} u, \frac{p_e}{\rho} \right)$$

is the left eigenvector dual to r_3^-,

$$[U] = ([\rho], 0, [\mathcal{E}])^{\mathrm{tr}},$$

and $\boldsymbol{q} = (0, 0, q)^{\mathrm{tr}}$. It is then straightforward to calculate

$$\Delta = \left(p_\rho - \frac{cm}{\rho} \right) [\rho] + \frac{p_e m^2}{2\rho} \left(\frac{[\rho]}{\rho^2} + [1/\rho] \right) + \frac{p_e}{\rho} \rho_+ [e] + \frac{p_e}{\rho} q. \tag{3.50}$$

Now, using

$$\frac{[\rho]}{\rho_-^2} + [1/\rho] = -\frac{1}{\rho_-} [\rho][1/\rho]$$

in our expression for Δ and simplifying, we obtain

$$\Delta = \left(-\frac{p_e m^2}{2\rho^2} [1/\rho] + p_\rho - \frac{cm}{\rho} \right) [\rho] + \frac{p_e}{\rho} \rho_+ [e] + \frac{p_e}{\rho} q. \tag{3.51}$$

Similarly as in [52], we have:

Claim 1. *The following equality holds:* $[e] + \langle p\rangle[1/\rho] = -\frac{qu_+}{m}$, *where* $\langle p\rangle := \frac{1}{2}(p_+ + p_-)$.

Proof. From (RH3),

$$m\left[\frac{\mathcal{E}}{\rho}\right] + m\left[\frac{p}{\rho}\right] = -u_+q,$$

or

$$[E] + \left[\frac{p}{\rho}\right] = -\frac{u_+q}{m}. \tag{3.52}$$

Using the relationship $E = \frac{u^2}{2} + e$, we can use (3.52) to calculate an expression for $[e]$, the jump in the specific internal energy. Thus, we find

$$[e] = -\left(\left[\frac{p}{\rho}\right] + \frac{m^2}{2}[\rho^{-2}]\right) - \frac{u_+q}{m}. \tag{3.53}$$

The term in parentheses in (3.53) can be simplified as

$$\left[\frac{p}{\rho}\right] + \frac{m^2}{2}[\rho^{-2}] = \langle p\rangle[1/\rho].$$

The claim then follows. □

From the claim, we find that

$$\frac{p_e}{\rho}\rho_+[e] = -\frac{p_e}{\rho}\rho_+\langle p\rangle[1/\rho] - \frac{p_e}{\rho}q, \tag{3.54}$$

and upon substituting (3.54) into the expression for Δ, we find that the q terms cancel out. Therefore,

$$\Delta = [\rho]\left(-\frac{p_e m^2}{2\rho^2}[1/\rho] + p_\rho - \frac{cm}{\rho} + \frac{p_e}{\rho^2}\langle p\rangle\right).$$

Since $[\rho] \neq 0$, the condition $\Delta = 0$ may be written as

$$p_\rho - \frac{cm}{\rho} + \frac{p_e}{\rho^2}\left(\frac{[p]}{2} + \langle p\rangle\right) = 0. \tag{3.55}$$

Next, we note that

$$\left(\frac{[p]}{2} + \langle p\rangle\right) = p_+,$$

so that (3.55) simplifies to

$$p_\rho - \frac{cm}{\rho} + \frac{p_e p_+}{\rho^2} = 0. \tag{3.56}$$

Using the fact that the sound speed c satisfies $c^2 = p_\rho + \rho^{-2}pp_e$, we find that

$$p_\rho + \frac{p_e p_+}{\rho^2} = c^2 + \frac{p_e}{\rho^2}[p].$$

Therefore, we successively reduce (3.56) to

$$c^2 + \frac{p_e}{\rho^2}[p] - \frac{mc}{\rho} = 0,$$

then (by (RH2))

$$c^2 - \frac{p_e m^2}{\rho^2}[1/\rho] - \frac{mc}{\rho} = 0,$$

and finally

$$1 - \frac{p_e m^2}{\rho^2 c^2}[1/\rho] - \frac{m}{\rho c} = 0.$$

Since the Mach number M satisfies $M = -\frac{m}{\rho c}$, we obtain an expression with the form of Majda's condition for inviscid shock instability:

$$M^2[1/\rho]p_e - M - 1 = 0. \tag{3.57}$$

See [43]. We remark that the jumps in the detonation formula above refer to end states of the whole wave and not of the Neumann shock at the leading edge. Using (3.57), we now show that Δ does not vanish when the equation of state is assumed to be of the form

$$p(\rho, e) = \Gamma \rho e.$$

We denote the compression ratio by

$$r = \frac{\rho_+}{\rho_-},$$

and we denote by r^* the compression ratio of the Neumann shock. It follows immediately from the Rankine-Hugoniot diagram, Fig. 1.2, that

$$1 < r < r^*. \tag{3.58}$$

Also, in the case of an ideal gas, we find, similarly as in [52], that

$$1 < r^* < \frac{\gamma + 1}{\gamma - 1} = 1 + \frac{2}{\Gamma}. \tag{3.59}$$

Combining (3.58) and (3.59), we find that

$$0 < r - 1 < \frac{2}{\Gamma}. \tag{3.60}$$

Specializing the Majda condition to the ideal-gas case yields

$$\begin{aligned} \Gamma(r-1)M^2 - M - 1 &< 2M^2 - M - 1 \\ &= (2M+1)(M-1). \end{aligned}$$

The quantity $(2M + 1)(M - 1)$ is nonpositive for $-\frac{1}{2} \leqq M \leqq 1$, and for strong detonations the Mach number M satisfies $0 < M < 1$. Finally, since $\Delta = [\rho](\Gamma(r-1)M^2 - M - 1)$ by our reduction above and since $[\rho] < 0$ for 3-shock detonations, we conclude that $\Delta > 0$ for all strong detonation waves under the ideal-gas assumption.

3.5. Evaluation of $\tilde\Gamma$ *in the ZND limit*

Here, we use the structure of the singular manifolds contructed in the existence argument of [18] to determine the sign of the transversality coefficient γ_d and the other terms in the stability index. Recall that this coefficient is defined by the determinant

$$\gamma_d = \det \begin{pmatrix} \zeta_{2,1}^+ & \zeta_{2,2}^+ & \zeta_{2,3}^+ & \zeta_{2,6}^- \\ \zeta_{3,1}^+ & \zeta_{3,2}^+ & \zeta_{3,3}^+ & \zeta_{3,6}^- \\ \zeta_{4,1}^+ & \zeta_{4,2}^+ & \zeta_{4,3}^+ & \zeta_{4,6}^- \\ \zeta_{4,1}^{+'} & \zeta_{4,2}^{+'} & \zeta_{4,3}^{+'} & \zeta_{4,6}^{-'} \end{pmatrix},$$

where $\zeta_{j,k}^\pm$ are defined as in (3.22). In order to use the structure of the singular manifolds, we must translate from the ζ-coordinates of our stability analysis to the (u, T, Y, Z)-coordinates of the existence argument. This is accomplished in two steps. First, using the original ζ-coordinate transformation (3.12) and the $\lambda = 0$ eigenvalue equation, we can connect the ζ-coordinates to a set of intermediate coordinates: $\rho_j, \mathcal{E}_j, z_j, z_j'$ via the linear transformation (dropping $\pm$)

$$\begin{pmatrix} \zeta_{2,j} \\ \zeta_{3,j} \\ \zeta_{4,j} \\ \zeta_{4,j}' \end{pmatrix} = \begin{pmatrix} b_{21} & 0 & 0 & 0 \\ b_{31} & b_{33} & 0 & 0 \\ \tilde d & 0 & d & 0 \\ \tilde d' + b_{21}^{-1}\alpha_{23}\tilde d & b_{21}^{-1}\alpha_{23}\tilde d & 0 & d \end{pmatrix} \begin{pmatrix} \rho_j \\ \mathcal{E}_j \\ z_j \\ z_j' \end{pmatrix}. \tag{3.61}$$

Note that the determinant of the linear transformation above is simply

$$b_{21} b_{33} d^2 = -\frac{\nu m}{\rho^3}\theta c_v^{-1} d^2 > 0.$$

To obtain (3.61), note that from (3.12) we know that

$$\zeta_{2,j} = b_{21}\rho_j, \tag{3.62}$$

$$\zeta_{3,j} = b_{31}\rho_j + b_{33}\mathcal{E}_j, \tag{3.63}$$

$$\zeta_{4,j} = \tilde d \rho_j + d z_j. \tag{3.64}$$

Also, from (3.64) it follows that

$$\zeta_{4,j}' = \tilde d' \rho_j + \tilde d \rho_j' + d z_j'.$$

Recall the second eigenvalue equation at $\lambda = 0$ is

$$(\alpha_{21}\rho_j + \alpha_{23}\mathcal{E}_j)' = (b_{21}\rho_j')'. \tag{3.65}$$

We are interested in the behavior of fast modes; therefore, we can integrate (3.65) once to obtain

$$\rho_j' = b_{21}^{-1}\alpha_{21}\rho_j + b_{21}^{-1}\alpha_{23}\mathcal{E}_j. \tag{3.66}$$

Finally, we use (3.66) to write ρ_j' in terms of ρ_j and $\mathcal{E}_j$. Substituting the above relation into the equation for $\zeta_{4,j}'$, we get

$$\zeta_{4,j}' = (\tilde{d}' + b_{21}^{-1}\alpha_{21}\tilde{d})\rho_j + b_{21}^{-1}\alpha_{23}\tilde{d}\mathcal{E}_j + dz_j'. \tag{3.67}$$

Combining equations (3.62)–(3.64) and (3.67) yields (3.61).

The second step is to connect $\rho_j, \mathcal{E}_j, z_j, z_j'$, the variations in the conserved quantities, to u_j, T_j, Y_j, Z_j, the coordinates in which the construction of the singular manifolds has been accomplished. We use the relationships

$$\rho u = m, \tag{3.68}$$

$$\rho(u^2/2 + c_v T) = \mathcal{E}, \tag{3.69}$$

$$\rho Y = z, \tag{3.70}$$

and we linearize (3.68)–(3.70) about the profile to obtain (note that subscript j indicates a variation while a bar indicates that a quantity is evaluated on the profile)

$$\rho_j = -\bar{\rho}\bar{u}^{-1}u_j, \tag{3.71}$$

$$\mathcal{E}_j = mu_j + c_v\bar{\rho}T_j - \bar{\rho}\bar{u}^{-1}u_j(\bar{u}^2/2 + c_v\bar{T}), \tag{3.72}$$

$$z_j = -\bar{\rho}\bar{u}^{-1}\bar{Y}u_j + \bar{\rho}Y_j. \tag{3.73}$$

Also, we linearize $z' = \rho' Y + \rho Y'$ about the profile and use known relationships among the variations to obtain

$$z_j' = \bar{\rho}' Y_j + \bar{Y}(b_{21}^{-1}\alpha_{21}\rho_j + b_{21}^{-1}\alpha_{23}\mathcal{E}_j) + \bar{Y}'\rho_j + \bar{\rho}\left(-\frac{m}{\bar{\rho}d}Z_j + \frac{m}{\bar{\rho}d}Y_j\right), \tag{3.74}$$

where Z is defined as in [18] and in the first section by

$$Z = Y - \frac{\rho d}{m}Y_x.$$

Finally, we can use (3.71)–(3.73) and (3.74) to write down the coordinate change as

$$\begin{pmatrix} \rho_j \\ \mathcal{E}_j \\ z_j \\ z_j' \end{pmatrix} = \begin{pmatrix} -\rho u^{-1} & 0 & 0 & 0 \\ m - \rho u^{-1}(u^2/2 + c_v T) & c_v\rho & 0 & 0 \\ -\rho u^{-1} Y & 0 & \rho & 0 \\ M_1 & M_2 & \rho' + \frac{m}{d} & -\frac{m}{d} \end{pmatrix} \begin{pmatrix} u_j \\ T_j \\ Y_j \\ Z_j \end{pmatrix}, \tag{3.75}$$

where

$$M_1 = -\rho u^{-1}(b_{21}^{-1}\alpha_{21}Y + Y') + (m - \rho u^{-1}(u^2/2 + c_v T))b_{21}^{-1}\alpha_{23}Y, \tag{3.76}$$

$$M_2 = b_{21}^{-1}\alpha_{23}Y c_v\rho, \tag{3.77}$$

and we have dropped the bars on all terms in the matrix. Moreover, we note that the determinant of the matrix of this second coordinate change is

$$c_v\rho^3 u^{-1}\frac{m}{d} > 0.$$

Therefore, we find, by virtue of the fact that the determinants of the two change of coordinates matrices are positive, that

$$\operatorname{sgn}\gamma_d = \operatorname{sgn}\det\begin{pmatrix} u_1^+ & \cdots & u_6^- \\ T_1^+ & \cdots & T_6^- \\ Y_1^+ & \cdots & Y_6^- \\ Z_1^+ & \cdots & Z_6^- \end{pmatrix}.$$

By a simpler calculation proceeding as above, we also find that

$$\operatorname{sgn}\gamma_{\mathrm{NS}} = \operatorname{sgn}\det\begin{pmatrix} \zeta_{2,1}^+ & \zeta_{2,2}^+ \\ \zeta_{3,1}^+ & \zeta_{3,2}^+ \end{pmatrix} = \operatorname{sgn}\det\begin{pmatrix} u_1^+ & u_2^+ \\ T_1^+ & T_2^+ \end{pmatrix}.$$

Finally, the sign of $\tilde{\Gamma}$ can be determined by a careful examination of the structure of the singular manifolds from which the solution is constructed.

Recall from Section 1 that the structure of the singular manifolds is as in Fig. 1.4. In this figure, the T-axis is perpendicular to the (u, Z)-plane on the page, while the fourth missing direction in the phase space is the Y direction. Also, the parabolic curve, $\mathcal{C}$, on which the slow (reactive) flow takes place is not in a $T =$ constant plane, see Fig. 1.3. The diagonal dotted line indicates the intersection of the $T = T_i$ plane and $\mathcal{K}$. Thus the upper segments of the branches of $\mathcal{C}$ are below ignition temperature and there are no slow dynamics on those portions of the curve. Furthermore, we note that the unburned state (u_+, T_+, Y_+, Z_+) is a degenerate rest point due to the ignition-temperature assumption with a 3-dimensional stable manifold. On the other hand, the burned state, $(u_-, T_-, 0, 0)$, has a 2-dimensional unstable manifold (featuring a reactive and a kinematic direction) and a 2-dimensional stable manifold. Note that the T and Y directions are both *stable* directions; see the discussion of existence in Section 1, particularly equations (1.17)–(1.19) and equations (1.25)–(1.28).

We evaluate

$$-\operatorname{sgn}\gamma_d\gamma_{\mathrm{NS}}\det(r_1^-, r_2^-, U_6^-)z_3^+\bar{z}_x|_{-\infty} \tag{3.78}$$

at the "corner" where the fast manifold which approaches the burned state intersects with the opposite branch of $\mathcal{C}$. See Fig. 3.1 for a schematic indicating the relevant vectors in the calculation. The 1, 7 arrow corresponds to the profile, while the 6 arrow corresponds to the kinematic direction at $-\infty$. The dashed 2 arrow represents the stable manifold in the T-direction, while the curve labeled with a 3 corresponds to the missing stable Y-direction. Then we find that $\operatorname{sgn}\gamma_d\gamma_{\mathrm{NS}}$ can be computed as

$$\operatorname{sgn}\det\begin{pmatrix} + & 0 & 0 & - \\ - & - & 0 & - \\ * & 0 & - & 0 \\ + & 0 & 0 & 0 \end{pmatrix}\det\begin{pmatrix} + & 0 \\ - & - \end{pmatrix} = -1. \tag{3.79}$$

These choices force

$$\operatorname{sgn}\det(r_1^-, r_2^-, U_6^-) = -1. \tag{3.80}$$

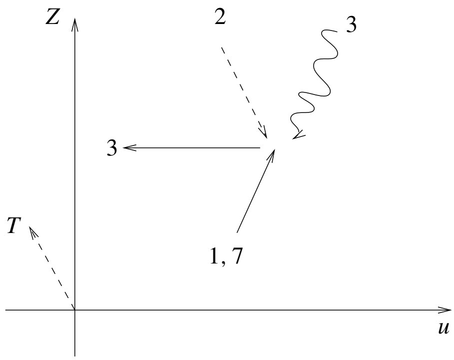

Fig. 3.1. The point of evaluation

To see (3.80), note that $\det(r_1^-, r_2^-, U_6^-|_{-\infty}) = l_3^- \cdot U_6^-$ where l_3^- is the appropriate left eigenvector, (simplified due to ideal-gas assumption (3.1))

$$l_3^- = \left(-cu + \frac{\Gamma u^2}{2}, c - \Gamma u, \Gamma\right)^{\mathrm{tr}}. \tag{3.81}$$

Then since

$$U_6^- = (\rho_6^-, m_6^-, \mathcal{E}_6^-)^{\mathrm{tr}} = (\rho_6^-, 0, \mathcal{E}_6^-)^{\mathrm{tr}}, \tag{3.82}$$

we obtain, by combining (3.81) and (3.82),

$$l_3^- \cdot U_6^- = \left(-cu + \frac{\Gamma u^2}{2}\right) \rho_6^- + \Gamma \mathcal{E}_6^-. \tag{3.83}$$

To take advantage of the signs we know, we translate into (u, T)-coordinates. Substituting (3.61) and (3.75) into (3.83) and performing some elementary simplification yields

$$l_3^- \cdot U_6^- = \left(c\rho - \frac{c_v p}{u}\right) u_6^- + \Gamma c_v \rho T_6^-. \tag{3.84}$$

To evaluate the sign of (3.84), we note that $u < 0$ because we consider a 3-shock, and thus the coefficient

$$\left(c\rho - \frac{c_v p}{u}\right)$$

is positive. Finally, (3.80) follows since both $\operatorname{sgn} u_6^-$ and $\operatorname{sgn} T_6^-$ are -1.

Also, we find then that

$$\operatorname{sgn} z_3^+ = \operatorname{sgn} \bar{\rho} Y_3^+ = -1. \tag{3.85}$$

Lastly, we find that

$$\operatorname{sgn} z_7^- = \operatorname{sgn} \bar{z}_x|_{-\infty} = \operatorname{sgn}(\bar{\rho}\bar{Y}_x|_{-\infty} - \bar{\rho}\bar{u}^{-1}\bar{Y}\bar{u}_x|_{-\infty}) = +1. \tag{3.86}$$

Tracking the signs computed in (3.79)–(3.86) and combining with (3.78) and the fact that we computed

$$\Delta > 0$$

for strong detonations satisfying the ideal-gas assumption, we discover that, given (3.1), the stability-index for a strong detonation with viscosity and Lax 3-shock structure satisfies in the ZND limit

$$\begin{aligned}\tilde{\Gamma} &= -\operatorname{sgn} \gamma_d \gamma_{\mathrm{NS}} \Delta \det(r_1^-, r_2^-, U_6^-) z_3^+ \bar{z}_x|_{-\infty} \\ &= (-1)(-1)(+1)(-1)(-1)(+1) = +1,\end{aligned}$$

which is consistent with stability. This completes the proof of Theorem 4.

Finally, we also remark that under the ideal-gas assumption we found sgn Δ by comparison to the Neumann Shock. Restating this, we found that if the Neumann Shock is "index stable," i.e., satisfies the stability-index necessary criterion, then the corresponding strong detonation is also "index stable." A natural question then is: under what conditions on the equation of state does this remain true? Or perhaps more important, what is the actual stability relationship, not just the relationship between stability-indexes, between the shock and the strong detonation? The partial information gathered from the stability-index approach definitely motivates further investigation into this question.

3.6. Stability index for multiple reactants

In actual combustion, the chemical reactions involved are typically more complicated than a single one-step reaction. It is natural to model reactions in a gas mixture of $s + 1$ components by a system of the general form

$$U_t + f(U)_x = (B(U)U_x)_x + Q\Phi(U)z, \tag{3.87}$$

$$z_t + (v(U)z)_x = (D^1(U)z_x)_x + (D^2(U,z)U_x)_x - \Phi(U)z, \tag{3.88}$$

where $U = (\rho, m, \mathcal{E})^{\mathrm{tr}}$ and f and B are as in the Navier-Stokes model for gas dynamics considered above. The vector $z = \rho Y \in \mathbf{R}^s$ measures the quantities of each of the reactants, and the constant matrix $Q \in \mathbf{R}^{3\times s}$ records the heat released in each reaction, hence

$$Q = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ q_1 & \cdots & q_s \end{pmatrix}.$$

The positive-definite matrix $\Phi(U) \in \mathbf{R}^{s\times s}$ incorporates the reaction rates, the diffusion matrices satisfy $D^1 \in \mathbf{R}^{s\times s}$ and $D^2 \in \mathbf{R}^{s\times 3}$, and the scalar function $v(U)$ is simply the velocity $\frac{m}{\rho} = u$.

It turns out that the number of equations required to model the chemistry can be reduced through the use of progress variables. See [14] or [40] for further discussion. Our analysis largely applies to the more general multiple-reactant case, with the following important exception below.

In the more complicated multi-species case the calculation for large λ for the reaction block is not so straightforward as in the single-species case; in particular, the connection to $\lambda = 0$ is not clear. We indicate here some partial results along these lines. Examining the reaction portion of the characteristic equation on the $+\infty$ side, we find, since the reaction function satisfies $\Phi = 0$, that

$$(\mu^2 D - \mu V + \lambda I)z = 0. \tag{3.89}$$

Since the convection in the reaction equation is just the background velocity, V is scalar, hence it commutes with the species diffusion. This implies that there is a symmetrizer V^0 such that $V^0 V$ and $V^0 D$ are symmetric, and $\operatorname{Re} V^0 D > 0$. Thus, arguments using Theorem 11 apply. On the other hand, things are trickier on the $-\infty$ side, where the characteristic equation reads

$$(\mu^2 D - \mu V + \lambda I - \Phi)z = 0. \tag{3.90}$$

Here, we point out that the argument of [2], Lemma 7.2 goes through word for word for constant-coefficient equations of the form $\lambda z + Az' + Bz'' + Cz = 0$ provided that A is symmetric and B and C are positive definite, i.e., $\operatorname{Re} B, \operatorname{Re} C > 0$, (more generally there is a coordinate transformation for which this is true) with the slightly modified Lyapunov function $H(v, w) = (1/2)\langle Aw, w\rangle + \operatorname{Re}\langle x, (\lambda + C)v\rangle$. Indeed, the conclusions become somewhat stronger, extending to all $\operatorname{Re}\lambda \geqq 0$ and not only $\operatorname{Re}\lambda > 0$ as in [2]. In our case, $A = v$ is scalar, $B = D$ and $C = \Phi$, so we find that the machinery of [2] may be applied provided that there exist coordinates in which D and Φ are both positive: in particular, if D is scalar.

3.7. Stability for small heat release q

Here, we examine the $q \to 0$ limit and take advantage of the simplification in the equations when $q = 0$. Using a continuity argument, we prove *strong spectral stability*, that is, nonexistence of eigenvalues with $\operatorname{Re}\lambda \geqq 0$ and $\lambda \neq 0$, as well as transversality of the connecting profile $\gamma_d \neq 0$, and low-frequency stability $\Delta \neq 0$. It is expected that these three properties should be sufficient to conclude full nonlinear orbital stability, by following the program of [66, 46, 48, 47]. Such a result would be an extension to the reacting Navier-Stokes model of the results of [38, 32], in which the authors show nonlinear stability of strong detonations in the Majda model as $q \to 0$. We note that in [38, 32], as here, working with the integrated equations is a key ingredient in the analysis. Finally, we recover the result of [60] in the case of no species diffusion, but by a much simpler argument.

We rewrite the system (1.1)–(1.4), using U to represent the vector of gas-dynamical variables and $z = \rho Y$ in the first three equations, as

$$(U + \boldsymbol{q}z)_t + (f(U) + \boldsymbol{q}uz)_x = (B(U)U_x)_x, \tag{3.91}$$

$$(\rho Y)_t + (\rho u Y)_x = (\rho d Y_x)_x - k\varphi(T)\rho Y. \tag{3.92}$$

The linearized eigenvalue equation from (3.91) is thus

$$\lambda(U + \boldsymbol{q}z) + (AU)' + (\boldsymbol{q}\bar{u}z + \boldsymbol{q}u\bar{z})' = (B(U)U')'. \tag{3.93}$$

Next, we define $\tilde{U}$ by $\tilde{U} = \int_{-\infty}^{x}(U + \boldsymbol{q}z)$, and then the linearized eigenvalue equation from (3.91) in the integrated variable $\tilde{U}$ is

$$\lambda\tilde{U} + A\tilde{U}' - (A - u)\boldsymbol{q}z = B\tilde{U}''. \tag{3.94}$$

The linearized eigenvalue equation from (3.92) takes the form

$$\begin{aligned}\lambda(\bar{\rho}Y + \rho\bar{Y}) + (\bar{\rho}\bar{u}Y + \bar{\rho}u\bar{Y} + \rho\bar{u}\bar{Y})' = (\bar{\rho}dY' + \rho d\bar{Y}_x)' - k\bar{\varphi}\rho\bar{Y}\\ - k\varphi\bar{\rho}Y - k\bar{\varphi}'(\bar{T})T\bar{\rho}\bar{Y}.\end{aligned} \tag{3.95}$$

When $q = 0$, we note that (3.94) reduces to the integrated eigenvalue equation for gas dynamics *about a gas-dynamical profile*. This observation will allow us to draw conclusions regarding the stability of small-q detonations in the case that the underlying gas-dynamical shock is stable. In particular, we remark that this is the case if the shock is of small amplitude.

Lemma 7. *The* $q = 0$ *limit of the integrated system* (3.94), (3.95) *has no eigenvalues with* $\operatorname{Re}\lambda \geqq 0$ *provided the limiting shock is stable.*

Proof. Since the limiting shock is spectrally stable, we have by standard considerations [66] that the integrated eigenvalue equation for gas dynamics supports no eigenvalues on $\operatorname{Re}\lambda \geqq 0$, and so $(\rho, u, \mathcal{E})$ identically vanish for any eigenfunction of the limiting eigenvalue equations as $q \to 0$. Thus, (3.95) has the simpler form

$$\lambda(\bar{\rho}Y) + (\bar{\rho}\bar{u}Y)' = (\bar{\rho}dY')' - k\bar{\varphi}\bar{\rho}Y.$$

We note that $\bar{\rho}\bar{u} = m$ is real and constant on the profile. Then, taking the standard complex L^2 inner product of the above equation with Y, we have

$$\langle\lambda(\bar{\rho}Y), Y\rangle + \langle(mY)', Y\rangle = \langle(\bar{\rho}dY')', Y\rangle - \langle k\bar{\varphi}\bar{\rho}Y, Y\rangle,$$

so that integrating by parts and taking real parts yields

$$\operatorname{Re}\lambda\langle\bar{\rho}Y, Y\rangle + 0 + \operatorname{Re}\langle k\bar{\varphi}\bar{\rho}Y, Y\rangle + \operatorname{Re}\langle Y', \bar{\rho}dY'\rangle = 0. \tag{3.96}$$

This follows since $\langle mY', Y\rangle = \langle(mY)', Y\rangle = -\langle mY, Y'\rangle = \overline{\langle mY, Y'\rangle}$ so that $\langle(mY)', Y\rangle$ is purely imaginary. But, for (3.96) to hold, we must have either $\operatorname{Re}\lambda < 0$ or $\operatorname{Re}\lambda = 0$ and also $Y' \equiv 0$ which implies that Y is constant. In that case, we need also $\operatorname{Re}\langle k\bar{\varphi}\bar{\rho}Y, Y\rangle = 0$, so that the constant value for Y must be 0. We conclude that any nontrivial solutions must correspond to $\operatorname{Re}\lambda < 0$.

In the case where species diffusion is neglected, things are even simpler. We write the reaction equation as

$$\lambda z + (vz)' = -\bar{\varphi}z, \tag{3.97}$$

where $z = \rho Y$ as usual. Applying the gap lemma to (3.97), we find that behavior at $+\infty$ is governed by the limiting constant-coefficient equations. These are easily seen to support no stable modes. Indeed, at $+\infty$, we have $\bar{\varphi} \equiv 0$, so there is no reaction at all, and the equation becomes simply

$$\lambda z = -(vz)'.$$

Rewriting this in terms of the new variable $w = vz$ as

$$w' = -\left(\frac{\lambda}{v}\right) w, \tag{3.98}$$

we see that all solutions blow up as $x \to \infty$ for $\operatorname{Re}\lambda > 0$. □

Corollary 1. *For sufficiently small q, the integrated eigenvalue equations* (3.94) *and* (3.95) *have no* $\operatorname{Re}\lambda \geqq 0$ *eigenvalues provided that the limiting shock is stable.*

Proof. The respective Evans functions vary continuously, and the limiting Evans function is nonvanishing on $\operatorname{Re}\lambda \geqq 0$. □

Proposition 6. *If the limiting shock is stable, then small-q detonations are strongly spectrally stable. That is, they have no eigenvalues for* $\operatorname{Re}\lambda \geqq 0$ *and* $\lambda \neq 0$*. Moreover, $D'(0) = \gamma_d \Delta \neq 0$, where γ_d and Δ are as defined in* (3.28) *and* (3.27).

Proof. When $\lambda \neq 0$, we may integrate the divergence-form gas eigenvalue equation to deduce that $(U + \boldsymbol{q}z)$ has zero integral, and thus $\tilde{U}$, defined as $\int_{-\infty}^{x}(U + \boldsymbol{q}z)$ lies in L^2 if U, z do. This follows by the gap lemma since functions decay exponentially (if at all) as do their integrals [66]. Thus, existence of an eigenfunction for the linearized eigenvalue equations (3.93) and (3.95) is equivalent to the existence of an eigenfunction for the integrated system (3.94), (3.95). Existence of a transverse connection $\gamma_d \neq 0$ follows likewise by continuity from the result for the limiting equations, provided there exists a transverse connection for the limiting gas-dynamical shock. Finally, $\Delta \neq 0$ follows by inspection from the corresponding property for the limiting gas-dynamical shock. For, the form of Δ for $q = 0$ reduces to this case, and nonvanishing of Δ is a necessary condition for stability of the gas-dynamical shock [66]. □

The results of this section clearly extend to the case of multiple reactants. Indeed, for the zero species diffusion case, the number and type of reactants plays no role. For the $D \neq 0$ case, the arguments above carry through if D and Φ are simultaneously positive, or if there exists a constant coordinate change making them both positive. In particular, the argument applies if the diffusion D is scalar and $\Phi = \varphi K$ for K constant and φ scalar.

Appendix A. Real viscosity

For completeness and the convenience of the reader, we provide here a general discussion of systems of conservation laws with real viscosity containing the facts relevant to our treatment of detonation in the main body of the paper. Our treatment is a revised and somewhat extended version of Appendix A.2 [65]. See also [48, 68].

Systems modeling gas dynamics have the general form

$$U_t + F(U)_x = (B(U)U_x)_x, \tag{A.1}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad F = \begin{pmatrix} f \\ g \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix},$$

and

$$u, f \in \mathbf{R}^{n-r}, \quad v, g \in \mathbf{R}^r, \quad b_1 \in \mathbf{R}^{r\times(n-r)}, \quad b_2 \in \mathbf{R}^{r\times r}.$$

We note that in the isentropic gas dynamics case $n = 2$ and $r = 1$ while in the Navier-Stokes case $n = 3$ and $r = 2$. Our interest is in traveling-wave solutions of the form

$$U = \bar{U}(x), \qquad \lim_{x\to\pm\infty} \bar{U}(x) = U_\pm = (u_\pm, v_\pm). \tag{A.2}$$

Standard assumptions for equations in this generality are:

(H0) $F, B \in C^2$;

(H1) for all $x \in \mathbf{R}$: $\begin{cases} \text{(i) } \operatorname{Re}\sigma(b_2(\bar{U}(x))) > 0, \\ \text{(ii) } \binom{\mathrm{d}f}{b}(\bar{U}(x)) \text{ full rank}, \\ \text{(iii) } \sigma(\mathrm{d}f_u - b_1(b_2)^{-1}\mathrm{d}f_v) \text{ is real;} \end{cases}$

(H2) $\sigma\,(\mathrm{d}F(U_\pm)\xi)$ real for $\xi \in \mathbf{R}$, $0 \notin \sigma(\mathrm{d}F(U_\pm))$;

(H3) $\operatorname{Re}\sigma\left(\xi\mathrm{d}F(U_\pm) - \xi^2 B(U_\pm)\right) \leqq 0$ for $\xi \in \mathbf{R}$;

(H4) solutions of (A.1)–(A.2) form a smooth manifold $\{\bar{u}^\delta\}$, $\delta \in \mathcal{U} \subset \mathbf{R}^\ell$.

These hypotheses are analogous to those of the strictly parabolic case considered in, e.g., [65], with (H1)(i) and (H1)(iii) ensuring local well-posedness. Indeed, they are the standard set of conditions identified by KAWASHIMA [30]; for further discussion, see [53]. The condition (H1)(ii), may be motivated by consideration of the traveling-wave equation

$$f(u, v) \equiv f(u_-, v_-), \tag{A.3}$$

$$b_1 u' + b_2 v' = g(u, v) - g(u_-, v_-). \tag{A.4}$$

For, (H1)(ii) is readily seen to be the condition that (A.4) describes a nondegenerate ordinary differential equation on the r-dimensional manifold described by (A.3); thus, this is a reasonable nondegeneracy condition to impose in the study of viscous profiles. Condition (H1)(iii) also arises in the analysis of the eigenvalue equation; see the discussion of consistent splitting in Appendix A2 of [65]. In the symmetrizable case it holds automatically.

We remark, finally, that (H1)(ii) (indeed, all of hypothesis (H1)) is satisfied for gas and plasma dynamics precisely when particle and shock velocities are distinct, which is always the case along a shock; for a study of viscous profiles in these contexts, see [21, 20, 16].

Let i_+ denote the dimension of the stable subspace of $\mathrm{d}f^1(u_+)$, i_- denote the dimension of the unstable subspace of $\mathrm{d}f^1(u_-)$, and $i := i_+ + i_-$. Let d_+ denote the dimension within the submanifold $f \equiv$ constant of the stable manifold at (u_+, v_+) of traveling-wave equation (A.4), and d_- the dimension of the unstable manifold at (u_-, v_-), and $d := d_- + d_+$. Then, we have the following result analogous to that of MAJDA & PEGO [44] in the strictly parabolic case.

Lemma 8 (See [47]). *Under assumptions* (H0)–(H3)*,* $(u_\pm, v_\pm)$ *are hyperbolic rest points of the reduced traveling-wave equation* (A.4)*. In particular, traveling-wave solutions satisfy*

$$|(\mathrm{d}/\mathrm{d}x)^k \big((\bar u(x), \bar v(x)) - (u_\pm, v_\pm)\big)| \leqq Ce^{-\theta|x|}, \quad k = 0, \dots, 4, \tag{A.5}$$

as $x \to \pm\infty$*. Moreover, the type of the connection agrees with the* (*hyperbolic*) *type of the shock, in the sense that*

$$d - r = i - n. \tag{A.6}$$

Proof. Integrating (A.4) from $-\infty$ to x and rearranging, we may write (A.3) and (A.4) in the alternative form:

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ g - g_- \end{pmatrix}. \tag{A.7}$$

Linearizing (A.7) about $U_\pm$, we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ g_u & g_v \end{pmatrix}_{|(U_\pm)} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{A.8}$$

or, setting

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}_{|(U_\pm)} \begin{pmatrix} u \\ v \end{pmatrix},$$

the pair of equations

$$z_1' = 0$$

and

$$z_2' = \begin{pmatrix} g_u & g_v \end{pmatrix} \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_r \end{pmatrix}_{|(U_\pm)} z_2, \tag{A.9}$$

the latter of which evidently describes the linearized ordinary differential equation on manifold (A.3). Observing that

$$\det \begin{pmatrix} g_u & g_v \end{pmatrix} \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_r \end{pmatrix}_{|(U_\pm)} = \det \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1}_{|(U_\pm)} \neq 0$$

by (H2) and (H1)(i), we find that the coefficient matrix of (A.9) has no zero eigenvalues. On the other hand, it can have no nonzero purely imaginarly eigenvalues $i\xi$, since otherwise

$$\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ v \end{pmatrix} = i\xi \begin{pmatrix} 0 \\ v \end{pmatrix},$$

and thus

$$\Big[-i\xi \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} - \xi^2 \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}\Big]\Big[\begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ v \end{pmatrix}\Big] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for $\xi \neq 0 \in \mathbf{R}$, in violation of (H3). Thus, we find that $U_\pm$ are hyperbolic rest points, from which (A.5) follows. Relation (A.6) now follows from Lemma 12, below. □

The linearized eigenvalue equations about $\bar{U}(\cdot)$ are:

$$(A_{11}u + A_{12}v)' = -\lambda u, \tag{A.10}$$

and

$$(b_1 u' + b_2 v')' = (A_{21}u + A_{22}v)' + \lambda v, \tag{A.11}$$

where

$$\begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix} := B(\bar{U}),$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} U := \mathrm{d}F(\bar{U})U - \mathrm{d}B(\bar{U})(U, \bar{U}'),$$

and '$'$' denotes ∂_x; in particular, note that

$$(A_{11}, A_{12}) = \mathrm{d}f(\bar{U}).$$

Utilizing the invertible change of variables

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{A.12}$$

we can write the eigenvalue equation as a first-order system

$$Z' = \mathbb{A}(x, \lambda)Z, \quad Z = (z_1, z_2, z_2')^{\mathrm{tr}}. \tag{A.13}$$

We note here that z_2 has r components.

The consistent-splitting hypothesis can be verified by a limiting analysis as $\lambda \to +\infty$, carried out without loss of generality in original coordinates W, for which the asymptotic characteristic equations become:

$$\det \begin{pmatrix} \mu A_{11} + \lambda & \mu A_{12} \\ \mu A_{12} - \mu^2 b_1 & \mu A_{22} - \mu^2 b_2 + \lambda \end{pmatrix}_\pm \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{A.14}$$

This yields $n - r$ roots $\mu \sim \tilde{\mu}\lambda$, $\tilde{\mu} = O(1)$, satisfying

$$\begin{pmatrix} \tilde{\mu}A_{11} + I & A_{12} \\ b_1 & b_2 \end{pmatrix}_\pm \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{A.15}$$

or

$$-\tilde{\mu}^{-1} \in \sigma(A_{11} - A_{12}b_2^{-1}b_1)_\pm, \tag{A.16}$$

and $2r$ roots $\mu \sim \tilde{\mu}\lambda^{1/2}$, $\tilde{\mu} = O(1)$, satisfying

$$\begin{pmatrix} I & 0 \\ -\tilde{\mu}^2 b_1 & -\tilde{\mu}^2 b_2 + I \end{pmatrix}_{\pm} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{A.17}$$

or

$$\tilde{\mu}^{-2} \in \sigma(b_2). \tag{A.18}$$

By assumption ($\tilde{\text{H}}1$)(iii), (A.16) yields a fixed number $k/(n-r-k)$ of stable/unstable roots, independent of x, and thus of $\pm$. Likewise, ($\tilde{\text{H}}1$)(i) implies that (A.18) yields r stable, r unstable roots. Combining, we find the desired consistent splitting, with $(k+r)/(n-k)$ stable/unstable roots at both $\pm\infty$. We can thus define an Evans function as usual as

$$D(\lambda) = \det(Z_1^+, \dots Z_{k+r}^+, Z_{k+r+1}^-, \dots Z_{n+r}^-)_{|x=0}, \tag{A.19}$$

where $\{Z_1^+, \dots Z_{k+r}^+\}$, $\{Z_{k+r+1}^-, \dots Z_{n+r}^-\}$ span the stable manifold at $+\infty$, unstable manifold at $-\infty$ of (A.13). Notice that the Evans function in Z coordinates is just a constant multiple of the corresponding Evans function defined in $W = (u, v, v')^t$ coordinates.

Appendix A.1. Stability index

The stability index is defined to be

$$\tilde{\Gamma} := \text{sgn}(\partial_\lambda)^\ell D(0) D(+\infty). \tag{A.20}$$

The low-frequency calculations of $D'(0)$ are detailed for the Navier-Stokes model in Section 3. Note that $\ell = 1$ in this case. Here, we evaluate the sign of $D(\lambda)$ as $\lambda \to +\infty$ along the real axis.

Lemma 9. *Let $\tilde{D}(\cdot)$ denote the alternative Evans function*

$$\tilde{D}(\lambda) := \det(W_1^+, \dots W_{k+r}^+, W_{k+r+1}^-, \dots W_{n+r}^-)_{|x=0} \tag{A.21}$$

computed in the original coordinates W. Then, for real λ sufficiently large, the following holds:

$$\text{sgn}\, \tilde{D}(\lambda) = \text{sgn} \det(\mathbb{S}^+, \mathbb{U}^+) \det(\pi\mathbb{W}^+, \varepsilon\mathbb{S}^+) \det(\varepsilon\mathbb{U}^-, \pi\mathbb{W}^-) \neq 0, \tag{A.22}$$

where π denotes projection of $W = (u, v, v')$ onto (u, v) components, and $\mathbb{S}(x)$, $\mathbb{U}(x)$ are real bases of the stable/unstable subspaces of $(A_{11} - A_{12}b_2^{-1}b_1)$ (note: $(n-r)$ dimensional), with $\varepsilon u := (u, -b_2^{-1}b_1 u)$ denoting extension.

Proof. Recalling ($\tilde{\mathrm{H}}1$)(ii), we know that the coordinate change$(u, v) \to (z_1, z_2)$ is invertible, and so we may work equivalently in (u, v, z_3) coordinates, where $z_3 := b_1 u' + b_2 v'$. Then, we find from (A.15), (A.17) that the stable/unstable manifolds of the frozen eigenvalue equation at any (fixed) x are spanned by vectors of form

$$\begin{pmatrix} u \\ -b_2^{-1} b_1 u \\ * \end{pmatrix},$$

with u an unstable/stable eigenvector of $(A_{11} - A_{12} b_2^{-1} b_1)$, $-\tilde{\mu}^{-1}$ the corresponding eigenvalue; and vectors

$$\begin{pmatrix} 0 \\ v \\ \mp\tilde{\mu}\lambda^{1/2} v \end{pmatrix},$$

with v an eigenvector of b_2, $-\tilde{\mu}^{-2}$ the corresponding eigenvalue. By the tracking lemma and its surrounding discussion in Section 1.5.3, we thus obtain

$$D(\lambda) \sim \det \begin{pmatrix} \underbrace{\begin{matrix} u & \cdots \\ -b_2^{-1} b_1 u & \cdots \\ * & \cdots \end{matrix}}_{n-r} & \underbrace{\begin{matrix} 0 & \cdots & 0 \\ v & \cdots & v \\ -\tilde{\mu}\lambda^{1/2} v & \cdots & \tilde{\mu}\lambda^{1/2} v \end{matrix}}_{2r} \end{pmatrix}$$
$$\times \det \left(\begin{array}{cc|cc} \alpha_+^1 & 0 & \alpha_+^2 & 0 \\ 0 & \alpha_-^1 & 0 & \alpha_-^2 \\ \hline \beta_+^2 & 0 & \beta_1^+ & 0 \\ 0 & \beta_-^2 & 0 & \beta_-^1 \end{array}\right), \tag{A.23}$$

provided that the right-hand side does not vanish, where both

$$\det \begin{pmatrix} \alpha_+^1 & \alpha_+^2 \\ \beta_+^2 & \beta_+^1 \end{pmatrix} \quad \text{and} \quad \det \begin{pmatrix} \alpha_-^1 & \alpha_-^2 \\ \beta_-^2 & \beta_-^1 \end{pmatrix}$$

are real, nonzero quantities. The right-hand side of (A.23) can be rewritten as

$$\det(\mathbb{S}^+, \mathbb{U}^+) \det(\mathbb{V})^2 \det \begin{pmatrix} \alpha_+^1 & \alpha_+^2 \\ \beta_+^2 & \beta_+^1 \end{pmatrix} \det \begin{pmatrix} \alpha_-^1 & \alpha_-^2 \\ \beta_-^2 & \beta_-^1 \end{pmatrix}, \tag{A.24}$$

where by $\det(\mathbb{V})$ we refer to the $r \times r$ determinant coming from the v component of (A.23). On the other hand, the term

$$\det(\pi \mathbb{W}^+, \varepsilon \mathbb{S}^+)$$

in (A.22) can be simplified to

$$\det(\pi \widetilde{\mathbb{W}}^+, \pi \mathbb{U}^+, \varepsilon \mathbb{S}^+) \det \left(\begin{array}{cc|c} \alpha_+^1 & \alpha_+^2 & 0 \\ \beta_+^2 & \beta_+^1 & 0 \\ \hline 0 & 0 & I \end{array}\right), \tag{A.25}$$

where $(\widetilde{\mathbb{W}}^+, \mathbb{U}^+) = \mathbb{W}^+$, but $\det(\pi\widetilde{\mathbb{W}}^+, \pi\mathbb{U}^+, \varepsilon\mathbb{S}^+)$ has the form

$$\det\left(\begin{array}{c|cc} 0 & \mathbb{U}^+ & \mathbb{S}^+ \\ \mathbb{V} & * & * \end{array}\right).$$

Finally, we see that (A.25) is simply

$$\det(\mathbb{V})\det(\mathbb{S}^+, \mathbb{U}^+)\det\begin{pmatrix} \alpha_+^1 & \alpha_+^2 \\ \beta_+^2 & \beta_+^1 \end{pmatrix}. \tag{A.26}$$

Similarly, the term

$$\det(\varepsilon\mathbb{U}^-, \pi\mathbb{W}^-)$$

simplifies to

$$\det(\mathbb{V})\det(\mathbb{S}^-, \mathbb{U}^-)\det\begin{pmatrix} \alpha_-^1 & \alpha_-^2 \\ \beta_-^2 & \beta_-^1 \end{pmatrix}. \tag{A.27}$$

Combining (A.26) and (A.27), we find that the expressions (A.22) and (A.23) agree modulo the real, positive factor $\det(\mathbb{S}^+, \mathbb{U}^+)^2$ since $\operatorname{sgn}\det(\mathbb{S}^+, \mathbb{U}^+) = \operatorname{sgn}\det(\mathbb{S}^-, \mathbb{U}^-)$. □

We further make the assumptions:

(A1) *semidissipativity* There exist symmetrizers $A^0_\pm$ such that $A^0_\pm A_\pm$ are symmetric and $\operatorname{Re} A^0_\pm B_\pm \leqq 0$.

(A2) *block structure* We have $(A^0_\pm)^{1/2} B_\pm (A^0_\pm)^{-1/2} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b}_2 \end{pmatrix}_\pm$.

Both of these assumptions hold for the compressible Navier-Stokes equations. When they hold, more can be said about the sign of $D(\lambda)$ for large real λ .

Lemma 10. *Let* (A1) *and* (A2) *hold. Then,*

$$\operatorname{sgn}\tilde{D}(\lambda) = \operatorname{sgn}\det(\mathbb{S}^+, \mathbb{U}^+)\det(\pi\mathbb{W}^+, \varepsilon\mathbb{S}^+)\det(\varepsilon\mathbb{U}^-, \pi\mathbb{W}^-)_{|\lambda=0} \neq 0 \tag{A.28}$$

for sufficiently large, real λ where $\tilde{D}(\cdot)$ as in (A.21) *denotes the Evans function computed in original coordinates $W = (u, v, v')^t$.*

Proof. Without loss of generality, we may take A symmetric, $B = \left(\begin{smallmatrix} 0 & 0 \\ 0 & b_2 \end{smallmatrix}\right)$, and $\operatorname{Re}\ b_2 > 0$, by the transformation

$$A \to (A^0)^{\frac{1}{2}} A (A^0)^{-\frac{1}{2}}, \quad B \to (A^0)^{\frac{1}{2}} B (A^0)^{-\frac{1}{2}}.$$

It is sufficient to show that quantity (A.28) does not vanish in the class (+)–(++). For, since $D(\lambda)$ does not vanish either, for real λ sufficiently large, we can then establish the result by homotopy of the symmetric matrix $A_\pm$ to an invertible diagonal matrix (straightforward, using the unitary decomposition $A = UDU^*$, $U^*U = I$, and the fact that unitary matrices are homotopic either to I or $-I$) and of $B_\pm$ to $\left(\begin{smallmatrix} 0 & 0 \\ 0 & I_r \end{smallmatrix}\right)$ (e.g., by linear interpolation of the positive-definite b_2 to I_r), in which case it can be seen by explicit computation that (A.28) is independent of $\lambda \in [0, +\infty]$. We note that the endpoint of this homotopy is on the boundary of, but not in, the

Kawashima class, since eigenvectors of A are in the kernel of B; indeed, our definition of semidissipativity is is *not* the "strict" dissipativity condition of Kawashima, but a nonstrict version. However, it suffices for the present, purely linear-algebraic purpose.

We begin by examining $\det(\pi\mathbb{W}^+, \varepsilon\mathbb{S}^+)$. When $\lambda = 0$, a bifurcation analysis as in [65] of the limiting constant-coefficient equations at $\pm\infty$ shows that the projections π of slow modes of $\mathbb{W}^+$ may be chosen as the unstable eigenvectors r_j^+ of A, corresponding to outgoing characteristic modes, and the projections of fast modes as the stable (i.e., $\operatorname{Re}\mu < 0$) solutions of

$$(A - \mu B)_\pm \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or without loss of generality

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - \mu b_2 \end{pmatrix}_\pm \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and thus of form

$$\begin{pmatrix} -(A_{11})^{-1} A_{12} v \\ v \end{pmatrix},$$

where

$$\left(b_2^{-1}(A_{22} - A_{21}(A_{11})^{-1} A_{12}) - \mu I \right) v = 0.$$

Likewise, using $b_1 = 0$, we find from the definitions of $\mathbb{S}$, ε in the statement of Lemma 9 that stable solutions $\mathbb{S}^+$ are in the stable subspace of A_{11}, with $\varepsilon\,\mathbb{S}^+ = \binom{\mathbb{S}^+}{0}$, hence vectors $\varepsilon\mathbb{S}^+$ lie in the intersection of the stable subspace of A and the kernel of B. Our claim is that these three subspaces are independent, spanning $\mathbf{C}^n$. Rewording this assumption, we are claiming that the stable subspace of $(A)^{-1}B$, the center subspace $\ker B$ intersected with the stable subspace of A, and the unstable subspace of A are mutually independent. (Note: that dimensions are correct follows by consistent splitting). But, this follows by Lemma 11 below. Similar considerations apply to $\det(\varepsilon\mathbb{U}^-, \pi\mathbb{W}^-)$. □

A key to the calculations is the following lemma established in [65], generalizing a result of Serre in the strictly parabolic case. (See [2] for the strictly parabolic version and related results.) For a matrix M, we denote by $\mathcal{S}(M)$ and $\mathcal{U}(M)$ the stable and unstable subspaces of M.

Lemma 11 (A modification of Serre's lemma [65])**.** *Let A be a symmetric, invertible matrix and let B be a positive-semidefinite matrix,* $\operatorname{Re}(B) \geqq 0$. *Then, the cones $\mathcal{S}(A^{-1}B) \oplus (N(A) \cap \ker B)$ and $\mathcal{U}(A)$ are transverse. Here $\mathcal{S}(M)$, $\mathcal{U}(M)$ refer to stable/unstable subspaces of M, and $N(M)$ refers to the cone* $\{v : \operatorname{Re}\langle v, Mv\rangle \leqq 0\}$.

Proof. Suppose to the contrary that $x_0 \neq 0$ lies both in the cone $\mathcal{S}(A^{-1}B) \oplus (N(A) \cap \ker B)$ and in $\mathcal{U}(A)$, i.e.,

$$x_0 = x_1 + x_2,$$

where $x_1 \in \mathcal{S}(A^{-1}B)$, $x_2 \in (N(A) \cap \ker B)$, and $x_0 \in \mathcal{U}(A)$. Define $x(t)$ by the ordinary differential equation $x' = A^{-1}Bx, \quad x(0) = x_0$. Then $x(t) \to x_2$ as $t \to +\infty$ and thus $\lim_{t\to+\infty}\langle x(t), Ax(t)\rangle \leqq 0$. On the other hand,

$$\langle x, Ax\rangle' = 2\langle A^{-1}Bx, Ax\rangle = 2\langle Bx, x\rangle \geqq 0$$

by assumption, hence $\langle x_0, Ax_0\rangle \leqq 0$, contradicting the assumption that x_0 belongs to $\mathcal{U}(A)$. □

Appendix A.2. Evaluation of $\tilde{\Gamma}$

Using z-coordinates, we may regard the traveling-wave equation as an r dimensional first-order dynamical system. We denote by $d_\pm$ the dimensions of the stable manifold at z_{2+} and the unstable manifold at z_{2-}. We also define d to be the sum $d = d_+ + d_-$. It follows then that $1 \leqq d_\pm \leqq r$. Also we denote by $i_\pm$ the number of characteristics entering the shock from the left $(-)$ and the right $(+)$. We put $i = i_+ + i_-$. Corresponding to [44], we have:

Lemma 12 (See [47]). *Provided there exists a connecting profile,*

(i) $n - i_+ = r - d_+ + \dim \mathcal{U}(\tilde{A}_+)$,
(ii) $n - i_- = r - d_- + \dim \mathcal{S}(\tilde{A}_-)$,

where $\tilde{A} = (A_{11} - A_{12}b_2^{-1}b_1)$. *Moreover,* $n - i = r - d$.

Proof. Equating the dimensions of $\mathbb{Z}^+$ at $\lambda = 0$ and as $\lambda \to \infty$ we find

$$\dim \mathcal{U}(A_+) + d_+ = \dim \mathcal{U}(\tilde{A}_+) + r,$$

or

$$(n - i_+) + d_+ = \dim \mathcal{U}(\tilde{A}_+) + r.$$

Similarly as $x \to -\infty$, we find

$$(n - i_-) + d_- = \dim \mathcal{S}(\tilde{A}_-) + r.$$

That $n - i = r - d$ follows by adding the two equations and noting that $\dim \mathcal{U}(\tilde{A}_+)$ and $\dim \mathcal{S}(\tilde{A}_-)$ are constant and sum to $n - r$. □

Corollary 2 (See [47]). *For* (*right*) *extreme shocks, for which* $i_+ = n$, *the equality* $d_+ = r$ *also holds. Thus, the connection is also extreme and* $\dim \mathcal{U}(\tilde{A}) \equiv 0$.

Proof. This follows at once from Lemma 12 due to the fact that $d_+ \leqq r$ and $\dim \mathcal{U}(\tilde{A}) \geqq 0$. □

The import of Lemma 12 is that the "parabolic" and "hyperbolic" types of connections agree. From Corollary 2, we may deduce that γ for an extreme right (i.e., n-shock) Lax profile consists of a Wronskian involving only modes from the

$+\infty$ side, and is therefore explicitly evaluable. For, working now in (z_1, z_2, z_2') coordinates, we obtain γ as a determinant[3] of z_2 components only.

Moreover, the expression (A.28) simplifies greatly. In the (z_1, z_2)-coordinates, we know that $\mathbb{U} = \emptyset$, $\mathbb{S}$ is full dimension $n - r$, and $\epsilon\mathbb{S}$ consists of vectors of the simple form $(z, 0)$. This means that $\det(\mathbb{S}, \mathbb{U})$ simplifies to just $\det \mathbb{S}$, while $\det(\pi\mathbb{Z}^+, \epsilon\mathbb{S}^+)$ simplifies to the product of $\det \mathbb{S}^+$ and γ. *Therefore, this term, similarly as in the strictly parabolic case, cancels with term γ in the computation of the stability index.*

Finally, $\det(\epsilon\mathbb{U}^-, \pi\mathbb{Z}^-)$ simplifies to $\det(r_1^-, \dots, r_{n-1}^-, \bar{u}')$, times the determinant of the coordinate transformation from W to Z coordinates, the latter determinant cancelling with a like factor appearing above. We are left in the end with the following very simple formula.

Proposition 7. *In the case of an extreme right* (n*-shock*) *Lax profile,*

$$\tilde{\Gamma} = \operatorname{sgn} \det(r_1^-, \dots, r_{n-1}^-, [u]) \det(r_1^-, \dots, r_{n-1}^-, \bar{u}'/|\bar{u}'|(-\infty)).$$

We emphasize that this is *identical with the stability index in the strictly parabolic case*. The only very weak information required from the connection problem is the orientation of $\bar{u}'$ as $x \to -\infty$, i.e., the direction in which the profile leaves along the one-dimensional unstable manifold. We remark that in the case of isentropic gas dynamics, the traveling-wave equation is scalar, and thus the orientation of $\bar{u}'$ is determined by the direction of the connection. See Section 2.4 or [40] for further details.

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[3] This γ measures the transversality of the connection $\bar{U}(\cdot)$ in (A.3), (A.4). Compare to terms γ_{NS} and γ_d in Sections 2 and 3.

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