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*Intermittency and Regularity Issues in 3D Navier-Stokes Turbulence**

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Abstract

Two related open problems in the theory of $3D$ Navier-Stokes turbulence are discussed in this paper. The first is the phenomenon of intermittency in the dissipation field. Dissipation-range intermittency was first discovered experimentally by Batchelor and Townsend over fifty years ago. It is characterized by spatio-temporal binary behaviour in which long, quiescent periods in the velocity signal are interrupted by short, active 'events' during which there are violent fluctuations away from the average. The second and related problem is whether solutions of the $3D$ Navier-Stokes equations develop finite time singularities during these events. This paper shows that Leray's weak solutions of the three-dimensional incompressible Navier-Stokes equations can have a binary character in time. The time-axis is split into 'good' and 'bad' intervals: on the 'good' intervals solutions are bounded and regular, whereas singularities are still possible within the 'bad' intervals. An estimate for the width of the latter is very small and decreases with increasing Reynolds number. It also decreases relative to the lengths of the good intervals as the Reynolds number increases. Within these 'bad' intervals, lower bounds on the local energy dissipation rate and other quantities, such as $\|\boldsymbol{u}(\cdot,\ t)\|_\infty$ and $\|\nabla\boldsymbol{u}(\cdot,\ t)\|_\infty$, are very large, resulting in strong dynamics at sub-Kolmogorov scales. Intersections of bad intervals for $n \geqq 1$ are related to the potentially singular set in time. It is also proved that the Navier-Stokes equations are conditionally regular provided, in a given 'bad' interval, the energy has a lower bound that is decaying exponentially in time.

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1. Introduction

The questions to be addressed in this paper concern the nature and behaviour of intermittent high Reynolds number solutions of the three-dimensional Navier-Stokes equations. Relatively quiescent flows can exist in nature at high Reynolds numbers. There are also flows that appear turbulent for all practical purposes but are nevertheless smooth at appropriately small length scales. Typically turbulent high Reynolds number Navier-Stokes flows, however, generally display a specific hallmark which is called dissipation-range intermittency. This was first discovered by BATCHELOR & TOWNSEND [1] and manifests itself in violent fluctuations of very short duration in the energy dissipation rate. These fluctuations away from the average are interspersed by quieter, longer periods in the dynamics. The data in Fig. 1 is an illustration of a typically intermittent signal representing a velocity derivative versus time recorded at a single point in space.

For theoretical studies of the Navier-Stokes equations it is usual to express energy dissipation in the L^2-volume-integrated sense. Postponing the full definition of system variables, if the global energy H_0 and enstrophy H_1 for a three-dimensional Navier-Stokes velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ are defined as

$$H_0(t) = \int_V |\boldsymbol{u}|^2 \, dV, \qquad\qquad H_1(t) = \int_V |\nabla \boldsymbol{u}|^2 \, dV, \tag{1}$$

then it is well known that while the energy H_0 is a uniformly bounded function of time, only the long-time averaged energy dissipation rate $\varepsilon_{av} = \nu L^{-3} \langle H_1 \rangle$ is known to be bounded [2], whereas the behaviour of $\varepsilon(t) = \nu L^{-3} H_1(t)$ pointwise

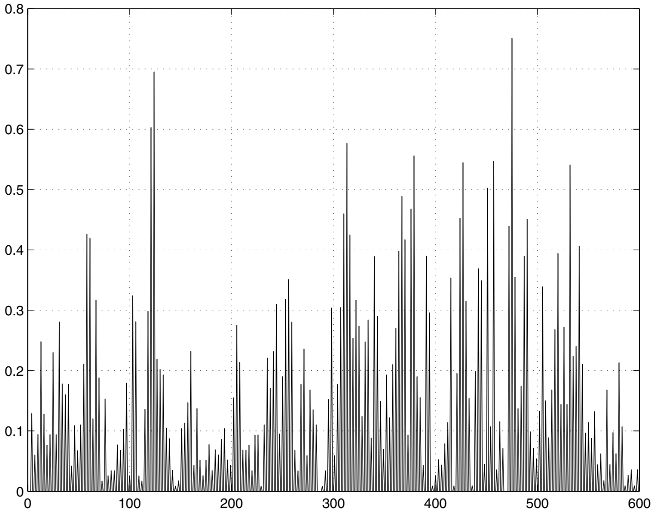

Fig. 1. A typical example of dissipation-range intermittency from wind tunnel turbulence where hot wire anemometry has been used to measure the longitudinal velocity derivative at a single point (D. Hurst and J. C. Vassilicos). The horizontal axis spans 8 integral time scales. The Taylor micro-scale-based Reynolds number is about 200.

in time may be wildly fluctuating or even singular. The long-time average $\langle\cdot\rangle$ is defined later in this section.

Whether $H_1(t)$ becomes singular in a finite time is an open question intimately related to the Navier-Stokes regularity problem. So long as $H_1(t)$ is finite the solution is smooth and unique; any finite time singularity must be accompanied by a divergence of $H_1(t)$ at that time. From LERAY onwards [2], this question has led to a long and rich literature on the nature of weak solutions [3–11]. In terms of physical length scales, the boundedness of $\langle H_1\rangle$ allows the time averaged turbulent energy dissipation rate per unit volume $\varepsilon_{av} = \nu L^{-3}\langle H_1\rangle$ to be used naturally in forming the inverse Kolmogorov length

$$\eta_K^{-1} = \left(\varepsilon_{av}/\nu^3\right)^{1/4}. \tag{2}$$

Spiky behaviour in $H_1(t)$, causing loss of resolution in large-scale computations, could mean that significant energy lies in wave-numbers $k > \eta_K^{-1}$ in the dissipation range of the energy spectrum. The ubiquity of dissipation-range intermittency in turbulent flows suggests that it should occur naturally in mathematical analyses of the Navier-Stokes equations. Strong temporal excursions in $H_1(t)$ are clear candidates for the formation of singularities and may be related to potentially singular solutions of the three-dimensional Euler equations [11–14], although nothing has been rigorously proved in this respect. Beginning with LERAY's seminal paper in 1934, rigorous methods of analysis on the full three-dimensional domain have led to seventy years of literature on the Navier-Stokes regularity problem but have yet to settle this question definitively [2]. Short-time regularity has also been known for many years, as have various interesting partial and conditional regularity results [3–11, 15–19]. BATCHELOR & TOWNSEND [1] suggested that energy dissipation is not distributed evenly across the full three-dimensional spatial domain but is clustered into smaller spots in the flow with the energy associated with the small-scale components being distributed unevenly in space and roughly confined to regions which become smaller with eddy size [20]. A similar observation has been made by EMMONS who observed turbulent spottiness in boundary layer flows [21]. In contrast to Kolmogorov's traditional theory that implies that energy dissipation is space-filling [22], MANDELBROT suggested that the spatial set on which it occurs is actually fractal [23]. In experimental investigations of the energy dissipation rate in several laboratory flows, and in the atmospheric surface layer, SREENIVASAN & MENEVEAU [24, 25] interpreted the evident intermittent nature of their signals in terms of multifractals (see also [26]). ZEFF *et al.* [27] have shown how more recent technical advances have made it possible to measure each derivative of all three velocity components to obtain a fuller experimental picture of the energy dissipation at a point in a flow. In general, numerical simulations and experiments suggest that respectively quasi-one- and two-dimensional tubes and sheets are the favoured low-dimensional sets on which vorticity and strain appear to accumulate [28–30] although, as GALANTI & TSINOBER [31] and TSINOBER [32] have pointed out, there are significant differences between these two sets. It is also true that explaining these phenomena in the simple geometrical terms of tubes and sheets is a visual over-simplification of much more complicated dynamical spatial structures at small scales. Examples of this are the spiral vortex structures introduced by LUNDGREN [33] and discussed in detail by VASSILICOS & HUNT [34], FLOHR & VASSILICOS [35] and ANGILELLA & VASSILICOS [36, 37].

Dissipation-range intermittency is a well established, experimentally observable phenomenon; its appearance in systems other than the Navier-Stokes equations has been discussed in an early and easily accessible paper by FRISCH & MORF [38]. One symptom of its occurrence is the deviation of the 'flatness' of a velocity signal (the ratio of the fourth-order moment to the square of the second-order moment) from the value of 3 that usually holds for Gaussian statistics [39]. More subtle is the phenomenon of inertial-range intermittency that has exercised the ingenuity of those of the physics community who focus on scaling methods. In the inertial range, Kolmogorov's theory predicts that the exponent, ζ_p, of the pth velocity structure function should vary linearly with p, whereas experimental data shows that the $(\zeta_p,\ p)$ relation is a concave curve lying below the line $p/3$ for $p \geqq 3$. This departure from Kolmogorov scaling in the inertial range, and therefore from the five-thirds law, is termed inertial-range intermittency. An extensive literature is quoted in FRISCH's book [22]. Studies in weak turbulence, applicable to predominantly dispersive systems, have been pursued by Zakharov. These ideas can be found in papers by ZAKHAROV, L'VOV & FALKOVICH [40] and ZAKHAROV [41].

To prove that solutions of the Navier-Stokes equations (usually taken in a periodic box [8–10, 42]) are typically intermittent in space-time poses formidable technical challenges to the mathematician. Analysis on time-evolving fractal domains with ill-defined boundary conditions is not advanced enough to gain rigorous results by concentrating on one fractal 'spot' in the flow. Historically, the partial regularity result of SCHEFFER [15], proving that the potentially singular set in time has zero half-dimensional Hausdorff measure, led to that of CAFFARELLI, KOHN & NIRENBERG [16] who showed that the potentially singular set in space-time has zero one-dimensional Hausdorff measure. This implies that if singularities do exist they must be relatively rare. LIN [43] and CHOE & LEWIS [44] have recently provided shorter proofs of this result.

The more realistic option adopted here is to show that solutions of the Navier-Stokes equations can have a *binary nature in time* in which the time-axis is divided into what are designated as *good* and *bad* intervals. On the good intervals the Navier-Stokes equations are uniformly regular. The bad intervals are shown to be very small in width with an upper bound that decreases with increasing Reynolds number and which also decreases relative to the widths of the good intervals. Within the bad intervals, very large lower bounds are shown to exist on both the local-in-time energy dissipation rate and several other quantities, such as $\|\boldsymbol{u}(\cdot,\ t)\|_\infty$ and $\|\nabla\boldsymbol{u}(\cdot,\ t)\|_\infty$. The corresponding local length scales within these intervals are, *at best*, comparatively much smaller than the Kolmogorov length. The regularity question within the bad intervals is still open, so only weak solutions are known to exist there. The great difficulties encountered by computational fluid dynamicists in resolving turbulent flows even for modestly high Reynolds numbers could be because of this binary behaviour.

These results, which are summarized in Section 2, have been obtained through the use of a set of quantities $\kappa_n(t)$ that have been introduced in previous papers

[42, 45, 46]. The more physical aspects of these ideas were laid out previously in a short paper [47]; the present paper gives a detailed and more advanced account of the methods and results reported there. The $\kappa_n(t)$ have the dimensions of inverse lengths and are formed from ratios of L^2 norms of derivatives of the velocity field. Together with the periodicity of the domain, the L^2-spatial integration within the $\kappa_n(t)$ means that spatially intermittent effects are included implicitly and cannot be averaged away. Clearly they are not the same quantities as those measured by experimentalists, such as the energy dissipation rate, but estimates for them are rigorous, making no appeal to any approximations, and ultimately lead to information on the energy dissipation. Physically they can be considered as a measure of the $2n$th moment of the energy spectrum. Their time-dependence is explicit and their long-time averages $\langle\kappa_n\rangle$ are uniformly bounded [46]. Pointwise in time their binary nature appears for each value of $n \geqq 2$.

The phrase 'solutions can have a binary nature in time' has been used above in the following sense: if no bad intervals occur, then solutions of the Navier-Stokes equations are bounded for all time. In this sense, the results in this paper are different from conventional short-time regularity proofs because loss of regularity can only occur in the bad intervals. This is consistent with the partial regularity result in time [15]: if singularities exist at points in time then these must be clustered within the intersection of the bad intervals. They are also consistent with the well-known problems of computational resolution in three-dimensional turbulence: the bounds indicate a structure so fine that it would be extremely difficult to distinguish between regular and singular solutions, despite the possibility of a lack of sharpness.

At this point it is appropriate to describe the Navier-Stokes system of partial differential equations considered on a periodic cube $V = [0,\ L]^3$,

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f}(\boldsymbol{x}), \qquad \nabla\cdot\boldsymbol{u} = 0, \tag{3}$$

where ν is the kinematic viscosity and p is the pressure. The applied body force $\boldsymbol{f}(\boldsymbol{x})$ is taken to be mean-zero and divergence-free so, without loss of generality, the solution $\boldsymbol{u}(\boldsymbol{x}, t)$ is mean-zero at all times. For simplicity narrow-band body forces with a single length scale ℓ are considered; that is, with Fourier components only at wave-number $k = \ell^{-1}$ and $\ell \leqq L/2\pi$. For finite energy initial data, the Navier-Stokes equations admit weak solutions in $L^2(V)$ at each instant of time with finite time integrals of the L^2 norms of the velocity gradients. With the assumption of narrow-band forcing, norms of gradients of $\boldsymbol{f}(\boldsymbol{x})$ are related to the norm of $\boldsymbol{f}(\boldsymbol{x})$; these can all be found in Table 1. Also found there are the definitions of the root-mean-square velocity scale U and the Reynolds number Re. The angled brackets $\langle\cdot\rangle$ denote the long-time average

$$\langle\Phi(\cdot)\rangle = \mathrm{LIM}_{t\to\infty}\left(\frac{1}{t}\int_0^t \Phi(s)\,ds\right). \tag{4}$$

LIM is a generalized long-time limit for functionals of (weak) statistical solutions of the Navier-Stokes equations [5, 6]. The square of the L^2-norm $\|\cdot\|_2^2$ is defined as

Table 1. Definitions of the main parameters in the paper.

Quantity	Definition	Comment
Box length	L	
Forcing length scale	ℓ	$\ell \leqq L/2\pi$
Average forcing	$f^2 = L^{-3}\|\boldsymbol{f}\|_2^2$	
Narrow-band forcing	$\|\boldsymbol{f}\|_2^2 \approx \ell^{2n}\|\nabla^n \boldsymbol{f}\|_2^2$	
Average velocity	$U^2 = L^{-3}\left\langle \|\boldsymbol{u}\|_2^2 \right\rangle$	
Grashof Number	$\mathrm{Gr} = f\ell^3\nu^{-2}$	
Reynolds Number [48]	$\mathrm{Re} = U\ell\nu^{-1}$	$\mathrm{Gr}^{1/2} \leqq c\,\mathrm{Re}$ as $\mathrm{Gr} \to \infty$

$$\|\boldsymbol{f}\|_2^2 = \int_V |\boldsymbol{f}|^2\, dV. \tag{5}$$

The Grashof number Gr is the natural control parameter, not the Reynolds number Re, but it is clear that high Reynolds number solutions may be achieved if the Grashof number Gr is sufficiently high; indeed, DOERING & FOIAS [48] have proved that for body-forced Navier-Stokes flows such as these $\mathrm{Gr}^{1/2} \leqq c\,\mathrm{Re}$ as $\mathrm{Gr} \to \infty$. Next we define the time dependent quantities

$$H_n(t) = \int_V |\nabla^n \boldsymbol{u}(\boldsymbol{x}, t)|^2\, dV = \sum_{\boldsymbol{k}} k^{2n} |\hat{\boldsymbol{u}}(\boldsymbol{k}, t)|^2, \tag{6}$$

where $\hat{\boldsymbol{u}}(\boldsymbol{k}, t)$ is the Fourier transform of $\boldsymbol{u}(\boldsymbol{x}, t)$. For higher derivatives of $\boldsymbol{u}$ and $\boldsymbol{f}$, the important quantities that will be used in this paper are

$$F_n = \int_V \left(|\nabla^n \boldsymbol{u}|^2 + \tau^2 |\nabla^n \boldsymbol{f}|^2\right) dV, \tag{7}$$

where τ is a characteristic time whose origin is discussed later in Section 3. In approaching this problem conventional L^2 norms have been used in (7) to avoid difficulties with the pressure field.

Not all interesting small-scale spatial behaviour can be averaged away on a finite periodic box; spatial events must show up in some temporal manner. The next step is to define the quantities

$$\kappa_{n,r}(t) = (F_n/F_r)^{1/2(n-r)}\,. \tag{8}$$

The particular one of interest is $\kappa_n = \kappa_{n,0}$ where the $r = 0$ label has been dropped for convenience. κ_n^{2n} can be interpreted as being related to the $2n$th moment of the energy spectrum [46]; this can be seen from writing $|v|^2 = |\hat{\boldsymbol{u}}|^2 + |\hat{\boldsymbol{f}}|^2$ and then applying Parseval's equality to (8) (with $r = 0$)

$$[\kappa_n(t)]^{2n} = \frac{\sum_k k^{2n} |v(\boldsymbol{k}, t)|^2}{\sum_k |v(\boldsymbol{k}, t)|^2}. \tag{9}$$

Moreover, the κ_n are ordered in magnitude for all $t \geqq 0$,

$$L^{-1} \leqq \kappa_1 \ldots \leqq \kappa_n \leqq \kappa_{n+1} \leqq \ldots \tag{10}$$

which is simply a result of Hölder's inequality. The full $\kappa_{n,r}$ are also ordered such that $\kappa_{n,r} \leqq \kappa_{n,r+1}$ for $r + 1 < n$.

Table 2. Definitions of the main variables and constants in the paper. The parameter δ, which lies in the range $0 < \delta < \frac{1}{6}$, is ignored hereafter.

	Definition	
H_n	$H_n(t) = \int_V \lvert\nabla^n \boldsymbol{u}\rvert^2 \, dV$	$n \geqq 0$
ε_{av}	$\varepsilon_{av} = \nu L^{-3} \langle H_1 \rangle$	
η_K	$\eta_K^{-4} = \varepsilon_{av}/\nu^3$	
F_n	$F_n(t) = \int_V \left(\lvert\nabla^n \boldsymbol{u}\rvert^2 + \tau^2 \lvert\nabla^n \boldsymbol{f}\rvert^2\right) dV$	
ω_0	$\omega_0 = \nu L^{-2}$	
τ	$\omega_0 \tau = \mathrm{Gr}^{-(1/2+\delta)}$	$0 < \delta < \frac{1}{6}$
κ_n	$\kappa_n(t) = (F_n/F_0)^{1/2n}$	
λ_n	$\lambda_n = 3 - \frac{5}{2n} + \frac{\delta}{n}$	

2. Summary of the main results of the paper

Sections 3–5 of this paper give a full account of the ideas with proofs from first principles. This section has been introduced for readers who prefer to peruse the proofs later. The next subsection on long-time averages is followed by a second describing where problems with regularity lie. The third describes how the time-axis can be split into two types of intervals, ‘good’ and ‘bad’. Section 2.4 summarises the dynamics on the bad intervals and the very large lower bounds that can be found within them.

2.1. Long-time averages

The best known long-time average in Navier-Stokes analysis is the explicit upper bound on the time averaged dissipation rate $\varepsilon_{av} = \nu L^{-3} \langle H_1 \rangle$ found from LERAY’s energy inequality [2]

$$\tfrac{1}{2}\dot{H}_0 \leqq -\nu H_1 + H_0^{1/2} \|\boldsymbol{f}\|_2. \tag{11}$$

Using DOERING & FOIAS’s result [48] that $\mathrm{Gr}^{1/2} \leqq c\,\mathrm{Re}$, it is easily shown that as $\mathrm{Gr} \to \infty$,

$$\langle H_1 \rangle \leqq c\, \nu^2 L^3 \ell^{-4} \mathrm{Re}^3. \tag{12}$$

Recall that the long-time average $\langle \cdot \rangle$ is defined in (4). The above Re^3 long-time average estimate for $\langle H_1 \rangle$ leads to an upper bound on the energy dissipation rate $\varepsilon_{av} \leqq c\, \nu^3 \ell^{-4} \mathrm{Re}^3$ and thence to the conventional estimate on the inverse Kolmogorov length η_K^{-1} defined in (2) ,

$$\eta_K^{-1} = \left(\frac{\varepsilon_{av}}{\nu^3}\right)^{1/4} \leqq \ell^{-1} \mathrm{Re}^{3/4}. \tag{13}$$

FOIAS, GUILLOPÉ & TEMAM’S generalization of (12), when the inequality $\mathrm{Gr}^{1/2} \leqq c\,\mathrm{Re}$ is applied [7], can be expressed as (see Theorem 2 in Section 3.3)

$$\ell \left\langle F_n^{\frac{1}{2n-1}} \right\rangle \leqq c_n (L\ell^{-1})^3 \nu^{\frac{2n}{2n-1}} \mathrm{Re}^3. \tag{14}$$

When $n = 1$, (14) recovers the sharp result of DOERING & FOIAS [48] for $\langle F_1 \rangle$, except for a spurious volume factor $(L\ell^{-1})^3$ on the right hand side.

A closely connected and important result used in this paper is the boundedness of the long-time averages of the κ_n, for $n \geqq 1$ taken with $\ell = L/2\pi$ (see [46] and Theorem 1 in Section 3.3),

$$\langle L\kappa_n \rangle \leqq c_n \mathrm{Re}^{\lambda_n}, \qquad \lambda_n = 3 - \frac{5}{2n} + \frac{\delta}{n}, \tag{15}$$

where δ is a small parameter lying in the range $0 < \delta < \frac{1}{6}$. From either (14) or (15) an upper bound can also found on $\langle \|\boldsymbol{u}\|_\infty \rangle$; see Theorem 2 in Section 3.3 and Table 5. Note that the case $n = 1$ gives $\langle L\kappa_1 \rangle \leqq c_1 \mathrm{Re}^{1/2}$, which is consistent with the traditional scaling of $\mathrm{Re}^{-1/2}$ for the Taylor micro-scale. Boundedness of the long-time average of κ_n does not, of course, imply that the κ_n are bounded pointwise in time.

2.2. *Problems with regularity: an illustration*

Normal practice has been to consider the time evolution of the F_n using differential inequalities. With variations, this has been the standard approach taken since the early days of the subject [8–10, 42]. One such inequality is

$$\tfrac{1}{2}\dot{F}_n \leqq -\tfrac{1}{2}\nu F_{n+1} + \left(c_n \nu^{-1} \|\boldsymbol{u}\|_\infty^2 + \nu \ell^{-2} \mathrm{Re} \right) F_n, \tag{16}$$

where $\|\boldsymbol{u}(\cdot,\, t)\|_\infty$ is, in effect, the peak velocity on the whole domain. The reader can find the precise derivation of (16) in Proposition 1 in Section 3.2. The right-hand side has two dominant terms: one negative term associated with the dissipation, and the dominant positive $\|\boldsymbol{u}\|_\infty^2$ term. Rewriting the F_{n+1} term in (16) in terms of the κ_n defined in (8), we obtain

$$F_{n+1} = \kappa_n^2 \left(\frac{\kappa_{n+1}}{\kappa_n} \right)^{2(n+1)} F_n. \tag{17}$$

Using a Sobolev inequality $\|\boldsymbol{u}\|_\infty^2 \leqq c\, \kappa_n^3 F_0$ for $n \geqq 2$, (16) becomes

$$\tfrac{1}{2}\dot{F}_n \leqq \left\{ -\tfrac{1}{2}\nu \kappa_n^2 \left(\frac{\kappa_{n+1}}{\kappa_n} \right)^{2(n+1)} + c_n \nu^{-1} \kappa_n^3 F_0 + \nu \ell^{-2} \mathrm{Re} \right\} F_n. \tag{18}$$

From (10) the ratio κ_{n+1}/κ_n has a lower bound of unity, $\kappa_{n+1}/\kappa_n \geqq 1$, for all periodic divergence-free functions. This reduces (18) to

$$\tfrac{1}{2}\dot{F}_n \leqq \left(-\tfrac{1}{2}\nu \kappa_n^2 + c_n \nu^{-1} \kappa_n^3 F_0 + \nu \ell^{-2} \mathrm{Re} \right) F_n. \tag{19}$$

The manifestly negative term $\sim \kappa_n^2$ in (19) is not sufficient to control the κ_n^3 term: arbitrarily large initial data on F_n can be chosen that makes the right hand side positive. Despite the finiteness of the time averages (14) and (15), this leads to a failure to control either F_n or κ_n, other than for short times or small initial data [3].

2.3. The potentially binary nature of the time-axis

One way of proceeding with this difficult problem is to ask whether the lower bound of unity on κ_{n+1}/κ_n could be raised, thereby effectively increasing the dissipation in (18). BATCHELOR & TOWNSEND [1] experimentally identified a similar quantity to κ_{n+1}/κ_n for $n = 2,\ 3$ that was larger than expected for Gaussian data. The principal result of this paper, proved in Section 4, is that an improved lower bound on κ_{n+1}/κ_n can be found which is valid only on sections of the time-axis. The estimates in Theorem 3 in Section 4, the key result of the paper, show that for Navier-Stokes weak solutions

$$\left\langle\left[c_n\left(\frac{\kappa_{n+1}}{\kappa_n}\right)\right]^{1/\mu-1} - \left[(L\kappa_n)^{\mu}\mathrm{Re}^{-\lambda_n}\right]^{1/\mu-1}\right\rangle \geqq 0, \tag{20}$$

where the real parameter μ can take any value in the range $0 < \mu < 1$. The c_n are the same as in (15). Given that the long-time average in (20) is nonnegative means that there must be intervals of the time-axis, called *good intervals*, where the inequality

$$c_n\left(\frac{\kappa_{n+1}}{\kappa_n}\right) \geqq (L\kappa_n)^{\mu}\,\mathrm{Re}^{-\lambda_n} \tag{21}$$

holds. It is easily seen that when (21) is applied to (18) on these intervals the strength of the dissipation is increased. This applies at small scales ($L\kappa_n > c_n\mathrm{Re}^{\lambda_n/\mu}$) where the lower bound on κ_{n+1}/κ_n in (21) is raised away from unity. The divisor (F_0) within κ_n is bounded both above and below. Thus (18) can be turned into a proper differential inequality in F_n; the reader can refer to Section 4.1 for details. The result, which is intuitively obvious from (18), is that the negative dissipation term is stronger than the positive nonlinear term when

$$\mu > \frac{1}{2(n+1)}, \tag{22}$$

so no singularities are possible on these intervals (see Section 4.1).

However, the integrand in (20) cannot be guaranteed to always be positive, so intervals on the rest of the time-axis, where it could be negative, are designated as *bad intervals*. On these the reverse inequality must be satisfied,

$$c_n\left(\frac{\kappa_{n+1}}{\kappa_n}\right) < (L\kappa_n)^{\mu}\,\mathrm{Re}^{-\lambda_n}. \tag{23}$$

Because its is always true that $\kappa_{n+1} \geqq \kappa_n$, then

$$L\kappa_n(t) > c_n\mathrm{Re}^{\lambda_n/\mu} \tag{24}$$

on these intervals. It is, of course, possible that there are no bad intervals and that the whole time-axis is 'good'. The positivity of the average in (20), however, ensures that the complete time-axis cannot be 'bad'. This paper is based on the worst-case supposition that bad intervals exist and need to be dealt with accordingly. There appears to be no further information within (20) as to their distribution, which may depend on the value of n.

2.4. Dynamics on the bad intervals

Various questions remain. Firstly, if the bad intervals exist, are they finite in width? The answer to this is in the affirmative. Estimates for the widths of bad intervals $(\Delta t)_b$ are displayed in Table 5 (see Theorem 5 in Section 5.2) with upper bounds on μ lying in two ranges together with a lower bound given in (21). It can be seen that these widths are exceedingly small; in fact $(\Delta t)_b \to 0$ as $\mathrm{Re} \to \infty$. The large lower bound on κ_n in (24) indicates the predominance of high wave-numbers within these very short intervals.

Fig. 2 is a descriptive picture of a typical distribution of good/bad intervals on the t-axis. It is drawn in such a way that $\kappa_n(t)$ looks like a relatively flat function. While this is probable, some artistic licence has been taken for the following reasons. The lower bound in (24) makes values of $\kappa_n(t)$ much larger than the upper bound on the time average (15), so it has to spend relatively long amounts of time in the good intervals to recompense. Nevertheless, this does not prove that it is quiescent in the central part of the interval. As is shown in Section 4, there is enough freedom within the upper bound on κ_n, under the constraint of the long-time average, to allow it to reach large enough values so that it can connect with the next bad interval. While we believe that it is likely that κ_n is flat in the central region, we have been unable to prove this. It is pathologically possible that enough fine structure might exist within the good interval without violating the average.

The positions of the intervals may differ as n varies so a new figure is needed for each n; Lemma 1 shows that if any *one* κ_n is bounded then all are bounded. Only if the intersection of all bad intervals is itself 'bad' is a singularity possible; Section 5.3 addresses this question. The set containing the intersection of all the bad intervals for all $n \geqq 1$ is designated there as $\mathcal{S}^{(\infty)}$ and must be related to the

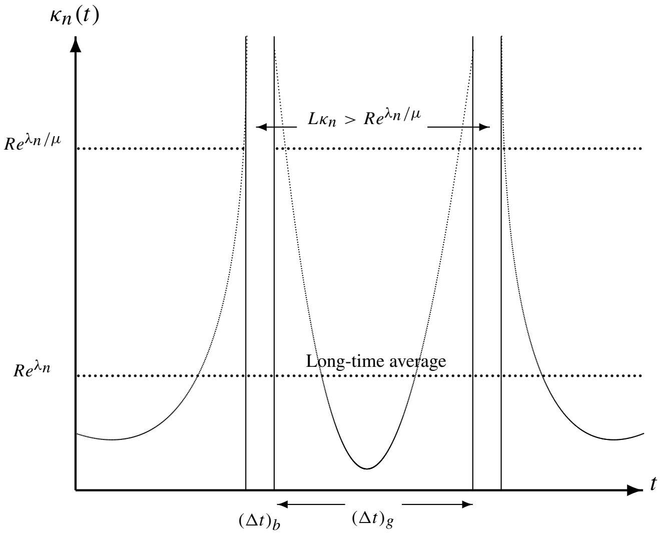

Fig. 2. A descriptive picture, not to scale, of good/bad intervals for some value of $n \geqq 2$; constants have been omitted.

potentially singular set [15]. As Theorem 6 of Section 5.3 shows, taking the limit to the potentially singular set narrows the range of μ.

The bad intervals (& their intersection) can be divided into sub-intervals; for the ith bad interval let us take the jth sub-interval on which $\dot{F}_n \geqq 0$. This we call the ijth 'dangerous' sub-interval of width $(\Delta t_+^{i,j})$. Singularities can occur on any interval within these because F_n is increasing, whereas for sub-intervals on which $\dot{F}_n \leqq 0$ no singularities can occur because F_n is bounded ($n \geqq 1$) by its initial value at the start of the sub-interval. Estimates for the width of dangerous sub-intervals, of width $(\Delta t_+^{i,j})$, are smaller than those for $(\Delta t)_b$ and can be found in Theorem 7 of Section 5.4 (see Table 5). It is then possible to find very large *lower* bounds within $(\Delta t_+^{i,j})$ on various quantities such as an equivalent of the inverse Kolmogorov length η_+^{-1}, based upon lower bounds on F_1 within $(\Delta t_+^{i,j})$; it is here where the intermittency of the dissipation field shows up. Additionally, the peak velocity and the peak velocity gradient matrix also have very large lower bounds. These are displayed in Table 3.

Conclusions that can be drawn from Tables 3, 4 and 5 at the end of this section are:

1. The bad intervals are exceptionally narrow for large Re.
2. The action within these intervals is intense; the lower bound on F_1 within them illustrates the strength of the dissipation field there.
3. The dynamics within these intervals is so fine, even if no singularities occur, that they would be exceptionally difficult to resolve numerically.

Theorem 4 bounds from below, in the average sense, the ratio of the widths of the good and bad intervals

$$\frac{\overline{(\Delta t)_g}}{\overline{(\Delta t)_b}} \geqq \mathrm{Re}^{\lambda_n\left(\frac{1}{\mu}-1\right)}, \tag{25}$$

showing that relative to the bad intervals, the widths of the good intervals increase with Re.

Table 3. A comparison between the upper bounds on the long-time averages in column two (see Theorems 1 and 2 in Section 3.3) and the very large lower bounds in dangerous sub-intervals $(\Delta t_+^{i,j})$ in column three (see Theorem 7 in Section 5.4). Notice there the very large lower bounds on F_1 illustrating the strong intermittency in the dissipation field. All multiplicative constants have been omitted to save space. In addition $\ell = L/2\pi$.

	Long-time average $\langle\cdot\rangle$	On $(\Delta t_+^{i,j})$ intervals
Energy moments	$\langle L\kappa_n\rangle \leqq \mathrm{Re}^{\lambda_n}$	$L\kappa_n \geqq \mathrm{Re}^{4+b_n}$
Enstrophy	$L^{-3}\langle F_1\rangle \leqq \omega_0^2\mathrm{Re}^3$	$L^{-3}F_1 \geqq \omega_0^2\mathrm{Re}^{4+b_n}$
(Kolmogorov scale)$^{-1}$	$L\eta_K^{-1} \leqq \mathrm{Re}^{3/4}$	$L\eta_+^{-1} \geqq \mathrm{Re}^{(4+b_n)/4}$
(Taylor micro-scale)$^{-1}$	$\langle L\kappa_1\rangle \leqq \mathrm{Re}^{1/2}$	$L\kappa_1 \geqq \mathrm{Re}^{b_n/2}$
Peak velocity	$\langle\|\boldsymbol{u}\|_\infty\rangle \leqq L\omega_0\mathrm{Re}^3$	$\|\boldsymbol{u}\|_\infty \geqq L\omega_0\mathrm{Re}^{4+b_n}$
Vel. gradient matrix	$\left\langle\|\nabla\boldsymbol{u}\|_\infty^{1/2}\right\rangle^2 \leqq \omega_0\mathrm{Re}^3$	$\|\nabla\boldsymbol{u}\|_\infty \geqq \omega_0\mathrm{Re}^{4+b_n}$

Table 4. Definitions of λ_n, a_n and b_n. On the intersection set $\mathcal{S}^{(\infty)}$ of Section 5.3, λ_n can be replaced by $\Lambda_n^{(\infty)}$.

	Definition	Range
λ_n	$\lambda_n = 3 - \frac{5}{2n}$	
a_n	$a_n = \frac{\lambda_{n+1}}{\mu}\left(\frac{2n-2}{2n-1}\right) - \frac{10n-1}{2n-1}$	$a_n > 0$
b_n	$b_n = \frac{\lambda_{n+1}}{\mu} - 4$	$b_n > a_n$

Table 5. Estimates of widths of bad intervals, their sub-intervals ($\Delta t_+^{i,j}$), and corresponding ranges of a_n, b_n & μ (see Theorem 5 in Section 5.2 and Theorem 7 in Section 5.4). It is always true that $b_n > a_n$ (see Table 4). For the case $n = 1$, $\mu > \frac{1}{2}$.

Range of μ	a_n, b_n	Width of interval $n \geqq 2$
$\mu \leqq \lambda_{n+1}\left(\frac{n-1}{6n-1}\right)$	$a_n \geqq 1$	$\omega_0(\Delta t)_b \leqq \mathrm{Re}^{-a_n}$
$\lambda_{n+1}\left(\frac{n-1}{6n-1}\right) < \mu < \lambda_{n+1}\left(\frac{2n-2}{10n-1}\right)$	$0 < a_n < 1$	$\omega_0(\Delta t)_b \leqq \mathrm{Re}^{-1}\ln \mathrm{Re}$
	$b_n > 1$	$\omega_0(\Delta t_+^{i,j}) \leqq \mathrm{Re}^{-b_n}$
$\frac{1}{2(n+1)} < \mu$		$\overline{(\Delta t)_g} \geqq \overline{(\Delta t)_b}\,\mathrm{Re}^{\lambda_n\left(\frac{1}{\mu}-1\right)}$

The final result of Section 6 displayed in Theorem 8 is a conditional regularity result. Assume that the energy $H_0(t)$ has a lower bound within the dangerous sub-intervals, $\Delta t_+^{(i,j)} = t - t_0^{i,j}$, of the form

$$H_0(t) \geqq H_0(t_0^{i,j})e^{-\omega_0 \mathrm{Re}\Delta t_+^{(i,j)}}, \tag{26}$$

then solutions of the Navier-Stokes equations are regular there. Note that $t_0^{i,j}$ is the initial time for the dangerous sub-interval. The very large initial conditions on the energy $H_0(t_0^{i,j})$ at the junction of the intervals is the main obstacle to completing a regularity proof.

3. Standard estimates

3.1. The forcing does not dominate the fluid

The technical parts of this paper revolve around the quantities

$$F_n(t) = H_n + \tau^2\|\nabla^n \boldsymbol{f}\|_2^2, \tag{27}$$

where the H_n are defined in (6). The F_n contain the fluid velocity derivatives and those of the forcing, although the latter has been assumed to have a narrow-band character, as shown in Table 1. They are included within F_n in order to circumvent problems that may arise when dividing by these (squared) semi-norms. Once these terms have been introduced it is necessary to demonstrate that they do not dominate the fluid [46], a point that has also made by TSINOBER [49] in numerical computations. The characteristic time τ will be chosen for convenience but as long as $\tau \neq 0$, the F_n are bounded away from zero by the explicit value $\tau^2 L^3 \ell^{-2n} f^2$. Moreover, τ may be chosen to depend on the parameters of the problem such that $\langle F_n \rangle \sim \langle H_n \rangle$ as $\mathrm{Gr} \to \infty$. To see how to achieve this, let us define

$$\tau = \ell^2 \nu^{-1} \mathrm{Gr}^{-(\delta+1/2)} \tag{28}$$

with $\delta > 0$, which is a parameter yet to be determined. Then the additional term in (27) is

$$\tau^2 \|\nabla^n \boldsymbol{f}\|_2^2 = L^3 \nu^{-2} \ell^{4-2n} f^2 \mathrm{Gr}^{-(2\delta+1)} = \nu^2 \ell^{-(2n+2)} L^3 \mathrm{Gr}^{1-2\delta}. \tag{29}$$

Now it has also been shown by DOERING & FOIAS [48] that the energy dissipation rate $\varepsilon_{av} = \nu L^{-3} \langle H_1 \rangle$ has a *lower* bound for high Gr:

$$\varepsilon_{av} \geqq c\, \nu^3 \ell^{-3} L^{-1} \mathrm{Gr}, \tag{30}$$

which can be used in the far right hand side of (29) ,

$$\begin{aligned} \tau^2 \|\nabla^n \boldsymbol{f}\|_2^2 &\leqq c\, \varepsilon_{av}\, \ell^{-(2n-1)} L^4 \nu^{-1} \mathrm{Gr}^{-2\delta} \\ &= c \left(L\ell^{-1}\right)^{(2n-1)} L^{-2(n-1)} \big\langle H_1 \big\rangle \mathrm{Gr}^{-2\delta} \\ &\leqq c \left(L\ell^{-1}\right)^{(2n-1)} \big\langle H_n \big\rangle \mathrm{Gr}^{-2\delta}, \end{aligned} \tag{31}$$

where Poincaré's inequality has been used at the last step. Hence, for any $\delta > 0$, the additional forcing term in (27) is seen to be negligible with respect to $\langle H_n \rangle$ as $\mathrm{Gr} \to \infty$. The parameter δ is left arbitrary at this stage, although it will be restricted further in the course of proving results in the next section. Our interest lies in results for high Gr so correction terms described above will be ignored and it can safely be said that for $\delta > 0$,

$$\langle F_1 \rangle \leqq c\, \nu^2 L^3 \ell^{-4} \mathrm{Re}^3. \tag{32}$$

This is Leray's result for weak solutions with narrow-band forcing included in a rational manner; the next section deals with long-time averages of other quantities.

3.2. The F_n ladder

In the calculations that follow the H_n are formally manipulated even though they are not known to be finite pointwise in time for weak solutions; the end results may be justified in the standard way by proceeding from a Galerkin approximation to the solutions and then removing the regularization in the final results. In the usual manner 'c' and c_n are used as generic constants.

Proposition 1. *For* Gr $\to \infty$ *and* $\delta > 0$*, the* F_n *satisfy the differential inequalities*

$$\tfrac{1}{2}\dot{F}_0 \leqq -\nu F_1 + c_1 \nu \ell^{-2} \mathrm{Re}^{1+2\delta} F_0, \tag{33}$$

$$\tfrac{1}{2}\dot{F}_1 \leqq -\tfrac{1}{4}\nu F_2 + c_2 \nu^{-3} F_1^3 + c\,\nu\ell^{-2}\mathrm{Re}^{1+2\delta} F_1, \tag{34}$$

and for $n \geqq 2$*, there are two alternative versions*

$$\tfrac{1}{2}\dot{F}_n \leqq -\nu F_{n+1} + c_{n,1}\left(\|\nabla \boldsymbol{u}\|_\infty + \nu\ell^{-2}\mathrm{Re}^{1+2\delta}\right) F_n, \tag{35}$$

$$\tfrac{1}{2}\dot{F}_n \leqq -\tfrac{1}{2}\nu F_{n+1} + c_{n,2}\left(\nu^{-1}\|\boldsymbol{u}\|_\infty^2 + \nu\ell^{-2}\mathrm{Re}^{1+2\delta}\right) F_n. \tag{36}$$

Remark 1. When the inequality $F_1^2 \leqq F_2 F_0$ is used in (34), the resulting differential inequality for F_1 demonstrates the inability of these methods, as they stand, to gain control over F_1 for arbitrarily large initial data.

Proof. The proof follows in four steps.

Step 1. Let us begin with the proof of (33). Leray's energy inequality is

$$\tfrac{1}{2}\dot{H}_0 \leqq -\nu H_1 + H_0^{1/2}\|\boldsymbol{f}\|_2. \tag{37}$$

Adding and subtracting the quantity $\nu\tau^2\|\nabla \boldsymbol{f}\|_2^2$, it is seen that

$$\tfrac{1}{2}\dot{F}_0 \leqq -\nu F_1 + \nu\tau^2\|\nabla \boldsymbol{f}\|_2^2 + H_0^{1/2}\|\boldsymbol{f}\|_2. \tag{38}$$

Because the forcing is narrow-band as in Table 1, it is possible to reduce a derivative on the $\|\nabla \boldsymbol{f}\|_2^2$ term. This, together with Young's inequality (using $g\tau^2 > 0$ as a parameter where g is to be suitably chosen below) to break up the last term yields

$$\tfrac{1}{2}\dot{F}_0 \leqq -\nu F_1 + \frac{1}{2g\tau^2} H_0 + \tau^2\left(\tfrac{1}{2}g + \frac{\nu}{\ell^2}\right)\|\boldsymbol{f}\|_2^2, \tag{39}$$

where g is determined by making the coefficients of H_0 and $\tau^2\|\boldsymbol{f}\|_2^2$ equal, giving

$$g = -\frac{\nu}{\ell^2} + \left\{\nu^2\ell^{-4} + \tau^{-2}\right\}^{1/2}. \tag{40}$$

With τ chosen as in (28) with $\delta > 0$, g becomes

$$g = \tau^{-1}\left(\left\{1 + \mathrm{Gr}^{-(2\delta+1)}\right\}^{1/2} - \mathrm{Gr}^{-(\delta+1/2)}\right). \tag{41}$$

Consequently, $g \sim \tau^{-1}$ as Gr $\to \infty$ in which case

$$\tau^{-1} = \nu\ell^{-2}\mathrm{Gr}^{\frac{1}{2}+\delta} \leqq c\,\nu\ell^{-2}\mathrm{Re}^{1+2\delta}. \tag{42}$$

In this limit (39) can be written as in (33).

Step 2. The proof of (34) is found directly from ($\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}$)

$$\tfrac{1}{2}\dot{H}_1 \leqq -\nu H_2 + \int_\Omega \boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}\,dV + H_1^{1/2}\|\nabla \boldsymbol{f}\|_2. \tag{43}$$

The middle term can be estimated as

$$\int_\Omega \boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}\,dV \leqq \|\boldsymbol{\omega}\|_4^2\|\nabla\boldsymbol{u}\|_2 \leqq c\,F_2^{3/4}F_1^{3/4}, \tag{44}$$

having used the Sobolev inequality $\|\boldsymbol{\omega}\|_4 \leqq c\,\|\nabla\boldsymbol{\omega}\|_2^{3/4}\|\boldsymbol{\omega}\|_2^{1/4}$. The procedure with the forcing is then used as in Step 1 to obtain

$$\tfrac{1}{2}\dot{F}_1 \leqq -\nu F_2 + c\,F_2^{3/4}F_1^{3/4} + c\,\nu\ell^{-2}\mathrm{Re}^{1+2\delta}F_1. \tag{45}$$

Young's inequality on the middle term finally gives (34).

Step 3. For a proof of (35), consider the ladder of differential inequalities satisfied by the H_n for $n \geqq 2$ (see [42, 50]):

$$\tfrac{1}{2}\dot{H}_n \leqq -\nu H_{n+1} + c_n\|\nabla\boldsymbol{u}\|_\infty H_n + H_n^{1/2}\|\nabla^n\boldsymbol{f}\|_2. \tag{46}$$

Then (35) is proved by following the procedure with the forcing as in Step 1.

Step 4. The alternative to the differential inequality (46) for $n \geqq 2$ is

$$\tfrac{1}{2}\dot{H}_n \leqq -\tfrac{1}{2}\nu H_{n+1} + c_n\nu^{-1}\|\boldsymbol{u}\|_\infty^2 H_n + H_n^{1/2}\|\nabla^n\boldsymbol{f}\|_2. \tag{47}$$

Then (36) is found by using the same procedure as in Step 1 except that the quantity $\frac{1}{2}\nu\tau^2\|\nabla^{n+1}\boldsymbol{f}\|_2^2$ is subtracted whereas $\nu\tau^2\|\nabla^{n+1}\boldsymbol{f}\|_2^2$ is added. □

3.3. *Long-time averages*

In Section 1 it was shown how the quantities κ_n are ordered such that $\kappa_n \leqq \kappa_{n+1}$. There is no κ_n that is known to be *a priori* bounded. What is known is the boundedness of the long-time averages defined in (4). The equivalent of Leray's bulk dissipation estimate in terms of κ_1 instead of F_1 is found from (33) by dividing through F_0 and long-time averaging:

$$\ell^2\left\langle \kappa_1^2\right\rangle \leqq c\,\mathrm{Re}^{1+2\delta}. \tag{48}$$

The first of the following two theorems states results on long-time averages for higher values of n. This estimate can be found in [46] with a wider range of δ.

Theorem 1. *For* Gr $\to\infty$ *and the parameter* δ *lying in the range* $0 < \delta < \frac{1}{6}$,

$$\ell\,\langle\kappa_n\rangle \leqq c_n\left(L\ell^{-1}\right)^{\frac{3(n-1)}{n}}\mathrm{Re}^{\lambda_n}, \qquad n \geqq 1, \tag{49}$$

where λ_n *is defined by*

$$\lambda_n = 3 - \frac{5}{2n} + \frac{\delta}{n}. \tag{50}$$

Proof. *Step 1.* Consider first $\kappa_{2,1}$:

$$\left\langle\kappa_{2,1}\right\rangle = \left\langle\left(\frac{F_2}{F_1}\right)^{1/2}\right\rangle \leqq \left\langle\frac{F_2}{F_1^2}\right\rangle^{1/2}\left\langle F_1\right\rangle^{1/2}$$
$$\leqq \frac{\nu^2}{2}\left\langle\frac{F_2}{F_1^2}\right\rangle + \frac{1}{2\nu^2}\left\langle F_1\right\rangle, \tag{51}$$

where Young's inequality has been used at the last step. Dividing inequality (34) in Proposition 1 in Section 3.2 by F_1^2 and long-time averaging give

$$\nu^2\left\langle\frac{F_2}{F_1^2}\right\rangle \leqq c\,\nu^{-2}\left\langle F_1\right\rangle + \nu\tau^{-1}\left\langle F_1^{-1}\right\rangle, \tag{52}$$

and so

$$\left\langle\kappa_{2,1}\right\rangle \leqq c\,\nu^{-2}\left\langle F_1\right\rangle + \nu\tau^{-1}\left\langle F_1^{-1}\right\rangle. \tag{53}$$

The last term is

$$\nu\tau^{-1}\left\langle F_1^{-1}\right\rangle \leqq \frac{\nu\tau^{-1}}{\tau^2\ell^{-2}L^3 f^2} = \ell^2 L^{-3}\mathrm{Gr}^{3\delta-\frac{1}{2}} \tag{54}$$

from which it is concluded that δ must lie in the range $0 < \delta < \frac{1}{6}$ to be certain that this term decreases as $\mathrm{Gr}\to\infty$. Because

$$\left\langle F_1\right\rangle \leqq \nu^2 L^3\ell^{-4}\mathrm{Re}^3 \tag{55}$$

it then follows that

$$\ell\left\langle\kappa_{2,1}\right\rangle \leqq c\left(L\ell^{-1}\right)^3\mathrm{Re}^3. \tag{56}$$

Step 2. Now consider the quantities $\left\langle\kappa_{n+1,n}\right\rangle$ for $n \geqq 2$:

$$\left\langle\kappa_{n+1,n}\right\rangle = \left\langle\left(\frac{F_{n+1}}{F_n^{2n/(2n-1)}}\right)^{1/2} F_n^{1/2(2n-1)}\right\rangle$$
$$\leqq \left\langle\frac{F_{n+1}}{F_n^{2n/(2n-1)}}\right\rangle^{1/2}\left\langle F_n^{1/(2n-1)}\right\rangle^{1/2}, \tag{57}$$

$$\left\langle F_n^{\frac{1}{2n-1}}\right\rangle = \left\langle\kappa_{n,1}^{(2n-2)/(2n-1)} F_1^{1/(2n-1)}\right\rangle$$
$$\leqq \left\langle\kappa_{n,1}\right\rangle^{(2n-2)/(2n-1)}\left\langle F_1\right\rangle^{1/(2n-1)}. \tag{58}$$

Having used the fact that $\kappa_{n,1} \leqq \kappa_{n+1,n}$, (57) and (58) give

$$\left\langle\kappa_{n+1,n}\right\rangle \leqq \left[\nu^{\frac{2}{2n-1}}\left\langle\frac{F_{n+1}}{F_n^{(2n-1)/2n}}\right\rangle\right]^{\frac{(2n-1)}{2n}}\left[\nu^{-2}\left\langle F_1\right\rangle\right]^{\frac{1}{2n}}, \tag{59}$$

so a Hölder inequality yields

$$2n\left\langle\kappa_{n+1,n}\right\rangle \leqq (2n-1)\nu^{\frac{2}{2n-1}}\left\langle\frac{F_{n+1}}{F_n^{2n/(2n-1)}}\right\rangle + \nu^{-2}\left\langle F_1\right\rangle . \tag{60}$$

To estimate the first long-time average on the right-hand side, consider the second F_n ladder in (36) ,

$$\tfrac{1}{2}\dot{F}_n \leqq -\tfrac{1}{2}\nu F_{n+1} + c_n\left(\nu^{-1}\|\boldsymbol{u}\|_\infty^2 + \nu\ell^{-2}\mathrm{Re}\right)F_n. \tag{61}$$

Now define

$$Y_n = F_n^{-\frac{1}{2n-1}} \tag{62}$$

and turn (61) into a differential inequality in Y_n which involves dividing by $F_n^{2n/(2n-1)}$. To achieve this we use $\|\boldsymbol{u}\|_\infty^2 \leqq c\,\kappa_{2,1}F_1$ and recall that $\kappa_{2,1} \leqq \kappa_{n,1}$. Then

$$\|\boldsymbol{u}\|_\infty^2 F_n^{-\frac{1}{2n-1}} \leqq c\,\kappa_{2,1}\left[\kappa_{n,1}^{-1}F_1\right]^{\frac{2n-2}{2n-1}} \leqq c\,\kappa_{2,1}^{\frac{1}{2n-1}}F_1^{\frac{2n-2}{2n-1}}. \tag{63}$$

Hence (61) can be rewritten as

$$(n-\tfrac{1}{2})(\dot{Y}_n + \nu\ell^{-2}\mathrm{Re}Y_n) \geqq \tfrac{1}{2}\nu\frac{F_{n+1}}{F_n^{\frac{2n}{2n-1}}} - c\,\nu^{-1}\kappa_{2,1}^{\frac{1}{2n-1}}F_1^{\frac{2n-2}{2n-1}}. \tag{64}$$

Setting the coefficient in ν to that in (60), a Hölder inequality on the last term gives

$$\begin{aligned}\nu^{\frac{2}{2n-1}}\frac{F_{n+1}}{F_n^{2n/(2n-1)}} &\leqq (2n-1)\nu^{\frac{3-2n}{2n-1}}\left[\dot{Y}_n + \nu\ell^{-2}\mathrm{Re}Y_n\right] \\ &\quad + \frac{1}{2n-1}\left\{\kappa_{2,1} + c\,(2n-2)\nu^{-2}F_1\right\}. \end{aligned}\tag{65}$$

Taking the long-time average of this in (60) we have

$$2n\left\langle\kappa_{n+1,n}\right\rangle \leqq \left\langle\kappa_{2,1}\right\rangle + c_n\,\nu^{-2}\left\langle F_1\right\rangle + \nu^{\frac{2}{2n-1}}(2n-1)\ell^{-2}\mathrm{Re}\left\langle Y_n\right\rangle . \tag{66}$$

The long-time average of $\dot{Y}_n$ has vanished and the last term $\langle Y_n\rangle$ is bounded above (because F_n is bounded below), so the long-time average is zero. Thus when (55) and (56) are used we have

$$\ell\left\langle\kappa_{n,1}\right\rangle \leqq \ell\left\langle\kappa_{n+1,n}\right\rangle \leqq c_n\left(L\ell^{-1}\right)^3\mathrm{Re}^3. \tag{67}$$

Note that the exponents of Re and L/ℓ are uniform in n; only the constant is not. So (67) can now be used to estimate $\langle\kappa_n\rangle$ in the final step.

Step 3. Rewrite $\langle\kappa_n\rangle$ in the following way:

$$\left\langle\kappa_n^{\frac{2n}{2n-1}}\right\rangle = \left\langle\left(\frac{F_n}{F_0}\right)^{\frac{1}{2n-1}}\right\rangle = \left\langle\left(\frac{F_n}{F_1}\right)^{\frac{1}{2n-1}}(\kappa_1^2)^{\frac{1}{2n-1}}\right\rangle$$
$$= \left\langle\kappa_{n,1}^{\frac{2n-2}{2n-1}}(\kappa_1^2)^{\frac{1}{2n-1}}\right\rangle \leqq \left\langle\kappa_{n,1}\right\rangle^{\frac{2n-2}{2n-1}}\left\langle\kappa_1^2\right\rangle^{\frac{1}{2n-1}}. \tag{68}$$

Using our estimate for $\left\langle\kappa_{n,1}\right\rangle$ from (67) and also that for $\left\langle\kappa_1^2\right\rangle$ from (48), the result in (49) is proved. □

The first infinite set of non-trivial, bounded, long-time averages were those found by FOIAS, GUILLOPÉ & TEMAM [7]. These are related to those in Theorem 1, and particularly to the estimates for $\kappa_{n,1}$ in (67).

Theorem 2. *For* Gr $\to\infty$*, the long-time averaged quantities of* FOIAS, GUILLOPÉ & TEMAM [7] *are estimated in terms of* Re *as*

$$\ell\,\langle\|\boldsymbol{u}\|_\infty\rangle \leqq c_1\nu\left(L\ell^{-1}\right)^3\mathrm{Re}^3, \tag{69}$$

$$\ell\left\langle F_n^{\frac{1}{2n-1}}\right\rangle \leqq c_{n,2}\,\nu^{\frac{2}{2n-1}}\left(L\ell^{-1}\right)^3\mathrm{Re}^3, \tag{70}$$

$$\ell\left\langle\|\nabla\boldsymbol{u}\|_\infty^{1/2}\right\rangle \leqq c_3\nu^{1/2}\left(L\ell^{-1}\right)^3\mathrm{Re}^3. \tag{71}$$

Remark 2. The case $n=1$ is distinct from the result in [48] because of the $\left(L\ell^{-1}\right)^3$ on the right-hand side.

Proof. The proof follows from the Sobolev inequalities

$$\|\boldsymbol{u}\|_\infty \leqq c\,\kappa_{n,1}^{1/2}F_1, \qquad \|\nabla\boldsymbol{u}\|_\infty \leqq c\,\kappa_{n,1}^{3/2}F_1^{1/2} \tag{72}$$

with the estimates (67) for $\kappa_{n,1}$ and (32) for $\langle F_1\rangle$. The quantities in (70) can be rewritten in terms of $\kappa_{n,1}$ and F_1, and the result follows. □

A lemma is now proved that will be useful in later sections:

Lemma 1. *If any F_m (κ_m) is bounded on a time interval* $[0,\,T]$ *for* $1\leqq m\leqq n$ *then so are all F_n (κ_n) for $n>m$.*

Proof. Consider (35) in Proposition 1 in Section 3.2 above: for $n\geqq 3$ a Sobolev inequality gives

$$\|\nabla\boldsymbol{u}\|_\infty \leqq c\,\|\nabla^n\boldsymbol{u}\|_2^a\|\nabla\boldsymbol{u}\|_2^{1-a} \leqq F_n^{a/2}F_1^{(1-a)/2}, \tag{73}$$

where $a=3/[2(n-1)]$. There is an inequality for the F_n of the form

$$F_N^{p+q} \leqq F_{N-p}\,F_{N+q}^p. \tag{74}$$

The choice of $N = n,\ p = n - 1$ and $q = 1$ gives

$$-F_{n+1} \leqq -F_n^{\frac{n}{n-1}} / F_1^{\frac{1}{n-1}} \tag{75}$$

so, as a consequence, (35) becomes

$$\tfrac{1}{2}\dot{F}_n \leqq -\nu F_n^{\frac{n}{n-1}} / F_1^{\frac{1}{n-1}} + c\, F_n^{1+a/2} F_1^{(1-a)/2} + c\,\nu\ell^{-2} F_n. \tag{76}$$

Because $n/(n-1) > 1 + a/2$, (76) makes it clear that if F_1 is bounded above at any time then all F_n are bounded. If any F_m is bounded for $m > 1$ then F_1 must also be bounded (from (74)), in which case all F_n are bounded for any $n > m$. The same results hold for the κ_n because the divisor F_0 is bounded from above and below. □

4. Intermittency: the binary form of the time-axis

In the summary section, Section 2, it was discussed how the effective viscosity could be increased by proving that the ratio κ_{n+1}/κ_n has a lower bound that is greater than unity under certain circumstances. This was discussed in the context of the ladder of differential inequalities (18) for the F_n, which is repeated here:

$$\tfrac{1}{2}\dot{F}_n \leqq \left(-\tfrac{1}{2}\nu\kappa_n^2 \left(\frac{\kappa_{n+1}}{\kappa_n}\right)^{2(n+1)} + c_n\nu^{-1}\kappa_n^3 F_0 + \nu\ell^{-2}\mathrm{Re}\right) F_n. \tag{77}$$

The task of this section is to investigate lower bounds on the ratio κ_{n+1}/κ_n. In the rest of this paper, the two lengths L and ℓ will be taken such that $\ell = L/2\pi$ to reduce algebra. Additionally, the parameter δ, lying in the range $0 < \delta < \frac{1}{6}$, that appears in the exponents of many of the estimates of the previous section, will be taken as arbitrarily small (but fixed) and ignored hereafter.

Theorem 3. *For the parameter μ taking any value in the range $0 < \mu < 1$, the ratio κ_{n+1}/κ_n obeys the long-time averaged inequality* ($n \geqq 1$)

$$\left\langle \left[c_n\left(\frac{\kappa_{n+1}}{\kappa_n}\right)\right]^{1/\mu - 1} - \left[(L\kappa_n)^{\mu}\mathrm{Re}^{-\lambda_n}\right]^{1/\mu-1} \right\rangle \geqq 0, \tag{78}$$

where the c_n are the same as those in (49). *Hence there exists at least one interval of time, designated as a 'good interval', on which the inequality*

$$c_n\left(\frac{\kappa_{n+1}}{\kappa_n}\right) \geqq (L\kappa_n)^{\mu}\,\mathrm{Re}^{-\lambda_n} \tag{79}$$

holds. Those other parts of the time-axis on which the reverse inequality

$$c_n\left(\frac{\kappa_{n+1}}{\kappa_n}\right) < (L\kappa_n)^{\mu}\,\mathrm{Re}^{-\lambda_n} \tag{80}$$

holds are designated as 'bad intervals'.

Remark 3. In principle, the whole time-axis could be a good interval, whereas the positive time average in (78) ensures that the complete time-axis cannot be 'bad'. This paper is based on the worst-case supposition that bad intervals exist, that they could be multiple in number, and that the good and the bad are interspersed. This is what is meant in this paper by a 'potentially binary character', although the precise distribution and occurrence of the good/bad intervals and how they depend on n remains an open question. It is even possible that the time-axis may take on a fractal structure. It will be left until later (Theorem 5) to prove that the bad intervals are finite in width.

Proof. Take two parameters $0 < \mu < 1$ and $0 < \alpha < 1$ such that $\mu + \alpha = 1$. The inverses μ^{-1} and α^{-1} will be used as exponents in the Hölder inequality on the far right-hand side of

$$\left\langle \kappa_n^\alpha \right\rangle \leqq \left\langle \kappa_{n+1}^\alpha \right\rangle = \left\langle \left(\frac{\kappa_{n+1}}{\kappa_n}\right)^\alpha \kappa_n^\alpha \right\rangle \leqq \left\langle \left(\frac{\kappa_{n+1}}{\kappa_n}\right)^{\alpha/\mu} \right\rangle^\mu \left\langle \kappa_n \right\rangle^\alpha , \tag{81}$$

thereby giving

$$\left\langle \left(\frac{\kappa_{n+1}}{\kappa_n}\right)^{\alpha/\mu} \right\rangle \geqq \left(\frac{\left\langle \kappa_n^\alpha \right\rangle}{\left\langle \kappa_n \right\rangle^\alpha}\right)^{1/\mu} = \left\langle \kappa_n^\alpha \right\rangle \left(\frac{\left\langle \kappa_n^\alpha \right\rangle}{\left\langle \kappa_n \right\rangle}\right)^{\alpha/\mu} . \tag{82}$$

Navier-Stokes information can be injected into these formal manipulations: the weak solution upper bound (49) and the lower bound $L\kappa_n \geqq 1$ can be used in the ratio on the far right-hand side of (82) to give (78), with the same c_n as in (49). □

4.1. Bounds within good intervals

On the good intervals, application of the improved lower bound (79) to the differential inequality (77) appears to imply that μ must satisfy $2\mu(n+1) > 1$ for the exponent of the negative term to be larger than the positive. To strengthen this argument it is necessary to convert (77) into a differential inequality in F_n alone. This can be achieved because the divisor within κ_n, namely F_0, is bounded above:

$$\begin{aligned} \tfrac{1}{2}\dot{F}_n \leqq & -\nu \mathrm{Re}^{-2\lambda_n(n+1)} L^{2\mu(n+1)} F_n^{\frac{(1+\mu)(n+1)}{n}} F_0^{-\frac{\mu(n+1)+1}{n}} \\ & + c\,\nu^{-1} F_n^{\frac{2n+3}{2n}} F_0^{\frac{2n-3}{2n}} + \nu L^{-2} \mathrm{Re} F_n . \end{aligned} \tag{83}$$

For $n \geqq 2$ and arbitrarily large initial data, a singularity can be prevented from forming if the exponent of the negative F_n term is greater than that of the positive:

$$\frac{(1+\mu)(n+1)}{n} > \frac{2n+3}{2n} \qquad \Rightarrow \qquad \mu > \frac{1}{2(n+1)} \tag{84}$$

as predicted. It is not possible to take the infinite time limit because of the finiteness of the interval, but the value of $F_n = F_{n,max}$ that turns the sign of the right-hand side of (83) is bounded by

$$F_{n,max} \leqq L^{-2n} \mathrm{Re}^{\gamma_n} F_{0,max} \equiv U_{bd} , \tag{85}$$

$$\gamma_n = \frac{4n[\lambda_n(n+1)+2]}{2\mu(n+1)-1}, \tag{86}$$

where the exponent $\gamma_n > 0$ when $2\mu(n+1) > 1$ and we have used the fact that $F_{0,max} = c\,L\nu^2\mathrm{Re}^4$. In terms of Fig. 2 in Section 2, it is necessary to prove that the solution in the good region can become large enough to form an initial condition for weak solutions in the bad region. This can be proved by the following argument: consider that on bad intervals the κ_n are bounded below uniformly by

$$[L\kappa_n(t)]^\mu > c_n\mathrm{Re}^{\lambda_n}, \tag{87}$$

where $c_{n,\mu} = c_n^{1/\mu}$. In terms of F_n this can be expressed as

$$F_n > c_{n,\mu}L^{-2n}\mathrm{Re}^{\frac{2n\lambda_n}{\mu}}F_{0,min} \equiv L_{bd}. \tag{88}$$

The question revolves around the relative sizes of the lower bound L_{bd} in (88) and U_{bd} in (85):

$$\frac{U_{bd}}{L_{bd}} = \left(\frac{F_{0,max}}{F_{0,min}}\right)\mathrm{Re}^{\frac{2n(\lambda_n+4\mu)}{\mu[2\mu(n+1)-1]}} > 1, \qquad \mathrm{Re} \gg 1. \tag{89}$$

Hence it is possible for F_n to reach magnitudes at the edges of the good region that lie above the lower bound in (88).

For the case $n = 1$, the following Lemma is applicable.

Lemma 2. *When $n = 1$ no singularity can form on good intervals provided $\mu > \frac{1}{2}$.*

Proof. This follows immediately by applying Theorem 3 to (34). □

Nothing has yet been proved with regards to the widths of the good and bad intervals, $(\Delta t)_b$ and $(\Delta t)_g$ respectively, nor have we any further information regarding their nature. While it is possible that they may form pathological fractal subsets of the time-axis it will be assumed that these intervals are simple open or closed sets; the next section is devoted to estimating upper bounds on $(\Delta t)_b$. Here it is shown that a *lower* bound can be found on the ratio of the average widths of the good and bad intervals. The argument is based on an elementary application of the Markov-Chebychev inequality. Consider an interval of time $[t_p,\, t_q]$ that contains an equal number N of good and bad intervals of widths $(\Delta t)_g^{(i)}$ and $(\Delta t)_b^{(i)}$ respectively. Define the average widths as

$$\overline{(\Delta t)}_g = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}(\Delta t)_g^{(i)}, \qquad \overline{(\Delta t)}_b = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}(\Delta t)_b^{(i)}. \tag{90}$$

Theorem 4. *Consider an interval of time $[t_p,\, t_q]$ containing N pairs of good and bad intervals. In the limits $N \to \infty$ and $[t_p,\, t_q] \to \infty$, provided $(\Delta t)_b > 0$, the ratio $\overline{(\Delta t)}_g/\overline{(\Delta t)}_b$ diverges as*

$$\frac{\overline{(\Delta t)}_g}{\overline{(\Delta t)}_b} \geqq c_n\mathrm{Re}^{\lambda_n\left(\frac{1}{\mu}-1\right)} \qquad \textit{and} \quad \mathrm{Re}\to\infty \tag{91}$$

Proof. Given (90), the fraction of time occupied by the bad intervals satisfies

$$\frac{\sum_{i=1}^{N}(\Delta t)_b^{(i)}}{\sum_{i=1}^{N}[(\Delta t)_g^{(i)}+(\Delta t)_b^{(i)}]} \leqq \frac{1}{t_q-t_p}\int_{T_{p,q}} dt \leqq \frac{1}{t_q-t_p}\left(\frac{\int_{[t_p,\,t_q]} L\kappa_n dt}{c_{n,\mu}\mathrm{Re}^{\lambda_n/\mu}}\right), \tag{92}$$

where $T_{p,q}=[L\kappa_n(t)\geqq c_{n,\mu}\mathrm{Re}^{\lambda_n/\mu}]\cap[t_p,\ t_q]$. So as $N\to\infty$ and $t_q-t_p\to\infty$, we have

$$\frac{\overline{(\Delta t)}_b}{\overline{(\Delta t)}_g+\overline{(\Delta t)}_b} \leqq \frac{\langle L\kappa_n\rangle}{c_{n,\mu}\mathrm{Re}^{\lambda_n/\mu}} \leqq \left[c_n\mathrm{Re}^{\lambda_n}\right]^{1-\frac{1}{\mu}}, \tag{93}$$

where we have used (87) and (49). Hence we have the result. □

5. What happens in the bad intervals?

It is necessary to prove that the bad intervals are of finite width: that is, an upper bound is required on $\Delta t = t - t_0$ where t_0 is the initial time of some arbitrary bad interval. Technically speaking, there should be a superscript label for the ith bad interval such that $\Delta t \equiv \Delta t^{(i)}$ and another on $t_0 \equiv t_0^{(i)}$, but these have been dropped for convenience. Recall that $\omega_0 = \nu L^{-2}$ and

$$\mathcal{E}(\Delta t) = \frac{e^{\omega_0\mathrm{Re}\,\Delta t}-1}{\omega_0\mathrm{Re}}. \tag{94}$$

It will become necessary to solve inequalities of the type

$$\mathcal{E}(\Delta t) \leqq \omega_0^{-1}\mathrm{Re}^{-\beta} \tag{95}$$

for $\beta > 0$ as $\mathrm{Re}\to\infty$. It is not difficult to show that when $\beta \geqq 1$, to leading order

$$\omega_0(\Delta t) \lesssim \mathrm{Re}^{-\beta}, \tag{96}$$

whereas when $0<\beta<1$ then, to leading order

$$\omega_0(\Delta t) \lesssim (1-\beta)\mathrm{Re}^{-1}\ln\mathrm{Re}. \tag{97}$$

The main task of this section is to show that the bad intervals have a finite widths and to find an upper bound on these. This requires two subsidiary estimates for

$$\int_{\Delta t} e^{\omega_0\mathrm{Re}\,\Delta t}F_1(t)\,dt \qquad \text{and} \qquad \int_{\Delta t} e^{\omega_0\mathrm{Re}\,\Delta t}\kappa_{2,1}(t)\,dt. \tag{98}$$

5.1. Two subsidiary estimates

Lemma 3. *An estimate for the exponentially weighted time integral of* F_1 *is*

$$\int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t} F_1(t)\,dt \leqq c_1 \nu L \mathrm{Re}^4 + c_2 \nu^2 L^{-1} \mathcal{E}(\Delta t) \left[\mathrm{Re}^5 + O(\mathrm{Re}^4)\right]. \quad (99)$$

Proof. Let us denote $H_0(t) = X^2(t)$ with $H_0(t_0) = X_0^2$ then Leray's energy inequality (11) for weak solutions,

$$\tfrac{1}{2}\dot{H}_0 \leqq -\nu H_1 + H_0^{1/2}\|\boldsymbol{f}\|_2 \quad (100)$$

in combination with Poincaré's inequality $k_1^2 H_0 \leqq H_1$, gives

$$\dot{X} \leqq -\nu k_1^2 X + \|\boldsymbol{f}\|_2. \quad (101)$$

Let us also denote X_f by ($k_1 = 2\pi/L$),

$$X_f = \frac{\|\boldsymbol{f}\|_2}{\nu k_1^2} = \left(\frac{\nu}{L^{3/2}k_1^2}\right)\mathrm{Gr} \leqq c\,\nu L^{1/2}\mathrm{Re}^2, \quad (102)$$

which has the same dimensions as X. Integration of (101) from t_0 to t results in

$$X(t) \leqq X_0 e^{-\nu k_1^2 \Delta t} + X_f\left(1 - e^{-\nu k_1^2 \Delta t}\right). \quad (103)$$

Because there is no specific knowledge of t_0 the upper bound on $H(t)$ is taken over the full time-range $0 \leqq t \leqq \infty$ which, from (103), is

$$H_0(t_0) \leqq \left(\frac{\nu^2}{L^3 k_1^4}\right)\mathrm{Gr}^2 \leqq c\,\nu^2 L \mathrm{Re}^4. \quad (104)$$

This is properly valid after the time when transients have died out. The exponential decay in (103) is trivial compared to $\exp(\omega_0 \mathrm{Re}\,\Delta t)$ so we obtain

$$\int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t} X(t)\,dt \leqq c\,\nu L^{1/2}\mathcal{E}(\Delta t)\mathrm{Re}^2. \quad (105)$$

Now multiply (100) by $e^{\omega_0 \mathrm{Re}\,\Delta t}$ and integrate by parts to obtain

$$\begin{aligned}\nu \int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t} H_1(t)\,dt &\leqq \int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t}\left(-\tfrac{1}{2}\dot{H}_0 + X\|\boldsymbol{f}\|_2\right)dt \\ &\leqq \tfrac{1}{2}H_0(t_0) - \tfrac{1}{2}H_0(t)e^{\omega_0 \mathrm{Re}\,\Delta t} + \tfrac{1}{2}\omega_0 \mathrm{Re}\int_{\Delta t} H_0 e^{\omega_0 \mathrm{Re}\,\Delta t}dt \\ &\quad + c\,\nu^3 L^{-1}\mathcal{E}(\Delta t)\mathrm{Re}^4. \end{aligned} \quad (106)$$

In the general case the negative term can be dropped, leaving

$$\begin{aligned}\nu \int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t} H_1(t)\,dt &\leqq c_1 \nu^2 L \mathrm{Re}^4 \\ &\quad + c_2 \nu^3 L^{-1}\mathcal{E}(\Delta t)\left(\mathrm{Re}^5 + O(\mathrm{Re}^4)\right)\end{aligned} \quad (107)$$

The predominant Re^5 term has a correction term of $O(\mathrm{Re}^4)$ from the fourth term in (106) and another of $O(\mathrm{Re}^2)$ from making up H_1 to F_1.

Note that the first term on the right hand side of (99) in Lemma 3 can be removed if the energy has the lower bound $H_0(t) \geqq H_0(t_0)e^{-\omega_0 \mathrm{Re}\,\Delta t}$ (see Section 6 for a discussion of this). $\square$

Lemma 4. *An estimate for the exponentially weighted time integral of* $\kappa_{2,1}$ *is*

$$\int_{\Delta t} \kappa_{2,1}(t) e^{\omega_0 \mathrm{Re}\,\Delta t}\, dt \leqq c\,\nu^{-2} \int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t} F_1 dt + c\,\mathcal{E}(\Delta t)(L\mathrm{Gr})^{-1}\mathrm{Re}. \tag{108}$$

Proof. The time integral of $\kappa_{2,1}$ can be estimated from

$$\begin{aligned}\int_{\Delta t} \kappa_{2,1} e^{\omega_0 \mathrm{Re}\,\Delta t}\, dt &= \int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t} (F_2/F_1^2)^{1/2} F_1^{1/2}\, dt \\ &\leqq \frac{\nu^2}{2}\int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t}(F_2/F_1^2)\, dt + \frac{1}{2\nu^2}\int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t} F_1\, dt.\end{aligned} \tag{109}$$

The first integral on the far right-hand side of (109) can be estimated by using the inequality for F_1 from (34) in Proposition 1 in Section 3.2,

$$\tfrac{1}{2}\dot{F}_1 \leqq -\frac{\nu}{4}F_2 + c\,\nu^{-3}F_1^3 + \omega_0 \mathrm{Re} F_1. \tag{110}$$

Dividing (110) by F_1^2, multiplying by $e^{\omega_0 \mathrm{Re}\,\Delta t}$ and integrating yield

$$\frac{\nu}{4}\int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t}(F_2/F_1^2) dt \leqq c\,\nu^{-3}\int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t} F_1 dt + \tfrac{1}{2}\left(F_1^{-1}(t)e^{\omega_0 \mathrm{Re}\,\Delta t} - F_1^{-1}(t_0)\right). \tag{111}$$

The last term can be rewritten in terms of $\mathcal{E}(\Delta t)$ which leaves a F_1^{-1} term. The upper bound on this is proportional to Gr^{-1}, which can be ignored as small. Then,

$$\int_{\Delta t} \kappa_{2,1} e^{\omega_0 \mathrm{Re}\,\Delta t}\, dt \leqq c\,\nu^{-2}\int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t} F_1 dt + c\,L^{-1}\mathcal{E}(\Delta t)\mathrm{Gr}^{-1}\mathrm{Re} \tag{112}$$

as in (108) above. $\square$

5.2. An estimate for $(\Delta t)_b$ when $n \geqq 2$

The two estimates above for the weighted time integrals of F_1 and $\kappa_{2,1}$ allow us to prove the main result of this section for $n \geqq 2$. Define

$$a_n = \frac{\lambda_{n+1}}{\mu}\left(\frac{2n-2}{2n-1}\right) - \frac{4}{2n-1} - 5. \tag{113}$$

Then $a_n \geqq 1$ if μ is chosen such that

$$\mu \leqq \lambda_{n+1}\left(\frac{n-1}{6n-1}\right), \tag{114}$$

whereas $0 < a_n < 1$ if μ is chosen to lie in the range[1]

$$\lambda_{n+1}\left(\frac{n-1}{6n-1}\right) < \mu < \lambda_{n+1}\left(\frac{2n-2}{10n-1}\right). \tag{115}$$

Theorem 5. *For $n \geqq 2$, if $a_n \geqq 1$ then the width of a bad interval is bounded by*

$$\tilde{c}_{n,1}\omega_0(\Delta t)_b \leqq \mathrm{Re}^{-a_n}, \tag{116}$$

whereas if $0 < a_n < 1$,

$$\tilde{c}_{n,2}\omega_0(\Delta t)_b \leqq \mathrm{Re}^{-1}\ln \mathrm{Re}. \tag{117}$$

Remark 4. There appears to be no obvious parallel result for the finiteness of bad intervals in the case $n = 1$, although Lemma 2 gives a lower bound $\mu > \frac{1}{2}$ for the prevention of singularities forming on good intervals.

Proof. Let us return to inequality (36) of Proposition 1 in Section 3.2, recalling that $\omega_0 = \nu L^{-2}$:

$$\tfrac{1}{2}\dot{F}_n \leqq -\tfrac{1}{2}\nu F_{n+1} + c_n\left(\nu^{-1}\|\boldsymbol{u}\|_\infty^2 + \omega_0 \mathrm{Re}\right)F_n. \tag{118}$$

This was manipulated in Section 3.3 to produce (64) which is restated here as

$$(n-\tfrac{1}{2})(\dot{Y}_n + \omega_0 \mathrm{Re}\,Y_n) \geqq \tfrac{1}{2}\nu\frac{F_{n+1}}{F_n^{\frac{2n}{2n-1}}} - c_2\nu^{-1}\kappa_{2,1}^{\frac{1}{2n-1}}F_1^{\frac{2n-2}{2n-1}} \tag{119}$$

The first term on the right-hand side of (119) can be estimated as

$$\frac{F_{n+1}}{F_n^{2n/(2n-1)}} \geqq \kappa_{n+1}^{\frac{2n-2}{2n-1}}F_0^{-\frac{1}{2n-1}} \geqq c\,\kappa_{n+1}^{\frac{2n-2}{2n-1}}(\nu^2 L\mathrm{Re}^4)^{-\frac{1}{2n-1}} \tag{120}$$

having used the fact that $\kappa_n \leqq \kappa_{n+1}$. This result, together with a Hölder inequality, gives

$$\begin{aligned}(n-\tfrac{1}{2})\frac{d}{dt}\left[Y_n e^{\omega_0 \mathrm{Re}\,\Delta t}\right] \geqq{}& c\,\nu^{\frac{2n-3}{2n-1}}(L\mathrm{Re}^4)^{-\frac{1}{2n-1}}e^{\omega_0\mathrm{Re}\,\Delta t}\kappa_{n+1}^{\frac{2n-2}{2n-1}}\\ &-c_2^{\frac{2n-2}{2n-1}}e^{\omega_0\mathrm{Re}\,\Delta t}\left\{\nu^{\frac{2n-3}{2n-1}}\kappa_{2,1} + \nu^{-\frac{2n+1}{2n-1}}F_1\right\}.\end{aligned} \tag{121}$$

[1] Note that for $n = 2$ the lower bound on μ in (115) is greater than $\frac{1}{2(n+1)}$.

So far this has just been a rearrangement of (118). The lower bound on κ_n is now applied to the first term on the right-hand side along with a time integration to yield

$$
\begin{aligned}
(n-\tfrac{1}{2})\left\{Y_n(t)e^{\omega_0 \mathrm{Re}\,\Delta t}\right\} \geqq\; & c_{n+1}^{\frac{2n-2}{2n-1}}\nu^{\frac{2n-3}{2n-1}}L^{-1}\mathcal{E}(\Delta t)\mathrm{Re}^{\frac{(2n-2)\lambda_{n+1}}{(2n-1)\mu}-\frac{4}{2n-1}} \\
& -c_2^{\frac{2n-2}{2n-1}}\nu^{\frac{2n-3}{2n-1}}\int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t}\kappa_{2,1}\,dt \\
& -c_2^{\frac{2n-2}{2n-1}}\nu^{-\frac{2n+1}{2n-1}}\int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t}F_1\,dt \\
& +(n-\tfrac{1}{2})Y_n(t_0).
\end{aligned} \tag{122}
$$

For the left-hand side it is sufficient to show that this is bounded above by a very small number on a bad interval, i.e.,

$$
Y_n = \kappa_{n+1}^{-\frac{2n}{2n-1}}F_0^{-\frac{1}{2n-1}} \leqq L\nu^{-\frac{2}{2n-1}}\mathrm{Re}^{-\frac{2n\lambda_{n+1}}{\mu(2n-1)}}\mathrm{Gr}^{-\frac{1}{2n-1}}. \tag{123}
$$

Using Lemmas 3 and 4, a comparison of the major terms in (122) shows that

$$
\omega_0\mathcal{E}(\Delta t)\left\{c_{n+1}^{\frac{2n-2}{2n-1}}\mathrm{Re}^{\frac{(2n-2)\lambda_{n+1}}{(2n-1)\mu}-\frac{4}{2n-1}} - c_3\mathrm{Re}^5\right\} \leqq c_4\mathrm{Re}^4. \tag{124}
$$

For $n \geqq 2$ the left-hand side is always positive provided μ is chosen in the range

$$
\frac{\lambda_{n+1}}{\mu}\left(\frac{2n-2}{2n-1}\right) - \frac{4}{2n-1} > 5, \tag{125}
$$

or in the range

$$
\mu < \lambda_{n+1}\left(\frac{2n-2}{10n-1}\right). \tag{126}
$$

To solve (124) use the definition of a_n in (113) to obtain

$$
\tilde{c}_n\,\omega_0\mathcal{E}(\Delta t) \leqq \mathrm{Re}^{-a_n}. \tag{127}
$$

The solution of this depends on whether a_n lies in the range $a_n \geqq 1$ or $0 < a_n < 1$. The estimates in (95) and (96) are appropriate. □

5.3. Intersection of bad intervals: the relation to the singular set

Fig. 2 of Section 2.4 is a representation of good and bad intervals for some $n \geqq 2$. Since it must be assumed that the position of the intervals changes with n, the intersection of all the bad intervals for $n \geqq 2$ is pertinent: only if this intersection is non-empty will singularities be possible. For each $n \geqq 2$, let us designate a bad interval as the set $\mathcal{B}_n$ on the time-axis on which

$$
c_n\frac{\kappa_{n+1}}{\kappa_n} < (L\kappa_n)^{\mu}\mathrm{Re}^{-\lambda_n}. \tag{128}
$$

Moreover, because $L\kappa_n \leqq L\kappa_{n+1}$, on this set there is a lower bound

$$(L\kappa_n)^{\mu} > c_n \mathrm{Re}^{\lambda_n}. \tag{129}$$

Now consider the set $\mathcal{B}_{n+1}$ on which

$$c_{n+1}\frac{\kappa_{n+2}}{\kappa_{n+1}} < (L\kappa_{n+1})^{\mu}\mathrm{Re}^{-\lambda_{n+1}} \Rightarrow (L\kappa_{n+1})^{\mu} > c_{n+1}\mathrm{Re}^{\lambda_{n+1}}. \tag{130}$$

Then on the intersection $\mathcal{I}_{n+1} = \mathcal{B}_n \cap \mathcal{B}_{n+1}$, we have

$$c_n^{1+\mu} c_{n+1}(L\kappa_{n+2}) < (L\kappa_n)^{(1+\mu)^2}\mathrm{Re}^{-\lambda_{n+1}-(1+\mu)\lambda_n}. \tag{131}$$

Using $(L\kappa_{n+2})^{\mu} \geqq (L\kappa_{n+1})^{\mu} > c_{n+1}\mathrm{Re}^{\lambda_{n+1}}$ on $\mathcal{I}_{n+1}$, a new lower bound is

$$(L\kappa_n)^{1+\mu} > \left(c_{n+1}\mathrm{Re}^{\lambda_{n+1}}\right)^{1/\mu}\left(c_n\mathrm{Re}^{\lambda_n}\right). \tag{132}$$

Now consider the intersection $\mathcal{I}_{n+2} = \mathcal{B}_n \cap \mathcal{B}_{n+1} \cap \mathcal{B}_{n+2}$. On this set there is a larger lower bound

$$(L\kappa_n)^{(1+\mu)^2} > \left(c_{n+2}\mathrm{Re}^{\lambda_{n+2}}\right)^{1/\mu}\left(c_{n+1}\mathrm{Re}^{\lambda_{n+1}}\right)\left(c_n\mathrm{Re}^{\lambda_n}\right)^{1+\mu}. \tag{133}$$

We wish to find a lower bound on $L\kappa_n$ on the set of p intersections

$$\mathcal{I}_{n+p} = \mathcal{B}_n \cap \mathcal{B}_{n+1} \cap \ldots \cap \mathcal{B}_{n+p}. \tag{134}$$

By inspection, the general formula for the lower bound of $L\kappa_n$ on $\mathcal{I}_{n+p}$ is

$$\begin{aligned}(L\kappa_n)^{(1+\mu)^{p+1}} &> \left(c_{n+p+1}\mathrm{Re}^{\lambda_{n+p+1}}\right)^{1/\mu}\left(c_{n+p}\mathrm{Re}^{\lambda_{n+p}}\right)\ldots\left(c_n\mathrm{Re}^{\lambda_n}\right)^{(1+\mu)^p}\\ &= \left(c_{n+p+1}\mathrm{Re}^{\lambda_{n+p+1}}\right)^{1/\mu}\left(\Pi_{i=0}^{p} c_{n+i}^{\xi_{p,i}}\right)\mathrm{Re}^{L_{n,p}},\end{aligned} \tag{135}$$

where

$$\xi_{p,i} = (1+\mu)^{p-i}, \qquad L_{n,p} = \sum_{i=0}^{p}\lambda_{n+i}\,\xi_{p,i}. \tag{136}$$

Now $\Lambda_n^{(p)}$ and $c_n^{(p)}$ are defined as

$$\Lambda_n^{(p)} = \frac{\lambda_{n+p+1} + \mu\, L_{n,p}}{(1+\mu)^{p+1}}, \quad c_n^{(p)} = \left\{c_{n+p+1}\left(\Pi_{i=0}^{p} c_{n+i}^{\xi_{p,i}}\right)^{\mu}\right\}^{\frac{1}{(1+\mu)^{p+1}}}, \tag{137}$$

and then, because $\lambda_{n+p} > \lambda_n$ and $c_{n+p} > c_n$, it follows that

$$\xi_p\lambda_n < L_{n,p} < \xi_p\lambda_{n+p}, \qquad c_n^{\xi_p} < \Pi_{i=0}^{p}\, c_{n+i}^{\xi_{p,i}} < c_{n+p}^{\xi_p}, \tag{138}$$

where ξ_p is defined by the sum

$$\xi_p = \sum_{i=0}^{p}\xi_{p,i} = \mu^{-1}\left\{(1+\mu)^{p+1} - 1\right\}. \tag{139}$$

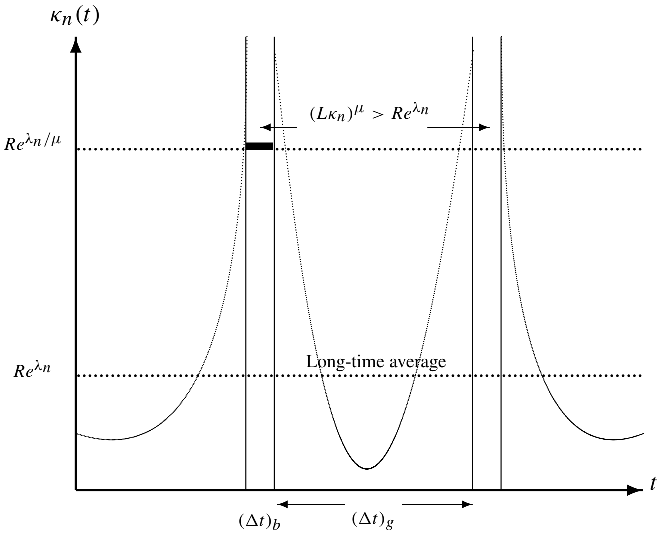

Fig. 3. Similar to Fig. 2, a representation of good/bad intervals for κ_n with a black strip representing the bad interval used in the intersection Table 6.

Then in the limit $p \to \infty$ we have

$$(L\kappa_n)^\mu > c_n^{(\infty)}\mathrm{Re}^{\Lambda_n^{(\infty)}} > c_n\mathrm{Re}^{\lambda_n}. \tag{140}$$

The potentially singular set $\mathcal{S}^{(\infty)}$, given by

$$\mathcal{S}^{(\infty)} = \mathcal{B}_1 \cap \mathcal{B}_2 \cap \ldots \cap \mathcal{B}_n \cap \ldots , \tag{141}$$

must necessarily include $\mathcal{B}_1$, the singular set of κ_1 (and therefore F_1). The range of values of μ expressed in (115) and Theorem 5 are valid for $n \geqq 2$. From (114) and (115), we define

$$M_n = \begin{cases} \lambda_{n+1}\left(\frac{n-1}{6n-1}\right) & a_n \geqq 1, \\ \lambda_{n+1}\left(\frac{2n-2}{10n-1}\right) & a_n < 1, \end{cases} \tag{142}$$

which gives

$$\lim_{n\to\infty} M_n \searrow \begin{cases} \frac{1}{2} & a_n \geqq 1, \\ \frac{3}{5} & a_n < 1. \end{cases} \tag{143}$$

As already pointed out in Lemma 2 of Section 4.1, a corresponding separate calculation for F_1 shows that μ lies in the range $\frac{1}{2} < \mu < 1$ for $n = 1$. When the allowed ranges of μ are taken into account for good and bad intervals we conclude

Theorem 6. *For all bad intervals to be finite for $n \geqq 2$ and for no singularities to form in good intervals for $n \geqq 1$, the allowed range of μ is*

$$\tfrac{1}{2} < \mu < \lim_{n\to\infty} M_n = \tfrac{3}{5}. \tag{144}$$

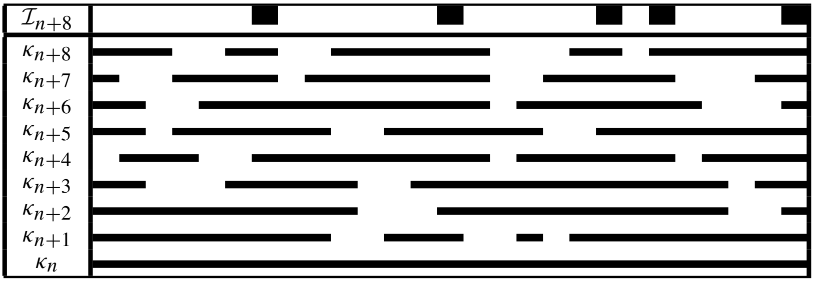

Table 6. The lowest continuous horizontal black strip is the bad interval of κ_n shown as the black strip in Fig. 3. The strips in the next 8 levels are a cartoon illustration of how some randomly chosen bad intervals ($\kappa_n \to \kappa_{n+8}$) could appear. The thicker strips at the highest level are the intersection of the 9 strips below.

Remark 5. In the infinite limit the upper and lower bounds coincide at $\frac{1}{2}$ for the narrower intervals, thereby creating a problem because of the inequalities in (144). It is necessary, however, to take into account the existence of wider intervals ($a_n < 1$) which accounts for the wider range $\frac{1}{2} < \mu < \frac{3}{5}$.

The set $\mathcal{S}^{(\infty)}$ is related to the potentially singular set of Scheffer [15]; this set is technically the union of all sets $\mathcal{S}^{(\infty)}$ associated with *every* bad interval. This set has zero $\frac{1}{2}$-dimensional Hausdorff measure and consists of, at most, points. Whether the κ_n actually become singular on this set is still an open question. From (137)-(139) we have

$$c_n \mathrm{Re}^{\lambda_n} < c_n^{(\infty)} \mathrm{Re}^{\Lambda_n^{(\infty)}} < \lim_{p\to\infty} c_{n+p+1} \mathrm{Re}^{\lambda_{n+p+1}}. \tag{145}$$

Divergence in this limit would guarantee singular behaviour if the set $\mathcal{S}^{(\infty)}$ is non-empty but there is no evidence that the sum and product in (137) diverges in the limit even though the upper bound in (145) is infinite. From (145) it is clear that $\Lambda_n^{(\infty)} > \lambda_n$ so all the estimates of the previous sections dependent upon λ_n should be replaced by $\Lambda_n^{(\infty)}$. This paper, however, provides no evidence on the distribution of the intervals; Table 6 is simply a pictorial cartoon representation of some randomly chosen bad intervals associated with $\kappa_n \to \kappa_{n+8}$ to illustrate how the final intersection may form.

5.4. Dangerous sub-intervals

In addition to the intersection idea of the last section, we consider the special set of sub-intervals within each bad interval on which $\dot{F}_n \geqq 0$. Consider the jth sub-interval within the ith bad interval: this is designated as a *dangerous* sub-interval of width $(\Delta t_+^{i,j})$ with an initial value of designated as $t_0^{i,j}$. It is on these sub-intervals where singularities are possible: they are not possible where any one of the F_n is decreasing. Ranges (114) and (115) show that the smallest lower bound on λ_n/μ

is $\lambda_n/\mu > 5$. Because $\lambda_{n+1} > \lambda_n$, and replacing λ_{n+1} by $\Lambda_{n+1}^{(\infty)}$, we define

$$b_n = \frac{\Lambda_{n+1}^{(\infty)}}{\mu} - 4 > 1. \tag{146}$$

Theorem 7. *Dangerous sub-intervals are bounded in width by ($n \geqq 2$)*

$$\omega_0(\Delta t_+^{i,j}) \leqq c_n \mathrm{Re}^{-b_n}, \qquad\qquad b_n > 1, \tag{147}$$

and on these sub-intervals,

$$L^{-3}F_1 \geqq \omega_0^2 \mathrm{Re}^{b_n+4}, \qquad \|\boldsymbol{u}\|_\infty \geqq L\omega_0 \mathrm{Re}^{b_n+4}, \tag{148}$$

$$\|\nabla \boldsymbol{u}\|_\infty \geqq \omega_0 \mathrm{Re}^{b_n+4}. \tag{149}$$

Remark 6. Note that $b_n > a_n$ so the upper bounds of these sub-intervals in (147) are smaller than those in Theorem 5.

Proof. Consider (16):

$$\tfrac{1}{2}\dot{F}_n \leqq -\tfrac{1}{2}\nu F_{n+1} + \left(c_n \nu^{-1}\|\boldsymbol{u}\|_\infty^2 + \omega_0^2 \mathrm{Re}\right) F_n. \tag{150}$$

Now use the Sobolev inequality $\|\boldsymbol{u}\|_\infty^2 \leqq c\,\kappa_{n,1}F_1$, and divide (150) by F_n. Then on these sub-intervals,

$$\kappa_{n+1,n}^2 \leqq c_n \nu^{-2}\kappa_{n,1}F_1 + c\,L^{-2}\mathrm{Re}. \tag{151}$$

Now we know that $\kappa_{n+1,n} \geqq \kappa_{n+1}$ so

$$\kappa_{n+1}^2 \leqq \left(c_n \nu^{-2}F_1\right)^2 + c\,L^{-2}\mathrm{Re}. \tag{152}$$

Now the lower bound $L\kappa_{n+1} \geqq \mathrm{Re}^{\Lambda_{n+1}^{(\infty)}/\mu}$ is invoked giving

$$\left(c_n^{(\infty)} \mathrm{Re}^{\Lambda_{n+1}^{(\infty)}/\mu}\right)^2 - c\,\mathrm{Re} \leqq \left(c_n L \nu^{-2}F_1\right)^2. \tag{153}$$

Because $\lambda_n/\mu > 1$ for $n \geqq 2$, the Re term is small in comparison, leaving

$$c_n^{(\infty)} \mathrm{Re}^{\Lambda_{n+1}^{(\infty)}/\mu} \leqq L\nu^{-2}F_1 \tag{154}$$

By multiplying by the exponential term, integrating over Δt^+, and then using the exponentially time-weighted integral of F_1 in Lemma 3, we obtain

$$c_n\,\omega_0 \mathcal{E}(\Delta t^+)\left[\mathrm{Re}^{\Lambda_{n+1}^{(\infty)}/\mu} - \mathrm{Re}^5\right] \leqq \mathrm{Re}^4. \tag{155}$$

Now we know that $\Lambda_{n+1}^{(\infty)}/\mu > 5$ and $b_n > 1$, so the result in (147) follows. The definition of a_n in (113) guarantees that

$$b_n = \frac{\Lambda_{n+1}^{(\infty)}}{\mu} - 4 > a_n, \tag{156}$$

which is the correct way round. Then (148) and (149) follow from (154). □

6. A conditional regularity result

The reader who has followed the proof of Theorem 5 will have noticed that attempts to prove Navier-Stokes regularity fail in the bad intervals. There, use was made of the variables $Y_n(t) = F_n^{-1/(2n-1)}$ defined in (62). To prevent the formation of singularities, it would be necessary to show that Y_n can never touch zero in a finite time. Within the dangerous sub-intervals of Section 5.4 this can be achieved provided the energy is bounded below in a certain manner. Specifically the result is

Theorem 8. *The Navier-Stokes equations are regular if, in dangerous sub-intervals $(\Delta t_+^{i,j})$, there is a lower bound on the energy*

$$H_0(t) \geqq H_0(t_0^{i,j})e^{-\omega_0 \mathrm{Re}\Delta t}. \tag{157}$$

Remark 7. Over the very short time interval Δt the exponent on the right-hand side of (157) is very small, so the right-hand side is almost $H_0(t_0^{i,j})$.

Proof. They key point preventing progress regarding bounding $Y_n(t)$ away from zero is the set of extra terms in Lemmas 3 and 4 that are not proportional to $\mathcal{E}(\Delta t)$. This creates negative terms on the right-hand side of (122) that cannot be controlled. To circumvent this problem it is necessary to remove two hurdles. The first is the last pair of terms within inequality (111)

$$\frac{\nu}{4}\int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t}(F_2/F_1^2)dt \leqq c\,\nu^{-3}\int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t}F_1 dt + \tfrac{1}{2}\left[F_1^{-1}(t)e^{\omega_0 \mathrm{Re}\,\Delta t} - F_1^{-1}(t_0)\right]. \tag{158}$$

On dangerous sub-intervals $t_0^{i,j}$ where F_1 is increasing, the last term in (158) can be rewritten as

$$\tfrac{1}{2}\left(F_1^{-1}(t)e^{\omega_0 \mathrm{Re}\,\Delta t} - F_1^{-1}(t_0^{i,j})\right) \leqq \tfrac{1}{2}\omega_0 \mathrm{Re}\,F_1^{-1}(t_0^{i,j})\mathcal{E}(\Delta t_+^{i,j}). \tag{159}$$

This term is now classed as one of the $\mathcal{E}(\Delta t_+^{i,j})$ terms and is merely a term of lower order than the dominant Re^5 term.

Secondly, in (106) if it assumed that on these sub-intervals (157) is true then the extra terms in (99) can be removed, leaving

$$\int_{\Delta t} e^{\omega_0 \mathrm{Re}\,\Delta t}F_1(t)\,dt \leqq c_2\nu^2 L^{-1}\mathcal{E}(\Delta t_+^{i,j})\left[\mathrm{Re}^5 + O(\mathrm{Re}^4)\right], \tag{160}$$

which again is proportional to $\mathcal{E}(\Delta t_+^{i,j})$. Thus (122) becomes

$$(n-\tfrac{1}{2})\left\{Y_n(t)e^{\omega_0 \mathrm{Re}\,(\Delta t_+^{i,j})}\right\} \geqq (n-\tfrac{1}{2})Y_n(t_0^{i,j}) + c_n \nu^{\frac{2n-3}{2n-1}}L^{-1}\mathcal{E}(\Delta t_+^{i,j})\left\{\mathrm{Re}^{\frac{(2n-2)\lambda_{n+1}-4\mu}{(2n-1)\mu}} - \mathrm{Re}^5\right\} \tag{161}$$

with no negative terms on the right-hand side. Given that μ is chosen in the restricted ranges of (114) and (115) within Theorem 5, and that $\mathcal{E} > 0$ for $t > 0$, then $Y_n(t)$ can never be zero. □

7. Discussion

To summarize the arguments of this paper, it has been shown that very strong fluctuations in the $\kappa_n(t)$ can occur in time, and lower bounds on these are much higher than the long-time average (15):

$$\langle L\kappa_n\rangle \leqq c_n \mathrm{Re}^{\lambda_n}, \qquad \lambda_n = 3 - \frac{5}{2n} + \frac{\delta}{n}. \tag{162}$$

This is based on the raising of the lower bound on the ratio κ_{n+1}/κ_n away for unity, a result which is expressed in Theorem 3 of Section 4, i.e.,

$$c_n \frac{\kappa_{n+1}}{\kappa_n} \geqq (L\kappa_n)^{\mu} \mathrm{Re}^{-\lambda_n}, \tag{163}$$

and is effective only on the good parts of the time axis. On those parts of the time-axis where the reverse of (163) is true, no upper bound has been found on the κ_n but very large lower bounds exist of the form

$$L\kappa_n > c_n \mathrm{Re}^{\lambda_n/\mu}. \tag{164}$$

The above results are valid for $n \geqq 2$. By including intervals at $n = 1$ the intersection set of bad intervals $\mathcal{S}^{(\infty)}$ is related to the set of potential singularities, in which case the right-hand side of (164) can be raised again by replacing λ_n by $\Lambda_n^{(\infty)}$. As inequality (144) of Theorem 6, makes clear, the constant[2] μ is then constrained to the range $\frac{1}{2} < \mu < \frac{3}{5}$.

A picture emerges of Navier-Stokes solutions that are regular on 'most' of the time-axis which is punctured by short, active intervals. While no upper bound on κ_n has yet been found within these intervals, to become singular κ_n would have to find its way through a non-empty intersection in a similar manner as the illustration in Fig. 3. Notwithstanding the widely held belief that the Navier-Stokes equations must be regular for arbitrarily long times, an equally credible alternative is that no formal upper bound exists on the κ_n, but the potentially singular set $\mathcal{S}^{(\infty)}$ in Section 5.3 allows only extremely rare singular events.

The results in this paper are consistent with ideas that have existed for many decades concerning intermittent flows [1, 20, 24, 25, 39] but it has to be acknowledged that our results are lacking in four areas:

1. It is possible that the bounds based on λ_n in (15) and Table 4 may not be sharp. Given this, the state of the analysis is such that it may be premature to suggest specific numerical tests.

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[2] In fact (163) breaks the dimensional scaling of the standard Sobolev and Gagliardo-Nirenberg inequalities, although how the introduction of the exponent μ affects this in a precise manner is not yet clear.

2. In addition to a lack of control over the κ_n within the bad intervals, their distribution and sensitivity to the value of n is an important but unanswered question.
3. The nature of solutions in the good intervals has yet to be properly established. Because solutions are bounded point-wise in time and are also constrained by the long-time average (162) it is to be expected that they should show a strong degree of quiescence, particularly within the central parts of these intervals. This has yet to be demonstrated.
4. How solutions at the junctions of the good and bad intervals connect to each other is not clear.

Results of this type derived in the manner of Section 4 are not confined to the three-dimensional Navier-Stokes equations but could be applied to simpler problems. All that is needed are long-time average bounds for weak solutions constructed in such a way that the equivalents of κ_n are bounded below. An example that springs to mind is the case of the two-dimensional Navier-Stokes equations. Not only are these regular, but tight estimates exist both for the attractor dimension [51] and the number of determining modes and nodes [52]. Other examples might be the alpha and Leray models of turbulence [53–55] or the complex Ginzburg-Landau equation [56, 57].

For many years the physics community has used scaling arguments based on Kolmogorov's original work. FRISCH's book gives a detailed factual and historical account of these arguments [22]. It has been argued in [46] that the scaling in the rigorous upper bounds on $\langle \kappa_n \rangle$ from (15) (repeated in (162) above) may be interpreted in terms of the Fourier spectrum $E_s(k)$ if a scaling of the form $E_s \sim k^{-q}$ is assumed in the inertial range, up to the cut-off wave-number $Lk_c \sim \mathrm{Re}^{q_c}$. Disregarding the correction from δ, the *a priori* bounds in (162) are consistent with $q = 8/3$ and $q_c = 3$. Such a $k^{-8/3}$ spectrum has arisen in at least two previous studies. SULEM & FRISCH [58] have shown that a $k^{-8/3}$ spectrum is the borderline steepness capable of sustaining an energy cascade in the Navier-Stokes equations when the total energy is finite. MANDELBROT [23], and later FRISCH, SULEM & NELKIN in their toy β-model [59, 22], came upon this same scaling exponent as an extreme limit of intermittency in the energy cascade. They found that if the energy dissipation is assumed to be concentrated on a fractal set (in space) of dimension $D = 8 - 3q$, then the energy spectrum scaling is of the form $E_s \sim k^{-q}$. Within this picture, the exponent $q = 8/3$ thus corresponds to dissipation concentrated at points in space. Interestingly, the conventional Kolmogorov $k^{-5/3}$ spectrum for homogeneous isotropic turbulence is associated with $D = 3$; that is, a complete lack of intermittency with dissipation spread uniformly in space is consistent with $q = 5/3$. Departures from Kolmogorov scaling, otherwise known as anomalous scaling, can be associated with intermittency in the inertial-range. These arguments have been applied to and tested on various models such as the β-model [59], and the bifractal and multi-fractal models [22]. While they suffer by comparison in not having the same degree of complexity as the Navier-Stokes equations – as FRISCH [22] and SREENIVASAN [60] have both pointed out – these models are both simple and capture the main essence of the phenomena in question. More recent work on anomalous scaling has centred on the role of the $SO(3)$ symmetry group in the expansion of the correlation functions [61, 62].

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