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ON THE PROPERTIES OF TUBES IN A CONSTANT MAGNETIC FIELD

by

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with

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MAJOR REPORTS ISSUED TO DATE

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Technical Report No. 1

H. W. Welch, Jr., "Space-Charge Effects and Frequency Characteristics of C-W Magnetrons Relative to the Problem of Frequency Modulation," November 15, 1948.

Technical Report No. 2

H. W. Welch, Jr., G. R. Brewer, "Operation of Interdigital Magnetrons in the Zero-Order Mode," May 23, 1949.

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H. W. Welch, Jr., "Effects of Space Charge on Frequency Characteristics of Magnetrons," Proc. I.R.E., 38, 1434-1449, December 1950.

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G. R. Brewer, "The Propagation of Electromagnetic Waves in a Magnetron-Type Space Charge," July 1951.

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J. R. Black, J. A. Boyd, G. R. Brewer, G. Hok, J. S. Needle, W. Peterson, S. Ruthberg, R. F. Steiner, H. W. Welch, "Theoretical Study, Design and Construction of C-W Magnetrons for Frequency Modulation," Final Report, January 1952.

PREFACE

During the past three years, several articles presenting theoretical considerations on the subject of the magnetron and the proposed magnetron travelling-wave tube have appeared in the Annales de Radioelectricite. A number of these articles have been translated from the French for use in this laboratory.

A series of four articles by O. Doehler et al. were of considerable interest to the personnel of the Electron Tube Laboratory, and it is thought that they would be of greater interest to other workers in this field if available in an English translation.

It is to be noted that for consistency with the coordinate system usually used in the United States, the x- and y-axes have been reversed in the course of this translation.

In the translation, all errors have been corrected according to "Errata," Vol. 3, p. 183.

This writer would like to express his appreciation for the assistance received from Dr. B. A. Uhlendorf, Editor of the Engineering Research Institute Publications, who carefully edited the translation.

Ann Arbor, Michigan
July, 1951

George R. Brewer

ON THE PROPERTIES OF TUBES IN A CONSTANT MAGNETIC FIELD

PART I

ON THE PROPERTIES OF TUBES IN A CONSTANT MAGNETIC FIELD

by O. Doehler

Annales de Radioélectricité
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Summary

This article treats the static and dynamic properties of the magnetron. Using the theories of L. Brillouin, the author gives the relations defining the space charge, the distribution of potential, and the characteristics for the magnetic field smaller than the critical magnetic field. The author then studies the oscillation frequencies of the multicavity magnetron and shows that the results of his calculations agree well with the measured values.

Introduction

The present work treats the static and dynamic behavior of the magnetron. Although much has been written on this question, this article seems justified, for it brings together the different ideas on certain points which have been published heretofore.

The first part treats the space charge, the distribution of potential, the trajectories of the electrons, and their characteristics; the second part studies the regions of oscillation by taking as a basis the preceding results. These results are the point of departure of the theory which considers the magnetron as a tube for the propagation of waves.

One begins with the idea of L. Brillouin in order to arrive at the relations concerning the space charge, the distribution of potential, and the characteristics for the magnetic field smaller than the critical magnetic field. However, in the use of the magnetron, these quantities are most important for the field larger than the critical field. Certain experimental results make one think that the usual ideas should be complemented in this domain from the ideas borrowed from other phenomena. One should study the motion of the electrons, not from the kinematic point of view, but from the statistical point of view; in fact, because of the very large current density, the interactions between the electrons are strong. It is therefore no longer permissible to regard the electronic current as a continuum but as having the properties corresponding to a discharge in a gas.

There does not exist an exact statistical theory of the electrons in the magnetron analogous to that of the discharge in the gas. This theory will not be developed here. Nevertheless, we can derive certain conclusions from the measurements of the distribution of potential and base the calculations on these. The measurements we conduct presuppose an approximately constant density of electrons, independent of the dimensions of the system in the static state. We can then calculate the trajectories of the electrons for a magnetic field 1.2 to 1.5 times greater than the critical magnetic field.

In the second part we will show that the oscillation frequencies of the multicavity magnetron, calculated according to this hypothesis, are in good agreement with the measured values. It is the same for the efficiency and the admittances of the oscillating magnetron.

Finally, the results of this treatment will give quantitative indications useful relative to a new type of amplifier tube: the magnetron travelling-wave tube, which will be discussed later.

PART I: CHARACTERISTICS AND TRAJECTORIES OF THE ELECTRONS IN THE MAGNETRON

When we study the dynamic operation of the magnetron, it is necessary to understand the static operation and in particular the characteristics and the electron trajectories in this type of behavior. This problem has already been treated by L. Brillouin,[1] who has shown the importance of the space charge; the distribution of potential is not the same as that in a saturated diode with space charge. We must distinguish three regions:

1. In the region below the critical point, the magnetic field is sufficiently weak so that the electrons reach the anode. This region is practically without interest.

2. In the region around the critical point, the magnetic field being equal or slightly greater than the critical field, we find ourselves between the knee above and the knee below on the curve of plate current as a function of the magnetic field. We know that in this region the electronic oscillations arise in the magnetron with or without gaps.

3. In the region above the critical point, the magnetic field being 1.2 to 1.5 times greater than the critical field, the plate current is small. In this region arise the oscillations of resonance in the magnetron with gaps and oscillations in the form of spirals in the magnetron without gaps.

1. Region Below the Critical Point

The characteristic in the region below the critical point has been calculated many times.[2-9] Here is the basis of the calculations:

(a) The equation of motion,

$$m \dot{\vec{v}} = e E + e(\vec{v} \times \vec{B}), \tag{1}$$

(b) Poisson's equation,

$$\nabla^2 \phi = \frac{\rho}{\epsilon_0}, \text{ and} \tag{2}$$

(c) the equation of conservation of charges,

$$\nabla \cdot (\rho \vec{v}) = 0 . \tag{3}$$

If one writes equation (1) in cylindrical coordinates (r, α, z), the magnetic field being in the direction Oz and the electric field radial, one obtains:

$$\ddot{r} - r\omega^2 = \eta E(r) + \eta r\omega B \tag{4}$$

$$r\dot{\omega} + 2\dot{r}\omega = - \eta B\dot{r}, \tag{5}$$

where $\eta = e/m$, $\omega = \dot{\alpha}$ (instantaneous angular velocity).
Equations (4) and (5) give after integration:

$$\dot{r}^2 + r^2\omega^2 = 2\eta\phi(r) \tag{6}$$

$$\omega = - \frac{eB}{2m} \left[1 - \frac{r_c^2}{r^2}\right]. \tag{7}$$

Equations (6) and (7) assume that the electrons leave the cathode ($\phi = 0$, $r = r_c$) with zero velocity.

For the plane magnetron, we find in the same manner, for a magnetic field in the direction Oz and an electric field in the direction y

$$\ddot{y} \quad = \quad \eta E(y) + \eta B\dot{x}, \tag{8}$$

$$\ddot{x} \quad = \quad - \eta B\dot{y} \; . \tag{9}$$

The integration of these equations, assuming that the electrons leave the cathode without initial velocity, gives

$$\dot{y}^2 + \omega_c^2 \, y^2 \quad = \quad 2\,\eta \phi(y) \tag{10}$$

$$\dot{x} \quad = \quad - \omega_c \, y \; , \tag{11}$$

where $\omega_c = eB/m$.

Equations (6) and (7) with (10) and (11) give the well-known value for the critical magnetic field B_{cr} for the cylindrical magnetron

$$B_{cr} \quad = \quad \frac{\sqrt{\frac{8m}{e}}\,\sqrt{U_p}}{r_p\left[1 - \left(\frac{r_c}{r_p}\right)^2\right]} \quad = \quad \frac{6.72\,\sqrt{U_p}}{r_p\left[1 - \left(\frac{r_c}{r_p}\right)^2\right]} \; , \tag{12}$$

where

B_{cr} = critical magnetic field in gauss

r_p = radius of the anode in centimeters

r_c = radius of the cathode in centimeters

U_p = plate voltage in volts,

and for the plane magnetron

$$\frac{\sqrt{\frac{2m}{e}}\sqrt{U_p}}{d} \quad = \quad \frac{3.36\,\sqrt{U_p}}{d} \; , \tag{13}$$

where d is equal to the distance between the anode and the cathode in centimeters.

If $B < B_{cr}$, all the electrons which are present at the same distance from the cathode have the same velocity, and equation (3) gives, for the plane magnetron

$$i_a = \rho v , \tag{14}$$

i_a being the current density, and for the cylindrical magnetron

$$I_a = 2\pi r \rho \dot{r} , \tag{15}$$

I_a representing the plate current (for unit height of the plate); $\dot{y}$ and $\dot{r}$ are determined by equations (10) and (6), respectively. If the current is sufficiently large for the space charge to limit the emission of the cathode, the electric field at the surface of the cathode is zero. We can in such a case, with the above equations, calculate the current as a function of plate voltage and the magnetic field.

Fig. 1 represents the curve obtained by Bethenod[2] for a plane magnetron. We have placed along the abscissa the ratio of the magnetic field to the critical magnetic field, on the ordinate the ratio of the plate current I_a to the plate current I_{ao} corresponding to B = 0. I_{ao} is given by the law of Langmuir-Child:

$$I_{ao} = 2.33 \times 10^{-6} \frac{S}{d^2} U_p^{3/2} . \tag{16}$$

I_{ao} is in amperes, U_p in volts, d distance between cathode and anode in centimeters, and S the surface area of the plate in square centimeters. Fig. 1 shows that the plate current diminishes by 28 per cent when B is increased from zero to B_{cr}.

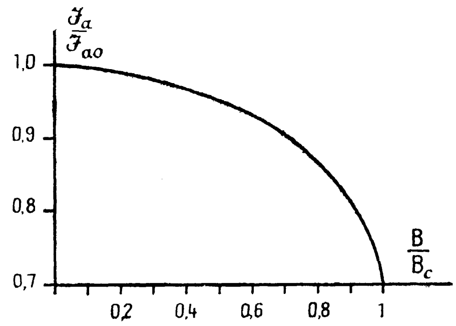

Fig. 1
Plate Current as Function of Magnetic Field. Plane Magnetron

For the cylindrical magnetron with a cathode of small diameter, we can calculate the plate current as a function of B from a development in series which Möller[3] has made use of for calculation of the distribution of potential.

Curve 1 (Fig. 2) represents the characteristic with the same coordinates as in Fig. 1. We can see that the plate current diminishes by 12 per cent when the magnetic field is increased from zero to B_{cr}.

A method due to Pidduck[7] permits the calculation of the plate current of a magnetron saturated with space charge for the critical magnetic field. This method gives the same results as the calculation carried out according to the method of Möller. Page and Adams[9] have likewise developed the calculations for the cathode of large diameter.

These authors find that, for r_p/r_c infinite, the plate current diminishes by 14 per cent when B varies from zero to B_{cr} for $r_p/r_c = 125$, by 10 per cent. In a very exact theory by L. Brillouin,[10] the diminution is by 20 per cent for the ratio r_p/r_c very large.

Curve 2 (Fig. 2) represents the results of the measurements of Hull,[11] Mulert,[12] and my personal measurements. All the latter give a diminution of current larger than that which we can expect according to the calculations. By itself, the theory of L. Brillouin appears to be in good accord with experience.

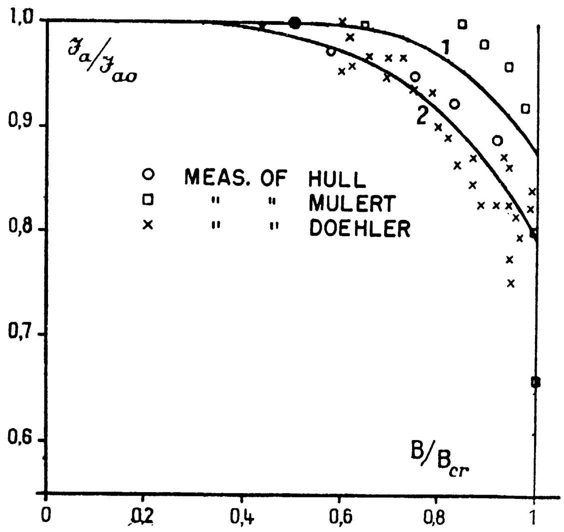

Fig. 2
Plate Current as Function of Magnetic Field. Cylindrical Magnetron.

Concerning the distribution of potential, the measurements of Engbert[13] are not in agreement with the theories of the various authors.

Theoretically, the distribution of potential for the critical magnetic field is given by (see Möller)

$$\phi = U_p \left[.932\left(\frac{r}{r_p}\right)^{2/3} + .053\left(\frac{r}{r_p}\right)^{2} + .014\left(\frac{r}{r_p}\right)^{10/3} + \ldots \right] . \qquad (17)$$

We should accordingly have for $r/r_p = 0.43$, $\phi = 0.71\ U_p$, and Engbert found, in this case, $\phi = 0.28\ U_p$. For a little smaller value of the magnetic field corresponding to $B/B_{cr} = 0.96$, Engbert finds only $\phi = 0.50\ U_p$ for $r/r_p = 0.43$.

2. Region Around the Critical Point

In this region, the plate current varies with I_a for $B = B_{cr}$, even to very small values. The magnetic field is enclosed between B_{cr} and 1.2

to 1.5 times B_{cr}. If $B > B_{cr}$, some electrons return to the cathode, and equation (15), which gives the density of the space charge, is no longer valid in this form. If one assumes that the electrons leave the cathode with zero velocity and if one holds to equations (1), (2), (3), the plate current must be zero for $B > B_{cr}$.

If one takes into account the Maxwellian distribution of velocities and if the space charge is not large, there can be no minimum of potential between anode and cathode, and we obtain, according to Linder,[14] a plate current of the form

$$I_a = I_a \exp\left[-\frac{e^2}{8m}\frac{r_p^2}{kT}(B^2 - B_{cr}^2)\right], \tag{18}$$

where

I_a = the plate current for $B = B_{cr}$

T = the temperature of the electrons

k = the Boltzmann constant.

If one plots $\ln I_a$ as a function of B^2, the slope of the curve obtained permits the calculation of the temperature of the electrons. Fig. 3 gives $\ln I_a = f(B^2)$ for various values of U_p. This function is not linear but we will replace, to a first approximation, the curve by a straight line. It is noteworthy that the slope of the straight line and, therefore, the temperature of the electrons depends on the applied plate voltage.

To calculate the distribution of potential, we can make use of equation (15) also in this region, providing we adopt the sum of the direct current and the current which returns instead of the plate current I_a; we then have

$$\rho = \frac{I_{a_1} + I_{a_2}}{2\pi r\dot{r}}, \tag{19}$$

I_{a1} being the direct current from the cathode toward the anode and I_{a2} the current which returns to the cathode.

For $\dot{r}$, we must take the absolute value, which we can obtain from equation (6).

We can then calculate the distribution of potential in the region below the critical point, and we find the same distribution of potential as in equation (17) if we introduce the distance covered to return instead of r_p.

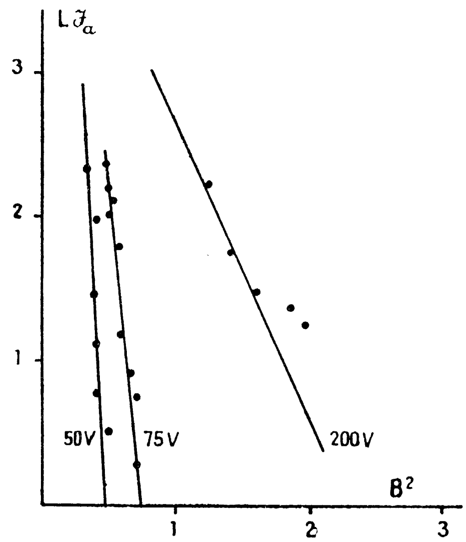

Fig. 3
$\ell n\ I_a = f(B^2)$

3. Region Above the Critical Point

The calculation of the distribution of potential is much more complicated if, at the same point, electrons are present with various radial velocities. This is the case in the region above the critical point, as we will show on the basis of some measurements. We will see that in this region the phenomena are more complex than in the others.

We can analyze the behavior of the magnetron:

(a) by the experimental determination of the characteristic;

(b) by the experimental determination of the potential distribution;

(c) by the measurement of the thermal velocity of the electrons.

<u>(a) Determination of the Characteristic.</u> Figs. 4 to 6 show the measured characteristics. The magnetic field is the abscissa and the plate current is the ordinate. In Fig. 6, we have plotted on the ordinate the ratio of the plate current corresponding to a given value of B to the plate current for B = 0.

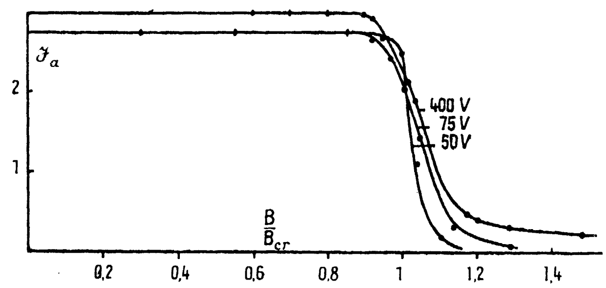

Fig. 4
Characteristic, with U_p as Parameter

In Fig. 4, the parameter is the plate voltage and in Fig. 6, the emission current.

We find that:

1. The plate current is never zero, even for a magnetic field $B \gg B_{cr}$. Moreover, as the plate voltage is raised, the plate current which passes is larger in spite of $B \gg B_{cr}$.

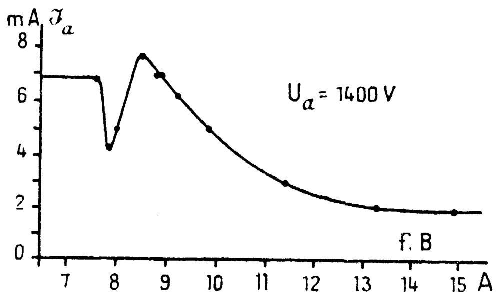

Fig. 5
Characteristic for a Large Plate Voltage

Harvey[15] points out that the plate current is not yet zero for some plate voltages if $B > 10\ B_{cr}$.

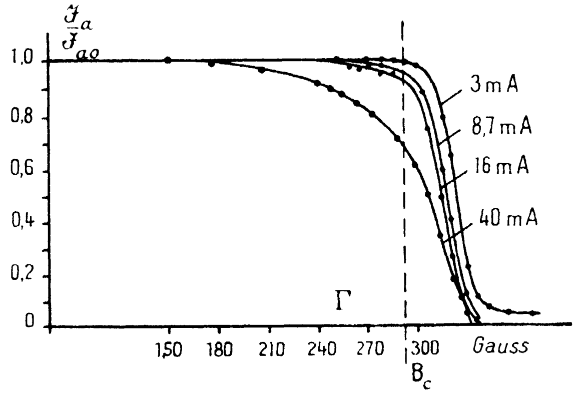

Fig. 6
Characteristic with the Emission Current as Parameter

2. The slope of the descending part of the characteristic $\Delta I_a/\Delta B$ decreases when the plate voltage increases, as we could see already in Fig. 3, as indicated by Linder.[14]

3. The relative slope $\frac{\Delta I_a}{I_a}\Big/\Delta B$ increases with emission current.

4. As the plate voltage is increased, I_a passes through a maximum for a magnetic field $B \cong 1.1\ B_{cr}$.

We observe, then, an increase of the temperature of the cathode. This well-known effect of the heating by returning electrons appears also in a vacuum and without oscillations. This is a particular characteristic of the magnetron. We will return to this in the second part of this paper.

(b) Distribution of Potential. This has been measured by Engbert.[13] Curve 1 (Fig. 7) gives the distribution of potential in the region above the critical point. This is independent of the magnetic field if $B \gtrsim 1.2$ to 1.5 times B_{cr}, is invariant with small angles of magnetic field with the axis, and independent of the emission current as long as it is larger than a well-determined value.

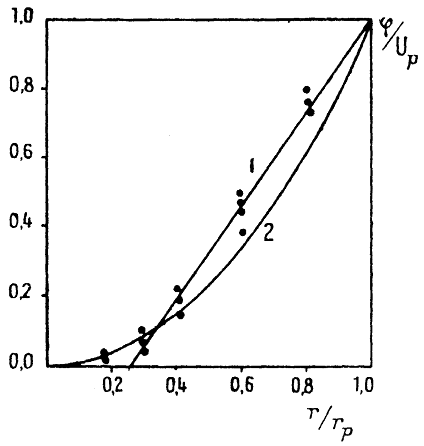

Fig. 7
Distribution of Potential in the Region Situated Above the Critical Point

The following empirical formula conveys pretty well the results of the measurements

$$\phi \quad = \quad U_p\left(\frac{r}{r_p}\right)^2 \qquad \text{(Fig. 7, Curve 2)} . \qquad (20)$$

With this distribution of potential, we have, according to (2), a constant space charge.

When we examine the region below and around this critical point, the distribution of potential (initially $\ln r$, if the emission current is small, and $\sim (r)^{2/3}$, if it is saturated with space charge) assumes progressively the form of the distribution of potential above the critical point given by equation (20).

L. Brillouin has demonstrated that, for the critical magnetic field and zero plate current, equation (20) is compatible with the equations of sections 1 to 3. In the region above the critical point, the electrons return toward the cathode for the values of r less than r_p, and one has, between the point of return and the anode, a logarithmic distribution of potential.

However, in the case of practical interest, the plate current, even above the critical point, is not zero, and one must take account of a density of charge in the entire space.

According to the measurements of Engbert, the result is that the constant density of space charge, calculated according to L. Brillouin for the critical point, is valid in the entire region above the critical point.

(c) Thermal Velocities of the Electrons. The temperature of the electrons in the magnetron has been studied by Slutzkin,[16] Wigdortshik,[17] and, in a very thorough manner, by Linder.[14] They have found that the thermal velocity of the electrons does not correspond to the temperature of the cathode, but to a temperature always much higher. We have established a distinction between these two temperatures.

1. For $B \gg B_{cr}$ the temperature of the electron is proportional to the plate voltage. The author's measurements have shown that the temperature of the electrons for a plate voltage of 1000 volts can be of the order of 10^6°K.

2. The temperature is a decreasing function of the emission current; beginning with a certain current, the temperature is constant (saturated space charge).

3. The temperature increases with the magnetic field in the region around the critical point; in the region below the critical point it decreases almost inversely proportional to the magnetic field.

The behavior of the characteristic, the distribution of potential, and the Linder effect can be explained in the following manner:

If $B > B_{cr}$, the point of return of the electrons is found between the electrodes. The electrons do not necessarily return to the cathode, but their trajectories are very long, that is to say, there is a large circulating current around the cathode even when the plate current is small. This current, which circulates around the cathode, is large enough to cancel the electric field at the cathode. This circulating current involves a large space charge, and the electrons which move in the space of the discharge traverse in this way a long path in an intense space charge. There is, therefore, an exchange of energy between the electrons because of the Coulomb forces, and one can no longer consider the space charge a continuum but we must take into account the electric field of the various electrons.* Some electrons gain energy by collision, others lose a corresponding quantity. In this manner, a part of the energy of the trajectory is transformed into disordered thermal energy. This problem is analogous to that of the discharge in a gas.[18] A plasma includes, in effect, a large number of electrons which take energy from the continuous field and transform it to thermal energy in giving it up, through dissipation, to the other electrons. In the discharge in the gas, the density of the electrons is of the order of 10^9 to 10^{13} per cubic centimeter, and this is of the same order of magnitude in the magnetron (see equation 55).

The problem of calculation of the temperature of the electrons in the plasma is not yet perfectly solved since one arrives at integrals which do not converge. (The Coulomb forces are inversely proportional to the square of the distance between electrons; the probability of collision is proportional to the square of the distance.) These analogous problems are posed in the theory of the widening of the line spectrum by pressure,[19] in the theory of the passage of electrons through matter,[20] and in the theory of the accumulation of the stars. All the calculations made up to the present lead to results which are too small.

If we take account of the interaction between electrons, we realize that the distribution of potential in the magnetron is profoundly modified

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* We can find a two-dimensional analogue to this phenomenon: that of a heavy point being displaced on a surface of disturbed water.

in relation to the considerations which permit the calculation of equations (6), (7), and (19).

For the values $B \gg B_{cr}$, we can attempt to consider the space charge as an electronic gas and apply to it the Maxwell-Boltzmann statistics by using the Hamiltonian function for the electrons. If one neglects the current absorbed by the plate, the density of electrons increases exponentially with the distance (the same as the molecular density in the barometric formula), as pointed out by Lueders in work which has not been published. However, this state is not stable because of the current absorbed by the anode.

We can admit as a first approximation to the distribution of potential in a cylindrical magnetron with small diameter:

$$\phi = U_p \left(\frac{r}{r_p}\right)^{\mu} . \tag{21}$$

We find, therefore, by integration of the equation of motion (30) that the electrons cannot leave the cathode for $\mu \leq 2$. For zero absorbed current, μ can become but little different from 2, as shown by the measurements of Engbert.

A distribution of potential like $(r/r_p)^2$ is, as we have said, equivalent to a constant density.

The same phenomena are to be found in the plane magnetron: if one takes for the distribution of potential an equation analogous to (21), we must have $\mu \leq 2$. For the plane magnetron or the cylindrical magnetron with small-diameter cathode, the distribution of potential $\phi \sim r^2$ fulfills the conditions at the cathode that $\phi = 0$ and $\partial\phi/\partial r = 0$. In the two cases, we have a uniform space-charge density. Thus we are led to assume also a uniform density for cylindrical magnetrons with large cathodes. This is merely a hypothesis that has not been verified up to the present; but we will point out that the latest of the experimental results are in good agreement with the calculations that result.

We can then calculate the distribution of potential in both these cases.

In the region above the critical point, we have ρ = constant. We then deduce:

for the cylindrical magnetron with small diameter cathode,

$$\emptyset = U_p\left(\frac{r}{r_p}\right)^2 , \tag{22}$$

for the plane magnetron,

$$\emptyset = U_p\left(\frac{y}{d}\right)^2 , \text{ and} \tag{23}$$

for the cylindrical magnetron with a large cathode,

$$\emptyset = U_p \frac{r^2 - r_c^2 - 2r_c^2 \, \ell n(r/r_c)}{r_p^2 - r_c^2 - 2r_c^2 \, \ell n(r_p/r_c)} . \tag{24}$$

In the region below the critical point, we obtain for a very small plate current (negligible space charge):

for the cylindrical magnetron,

$$\emptyset = U_p \frac{\ell n(r/r_c)}{\ell n(r_p/r_c)} , \tag{25}$$

for a very thin cathode,

$$\emptyset \sim U_p, \text{ and} \tag{26}$$

for the plane magnetron,

$$\emptyset = U_p \frac{y}{d} . \tag{27}$$

With a dense space charge, we obtain in the region below the critical point the following approximate values for distribution of potential:

for the cylindrical magnetron with a thin cathode,

$$\phi = U_p \left(\frac{r}{r_p}\right)^{2/3} , \qquad (28)$$

(a more precise result should be obtained beginning with equation (17)), and

for the plane magnetron,

$$\phi = U_p \left(\frac{r}{r_p}\right)^{4/3} , \qquad (29)$$

(a more precise result is given in (2)).

When we pass through the region around the critical point, we pass progressively in the distribution of potential below the critical point (equations 25 to 29) to the one in the region above the critical point (equations 22 to 24).

In the region around the critical point, we can represent the distribution of potential by equation (21). For a cylindrical magnetron with small diameter cathode,

$$0 \leq \mu \leq 2 ,$$

and for a plane magnetron,

$$1 \leq \mu \leq 2 .$$

4. The Electron Trajectories

The trajectories of the electrons in a space where we know the distribution of potential can be calculated from the equations of motion.

<u>(a) Cylindrical Magnetron with Very Thin Cathode</u>. From (6) and (7) we find, for the electrons leaving the cathode with zero velocity,

$$\omega_L t = \int_0^{r/r_r} \frac{d(r/r_r)}{\sqrt{(r/r_r)^\mu - (r/r_r)^2}} , \qquad (30)$$

r_r being the distance to the point of return;

$$\omega_L = -\frac{eB}{2m} .$$

Equation (30) shows that the electrons can leave the cathode only for $\mu < 2$. This is the reason that we cannot have a distribution of potential with $\mu > 2$.

Equation (30) gives an integration

$$r = r_r \left[\sin \frac{2-\mu}{2} \omega_L t\right]^{\frac{2}{2-\mu}} , \qquad (31)$$

and

$$\alpha = \omega_L t = -\frac{eB}{2m} t . \qquad (32)$$

Fig. 8 represents the trajectories of the electrons for various exponents. For $\mu = 2/3$, we obtain the well-known curve of Hull. For the electronic oscillations it is important to know the time of transit τ , i.e., the time between the emission of the electron and its return to the cathode.

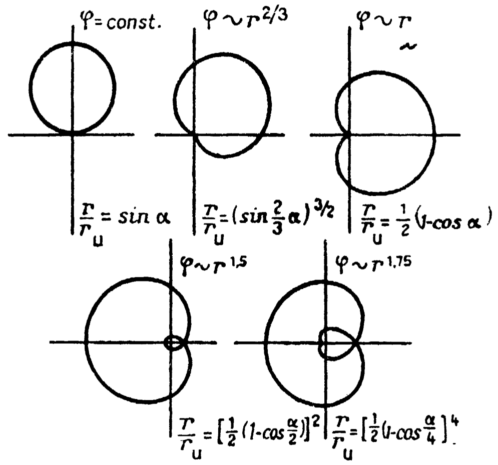

Fig. 8
Trajectories of the Electrons in the Region Around the Critical Point

We calculate it from (31)

$$\tau = \frac{2\pi}{2-\mu}\,\frac{2m}{eB}\,. \tag{33}$$

For $\mu = 2/3$ we obtain

$$\tau = 4.72\,\frac{2m}{eB}\,.$$

We can calculate τ with the aid of equations (1), (2), and (3) without use of the hypothesis about the distribution of potential, and one obtains, from Pidduck[6]

$$\tau = 5.04\,\frac{2m}{eB}\,,$$

a value 6 per cent larger than that which we obtain with the distribution of potential by Langmuir ($\mu = 2/3$).

For the quadratic distribution of potential, $\mu = 2$, equation (30) no longer has meaning, that is to say, the electrons cannot leave the cathode with zero velocity. This distribution is a limiting case which, therefore, cannot occur. Practically, the true distribution deviates slightly, the electrons can leave the cathode, and μ is slightly less than 2.

The trajectories of the electrons cannot be known exactly and are defined only statistically because of the extraordinary density of the charge and of the interactions between electrons.

What are the possible trajectories for the electrons which have left the cathode with the initial conditions given?

In rectangular coordinates, we obtain for $\mu = 2$,

$$\ddot{y} = 2\eta \frac{U_p}{r_p^2} y + \eta B\dot{x} , \tag{35}$$

$$\ddot{x} = 2\eta \frac{U_p}{r_p^2} x - \eta B\dot{y} , \qquad \eta = \frac{e}{m} \tag{36}$$

which give a differential equation of the fourth order in y.

$$\ddot{\ddot{y}} + \omega_c(\omega_c - 2\omega_p)\ddot{y} + \omega_c^2 \omega_p^2 y = 0 , \tag{37}$$

$$\omega_c = \frac{eB}{m} , \tag{38}$$

$$\omega_p = \frac{2U_p}{r_p^2 B} . \tag{39}$$

Equation (37) gives

$$y = A_1 \sin (\omega_1 t + \phi_1) + A_2 \sin (\omega_2 t + \phi_2) ; \tag{40}$$

the frequencies ω_1 and ω_2 are determined by the equation

$$\omega^2 = \frac{\omega_c^2}{2}\left[1 - 2\frac{\omega_p}{\omega_c} \pm \sqrt{1 - 4\frac{\omega_p}{\omega_c}}\right] , \tag{41}$$

which gives, for ω_1 and ω_2

$$\omega_1 = \frac{\omega_c}{2}\left(1 + \sqrt{1 - 4\frac{\omega_p}{\omega_c}}\right) , \tag{42}$$

and

$$\omega_2 = \frac{\omega_c}{2}\left(1 - \sqrt{1 - 4\frac{\omega_p}{\omega_c}}\right) . \tag{43}$$

A_1, A_2, $\emptyset_1$, $\emptyset_2$ are given by the initial conditions.

If one assumes $\omega_2 \ll \omega_c$ (from 38, 39, and 12, this is equivalent to $B \gg B_{cr}$), we obtain as a first approximation

$$\omega_1 = \omega_c = \frac{eB}{m} , \tag{38}$$

$$\omega_2 = \omega_p = \frac{2U_p}{r_p^2\ B} \text{ (formula by Posthumus).} \tag{39}$$

As a second approximation,

$$\omega_1 = \omega_c\left(1 - \frac{\omega_p}{\omega_c}\right), \tag{45}$$

$$\omega_2 = \omega_p\left(1 + \frac{\omega_p}{\omega_c}\right). \tag{44}$$

In the same manner, we obtain, for x, if $B \gg B_{cr}$,

$$x = A_1 \cos(\omega_1 t + \emptyset_1) + A_2 \cos(\omega_2 t + \emptyset_2) . \tag{46}$$

Equations (41) and (46) show that, in the region above the critical point, the electrons describe epicycloidal trajectories around the cathode, resulting in impulsive motion whose frequency is given by (43, (39), or (45), depending on the case, and a relative motion whose frequency is given by (42, (38), or (44). The space charge is thus represented as a gas which rotates with a constant angular velocity, ω_2, around the cathode. The phase and amplitude of the motion are essentially determined by the scattering of the electrons.

We obtain the same angular velocities, ω_1 and ω_2, if we use, in equation (4), the value of E_r corresponding to $\mu = 2$ and if we calculate the angular velocity of the electrons which describe circles around the cathode. The velocities found are those of the equations (42) and (43).

Physically, the existence of a circular motion around the cathode implies an equilibrium between the electrical force eE, the centrifugal force $mr\omega^2$, and the Lorentz force. If the intensity of the field is proportional to r, the angular velocity is independent of the radius. There are two solutions: (1) the electrons can circulate sufficiently fast so that the electric force is negligible compared with the centrifugal force; we then obtain equation (38) (the cyclotron frequency). Or else (2) the electrons circulate sufficiently slowly for the centrifugal force to become negligible compared with the electric force; we then have equation (39) (formula of Posthumus). This equation is important for the oscillations of resonance. Brillouin has made analogous calculations.[1]

(b) Cylindrical Magnetron with Large Cathode. This case is important since, in particular, all the multicavity magnetrons have large cathodes.

We must put into (4) the electric field of (24) and we deduce for

$\ddot{r} = 0$

$$\omega = \frac{\omega_c}{2}\left[1 \pm \sqrt{1 - \left(\frac{B_{cr}}{B}\right)^2 \frac{\left[1 - \left(\frac{r_c}{r}\right)^2\right]\left[1 - \left(\frac{r_c}{r_p}\right)^2\right]^2}{1 - \left(\frac{r_c}{r_p}\right)^2 \left(1 + 2\ln\frac{r_p}{r_c}\right)}}\right] . \tag{47}$$

The negative sign corresponds to the frequency of Posthumus and the positive sign to the cyclotron frequency. This equation, (47), which shows that the angular velocity depends on the radius, determines the frequency of oscillation in the multicavity magnetron.

(c) Plane Magnetron. We obtain analogous relations for the plane magnetron. In the region around the critical point, equations (10) and (11) give

$$y = \frac{y_r}{2}\left[\sin\frac{2-\mu}{2}\omega_c t\right]^{\frac{2}{2-\mu}} , \tag{48}$$

$$\dot{x} = -\omega_c y \tag{49}$$

$$\omega_c = \frac{eB}{m} ,$$

where y_r is the abscissa of the point of return of the electron.

For $\mu = 1$, we obtain, for example,

$$y = \frac{y_r}{2}(1 - \cos\omega_c t) , \tag{50}$$

$$x = -\frac{U_p}{dB} t + x_o + \frac{y_r}{2}\sin\omega_c t , \tag{51}$$

that is, the electrons describe cycloids.

For the region above the critical point ($\mu = 2$), we obtain the trajectories of the electrons

$$y \quad = \quad \rho_0 \sin \omega t + y_0 \ , \tag{52}$$

$$x \quad = \quad \rho_0 \cos \omega t + \frac{2U_p}{d^2 B} y_0 t + x_0 \tag{53}$$

with

$$\omega \quad = \quad \frac{eB}{m} \sqrt{1 - \frac{U_p}{U_{cr}}} \cong \frac{eB}{m} \left(1 - \frac{1}{2} \frac{U_p}{U_{cr}}\right), \tag{54}$$

U_{cr} being the critical voltage.

Equation (54) is verified by the measurements of Gutton and Ortusi.[22] These authors use the proper oscillations of the electrons in the magnetron as variable capacity. This capacity is very large if the pulsation of the high-frequency oscillation is equal to ω. Their measurements justify indirectly our hypotheses.

Note that the constant velocity in the x direction depends oppositely to the case $\mu = 1$ on the parameter y_0.

5. Density of Space Charge and Electronic Current in the Magnetron

From equation (2) and the distribution of potential (21), the density of electrons per cubic centimeter is given by:

$$N \quad = \quad 5.6 \times 10^5 \frac{U_p}{r_p^2} \mu^2 \left(\frac{r}{r_p}\right)^{\mu-2} . \tag{55}$$

For example, for $U_p = 1000$ volts, $r_p = 0.2$ cm, and $\mu = 2$, the density of electrons is $6.1 \times 10^{10}/cm^3$, of the same order of magnitude as in a plasma.

In the magnetron, the electrons produce a flux of magnetic induction that one can measure and which Möller has calculated, in the case where

$\phi \sim r^{2/3}$. The measurement of this flux is a means of verification of the theoretical studies of the magnetron. We will call annular current the current that would induce, while moving in the anode, the same magnetic flux as that of the rotation of the space charge. It is given by:

$$I_r = 5.2 \times 10^{-6} \frac{\mu^2}{1+\mu^2} \frac{B}{B_{cr}} \frac{\ell}{r_p} U_p^{3/2} , \qquad (56)$$

where I_r is in amperes and U_p in volts.

Möller[23] has measured the annular current around the critical point. These measurements lead to $\mu = 1$ for the distribution of potential.

We mean by "magnetron current" the current of the space charge circulating around the cathode above the critical point. It is given by

$$I_m = \int_{r_c}^{r_p} \rho r \omega \, dr . \qquad (57)$$

If one puts $r = r_p$ in equation (47), we obtain, by first approximation

$$I_m = 2 \epsilon_o U_p \omega_p \frac{1 - \left(\frac{r_c}{r_p}\right)^2}{1 - \left(\frac{r_c}{r_p}\right)^2 \left[1 + 2 \ln \frac{r_p}{r_c}\right]} . \qquad (58)$$

This current of the magnetron is related to the continuous plate current of a magnetron which oscillates in the region above the critical point.

6. Conclusions

We have examined the trajectories of the electrons and the influence of the space charge to the extent that is necessary for studying the mechanism of oscillation of the magnetron. We have made the hypothesis that the electronic density is constant in the region above the critical point. For the plane magnetron or the cylindrical with small cathode, Brillouin has obtained from the calculations a constant density for the critical field. The hypothesis which we have made leads to the following differences with respect to the calculations of Brillouin:

1. The space charge ρ is not constant for $B = B_{cr}$ but only for $B \gtrsim 1.2$ to $1.5\ B_{cr}$.

2. ρ is independent of the magnetic field for $B \gtrsim 1.2$ to $1.5\ B_{cr}$.

3. The distribution of constant space charge is valid for any r_p/r_c.

The two first points are verified experimentally by the measurements of the distribution of potential.

In a second article we will show that the third point is in good agreement with the dynamic behavior of the magnetron.

We have thus been able to treat all the cases of practical interest. We can distinguish two important cases: (1) the performance in the region around the critical point and (2) in the region above the critical point.

In the region above the critical point, the electronic trajectories are normal to the continuous field. We use the energy of tangential motion in introducing an alternating tangential electric field. These are the oscillations of resonance with large amplitude.

In the region around the critical point, we use the radial motion of the electrons to produce the oscillations. These are the electronic oscillations of small amplitude.

Since in the second case the oscillations are the oscillations of transit time, we see that the wavelength of the electronic oscillations is inversely proportional to B' (equation 33), while that of the oscillations of resonance is proportional to B (equation 39).

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PART II

ON THE PROPERTIES OF TUBES IN A CONSTANT MAGNETIC FIELD

by O. Doehler

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Summary

Starting with the results established in the first part of his article, the author considers the oscillations of resonance with high efficiency, that is to say, excited in the region above the critical point. He examines the differences existing between these oscillations and the electronic oscillations, then establishes a quantitative relation giving the conditions for optimum operation. The efficiency and the input impedances are calculated, and an empirical relation between the anode current and the current of rotation is given. Finally, the dynamic electronic trajectories of a plane magnetron without space charge are calculated to serve as an introduction to the study of the travelling-wave tube in a magnetic field.

PART II: THE OSCILLATIONS OF RESONANCE

The static behavior of the magnetron has been treated in the first part of this article.* According to the measurements of the distribution of

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* This first part is designated I (Annales de Radioélectricité, Vol. 3, 1948, p. 39.

potential, of the distribution of the velocity of the electrons, and of the characteristics, we have concluded that the magnetron is the seat of a phenomenon which is negligible in ordinary electronic tubes. There exists a dispersion of the velocity among the electrons in the space of the discharge because of their high density and their long time of existence. In the extreme case (very large magnetic field), we can consider the discharge as an electronic gas. For a stable discharge, the result therefore is that one has, in first approximation, a constant electronic density for the cylindrical magnetron with a very slender cathode and for the plane magnetron.* We assume therefore a constant density for the cylindrical magnetron with a cathode of finite diameter.**

This constant space charge exists only in the region above the critical point (small anode current). In the region below the critical point, there exists a distribution of potential which corresponds (without space charge) for a small emission current to the electrostatic distribution and for a large emission current to the distribution of potential of a diode of the same dimensions without magnetic field. In the region around the critical point ($B_{cr} \leq B \lesssim 1.2$-$1.5\ B_{cr}$), the distribution of potential of the region below the critical point is transformed gradually into that of the region above the critical point.

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* According to the unpublished measurements of Mr. Gutton, it is probable that the space charge is not constant but has a maximum. This maximum depends on the conditions of operation. For our calculations it is the distribution of potential which is important. This is why the small deviations of the density from a constant value do not have a large influence on the dynamic behavior of the magnetron. I thank Mr. Gutton for this information.

** Ponte[3] assumed as early as 1934, because of the large time of existence of the electrons, a distribution of potential which is approximately quadratic (see Fig. 7 of his article[3]).

The resulting trajectories of the electrons show that, in the region above the critical point, the electrons encircle, in general, the cathode. The angular velocity ω is independent of the distance if the cathode is very slender; for a large cathode, ω depends on the distance. In the region around the critical point, the electrons describe cardoidal trajectories. The time of transit of the electrons from the cathode out to the point of return depends on the distribution of potential, i.e., on the magnetic field.

As we know, the electronic oscillations of the magnetron with slots and without slots begin in the region around the critical point. Their frequency is independent of the frequency of resonance of the coupled circuit to the extent which the natural frequency of an oscillating system depends on the coupling to a second circuit. The efficiency is small. The wavelength is given by

$$\lambda = \frac{C}{B}. \tag{1}$$

C is here a constant which depends on the number of slots, on the type of oscillation, and on the ratio B/B_{cr}. We shall not occupy ourselves in this article with the mechanism of excitation of the electronic oscillations except to the extent to which we have used them compared to the oscillations of resonance.

The present article treats the oscillations of resonance excited in the region above the critical point, that is, the form of oscillations generally employed at present for technical reasons (for example in the multicavity magnetron or in the magnetron with resonance of the segments). The frequency of the oscillations of resonance is given, in general, by the oscillating circuit. Unlike the oscillations excited by the static negative resistance of the magnetron, the conditions for optimum operation of the oscillations of resonance depend on the wavelength. The efficiency is large (theoretically up to 100 per cent); in practice one has obtained efficiencies up to 70-80 per cent. The oscillations of resonance have been studied experimentally and theoretically in numerous works.[1-15] We shall not occupy ourselves in the following with the electronic mechanism. In this second part, we present a kinetic theory of the magnetron taking account of the influence of the space charge on the distribution of potential. In a third part, we will treat the dynamic behavior for small amplitudes, proceeding from the work of L. Brillouin.[16-18] In Chapter 1 of the present article, the essential differences between the oscillations of resonance and the electronic oscillations will be examined. In Chapter 2, a relation will be established to give the optimum operation of the oscillations of resonance. This relation will be compared with the measurements published up to the present. The efficiency and the input impedances will be calculated in Chapters 3 and 4. In Chapter 5, we shall give an empirical relation between the anode current and the current of a magnetron (1st par., Chapter 5). The dynamic electron trajectories of a plane magnetron without space charge will be treated as introduction to Part III, in Chapter 6.

This article does not treat the problems of the oscillating circuit. The latter, in particular for the multicavity magnetron, have been studied, among others, by Goudet,[19] Fisk, Hagstrum, and Hartmann,[20] Crawford and Hare.[21]

1. Differences Between the Electronic Oscillations and the Oscillations of Resonance

If the electrons describe a circle around the cathode, the angular velocity ω is such that there exists an equilibrium between the electric force eE, the centrifugal force $mr\omega^2$ and the Lorentz force $er\omega B$. Two limiting cases are then possible:

<u>First Limiting Case (Electronic Oscillations)</u>. The centrifugal force is sufficiently large so that, if one neglects the electric force, all the effects of the space charge are equally negligible. This limiting case is present if one uses for the excitation of the oscillations the arrangement of an inverted cyclotron. In the cyclotron, the following process takes place: an electron, which is accelerated in the direction of the tangent to its trajectory by an alternating field, attains a larger angular velocity and describes a larger circle, and therefore a larger centrifugal force. An electron retarded by the tangential alternating field describes, on the contrary, a smaller circle. The electrons retaining their energy will arrive therefore at the anode, and the electrons which lost energy, to the center of the arrangement.

This principle is used to obtain oscillations. With an electron gun and a magnetic field there is produced, in the vicinity of the anode, a circulating electronic current in a box split as in a cyclotron. The electrons of unfavorable phase reach the anode and are eliminated. The electrons of favorable phase give up the energy during their travel if the angular velocity of the electrons is in resonance with the pulsation of the oscillating circuit present between the slots. Since there does not exist a continuous radial electric field, there is produced, as a result of the equilibrium between the centrifugal force and the Lorentz force, a pulsation

$$\omega_H = \frac{eB}{m} k, \tag{2}$$

k being the mode of oscillation which is excited.[19-20]

In the usual magnetron, the radial electric field is not negligible. For a quadratic distribution of potential, it follows therefore, according to equation (1.45) in second approximation,

$$\omega_H = \frac{eB}{m} k \left(1 - \frac{\omega_p}{\omega_c}\right) = \frac{eB}{m} k \left(1 - \frac{2\,U_p}{r_p^2 \frac{e}{m} B^2}\right). \tag{3}$$

This equation (3) is confirmed by the measurements of the proper frequencies in the magnetron by Gutton and Ortusi[22] which we have already mentioned in the first part.

The oscillations of the inverted cyclotron have not been studied experimentally up to the present. The most important difficulty is to be found in the production of the electronic bunches. The modulation of the density of unfavorable phase for the radial alternating field can be avoided by an arrangement analogous to that of the cyclotron.* The oscillations of the inverted cyclotron possess the advantage that the cathode is found outside of the space of the discharge. The backheating of the cathode is therefore avoided.

The Second Limiting Case (Oscillations of Resonance). The second limiting case is characterized by the fact that, the velocity of the electrons being very small, the centrifugal force is negligible compared to the electric force. There is therefore equilibrium between the electric force eE and the Lorentz force $er\omega B$. For the cylindrical magnetron with a small cathode, the intensity of the electric field is proportional to r in the region above the critical point, from which we deduce

$$\omega = \frac{2\,U_p}{r_p^2\,B} = \omega_p . \tag{4}$$

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* See end of Chapter.

If there is a tangential alternating electric field and if an electron is accelerated in the alternating field, the Lorentz force is larger and the electron returns to the cathode. If an electron is retarded in the alternating field, it goes toward the anode.* In the region above the critical point, the electrons return toward the cathode to within a small distance from the latter (theoretically zero in the limiting case of a distribution of potential exactly quadratic), and the electrons of unfavorable phase move only in the neighborhood of the cathode, while the electrons of favorable phase reach, in the intense alternating field, to the neighborhood of the anode. The high efficiency of the oscillations of resonance rests on the fact that only the electrons giving up energy can pass through an intense alternating field. The energy which they lose in the alternating field is provided by the continuous radial field.

There exists still an essential and very great difference between the electronic oscillations and the oscillations of resonance, namely, the action of the radial alternating field. If an electron is moving under the action of a radial alternating field, this field acts as if the anode voltage [U_p in equation (3)] possessed an increased or a diminished value, according to the phase of the field. In the first limiting case, an increase of the voltage produces a decrease of the angular velocity and, inversely, a depression of the voltage and increase of the angular velocity [equation (3)]. In Fig. 1, the lines of the field are drawn schematically. It follows therefore that, for example, an electron present at point A (radial accelerating field) will be retarded in the tangential direction, and an electron at point B (radial retarding field) will be accelerated. That is, there is produced,

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* A change of the energy of the electron has therefore an exactly opposite action on the motion of the electron to that in the first case.

at point C, a focussing of the phase and therefore an increase of the density in the field of acceleration, in a phase unfavorable for the transfer of energy.

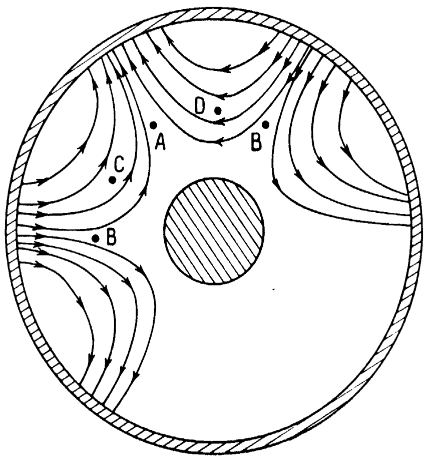

Fig. 1
Effect of Focussing on the Oscillations of Resonance
and the Electronic Oscillations

On the other hand, in the case of the oscillations of resonance, the angular velocity, according to equation (4), is proportional to the anode voltage, and therefore, for example, an electron present at point A will be accelerated in the tangential direction. An electron at point B is retarded, that is to say, there is produced a focussing of phase at point D and an increase of density in the retarding field, which is in a favorable phase.

2. The Conditions for the Frequency of Oscillations of Resonance

From these qualitative considerations, we can derive a quantitative relation for the optimum conditions. A synchronism must exist between the angular velocity ω of the electrons and the pulsation ω_H of the alternating field. If k is the mode of the oscillations, we obtain

$$\omega = \frac{\omega_H}{k}, \tag{5}$$

for a very small cathode, ω being independent of the distance. There results, according to equation (1.43):

$$\omega_H = \frac{eB}{2m} k \left[1 - \sqrt{1 - \left(\frac{B_{cr}}{B}\right)^2} \right] . \tag{6}$$

We deduce the well-known formula of Posthumus[5] for $B \gg B_{cr}$. In the practical system of units equation (4) gives

$$\lambda = 946 \frac{B\, r_p^2}{k\, U_p} , \tag{7}$$

where B is measured in gauss, r and λ in centimeters, and U_p in volts. Herriger and Hülster[8] find experimentally 1100 for the constant in equation (7). This disagreement arises from the difference of the distribution of potential in the magnetron as compared with the quadratic distribution of potential of equation (1.25). This distribution of potential is only a first approximation, which is not sufficiently exact. In reality, the intensity of the field is very small and therefore the constant of equation (7) is very large. We can deduce from the measurements of Herriger and Hülster a distribution of potential $\phi \sim r^{1.75}$ which, as shown in Fig. 2, is approached better by the measurements of Engbert than the quadratic distribution of potential.

Equation (6) (see 1), gives, in second approximation:*

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* We can arrive experimentally at the second term (term in parentheses) of equation (8) if we excite a form of oscillations which depends less on the oscillating circuit. This is therefore the case for the spiral oscillations (14,23,25). The spiral oscillations appear in the region above the critical point for the magnetron with and without slots. These are excited by the oscillations of the electrons in the direction of the axis of the cathode in an inclined magnetic field. Under the hypothesis of a quadratic distribution of potential, we obtain a condition of the frequency which is given with a constant by equation (6). In Fig. 3, the curves representing equations (7) and (8) are drawn as a function of the magnetic field. The measurements are derived from (24). The constant of equations (7) and (8) have been adjusted according to experimental results.

$$\lambda = 946 \frac{B\, r_p^2}{k\, U_p} \left[1 - \frac{1}{4} \left(\frac{B_{cr}}{B} \right)^2 \right] . \qquad (8)$$

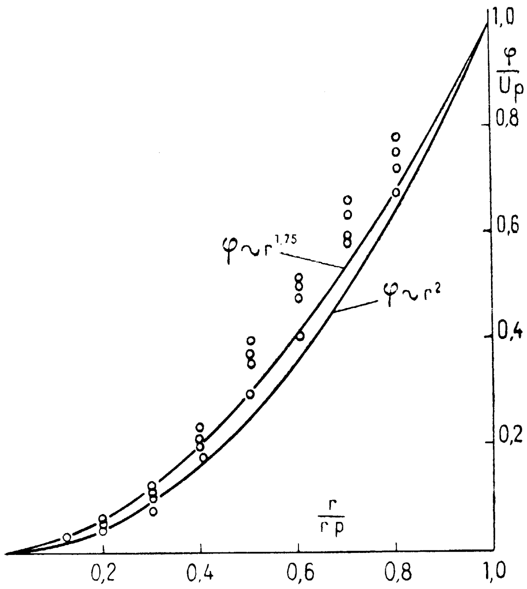

Fig. 2
Distribution of Potential $\phi \sim r^2$ and $\phi \sim r^{1.75}$
Compared with the Measurements.

The magnetron with large cathode is particularly important. It was used in the first place by Gutton[26] and employed most recently in the multicavity magnetron. It results from equation (1.47) that the angular velocity depends on the radius. In the multicavity magnetron the alternating electric field is found in the neighborhood of the anode, so that it is essential for optimum conditions that there be synchronism between the waves and the angular velocity of the electrons in the vicinity of the anode. If one substitutes for $r = r_p$ in equation (1.47) we obtain with (1.47) and with (5) for the optimum anode voltage

$$U_p = \left[\frac{\pi f}{|k|} r_p^2 B - \frac{2m}{e}\left(\frac{\pi f \ r_p}{k}\right)^2\right] F\left(\frac{r_c}{r_p}\right), \tag{9}$$

f being the frequency of the alternating field. The function $F\ (r_c/r_p)$ is given by

$$F\left(\frac{r_c}{r_p}\right) = \frac{1 - \left(\frac{r_c}{r_p}\right)^2 \left(1 + 2\ \ell n\left(\frac{r_p}{r_c}\right)\right)}{1 - \left(\frac{r_c}{r_p}\right)^2}. \tag{10}$$

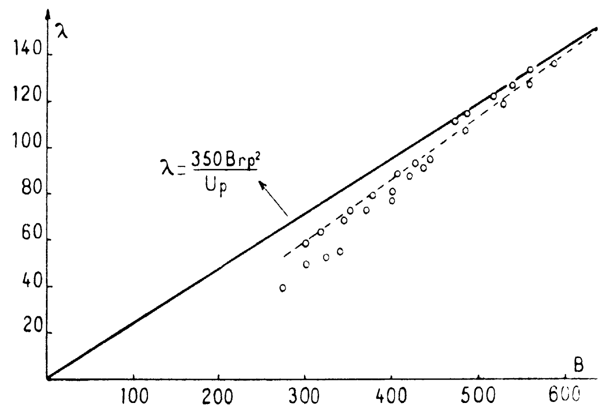

Fig. 3
Wavelength of the Spiral Oscillations

Slater,[20] Hartree,[20] and Bloch[27] have obtained analogous relations. Equation (9) corresponds to the equations of Slater. It differs in the form of the function $F\ (r_c/r_p)$. The function $F\ (r_c/r_p)$ is, according to Slater:

$$F\left(\frac{r_c}{r_p}\right) = 1 - \left(\frac{r_c}{r_p}\right)^2. \tag{11}$$

We can verify experimentally equation (9) for the results concerning the multicavity magnetron[20] and the donutron.[27] The mode k is considered as unknown in equation (9) with the result that in the formulas given by Hartree and Slater, the calculation of the optimum anode voltage is not possible with a large number of slots. In order to compare the theoretical and experimental values, Fisk, Hagstrum, and Hartmann[20] evidently proceeded in like fashion when they introduced for k the value of the mode which corresponds closest to experiment into the equations of Hartree and Slater. We proceeded in an analogous manner for our relations (9) and (10), and we find, therefore, the values for k are occasionally a little different. The differences are indicated below. Table I, 4th line, gives the measured anode voltage from the American multicavity magnetrons in the region enclosed between $\lambda = 50$ cm and $\lambda = 3$ cm. The third line contains the optimum anode voltage calculated according to equations (9) and (10). The first and second lines reproduce, by way of comparison, the corresponding values according to the formulas of Slater and Hartree.

In that which concerns the mode, there exist, with respect to (20), the following differences:

In the formulas of Slater, Hartree, and in equation (9), we use:

for type 4J42	the mode $k = 2$	instead of $k = 1$
for type 4J51	the mode $k = 3$	instead of $k = 1$

(We believe that there is, in this case, a mistake or typographical error in (20), the value $k = 1$ leading to considerable deviation.)

In equation (9), we use:

for type 5J26	the mode $k = 2$	instead of $k = 3$
for type 725A	the mode $k = 4$	instead of $k = 5$
for type 2J48-50	the mode $k = 4$	instead of $k = 5$
for type 4J50	the mode $k = 5$	instead of $k = 7$
for type 4J52	the mode $k = 6$	instead of $k = 7$

We observe that, according to the comparison between the calculations and the experiment, no multicavity magnetron oscillates in the π mode. Also, in the table of Fisk, Hagstrum, and Hartmann,[20] the mode calculated according to the equations of Slater and Hartree is not the π mode. In the region of

TABLE I

MAGNETRON BETWEEN ~ 20 — 45 cm.

Type	700 A-D.	728 A.			5 J 23.			4 J 21-25			4 J 26-30.			4J42.	4J51.	5J26.
Wavelength	43.0	32.1			28.6			22.8.			24.0.			43.0.	32.1.	23.4.
p (Slater) (kV)	15.8	21.7	24.3	26.5	19.5	21.8	28.8	18,4	25.9	30.9	19.6	27.4	32.6	15,4	24,0	23,6
p (Hartree) "	15.6	21,2	23.8	26.2	18.9	21.2	27.5	17.8	25.2	30.2	18.9	26.7	31.9	14,7	23.6	21.9
p [Eq. (9)] "	13.9	17.6	19.6	21.6	15.0	17.2	22.0	14.4	20.4	24.4	15.6	21.0	25.9	13.6	18.7	22.5
p (measured) "	12.0	19	21.0	24.5	16.5	19.0	24.5	15.5	22.0	26.5	16.5	23.0	27.0	12.0	23.0	27.0
calculated (o/o)	66	82	84	86	72	77	82	70	79	82	72	80	83	66	86	65
measured (o/o)	35	61	65	61	58	58	62	48	53	53	51	54	60	32	65	58
calculated (A)	7.2	13.6	15.0	17.5	18.3	21.0	27.0	17.7	25.0	30.2	18.8	26.1	30.6	8.1	20.2	62.0
measured (A)	10.0	19	20	28	20	24	33	25	40	48	25	40	46	9	20	46

MAGNETRON λ ~ 10 cm.

Type	706 A-C.	714 A.	706 A.			714 A.			720 A-E.			4 J 45-47.			718 A-E.		
Wavelength	9.8.	9.1.	9.8.			9.1.			10.7.			10.7.			10.7.		
p (Slater) (kV)	15.6	16.1	13,8	22.8	25,4	14.8	22.8	24.8	24,7	28,4	32.3	24,7	32,3	28,4	14,2	21.7	24.9
p (Hartree) "	14.5	14.9	12.9	21.9	23.3	13.9	21.9	23.9	23.9	27.6	31.5	23.9	31.5	27.6	13.2	20.7	23.9
p [Eq. (9)] "	12,4	12.8	11.2	18.2	20.5	11.7	18.1	19.8	19.7	22.7	25.7	19.7	25.7	22.7	11.3	17.3	19.0
p (measured) "	11.0	11.0	11.4	18.5	20.9	11.6	17.7	19.5	21.0	24.0	27.0	21.0	27.0	23.0	11.9	17.9	20.0
calculated (o/o)	35	25	44	64	70	52	70	73	77	80	82	77	80	82	50	67	70
measured (o/o)	19	17	31	55	55	31	53	47	62	68	68	62	68	68	34	53	54
calculated (A)	11.8	12.7	12.2	19.8	22.4	13.4	20.5	22.5	41	47	53	41	53	47	11.7	17.6	19.6
measured (A)	12.5	12.5	12.5	16	20	12.5	16	20	45	53	65	45	65	45	12.5	16.0	20.0

MAGNETRON λ ~ 3 cm.

Type	725-730 A.			2 J 48-50			2 J 55-56.	4 J 50-4 J 78.	4 J 52.	2 J 51.			3 J 21.
Wavelength	3,2.			3.22.	3.3.	3.4.	3.2-3. 24.	3. 2-3. 3.	3.2.	3.33.			1,25.
p (Slater) (kV)	10.9	13.5	14.3	10.6	13.1	13.8	11.6	24.0	16.7	9.9	11.1	13.7	16.8
p (Hartree) "	10.2	12.8	13.6	10.0	12.5	13.2	10.7	22.7	15.3	9.1	10.3	12.9	15.4
p [Eq. (9)] "	9.5	11.8	12.5	9.3	11.5	12.2	9.9	21.2	12.8	8.5	9.6	12.1	13.3
p (measured) "	10.0	12	13.0	10	12.0	13.0	12.0	22.0	15.0	11.0	12.0	14.3	15.0
measured (o/o)	44	52	51	44	52	51	50	66	69	42	43	46	37
calculated (o/o)	60	68	70	61	69	71	52	73	72	50	56	61	51
calculated (A)	9.3	11.5	12.2	9.3	11.2	12.1	11.4	25.4	14.5	10.1	11	13.1	25
measured (A)	10	10	12	10	10	12	12.0	27	15.0	11	12	14	15

3 cm, Sayers and Sixmith[35] demonstrated experimentally that the tube does not oscillate in the π mode. Analogous observations have been made already by Herriger and Hülster[8] for a tube with six slots with a small cathode. They find, for the π mode, an efficiency $\eta = 18$ per cent with a ratio $B/B_{cr} = 1.5$, while for the mode $k = 2$, they find more than 50 per cent for $B/B_{cr} = 1.5$. They have not yet given physical explanations. The alternating electric field, for the modes lower than the π mode, apparently penetrates more deeply into the space of the discharge, with the result that for this mode both the exchange of energy between the electronic current and the oscillating circuit, and therefore the conditions of oscillation, are more favorable.

The measurements made on the donutron[27] furnish a better check of equations (9) and (10); the possible mode in this case is determined by a preliminary experiment. In Figs. 4 and 5, the straight lines are calculated according to equations (9) and (10) for the mode and for the indicated wavelength. The experimental points are borrowed from the work of Crawford and Hare.[27] Fig. 5 of this article shows that the mode "C" corresponds to the π mode, the mode "B" to the mode $k = 4$.

Let us compare a value of these curves with the calculated values. For $B = 1200$ gauss, we have measured $U_p = 1600$ volts, a value equal to that calculated according to our theory in equation (9). According to Slater, we obtain $U_p = 2700$ volts, and according to Hartree, $U_p = 2400$ volts.

In another example of Kilgore, Shulmann, and Kurshan,[28] a magnetron modulated in frequency for a wavelength $\lambda = 7.5$ cm, has the following conditions: $r_c = 0.97$ cm, $r_p = 1.93$ cm, $B = 1600$ gauss. For the mode $k = 4$, we should obtain from equation (9), an anode voltage of 825 volts, as compared to a measured voltage of 800 volts.

Equations (9) and (10) should give the optimum anode voltages which are too small to compare with these measurements. These equations are actually established under the hypothesis of a resonance between the HF field and the angular velocity at the surface of the anode. But we must, in fact, look for a resonance between the HF field and the angular velocity of the electron at a certain distance from the anode. The effective voltage is then increased, as shown by Table I and Figs. 4 and 5. In neglecting the space charge, Slater assumes a synchronism between the wave and the velocity of the electrons at the center of the space of the discharge. The optimum anode voltages calculated according to the formula of Slater will then be too large.

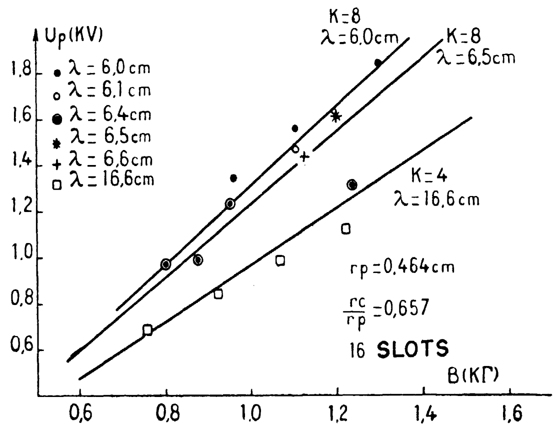

Fig. 4
U_p as a Function of B

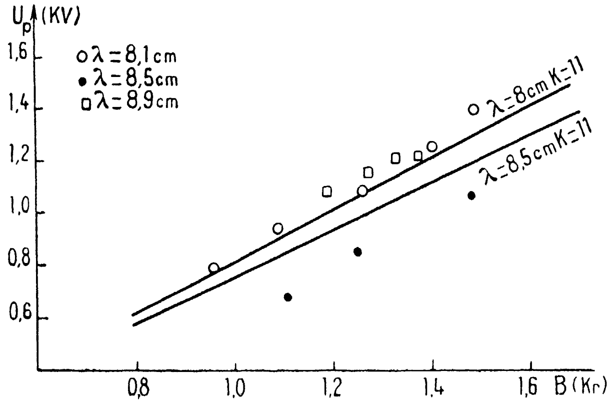

Fig. 5
U_p as a Function of B

3. The Efficiency

We can evaluate the efficiency from the considerations given in paragraph 1 for the mechanism of the oscillations of resonance. According to the representations given here, the electrons describe, around the cathode, epicycloidal trajectories with a frequency given, respectively, by equations (1.43), (1.39), and (1.44) or (1.47).* The frequency of relative circular motion which is superposed on the circular motion revolving around the cathode is given, respectively, by (1.42), (1.38), (1.45), and (1.47). If we neglect this relative circular motion, the electrons reach the anode with a kinetic energy $(m/2)\ r_p^2\ \omega^2$. The electronic efficiency is therefore

$$\eta = 1 - \frac{m}{2}\frac{r_p^2\ \omega^2}{e\ U_p}, \tag{12}$$

or

$$\eta = 1 - \frac{10^7\ r_p^2}{\lambda^2\ k^2\ U_p}, \tag{13}$$

r_p and λ are measured in centimeters, U_p in volts.

According to equations (12) and (6), we find, for small cathodes:

$$\eta = 1 - \left(\frac{B}{B_{cr}}\right)^2 \left[2 - \left(\frac{B_{cr}}{B}\right)^2 - 2\sqrt{1 - \left(\frac{B_{cr}}{B}\right)^2}\ \right]. \tag{14}$$

It follows from equation (14) that the efficiency for the critical magnetic field is zero. The oscillations of resonance cannot be excited for the critical magnetic field, even if the anode current is zero in the static state. We believe therefore that a theory seeking todetermine the mechanism of

* The "1" refers to Article No. 1 of this report.

excitation of the oscillations of resonance beginning with the behavior of the magnetron for the critical magnetic field, is not valid. According to equation (12), the efficiency is determined only by the ratios r_c/r_p and B/B_{cr}, if the plate voltage has its optimum value given by (9) and (10). In Fig. 6, the efficiency is drawn as a function of B/B_{cr} for different values of r_c/r_p. The measured points represent the electronic efficiencies measured on American multicavity magnetrons with a ratio of radii r_c/r_p = 0.375 in the region λ = 50 cm to λ = 10 cm. The measured electronic efficiencies prove to be below the calculated curve, as is to be expected. Lines 5 and 6 (Table I) show the electronic efficiencies of American multicavity magnetrons, on the one hand, calculated according to equation (12), and on the other hand, drawn from Fisk, Hagstrum, and Hartmann. It is noted that the theoretical efficiency is larger than the measured values, particularly for λ = 40 cm (types 700-AD and 4J42).

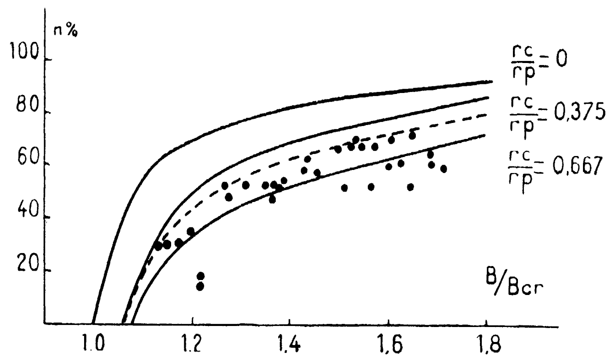

Fig. 6
Efficiency As a Function of B/B_{cr}

To a large extent we can explain the difference and obtain a better agreement between the calculations and the experiments. It results, in fact, from the measurements on an English magnetron[29] that the back heating for a magnetron in oscillation reaches 3 per cent to 6 per cent of the continuous power. In Fig. 6, the curve of the efficiency is drawn dotted for r_c/r_p = 0.375 in consideration of the fact that there is a loss of 6 per cent due to the electrons of unfavorable phase. We see that we reach therefore nearly theoretical efficiency.

It is important to remark that, according to our theory, the electronic efficiency does not depend on the value of the impedance of the load. The latter changes only the alternating voltage and the power. A variation of the alternating voltage modifies the time of transit of the electrons in the space of the discharge but does not vary the velocity of the electron on its arrival at the anode, largely determining, from (12), the efficiency. These results are justified by the measurements of Fisk, Hagstrum, and Hartmann (see Fig. 19 of this article [20]). The negligible influence of the alternating voltages on the angular velocity explains the fact that the shunt impedance of the cavities and of the load does not have an appreciable influence on the continuous voltage and the magnetic field corresponding to optimum operation. This explains that, for very different relative values of the loss, the optimum conditions correspond well to the indicated experimental results.

The important differences mentioned between the calculations and the experiments for the magnetrons 700 AD and 4J42 are, in our opinion, bound to a construction that does not correspond to optimum. We shall return to this in Chapter 4.

Posthumus,[5] Herriger and Hülster,[8] Fischer and Lüdi,[12] and the Hartree group[29] arrive at analogous conclusions for the electronic efficiency.

4. The Impedance of the Magnetron

From the measure of theimpedance of the magnetron, we can obtain conclusions on the behavior of the magnetron in oscillation, this impedance being measured between the slots of a magnetron which does not oscillate. Jänke[30] and Harvey[31] have carried out the measurements of the impedance of a cylindrical magnetron with slots and a small cathode by exciting the magnetron from an external generator. We can establish quantitative results giving the order of magnitude of the impedance and its variation with the data of the operation for a cylindrical magnetron with a small cathode. In this case, the angular velocity is independent of the radius. The current of the magnetron calculated in Chapter 5, 1st par., undergoes a phase-focussing action produced by the resonance between the angular velocity of the electrons and the phase velocity of the wave which is propagated around the cathode in such a manner that the electronic current gives the energy to the HF field.

Supposing the condition of resonance to be realized, we make two hypotheses in order to calculate the impedance:

(1) We assume that, in the radial plane which passes through a slot, the potential undergoes a discontinuity equal to $2\,\Delta U_p(r/r_p)^2$ and that between these radial planes, it depends only on the radius. We have, therefore, the distributions of potential given alternatively by

$$\phi = U_p\,(1+\Delta)\left(\frac{r}{r_p}\right)^2 \qquad \text{or} \qquad \phi = U_p\,(1-\Delta)\left(\frac{r}{r_p}\right)^2 .$$

This condition is not fulfilled for the multicavity magnetrons because the breadth of the slots is of the same order of magnitude as the breadth of the segments. In the magnetrons with small cathode and a small number of slots, this condition is only fulfilled in the vicinity of the anode. But the error is not very large because the exchange of energy is accomplished principally in the neighborhood of the anode.

(2) All the space charge present in the space of the discharge is focussed in such a way that it is found only in the negative phase of the alternating field propagating around the cathode. The current density at point r is $\rho r \omega$. According to our hypothesis we find in the negative phase a current density twice this value. In the positive phase there is no current. The current of the magnetron as a function of the time of transit is given therefore by Fig. 7. The amplitude is $2\, J_m$; J_m is given by equation (1.57).

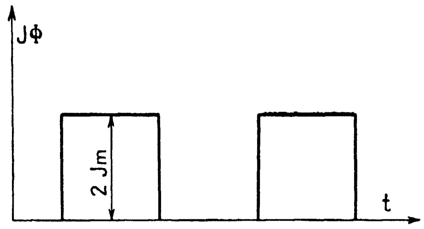

Fig. 7
Current as a Function of Time

The Fourier analysis of the current (Fig. 7) gives $(4/\pi)\, J_m$ for the fundamental. The balance of the power is therefore

$$\Delta U_p^2 \quad G_m \quad = \quad \frac{2}{\pi} \ell \omega_H \int_0^{r_p} \rho r \quad \Delta U_p \left(\frac{r}{r_p}\right)^2 dr \;, \tag{15}$$

where ρ is the magnetron space-charge density, G_m the negative admittance, and ℓ the length of the anode.

ρ is, from equation (1.22)

$$\rho \quad = \quad \frac{4\varepsilon_o}{r_p^2} \quad U_p \tag{16}$$

It follows from equations (15) and (16) that the resistance is negative.

$$R_m \quad = \quad \frac{1}{G_m} \quad = \quad - \; \frac{\pi}{2} \frac{\Delta U_p}{\omega_H U_p \varepsilon_o \ell} \quad = \quad - \; \frac{94\lambda \;\; \Delta U_p}{U_p \;\ell} \tag{17}$$

From this it follows that:

(1) the admittance G_m is inversely proportional to the amplitude of the alternating voltage;

(2) from equation (7), for $B \gg B_{cr}$, λ and R_m are proportional to the magnetic field for a constant amplitude of the alternating voltage.

These conclusions are valid only in first approximation. With the aid of the method of Brillouin, we will calculate exactly, in the third part of this article, the impedance for small amplitudes. Equations (15) and (17) are confirmed by the measurements of Harvey and of Jänke. In Fig. 8, the admittance of a magnetron with an external HF source is drawn as a function of U_p from the measurements of Jänke.

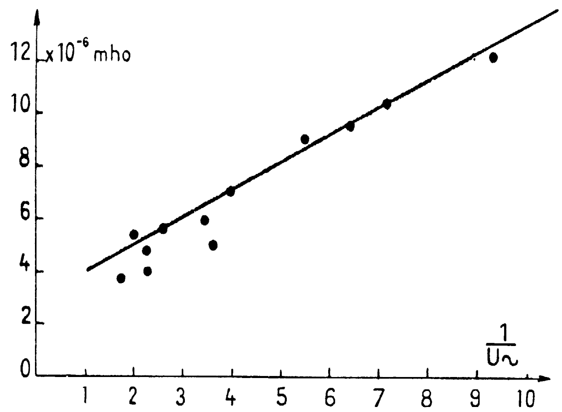

Fig. 8

Admittance of a Magnetron as a Function of Amplitude According to Janke

We obtain the desired straight line. Fig. 9 represents the negative resistance of resonance as a function of the magnetic field for a constant amplitude of the alternating voltage, from the measurements of Harvey. According to equations (7) and (17), R_m must yield a straight line as a function of the magnetic field. This straight line passes through the origin. In Table II, we have compared the values measured by Harvey with equation (17) for a magnetron with $r_p = 0.5$ cm, $\ell = 2$ cm. A coincidence, if exact, can be only fortuitous. We cannot interpret quantitatively the measurements of Jänke, the length of the anode not being known. From equation (17), the length of the anode should be 1.2 cm, a length certainly too small.

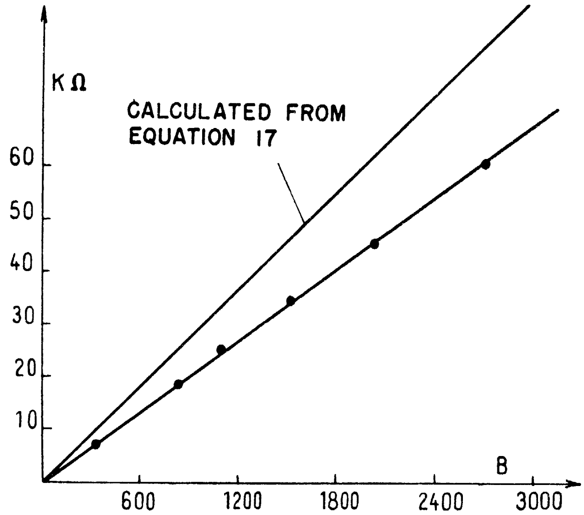

Fig. 9
Impedance of the Magnetron as Function of B, Measured by Harvey

TABLE II

U_p (V).	ΔU_p (V).	λ (m).	R_m measured (kΩ).	R_m calculated (kΩ).
100	29	9,27	7,0	9
100	29	14,7	12	14,2
100	29	23,2	18	22,8
100	29	33,1	25	32,5
100	29	43,3	35	42,5
100	29	61,8	45	61,0
100	29	80,3	60	79
100	13,6	10,0	3,5	4,6
100	13,6	19,0	5,5	8,3
100	13,6	60,0	19	25
120	78	61,8	115	127
97	19	61,8	24,8	27,4

The behavior of the cylindrical magnetron with small cathode for an HF which is not in resonance with the angular velocity has been studied experimentally by Jänke. Fig. 10 represents, according to Jänke, the admittance of a magnetron with four slots, in the complex plane, for different amplitudes. We find first that the largest real admittance is associated with an inductive component. As we will show in Chapter 6, this inductive component can be explained by a focussing of the electrons in the radial direction. This focussing of the electrons has the same phase as the electrons which induce an inductive component. Unlike other well-known transit-time generators, the oscillations of resonance have a negative resistance for all frequencies. This fact is one of the most remarkable qualities of the oscillations of resonance. For the electronic oscillations, for example, this impedance is negative only for certain ranges of frequency. In first approximation, Jänke substitutes for the experimental curve a circle, and thus finds a circuit equivalent to a resonant circuit with an admittance given approximately by equation (17). The difference between the circle and the experimental curve is considerable. Malter and Moll[32] assume the same equivalent circuit for a magnetron with large cathode.

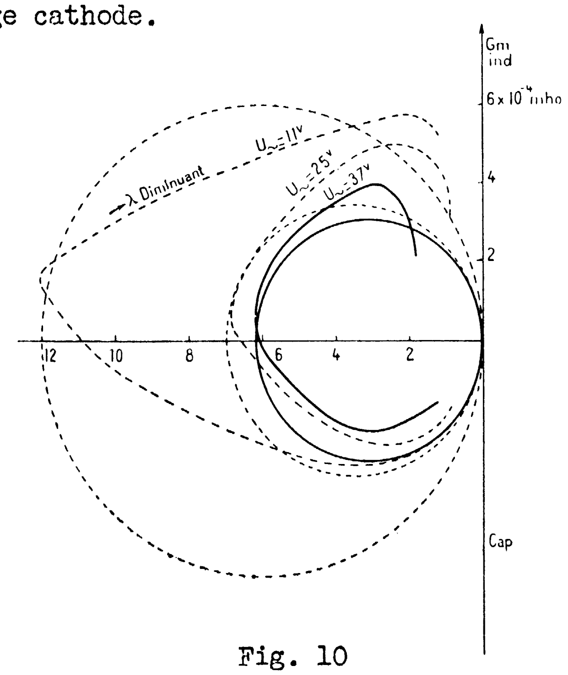

Fig. 10
Admittance of the Magnetron According to Jänke

The difference between the experimental curve and a circle is qualitatively comprehensible. A circle would signify that the magnetron current would have the same intensity as if there were no resonance between the angular velocity and the HF alternation. There should exist only a change of phase between the alternating field and the current of the magnetron if the curve is a circle.

It is not easy to calculate the impedance of a magnetron with large cathode, the angular velocity depending on the distance. The admittance is probably proportional to the magnetron current, calculated in the first part. The admittance depends on the amplitude in a very complicated manner because the electrons travel a shorter time in the space charge when the amplitude increases,and, therefore, the change-of-phase of the electrons, with the alternating field propagating around the cathode, diminishes.

Qualitatively, we find that the efficiency and the negative resistance of the magnetron increase with B/B_{cr}, but thus, the efficiency of the circuit decreases with increase of magnetic field. It follows that there is, for a given circuit resistance, an optimum magnetic field for the maximum total efficiency.

On the other hand, the negative resistance of the magnetron must increase and the efficiency decrease if one increases r_c/r_p. If the distance between the cathode and the anode becomes very small, larger alternating fields are necessary in order that the space charge not be out of phase during its travel in the space of the discharge. The measurements published in (20) confirm this result. The coupling of the load for a maximum efficiency must be so much less when r_c/r_p is larger. In (20) the Q of the loaded circuit has the following values:

r_c/r_p	Q
0.27	280
0.37	100-150
0.50	280
0.67	350

Except when the value of $r_c/r_p = 0.27$, Q is proportional to r_c/r_p. The tubes 700 AD and 4J42 correspond to the value $r_c/r_p = 0.27$, $\lambda = 40$ cm. As we have already mentioned, these tubes show an efficiency deviating very considerably from the theoretical possibilities.

Apparently, in these tubes the different modes are very near to each other. The optimum coupling of the exterior load cannot be obtained without the danger of "demoding" (variation of the mode during normal operation).

The impedance of a magnetron with large cathode calculated according to the method for the magnetron with small cathode, deviates from the measurements more as r_c/r_p is larger. According to the unpublished measurements of Brück and Hülster, the alternating voltage in a magnetron with 8 cavities is 10 kv for a continuous voltage of 14 kv, $r_c/r_p = 0.375$ and a power of 10 kw. It follows that the impedance is of the order of magnitude of 1000Ω. From the calculation with the method for the magnetron with small cathode, we obtain the impedance of the order of magnitude of 100 - 500 Ω.

5. The Anode Current

Kilgore, Shulmann, and Kurshan[33] calculate from the measurements the anode current in an oscillating magnetron. They find in this way that the anode current is of the order of magnitude of 1/10 to 1/50 of the anode current of a space-charge-limited diode, without magnetic field and with the same dimensions. The anode current depends on a large number of parameters. We can, nevertheless, expect that it will be proportional to the magnetron current (1.51). The anode current should therefore be in practical units:

$$J_a = \frac{11\,\ell\ U_p}{\lambda k}\ \frac{1-\left(\frac{r_c}{r_p}\right)^2}{1-\left(\frac{r_c}{r_p}\right)^2\left(1+2\ln\frac{r_p}{r_c}\right)}, \tag{18}$$

J_a being in amperes, U in kilovolts, and the wavelength, λ, and the anode length, ℓ , in centimeters. The numerical constant of equation (18) has been chosen in such as way as to represent the experimental results. We find therefore that the anode current is of the same order of magnitude as the magnetron current, as is in agreement with the measurements made with an American multicavity magnetron.

The anode current measured and calculated from equation (18) is shown in the 7th and 8th lines of Table I. Donal, Bush, Cuccia, and Hegbar[34] give an anode current of 0.73 A for a magnetron of 1 kw with frequency modulation. We obtain 0.81 A from equation (18). The magnetron of Kilgore, Shulmann, and Kurshan[28] has an anode current of 62 mA. Equation (18) gives us a value of 310 mA. The error is considerable. This shows that equation (18) is only a first approximation. We can give an approximation to the trajectory of an electron in an oscillating magnetron. The amplitude of the current is in the neighborhood of 2 J_m (Fig. 7). The anode current is approximately J_m. It follows that the electron travels in a tangential direction, double the distance between the anode and the cathode.

6. Calculation of the Trajectories of the Electrons in a Plane Magnetron Without Space Charge for Small Amplitude of the Alternating Voltage

The relative circular motion of the electrons of angular velocity ω_c has been neglected in our preceding considerations. For $B \gg B_{cr}$, this amplitude is small. In practical cases, $B \sim 1.5\ B_{cr}$, with the result that the motion again has a positive influence on the behavior of the tube. Therefore, confining ourselves to the case without space charge and with a small alternating electric field, we shall calculate the electronic trajectories in the plane magnetron, a case where the relative motion plays an essential role and where the calculation is still rather simple. But it is not possible to reach the conclusions for the case with space charge since in this case the velocity of the electrons depends on the distance from the cathodes. In spite of all, the results give a first general idea.

In the stationary case, the motion of the electrons is given, from equations (1.50) and (1.51), by

$$y = \rho(1 - \cos \omega_c t) \tag{19}$$

$$x = \rho \sin \omega_c t - \rho \omega_c t \quad , \tag{20}$$

where

$$\rho = \frac{U_p \eta}{d \omega_c^2} \quad , \quad \omega_c = \frac{eB}{m} \ .$$

An electromagnetic wave with phase velocity $V_{ph} = \frac{\omega_H}{\Gamma}$ is moving in the x direction (perpendicular to the continuous electric field and the magnetic field). The alternating voltage at the anode $y = d$ would be

$$U_p \, e^{i(\omega t - \Gamma x)} \ .$$

The alternating field $\tilde{\phi}$ is therefore given in the space of the discharge by

$$\tilde{\phi} = \tilde{U}_p \frac{\operatorname{sh} \Gamma y}{\operatorname{sh} \Gamma d} \, e^{i(\omega t - \Gamma x)} \ . \tag{21}$$

For $\Gamma d \ll 1$, we obtain

$$E_y = \frac{\tilde{U}_p}{d} \, e^{i(\omega_H t - \Gamma x)} \tag{22}$$

$$E_x = - i \Gamma y \frac{\tilde{U}_p}{d} \, e^{i(\omega_H t - \Gamma x)} \quad . \tag{23}$$

We can now obtain the equations of motion

$$\dot{y} = \frac{\eta U_p}{d} + \omega_c \dot{x} + \eta \frac{\tilde{U}_p}{d} \, e^{i(\omega_H t - \Gamma x)} \tag{24}$$

$$\ddot{x} = -\omega_c\dot{y} - i\frac{\eta\tilde{U}_p}{d}\Gamma y\, e^{i(\omega_H t - \Gamma x)} \quad . \tag{25}$$

If one introduces in the alternating term of equations (24) and (25) for x and y, the static trajectory (equations (19) and (20), we obtain a factor $e^{i\Gamma(\rho \operatorname{Sin} \omega_c t)}$, which, developed into a Fourier series, gives

$$e^{-i\Gamma\rho \sin\omega_c t} = \sum_{n=o}^{\infty} \varepsilon_{2n} J_{2n} (\Gamma\rho) \cos 2n\,\omega_c t$$
$$- i\varepsilon_{2n+1} J_{2n+1} (\Gamma\rho) \sin (2n+1)\,\omega_c t, \tag{26}$$

where $\varepsilon_n = 2$, for $n \geqq 1$, $\varepsilon_o = 1$, J_n = Bessel function of order n. If we assume a synchronism of the velocity of the electrons in the x direction with the phase velocity of the progressive wave, we then obtain

$$\omega_H = -\Gamma\rho\,\omega_c \; . \tag{27}$$

Equations (19), (20), (26), and (27) will be introduced into equations (24) and (25). The initial conditions are

$$\dot{x} = 0 \quad x = 0 \quad \dot{y} = 0 \quad y = 0 \quad \text{for } t = 0 \; .$$

Equations (24) and (25) are then integrable. If we take into consideration only the first two terms of equation (26), we obtain the solution

$$\frac{x}{\rho} = \left(1 + \varepsilon J_o(\Gamma\rho) + \frac{J_1(\Gamma\rho)}{2}\right) \sin\omega_c t - \left(1 + \varepsilon J_o(\Gamma\rho)\right) \omega_c t$$
$$- i\varepsilon\left[J_1(\Gamma\rho) + \frac{\Gamma\rho}{4} J_o(\Gamma\rho)\right] \omega_c t \sin\omega_c t - \varepsilon\Gamma_\rho J_1(\Gamma\rho)\, \omega_c t \cos\omega_c t \tag{28}$$
$$- 2i\varepsilon\left[-J_1(\Gamma\rho) + \frac{\Gamma\rho}{4} J_o(\Gamma\rho)\right] (1 - \cos\omega_c t) + \varepsilon\Gamma\rho\,\frac{J_1(\Gamma\rho)}{3} \sin\omega_c t$$

$$\frac{y}{\rho} = \left[1+\varepsilon J_o(\Gamma_\rho) + \frac{5}{3}\varepsilon(\Gamma_\rho)\, J_1(\Gamma_\rho) + \varepsilon\, i\, J_1(\Gamma_\rho) + \varepsilon\, i\, \frac{\Gamma_\rho}{4} J_o\, (\Gamma_\rho)\right] (1 - \cos \omega_c t)$$

$$-i\varepsilon(\Gamma_\rho)J_o(\Gamma_\rho)\omega_c t + \frac{\varepsilon}{2}J_1(\Gamma_\rho)(1-\cos 2\,\omega_c t) - \varepsilon\Gamma_\rho\, J_1(\Gamma_\rho)\omega_c t\, \sin \omega_c t \qquad (29)$$

$$+i\varepsilon\left[-J_1(\Gamma_\rho) + \frac{3}{4}\Gamma_\rho\, J_o(\Gamma_\rho)\right] \sin \omega_c t + i\varepsilon\left[J_1(\Gamma_\rho) + \frac{\Gamma_\rho}{4}\, J_o(\Gamma_\rho)\right] \omega_c t \cos \omega_c t \;,$$

where $\varepsilon = \frac{\tilde{U}_p}{U_p}\, e^{i\phi}$, ϕ being the phase at time $t = 0$ and for $x = 0$.

Equations (28) and (29) are represented in Figs. 11 to 14 for $\varepsilon = 0.1$, and $\Gamma\rho = 0.5$ and 1. These values $\Gamma\rho = 0.5$ to 1 are not compatible with the hypothesis $\Gamma\rho \ll 1$. The fields being proportional to $sh(\Gamma\rho)$ or to $ch(\Gamma\rho)$ and the value of these functions not departing appreciably from $\Gamma\rho$ and from 1 for $\Gamma\rho < 1$, we have chosen these large values to make evident the influence of $\Gamma\rho$ on the motion of the electrons.

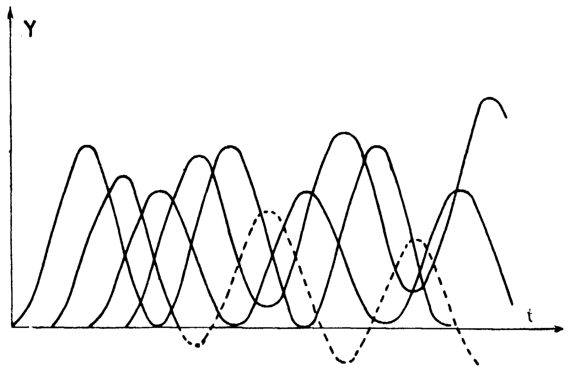

Fig. 11
Trajectories of the Electrons in the Direction y
$\Gamma_\rho = 0.5 \qquad \varepsilon = 0.1$

In the multicavity magnetron, we have for the π mode: $\Gamma = \pi/b$, where b is the distance between two slots[19]. For the k mode: $\Gamma = (2\pi/b)(k/N)$ (N = number of slots). The distance being the difference of the radius between the anode and the cathode, we find therefore empirically that, for the American multicavity magnetrons

$$\Gamma d \sim 1.9 \pm 0.5 \;,$$

that is to say,

$$k\left(1 - \frac{r_c}{r_p}\right) \quad = \quad 1.9 \qquad (\pm\ 0.5) \tag{30}$$

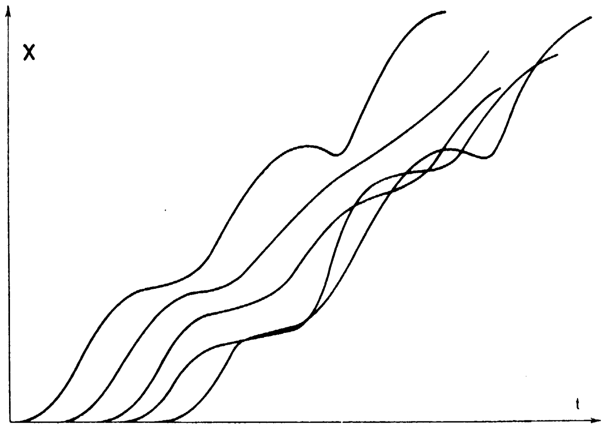

Fig. 12
Trajectories of the Electrons in the Direction X
Γ_ρ = 0.5 $\quad \varepsilon$ = 0.1

Figs. 12 and 14 show the motion of the electrons in the direction of the travelling wave for different phases ϕ. There is a focussing of the electrons in the negative phase of the travelling wave, as we have already explained qualitatively in Chapter 1.

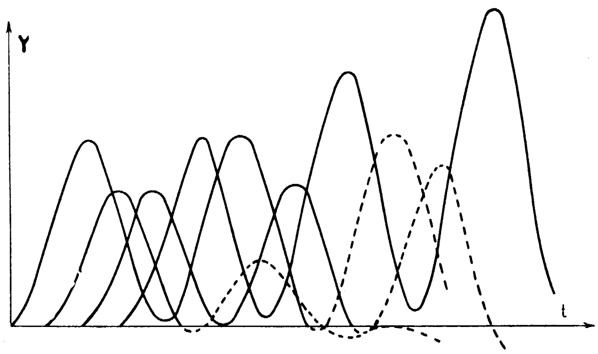

Fig. 13
Trajectories of the Electrons in the Direction Y
Γ_ρ = 1.0 $\quad \varepsilon$ = 0.1

Figs. 11 and 13 show the motion of the electrons in the direction of the anode. The electrons of favorable phase go to the anode and the electrons of unfavorable phase to the cathode. For large Γ_ρ the amplitude of oscillations of the electrons of unfavorable phase (Fig. 13) is small; the electrons reach the cathode with a small velocity and cause therefore little back-heating. The electrons of favorable phase, on the other hand, reach the anode with a large kinetic energy and therefore decrease the efficiency. Moreover, for small values of $\Gamma\rho$, the back-heating is larger, and the kinetic energy of the electrons which reach the anode smaller. We can therefore find an optimum value of $\Gamma\rho$.

There are other reasons leading likewise to an optimum value of $\Gamma\rho$, as Goudet has already indicated. If, for a constant number of slots, the cathode-anode distance is too great, the HF field cannot penetrate very far into the cathode space. If the cathode-anode space is too small, the capacitance of the circuit is too large and the strength of resonance too small; the tangential alternating field disappears. Goudet indicated that the optimum distance is of an order of magnitude equal to the distance of the slots. We obtain, therefore, the ratio

$$\frac{r_c}{r_p} \approx 1 - \frac{2\pi}{N} , \tag{31}$$

(N = number of slots).

Slater[33] gives an empirical formula for the optimum ratio:

$$\frac{r_c}{r_p} = \frac{N-4}{N+4} . \tag{32}$$

Equations (31) and (32) do not differ much from each other.

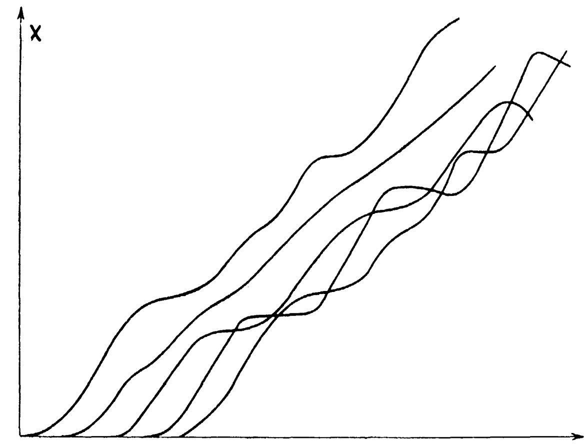

Fig. 14
Trajectories of the Electrons in the Direction X
$\Gamma\rho = 1.0 \qquad \varepsilon = 0.1$

There results from the motion of the electrons, in Figs. 11 and 13, that there is a weak focussing in the y direction. This focussing has an inductive phase which can explain the inductive component of the magnetron impedance for the case of resonance (Fig. 10).

7. Conclusions

In this report, we have obtained quantitative relations for the efficiency (13), the optimum conditions for operation (9), and the magnetron impedance with small cathode (17). We have developed semiempirical considerations for the anode current (18) and for the optimum ratio r_c/r_p (30). We are now going to indicate briefly the method of calculation of a magnetron, the wavelength of the anode, the plate voltage, and the HF power being given. We choose for examples the American multicavity magnetrons, indicated in (20) and (34), i.e., magnetrons with wavelength and construction very different.

The ratio r_c/r_p may be calculated from equation (30) for the different modes, that is to say, for the different number of slots. The continuous power and therefore the anode current are derived from Fig. 6, for a given ratio B/B_{cr} with a circuit efficiency of 90 to 95 per cent. The anode length is calculated from equation (18) and the radius of the anode from equation (9), for a given mode and a given ratio B/B_{cr}. According to this method, some tubes have been calculated in Table III for $B/B_{cr} = 1.4$ to 1.5.

The dimensions and the information of the operation of the American multicavity magnetrons are shown in the last column; the information on the operation and the dimensions for the different modes, in the other columns. The agreement is satisfactory.

Apart from this information, other parameters influence also the design of the magnetron: the strength of the magnetic field, the specified cathode density, the back-heating of the cathode, the maximum anode load, and, for large ratios of r_c/r_p, the efficiency of the circuit.

TABLE III

TYPE 720 A-E.

$\lambda = 10$ cm, $\frac{B}{B_{cr}} = 1,4.$

Mode	3.	4.	720 A-E.
U_p (kV)	21	21	21
P (kW)	550	550	550
J_a (A)	47,5	47,5	42
ℓ (cm)	4,1	4,1	4,0
r_p "	0,64	0,97	0,70
r_c "	0,24	0,36	0,27
B (Γ)	2500	1900	2300

TYPE 4 J 21-25.

$\lambda = 22,8$, $\frac{B}{B_{cr}} = 1,5.$

Mode	2.	3.	4.	4 J 21-25.
U_p (kV)	22	22	22	22
P (kW)	50	50	50	50
J_a (A)	40	40	40	40
ℓ (cm)	7,5	7,5	7,5	4,9
r_p "	1,05	2,0	3,2	1,5
r_c	0,05	0,75	1,66	0,55
B (Γ)	1620	880	620	1200

TYPE 2 J 53-56

$\lambda = 3$ cm, $\frac{B}{B_{cr}} = 1,5.$

Mode	4.	5.	6.	2 J 55-56.
U_p (kV)	12	12	12	12
P (kW)	50	50	50	50
J_a (A)	9,3	9,3	9,3	12
ℓ (cm)	0,42	0,42	0,38	0,62
r_p "	0,23	0,26	0,29	0,31
r_c	0,12	0,18	0,18	0,16
B (Γ)	6400	4350	4750	3350

TABLE III (cont'd)

1 kW MAGNETRON ACCORDING TO [34].

$$\lambda = 34{,}5\ \text{cm}, \quad \frac{B}{B_{cr}} = 1{,}5.$$

Mode	3.	4.	5.	Measured
U_p (kV)	2,5	2,5	2,5	2,5
P (kW)	1	1	1	1
J_a (A)	0,80	0,80	0,80	0,73
ℓ (cm)	2,0	2,0	1,8	3,3
r_p "	1,15	1,62	2,25	1,63
r_c	0,42	0,82	1,40	0,83
B (Γ)	580	410	360	400

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PART III

THE TRAVELLING-WAVE TUBE IN A MAGNETIC FIELD

by J. Brossart and O. Doehler

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Summary

In this article the authors study the behavior of the magnetron as a travelling-wave tube; they describe a new type of tube, the magnetron travelling-wave tube, and, neglecting the influence of space charge, they calculate the gain of this tube used as an amplifier. Finally, they point out the essential differences between the traveling-wave tube of the Kompfner-Pierce type and the magnetron travelling-wave tube.

The first part of this investigation was devoted to the study of the static characteristics of the magnetron; we have shown, in particular, that for the plane magnetron, in the absence of space charge, the electrons move with a velocity v_s perpendicular to the electric and magnetic fields, and it has been found that

$$v_s = \frac{U_p}{dB}, \tag{1}$$

where U_p is the anode voltage

B is the magnetic field

d is the anode-cathode distance

On this motion is superposed also an oscillatory motion of pulsation $\omega_c = \frac{eB}{m}$. It is interesting to note that v_s is independent of y, the distance of the electron from the cathode. When the space charge is not negligible - always for the plane magnetron - we have seen that this space charge has a constant density in first approximation; the electronic trajectories remain similar to the preceding case where we neglect the space charge, but the velocity of the electrons perpendicular to the electric and magnetic fields is written as

$$v_c = \frac{2U_p}{dB} \frac{y}{d} \tag{2}$$

and depends on the distance y of the electron from the cathode; in addition, the oscillatory motion superposed on the continuous motion depends slightly on U_p.

The oscillations of resonance are excited by a high-frequency electric field that follows each of the electrons and constantly takes energy from the aggregate of the space charge which is of the same rotation. The principle of the travelling-wave tube is analagous[3,4]; we obtain amplification of a high-frequency wave owing to the interaction between the bunches of electrons and the electric field of the wave which is propagated in the same direction as the electrons and with a velocity similar to the latter. We are therefore led to believe that it can be possible to use the magnetron combined with the principle of the travelling-wave tube and obtain in this way an amplifier tube for U.H.F. In the remainder of this study we shall call the new tube Magnetron Travelling-Wave Tube and we shall describe two modes of possible realization, for which we will calculate the gain, limiting ourselves to small signals.

Fig. 1[6] gives a first realized form. C is the cathode, A is the anode; which is made up of a delay line, for example, a helical plate; E is the entrance for the electromagnetic waves; S is the exit. In the direction of the axis is found a constant homogenous magnetic field; O is an arrangement for focussing. When the velocity of the electrons leaving the gun is so great that the force due to the electric field between A and C, augmented by the centrifugal force, balances exactly the Lorentz force, we have

$$\eta E - r\omega^2 = \eta r\omega B \qquad \left(\eta = \frac{e}{m}\right) ,$$

and the electrons rotate around the cathode. If, in addition, we can neglect the centrifugal force, that is to say if $r\omega^2 < E$, or if we are considering a plane magnetron, we obtain for the velocity of the electrons the relation (1); the electronic trajectories are those of circles concentric to the cathode, and the motion relative to the frequency $\omega_c = \frac{eB}{m}$ is negligible. The influence of the centrifugal force and the deviations of the trajectory beginning with the trajectory of equilibrium have been examined in the first part of this investigation.

The electronic bunch is finally captured by the collector, K, which is serving likewise as an electromagnetic screen between the entrance and the exit, so as to prevent parasitic coupling.

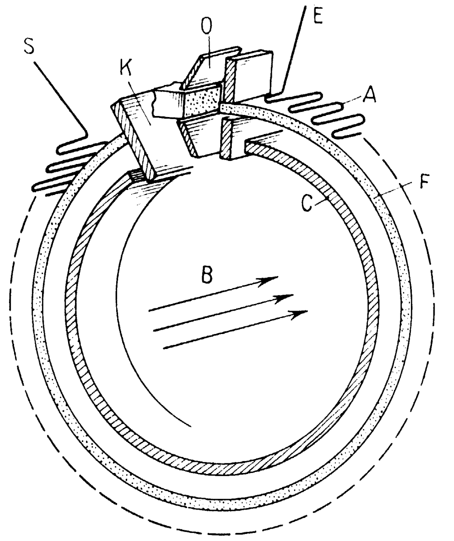

Fig. 1
Magnetron Travelling-Wave Tube
with Circular Trajectories

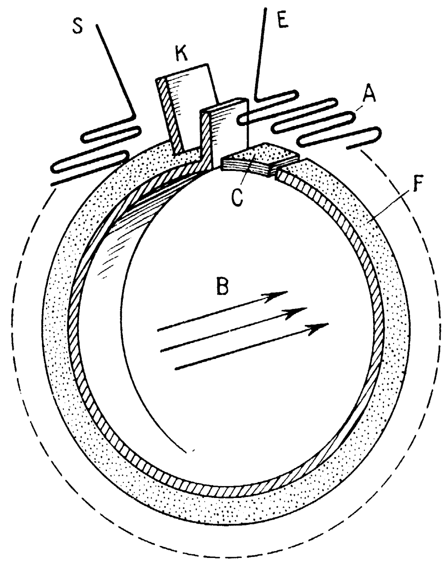

Fig. 2
Magnetron Travelling-Wave Tube
with Epicycloidal Trajectories

Fig. 2 gives a second realized form. The cathode C is emissive in many places. The anode is made up of a delay line with entrance, E, and exit, S, and a collector, K, to absorb all the bunches and form a screen between the entrance and the exit. Experience gained on such a tube with emissive cathode at the entrance has shown the influence of the space charge to be negligible when the current does not exceed approximately 10 mA. On the other hand, for currents much larger, we can imagine that the space charge plays a considerable role, and we will take account of this in assuming that the space charge is constant (except perhaps in the neighborhood of the collector because of the absorption of the electrons), as has been established in reference (1). For this arrangement (Fig. 2), we will have epicyloidal electronic trajectories, and unlike the preceding type of tube, it will be necessary to take account of the motion relative to the frequency $\omega_c = \frac{e\,B}{m}$ when the space charge is negligible.

The magnetron travelling-wave tube presents important advantages in relation to the travelling-wave tube, and it seems justifiable to study the practical realizations of this tube in spite of the many technical difficulties which one will have to overcome. In effect, we have shown[5] that the travelling-wave tube is not usable for an amplifier of high power, for its efficiency cannot be greater than a few per cent. This is due to the fact that, if one wishes to obtain a gain, one cannot give to the electrons a velocity much larger than the phase velocity of the forced wave which they accompany, and only the surplus of kinetic energy corresponding to this difference of velocity can be transformed into electromagnetic energy. In the magnetron travelling-wave tube, on the other hand, the electrons that would have to give up energy to the high-frequency field constantly absorb energy from the constant field, as we have shown in the second part of this investigation. We would thus have an output very much larger. On the other hand, we could use, in the magnetron travelling-wave tube, an electronic current very much larger (of the order of one ampere) than in the travelling-wave tube (10 to 20 mA.).

We are going to study, in this article, the gain of the magnetron travelling-wave tube for small signals. The method used will make possible, in addition, the solution of a problem which has been posed, but not solved, in the second part of this study[2], namely: the determination of the starting resistance for a magnetron with large cathode. The calculation of the principal characteristics of operation of a magnetron travelling-wave tube will be done following the same method as that given at the end of the second part.

1. Hypotheses

Compared with the travelling-wave tube, the study of the magnetron travelling-wave tube is more complicated because, in the first place, it concerns a cylindrical problem in two dimensions and in the second place it is necessary to take into account the essential role that the space charge plays (see the first and second parts of this study), particularly for the second form of construction (Fig. 2). In consequence, we will not be able to treat the question without a number of important simplifications and approximations:

1. We shall restrict the study to small signals.

2. We shall limit ourselves to the case of the plane magnetron, and to the case of the cylindrical magnetron when one has $r_p/r_c \simeq 1$ (r_p radius of anode, r_c radius of cathode).

It is necessary, nevertheless, to note for the magnetron travelling-wave tube, that one must already realize this to be considerably different from the limiting case of the plane magnetron.

3. For the arrangement represented in Fig. 1 and for the arrangement represented in Fig. 2, when the electronic current is not very large, we shall assume that the influence of the space charge is negligible. In the arrangement represented in Fig. 2 with large current, we shall take account of the space charge by assuming that the density is constant.

4. In the arrangement represented in Fig. 1 we shall neglect motion relative to the frequency $\omega_c = \eta B$, and assume that the electrons describe circles concentric with the cathode.

5. For the arrangement represented in Fig. 2:

a) We shall neglect the influence of the space charge but take into account the relative motion of the electrons.

b) We shall take account of the space charge by assuming that the density is constant, but we shall neglect the relative motion of the electrons.

6. We assume

$$\eta B \gg |\omega - \Gamma v_c| ,$$

v_c, being given by equations (1) and (2), is the velocity of the electrons. Γ is the phase constant of the wave which is propagated in the anode-cathode space. ω is the angular frequency of the high-frequency wave.

In the case where the space charge is negligible, the preceding inequality is always valid, because we have a gain only when the velocity of the electrons is approximately equal to the phase velocity of the wave. On the other hand, when the space charge is important, the velocity of the electrons depends on their distance from the cathode, so that in the vicinity of the cathode, the preceding inequality is valid only for a very intense magnetic field, $B \gg B_{cr}$. On the other hand, in the vicinity of the anode, the electric fields are important and the inequality is always valid.

7. We shall neglect the effect of the boundaries and assume the system to be indefinitely extended along the axis.

8. For the case with space charge, we shall assume $\Gamma d \ll 1$, which signifies that the wave length of propagation of the wave is large compared with the product of 2π times the anode-cathode distance. We note that the preceding hypothesis is not valid in the multicavity magnetron. The influence of Γd on the electronic trajectories has been discussed in the second part (Section 6).

2. Study of the Magnetron Travelling Wave Tube with Space Charge Neglected

A. Study of the Arrangement of Fig. 1: We Neglect the Motion Relative to Frequency $\omega_c = \eta B$. The calculation of the gain is done according to the method used in (4).

1. Determination of the Electronic Trajectories. - The system is represented schematically in Fig. 3, with the three axes of reference. A constant electric field, U_p/d, is directed along - y and a constant magnetic field, B, along + Z.

We assume that the electrons enter in the system parallel to the direction + x with a velocity v_o given by (1):

$$v_o = \frac{U_p}{dB}$$

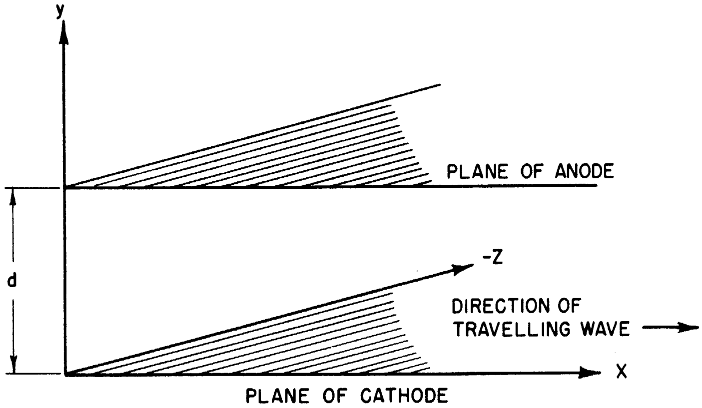

Fig. 3
Coordinates of the System

In the absence of the high-frequency field their trajectories are then defined by

$$
\begin{aligned}
x &= v_o \tau \\
y &= y_o \ ,
\end{aligned}
\tag{3}
$$

with $\tau = t - t_o$, t_o being the time of passage of the electron considered in the plane $x = o$, and t being the instant when this electron is found at X.

If now we assume that a high-frequency wave be propagated in the direction x with a phase constant $\Gamma = \bar{\gamma} - i\bar{k}$, we can obtain the high-frequency electric fields with the aid of a simple theory which neglects the influence of the potential vectors* and we arrive at the expressions:

$$
\begin{aligned}
E_x &= -\Gamma \Delta U_p \frac{sh(i\Gamma y)}{sh(i\Gamma d)} e^{i\omega t_o} e^{i\omega\tau + \Gamma x}, \\
E_y &= -i\Gamma \Delta U_p \frac{ch(i\Gamma y)}{sh(i\Gamma d)} e^{i\omega t_o} e^{i\omega\tau + \Gamma x},
\end{aligned}
\tag{4}
$$

with

$$
\Gamma = \bar{\gamma} - i\bar{k}, \qquad \bar{k} = \frac{\omega}{\bar{v}} ,
$$

$\bar{v}$ being the phase velocity of the wave.

ΔU_p, having the dimensions of a voltage, serves to define the amplitude of the field; when this amplitude is small compared to that of the electric field U_p/d, the trajectories of the electrons in the presence of the high-frequency wave can be considered to be the trajectories (3), perturbed as

$$
\begin{aligned}
y &= y_o + \delta y, \\
x &= v_o\tau + \delta x.
\end{aligned}
\tag{5}
$$

Supposing the phase velocity of the wave approximately equal to v_o, δx and δy

---

* A more complete theory beginning with the equations of Maxwell is developed in the appendix; for $|\Gamma|^2 \gg {}^2/c^2$, the case which interests us here, this gives practically the same expressions for E_x and E_y as the simple theory.

satisfy to first approximation the equations

$$\delta\ddot{y} + \omega_c\delta\dot{x} = i\eta\Gamma\Delta U_p \frac{ch(i\Gamma yo)}{sh(i\Gamma d)} e^{i\omega to} e^{i\xi\tau},$$

$$\delta\ddot{x} - \omega_c\delta\dot{y} = \eta\Gamma\Delta U_p \frac{sh(i\Gamma yo)}{sh(i\Gamma d)} e^{i\omega to} e^{i\xi\tau}, \tag{6}$$

where we have used

$$\eta = \frac{e}{m}, \quad \omega_c = \eta B, \quad \xi = \omega\rho - iv_o\bar{\gamma}, \quad \rho = 1 - \frac{v_o}{\bar{v}}.$$

If we have $|\xi| \ll \omega_c$, it follows that

$$\delta y = \frac{i\eta\Gamma\Delta U_p}{\omega_c\xi} \frac{ch(i\Gamma yo)}{sh(i\Gamma d)} e^{i\omega to} e^{i\xi\tau},$$

$$\delta x = \eta\frac{\Gamma\Delta U_p}{\omega_c\xi} \frac{sh(i\Gamma yo)}{sh(i\Gamma d)} e^{i\omega to} e^{i\xi\tau}, \tag{7}$$

$$\delta v_x = i\xi\delta x, \quad \delta v_y = i\xi\delta y \tag{8}$$

We must remark here that δx and δy are proportional to $1/\xi$, whereas, in the travelling-wave tube, the perturbations in the trajectories are proportional to $1/\xi^2$ [see (3), (4)]. On the other hand, δy is proportional to the field Ex and δx to field Ey.

2. Calculation of the High-frequency Current. - The unmodulated electronic beam is supposed to occupy the interval yo, yo + Δyo and carry a constant current I. The constant density of charge is given by

$$\rho_o = \frac{I}{v_o\Delta yo}.$$

Let us now apply to the modulated beam the equation of conservation of electricity:

$$\nabla\cdot(\rho\vec{v}) = -\frac{\partial\rho}{\partial t}.$$

Restricted to small motions, this equation is reduced to

$$v_0 \frac{\partial}{\partial x}(\rho - \rho_0) + i\omega(\rho - \rho_0) = 0$$

and shows that one has everywhere $\rho = \rho_0$. In other words, in the modulated beam the density of charge remains constant and equal to the density of the unmodulated beam.

The alternating current i_x is therefore due only to the variation of the cross section of the beam, and one can write

$$I + i_x = \int_{y_0 + \delta(y_0)}^{y_0 + \Delta y_0 + \delta(y_0 + \Delta y_0)} \rho_0 [v_0 + \delta v_x(y)] \, dy$$

so, since $|\xi| \ll \omega$ and Δy_0 is very small

$$i_x = iId\Gamma \frac{\Delta U_p}{U_p} \frac{\omega}{\xi} \frac{ch(i\Gamma y_0)}{sh(i\Gamma d)} e^{i\omega t_0} e^{i\xi\tau}. \qquad (9)$$

Remarks:

a) We can find directly the expression for the current by applying the conservation of electricity in the direction x, that is to say, by writing

$$I + i_x = I \frac{dt_0}{dt},$$

but this method does not show directly the interesting physical fact that $\rho = \rho_0$ throughout the beam.

b) The alternating current i_y can be neglected in first approximation; in effect, we do not have the continuous component of velocity along y, and the current crossing an element of height, q, situated in the ordinate y is written simply

$$i_y = \int_x^{x+q} \rho_0 \delta v_y dx,$$

a term which is negligible compared to i_x, since ξ is very small compared to ω.

3. The Energy Balance. - Let: - dP be the apparent power given up by the electronic current along a short distance d x; + dP_1, the apparent power consumed by the line along the distance d x; + d P_2, the increase, along d x, of the apparent power which is propagated in the direction of the wave.

From the law of conservation of energy, we can write

$$dP + dP_1 + dP_2 = 0 , \tag{10}$$

the equation which is going to permit the calculation of the propagation constant of the forced wave, beginning with the characteristics of the tube (current, voltage, magnetic field, etc.) and the propagation constant γ -ik of the free wave capable of being propagated along the delay line in the absence of electronic current.

- d P is given by

$$-dP = \frac{E_x i_x^* dx}{2} \tag{11}$$

and is obviously written, by neglecting $\bar{\gamma}$ in comparison with $i\bar{k}$ in the hyperbolic functions

$$-dP = \frac{e^{2\gamma y}}{2} \frac{\Delta U_p^2}{U_p} I \, d\bar{k}^2 \frac{sh(\bar{k}y)ch(\bar{k}y)}{sh^2(\bar{k}d)} \frac{v_o\bar{\gamma}\omega + i\omega^2\rho}{\rho^2\omega^2 + v_o^2\bar{\gamma}^2} . \tag{12}$$

For the calculation of d P_1 and d P_2, it is necessary to recall that the apparent energy carried by the wave can be written

$$P = \frac{E_x E_x^*}{2R_x} , \tag{13}$$

R_x being a constant characteristic of the delay line (the coupling resistance). (In the Appendix, we shall calculate the electromagnetic fields and the coupling

* indicates complex conjugate.

resistance for a helical plate.)

Under these conditions, we have

$$dP_1 = -\frac{E_x E_x^*}{R_x} (\gamma + ik)\, dx, \tag{14}$$

$$dP_2 = \frac{E_x E_x^*}{R_x} (\bar{\gamma} + i\bar{k})\, dx. \tag{15}$$

Cancelling separately the real and imaginary parts of equation (10), we finally obtain the two relations

$$(\bar{\gamma} - \gamma)\left(\bar{\gamma}^2 + \frac{\omega^2}{v_0^2}\rho^2\right) = \frac{dR_x \coth(\bar{k}y_0)}{2Z_0} \frac{\omega}{v_0} \bar{\gamma}, \tag{16}$$

$$(\bar{k} - k)\left(\gamma^2 + \frac{\omega^2}{v_0^2}\rho^2\right) = \frac{dR_x \coth(\bar{k}y_0)}{2Z_0} \frac{\omega^2}{v_0^2}\rho, \tag{17}$$

where $Z_0 = \frac{Up}{I}$ represents the impedance of the beam.

Relations (16) and (17) now permit the calculation of the gain of the tube and the phase velocity of the forced wave. Relation (17) shows, in particular, that if $\bar{k} = k$, we have $\rho = o$; in other words: if the velocity v_0 of the electrons is equal to the phase velocity v of the free wave, the phase velocity $\bar{v}$ of the forced wave will be the same as that of the free wave. This is the profound difference in the mechanism of the operation of the magnetron travelling-wave tube and that of the travelling-wave tube where the phase velocity of the forced wave is always different from that of the free wave.

Relations (16) and (17) are discussed at the end of paragraph B.

B. Study of the Arrangement in Figure 2: We Will Take into Account the Relative Motion of the Electrons. -- In the preceding paragraph we have neglected the relative motion of the electrons, because we assume that an appropriate arrangement of focussing injected the electron beam into the cathode-anode space of the magnetron with a velocity v_0 parallel to the cathode and

* indicates complex conjugate.

equal to U_p/dB. Now, it is evident that this arrangement will be difficult to realize, and, as a practical matter, we will rather put the source of electrons in the plane of electrode C (Fig. 1). In this case and in the absence of space charge, the electrons describe the static cycloidal trajectories calculated in the first part of this study [equations (50) and (51)]:

$$y = \frac{Y_r}{2} (1 - \cos \omega_c \tau), \tag{18}$$

$$x = v_o \tau + \frac{Y_r}{2} \sin \omega_c \tau , \tag{19}$$

where v_o is given by (1), $Y_r = d \left(\frac{B_{cr}}{B}\right)^2$ being the amplitude of the cycloid and $\tau = t - t_o$ is the time of transit of the electron entering at time t_o. In the Second Part, paragraph 6, we have calculated the dynamic trajectories of the electrons when the anode-cathode distance is small, for example $\Gamma d = \frac{2\pi d}{\lambda_o} < 1$ (λ_o is the wavelength of the RF in the cathode-anode space); we have found in particular that for small values of Γd, the amplitude of the relative motion of the electrons in favorable phase is small, also that the electrons in unfavorable phase have a relative motion of large amplitude (see Part II, Fig. 11). For the values of Γd of the order of 1, we have likewise found that the amplitude of the relative motion of the electrons of unfavorable phase is large, which entail a small efficiency, an important part of the electronic current being absorbed by the cathode (see Part II, Fig. 13). As a practical matter, the American magnetrons have the values of Γd of the order of 1.9 ± 0.5. For such values of Γd the calculation of the dynamic trajectories of the electrons is complicated; these trajectories have components of pulsation ω (pulsation of the HF wave), $\omega_c = \eta B$, $2\omega_c$ etc., also $p\omega_c + q\omega$ (linear combinations of the coefficients of integral number of pulsations ω_c and ω). In that which follows, we will take account only of the components of pulsation ω, since only these components will give rise to a current frequency $F = \frac{\omega}{2\pi}$. Once the HF current is calculated, we will be able to determine the gain of the tube, providing we do not take account of the absorption of the electrons by the interior and exterior cylinders. Also, we have seen (Part II, paragraph 6) that the components of frequency $F_c = \frac{\omega_c}{2\pi}$ should have a large influence on the electron trajectories, for these are the ones which cause an absorption by the electrodes. We can neglect this absorption when Y_r is smaller than d and when the interior cylinder is at a negative potential with respect to the cathode.

Suppose that these conditions are fulfilled and assume that an HF wave is propagated in the direction + x with a phase constant $\Gamma = \bar{\gamma} - i\bar{k}$, then the electric fields E_y and E_x are given by the relations (4) and the trajectories of the electrons will be the trajectories (18) and (19), perturbed, and the perturbations δy and δx must then satisfy the relations:

$$\begin{aligned} \delta\ddot{y} + \omega_c\delta\dot{x} &= i\eta\Gamma\Delta U_p \frac{ch(i\Gamma y)}{sh(i\Gamma d)} e^{i\omega t_o} e^{i\omega\tau+\Gamma x} \\ \delta\ddot{x} - \omega_c\delta\dot{y} &= \eta\Gamma\Delta U_p \frac{sh(i\Gamma y)}{sh(i\Gamma x)} e^{i\omega t_o} e^{i\omega\tau+\Gamma x} . \end{aligned} \tag{20}$$

Substituting in equation (20) x and y from the non-perturbed values (18) and (19) and neglecting in the development of $ch(i\Gamma y)e^{\Gamma x}$ and $sh(i\Gamma y)e^{\Gamma x}$ all the terms which contain ω_c, we can write equation (20) in the form

$$\begin{aligned} \delta\ddot{y} + \omega_c\delta\dot{x} &= A e^{i\omega t_o} e^{i\xi\tau}, \\ \delta\ddot{x} - \omega_c\delta\dot{y} &= B e^{i\omega t_o} e^{i\xi\tau}, \end{aligned} \tag{21}$$

where

$$A = i\eta\,\Delta U_p \frac{Q}{sh(i\Gamma d)} , \tag{22}$$

$$B = \eta\,\Gamma\,\Delta U_p \frac{T}{sh(i\Gamma d)} , \tag{23}$$

$$\begin{aligned} Q = {} & J_o\left(\frac{i\Gamma Y_r}{2}\right)\left[1 + \frac{3}{16}(i\Gamma Y_r)^2 + \frac{35}{24.128}(i\Gamma Y_r)^4 + \frac{231}{720.2^{10}}(i\,Y_r)^6\right] \\ & + J_2\left(\frac{i\Gamma Y_r}{2}\right)\left[\frac{(i\Gamma Y_r)^2}{16} + \frac{7}{24.32}(i\Gamma Y_r)^4 + \frac{495}{720.2^{11}}(i\Gamma Y_r)^6\right] . \end{aligned} \tag{24}$$

$$T = J_0\left(\frac{i\Gamma Y_r}{2}\right)\left[\frac{i\Gamma Y_r}{2} + \frac{5}{6.16}(i\Gamma Y_r)^3 + \frac{63}{120.256}(i\Gamma Y_r)^5\right]$$

$$+J_2\left(\frac{i\Gamma Y_r}{2}\right)\left[\frac{(i\Gamma Y_r)^3}{2} + \frac{15}{120.64}(i\Gamma Y_r)^5\right], \tag{25}$$

where

$\xi = +\ \omega\ -i\Gamma v_o$ (same as the preceding value).

Under these conditions the forced solution of (21) gives:

$$\delta y = \frac{-A-i\frac{\omega_c}{\xi}B}{\xi^2 - \omega_c^2}\, e^{i\omega t_o}\, e^{i\xi\tau},$$

$$\delta x = \frac{-B+i\frac{\omega_c}{\xi}A}{\xi^2 - \omega_c^2}\, e^{i\omega t_o}\, e^{i\xi\tau},$$

or, practically, when $|\xi| \ll \omega_c$,

$$\delta y = \frac{iB}{\xi\omega_c}\, e^{i\omega t_o}\, e^{i\xi\tau}, \tag{26}$$

$$\delta x = \frac{-iA}{\xi\omega_c}\, e^{i\omega t_o}\, e^{i\xi\tau}. \tag{27}$$

Applying the conservation of electricity in the direction x, we can then calculate the HF current i_x in the same manner as in paragraph 2-A. But at first it is necessary to clarify the following point: as a result of the cycloidal motion of the electrons, we no longer have a density of continuous current constant in the x direction, but the density depends on y, the distance to the cathode. In particular, for y = o, the continuous density in the x direction is zero, so that, in the vicinity of y = o, the approximations made for the calculation of i_x are no longer justified.

In that which follows, we will admit, in any case, that the density of continuous current is constant in each section and equal to the mean value $\frac{I}{Y_r}$. This permits us to simplify the calculation of i_x without involving too large an error being made in the remainder under the hypothesis of small signals.

We find then

$$i_x = \int_{y=o}^{y=Y_r} di_x \,, \qquad (28)$$

with

$$di_x = i\frac{I}{Y_r} d\Gamma \frac{\Delta U_p}{U_p} \frac{\omega}{\xi} \frac{Q}{sh(i\Gamma d)} e^{i\omega t_o} e^{i\xi\tau} dy \,. \qquad (29)$$

The energy balance is written as before:

$$dP + dP_1 + dP_2 = o, \qquad (30)$$

with

$$-dP = dx \int_{y=o}^{y=Y_r} \frac{E_x}{2} di_x^* \,, \qquad (31)$$

$$dP_1 = -\frac{E_x E_x^*}{R_x} (\gamma + ik) \, dx, \qquad (32)$$

$$dP_2 = \frac{E_x E_x^*}{R_x} (\bar{\gamma} + i\bar{k}) \, dx. \qquad (33)$$

Let us recall that we have

$$E_x = -\Gamma \Delta U_p \frac{sh(i\Gamma y)}{sh(i\Gamma d)} e^{i\omega t_o} e^{i\xi\tau} \qquad \text{see equation (4)}$$

$$R_x = \frac{k}{h} \sqrt{\frac{\mu_o}{\varepsilon_o} \frac{\sin\psi}{\cot\psi}} e^{-kd} \frac{sh^2(ky)}{sh(kd)} \qquad \text{see Appendix, (15')}$$

---

* indicates complex conjugate.

Cancelling separately the real and imaginary parts of equation (30), we obtain finally

$$(\bar{\gamma} - \gamma)(\bar{\gamma}^2 + \frac{\omega^2}{v_o^2}\ \rho^2) = D\bar{\gamma}, \tag{34}$$

$$(\bar{k} - k)(\gamma^2 + \frac{\omega^2}{v_o^2}\ \rho^2) = D\ \frac{\omega}{v_o}\ \rho, \tag{35}$$

with

$$D = \frac{Q}{2Y_r Z_o}\ \frac{ch(\bar{k}Y_r)-1}{sh(\bar{k}\ d)}\ \frac{\sin\psi}{\cot\psi}\ \sqrt{\frac{\mu_o}{\varepsilon_o}}\ \frac{de^{-\bar{k}d}}{h}\ \frac{\omega}{v_o},$$

$$\rho = 1 - \frac{v_o}{\bar{v}} = u - \frac{v_o}{v}\ \frac{\Delta k}{k}, \tag{36}$$

$$u = 1 - \frac{v_o}{v},$$

$$\Delta k = \bar{k} - k .$$

The relations (34) and (35) are identical to the relations (16) and (17) obtained for the case where we neglect the relative motion of the electrons. Only the value of D is different in the two cases.

Discussion of the Results: Let us assume from the first that $\rho = o$. We have then propagation of an amplified wave of amplification constant:

$$\bar{\gamma} = \frac{\gamma}{2} + \sqrt{\frac{f(kY_r)}{2Y_r Z_o sh(kd)}\ \frac{\sin\psi}{\cot\psi}\ \sqrt{\frac{\mu_o}{\varepsilon_o}}\ \frac{de^{-kd}}{h}\ \frac{\omega}{v_o} + \frac{\gamma^2}{4}}, \tag{37}$$

with

$$f(kY_r) = \left[ch(kY_r) - 1\right] \left\{ J_o\left(\frac{kY_r}{2}\right)\left[1 + \frac{3}{16}(kY_r)^2 + \frac{35}{24.128}\ (kY_r)^4 + \frac{231}{72.20^{10}}\ (kY_r)^6\right]\right.$$

$$\left. + J_2\left(\frac{kY_r}{2}\right)\left[\frac{(kY_r)^3}{16} + \frac{7}{24.32}\ (kY_r)^4 + \frac{495}{720.2^{11}}\ (kY_r)^6\right]\right. .$$

For variation in the ratio B/B_{cr} from 1 to 1.3, we find that $\bar{\gamma}$ is maximum for values of kd near 1.

Let us assume that γ = o (the delay line has no attenuation). There are three possible values for $\bar{\gamma}$, and to each of these corresponds a value of Δk:

$$
\begin{aligned}
\bar{\gamma} &= o, \text{ with } \Delta k = \frac{\omega}{v_o}\frac{u}{2} \pm \sqrt{\left(\frac{\omega}{v_o}\frac{u}{2}\right)^2 - D}\ , \\
\bar{\gamma} &= -\sqrt{D - \left(\frac{\omega}{v_o}\ \frac{u}{2}\right)^2}\ , \qquad \text{with } \Delta k = \frac{\omega}{v_o}\ \frac{u}{2}\ , \\
\bar{\gamma} &= +\sqrt{D - \left(\frac{\omega}{v_o}\ \frac{u}{2}\right)^2} \qquad \text{with } \Delta k = \frac{\omega}{v_o}\ \frac{u}{2}\ .
\end{aligned}
\tag{38}
$$

If we are in the region where we have amplification, we will necessarily have $D > \left(\frac{\omega u}{v_o 2}\right)^2$; consequently, the wave corresponding to $\bar{\gamma}$ = o cannot exist, for this would correspond to Δk imaginary, and this is contrary to the hypotheses of the calculation, and there is no possibility of propagation except for two waves:

the one strongly attenuated,

$$\bar{\gamma} = -\sqrt{D - \left(\frac{\omega u}{v_o 2}\right)^2}\ ,$$

the other strongly amplified,

$$\bar{\gamma} = +\sqrt{D - \left(\frac{\omega u}{v_o 2}\right)^2}\ ,$$

each one of these having a phase velocity $\bar{v}$ equal to the arithmetic mean $\frac{v_o + v}{2}$ between the velocity v_o of the electrons and the velocity v of the free wave.

This constitutes, therefore, one more essential difference between the operation of the magnetron travelling-wave tube and that of the travelling-wave tube where one always has three waves. The existence of only two waves coincides also with the fact that the gain of the magnetron travelling-wave tube is proportional to $I^{1/2}$ and not to $I^{1/3}$, as in the travelling-wave tube, and that the gain is reduced to the value $\frac{\gamma}{2}$ and no longer $\frac{\gamma}{3}$.

Study of the Initial Conditions: In the travelling-wave tube, the three waves fulfill, at the entrance, the following conditions:

Σ fields = field of the injected wave,

Σ HF currents = 0

Σ HF velocities = 0.

In the magnetron travelling-wave tube, the two waves will fulfill, at the entrance, the conditions:

Σ fields = field of the injected wave,

Σ HF currents = 0,

Σ HF velocities = 0.

The two first conditions give

$$\vec{E}_1 + \vec{E}_2 = \vec{E}_o \qquad \frac{\vec{E}_1}{\xi_1} + \frac{\vec{E}_2}{\xi_2} = 0, \tag{39}$$

so that

$$\vec{E}_1 = \frac{E_o}{1 - \frac{\xi_2}{\xi_1}} \quad .$$

For $\rho = o$, we obtain

$$\vec{E}_1 = \vec{E}_2 = \frac{\vec{E}_o}{2} ,$$

and the gain in decibels is written

$$G_{db} = 8.7\,\bar{\gamma}\,\ell - 6 \tag{40}$$

$\bar{\gamma}$ being given by (37) and ℓ being the length of the delay line.

Remarks: The condition Σ velocities at entrance = 0 is fulfilled automatically; in effect, in a magnetron travelling-wave tube, there is always, in addition to motion parallel to the cathode, a relative motion a function of ω_c, $2\omega_c$, etc., and the amplitude of this relative motion depends on the phase of the HF wave and is determined by the condition Σ velocities at entrance = 0.

Conclusions

In closing, we shall recapitulate the principal differences between the magnetron travelling-wave tube and the travelling-wave tube.

1. In the magnetron travelling-wave tube, the current is proportional to $1/\xi$, while it is proportional to $1/\xi^2$ in the travelling-wave tube. This involves a smaller gain for the magnetron travelling-wave tube than for the travelling-wave tube.

2. In the magnetron travelling-wave tube, the gain is maximum when the electrons have the same velocity as the forced wave, and, in these conditions, this velocity is likewise that of the free wave. In the travelling-wave tube, on the contrary, we have a maximum gain when the velocity of the electrons is equal to the velocity of the free wave.

3. There is no modulation of density in the magnetron travelling-wave tube, while this modulation exists in the travelling-wave tube and has important consequences (9).

4. The efficiency of the magnetron travelling-wave tube is large, while that of the travelling-wave tube is small.

5. In the magnetron travelling-wave tube, there exist two forced waves; in the travelling-wave tube there are three.

6. The existence of two waves instead of three involves:

a) that the gain of the magnetron travelling-wave tube is proportional to $I^{1/2}$ instead of $I^{1/3}$ for the travelling-wave tube;

b) that the attenuation of the delay line comes into play appreciably by the factor $\gamma/2$ (instead of $\gamma/3$ for the travelling-wave tube) to reduce the gain;

c) that the amplitude of the injected wave is divided into half for each of the forced waves (in the case of maximum gain), while this is divided by three for each of the three forced waves of the travelling-wave tube.

APPENDIX

In the following pages we shall calculate the resistance of coupling (see equations (13), (32) with the aid of an approximate method used by Rudenberg[8]. Fig. 4 shows a delay line such as is practically realizable for a magnetron travelling-wave tube: it acts simply as a helical plate wound on a cylinder of rectangular cross section; the wires of the surface of the helix turned toward the cathode make a constant angle ψ close to $\pi/2$ with the direction x. In order to simplify the theory we suppose that the HF fields in the cathode-anode region are due only to the HF currents circulating on the surface closest to the cathode, the influence of the upper surface and the lateral surfaces being considered as negligible. We can, therefore, to a first approximation, replace the helix by a plane located at $y = d$ indefinitely extended in the directions x and z and infinitely conducting in the direction making the angle ψ with x; moreover, it is necessary to impose on the fields the following conditions in the phase to take account of the geometrical structure of the helix:

p being the pitch of the helix,

s being the length of a turn of the helix, we should obtain

$$\vec{E}\,(x + s\cos\psi,\ z + s\sin\psi) = \vec{E}\,(x + p,\ z)$$
$$H\,(x + s\cos\psi,\ z + s\sin\psi) = \vec{H}\,(x + p,\ z)$$

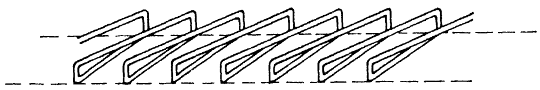

Fig. 4a Helical Plate

Finally, we have to find the HF fields produced by the surface of current described above which separates the space into two regions:

Region I is the interior region defined by $0 \leqslant y \leqslant d$

Region II is the exterior region defined by $y > d$

The boundary conditions, which it is necessary to take into account, are:

a) On the cathode (for $y = 0$)

$$E_x = 0\ ,\ E_z = 0, \qquad (1')$$

conditions which express that the electrical field is normal at the cathode.

b) At the anode ($y = d$)

$$E_x \cos\psi + E_z \sin\psi = 0, \qquad (2')$$

a condition which expresses that the electric field is to be normal to the current in the crosspiece surface.

$$F_{xI} = E_{xII}\ ,\ E_{zI} = E_{xII}\ ; \qquad (3')$$

conditions which express that the tangential electric field is to be continuous at the crosspiece of the surface of the current.

$$H_{zI}\ \sin\ + H_{xI}\ \cos\ = H_{zII}\ \sin\ + H_{xII}\ \cos\ , \qquad (4')$$

a condition which expresses that the current of the surface is parallel to the direction making an angle ψ with x.

$$\vec{E}\,(x + s\cos\psi\,,\ z + s\sin\psi\,) = \vec{E}\,(x + p, z), \tag{5'}$$

a condition already expressed above.

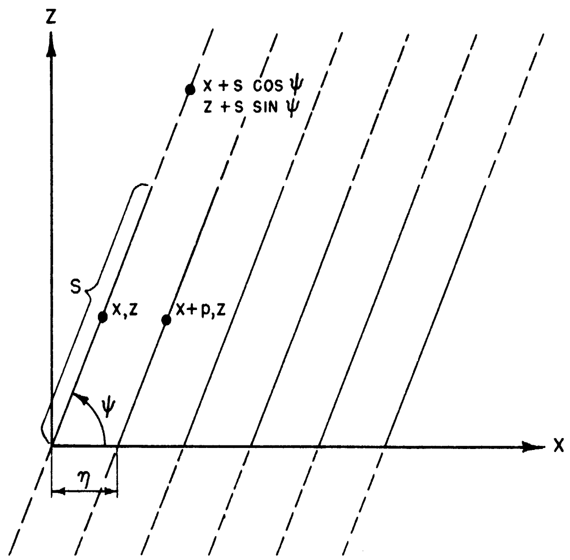

Fig. 4b Replacing the Helix by a Plane System

c) At infinity ($y = \infty$). The fields should be zero

$$\begin{aligned} \vec{E}\,(y = \infty) &= 0 \\ \vec{H}\,(y = \infty) &= 0\,. \end{aligned} \tag{6'}$$

It remains, therefore, to integrate the Maxwell equations:

$$\nabla \times \vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t}\,, \qquad \nabla \times \vec{H} = \varepsilon_o \frac{\partial \vec{E}}{\partial t}\,,$$

$$\nabla \cdot \vec{E} = 0, \qquad \nabla \cdot \vec{H} = 0\,,$$

in the case where the fields depend only on x and t by the factor

$$e^{i(\omega t - kx)}\,,$$

Letting $\mu_o\, \varepsilon_o\, \omega^2 = \dfrac{\omega^2}{c^2} = K^2$, we obtain in particular for E_x and H_x,

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + (K^2 - k^2)\, E_x = 0,$$

$$\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} + (K^2 - k^2)\, H_x = 0,$$

equations which show that the fields depend only on z by the factor $e^{-i\beta z}$. Finally, taking account of conditions (1'), the fields in the interior are written

$$\left.\begin{aligned}
E_{yI} &= \left[\frac{ik\alpha}{k^2 - K^2} B_1 - \frac{\mu_0 \omega \beta}{k^2 - K^2} A_1\right] \mathrm{ch}\, \alpha y \\
E_{xI} &= B_1 \,\mathrm{sh}\, \alpha y \\
E_{zI} &= \left[\frac{k\beta}{k^2 - K^2} B_1 + i \frac{\mu_0 \omega \alpha}{k^2 - K^2} A_1\right] \mathrm{sh}\, \alpha y \\
H_{yI} &= \left[\frac{ik\alpha}{k^2 - K^2} A_1 + \frac{\omega \varepsilon_0 \beta}{k^2 - K^2} B_1\right] \mathrm{sh}\, \alpha y \\
H_{xI} &= A_1 \,\mathrm{sh}\, \alpha y \\
H_{zI} &= \left[\frac{k\beta}{k^2 - K^2} A_1 - i \frac{\omega \varepsilon_0 \alpha}{k^2 - K^2} B_1\right] \mathrm{ch}\, \alpha y
\end{aligned}\right\} \epsilon^{i(\omega t - kx - \beta z)}.$$

Taking account of conditions (6'), the fields in the outside are written

$$\left.\begin{aligned}
E_{yII} &= \left[\frac{-ik\alpha}{k^2 - K^2} B_2 - \frac{\mu_0 \omega \beta}{k^2 - K^2} A_2\right] \\
E_{xII} &= B_2 \\
E_{zII} &= \left[\frac{k\beta}{k^2 - K^2} B_2 - i \frac{\mu_0 \omega \alpha}{k^2 - K^2} A_2\right] \\
H_{yII} &= \left[\frac{-ik\alpha}{k^2 - K^2} A_2 + \frac{\omega \varepsilon_0 \beta}{k^2 - K^2} B_2\right] \\
H_{xII} &= A_2 \\
H_{zII} &= \left[\frac{k\beta}{k^2 - K^2} A_2 + i \frac{\omega \varepsilon_0 \alpha}{k^2 - K^2} B_2\right]
\end{aligned}\right\} e^{-\alpha y}\, i(\omega t - kx - \beta z),$$

α, β, k being related by the equation

$$\beta^2 = \alpha^2 + K^2 - k^2. \tag{7'}$$

The conditions (3'), (5'), (1') and (4') give, respectively,

$$A_2 = - A_1 \; e^{\alpha d} \, sh\,(\alpha d) \quad \text{and} \quad B_2 = B_1 \; e^{\alpha d} \, sh\,(\alpha d), \tag{8'}$$

$$\beta = k\left(\frac{p}{s \sin\psi} - \cot\psi\right), \tag{9'}$$

$$\frac{i\mu_o \omega\alpha}{k^2 - K^2} \sin\psi \; A_1 + \left[\frac{k\beta}{k^2 - K^2} \sin\psi + \cos\psi\right] B_1 = 0, \tag{10'}$$

$$\left[\frac{k\beta}{k^2 - K^2} \sin\psi + \cos\psi\right] A_1 - i\,\frac{\omega\,\varepsilon_o\alpha}{k^2 - K^2} \sin\psi \; B_1 = 0. \tag{11'}$$

The fields are now known without ambiguity and in combining their expressions with relations (7') and (11'), we can express them as a function of a single constant which, in addition, we will be able to relate to the current density of the surface with the aid of Ampere's law.

We have therefore:

a) In the interior

$$\left.\begin{aligned}
E_{yI} &= \frac{1}{\sin\psi} \sqrt{\frac{\mu_o}{\varepsilon_o}} \; A_1 \;\; ch\,\alpha y \\
E_{xI} &= -\, i \sqrt{\frac{\mu_o}{\varepsilon_o}} \; A_1 \;\; sh\,\alpha y \\
E_{zI} &= i \sqrt{\frac{\mu_o}{\varepsilon_o}} \; \cot\psi \;\; A_1 \;\; sh\,\alpha y \\
H_{yI} &= \frac{iA_1}{\sin\psi} \;\; sh\,\alpha y \\
H_{yI} &= A_1 \;\; ch\,\alpha y \\
H_{zI} &= -\, \cot\psi \;\; A_1 \;\; ch\,\alpha y
\end{aligned}\right\} \; e^{i(\omega t - kx - \beta z)}. \tag{12'}$$

b) In the exterior

$$\left.\begin{aligned}
E_{yII} &= -\sqrt{\frac{\mu_o}{\varepsilon_o}}\,\frac{e^{\alpha d}\,\mathrm{sh}\,\alpha d}{\sin\psi}\,A_1 \\
E_{xII} &= -\,i\sqrt{\frac{\mu_o}{\varepsilon_o}}\,e^{\alpha d}\,\mathrm{sh}\,\alpha d\,A_1 \\
E_{zII} &= i\sqrt{\frac{\mu_o}{\varepsilon_o}}\,e^{\alpha d}\,\mathrm{sh}\,\alpha d\,\cot\quad A_1 \\
H_{yII} &= \frac{ie^{\alpha d}\,\mathrm{sh}\,\alpha d}{\sin\psi}\,A_1 \\
H_{xII} &= -\,e^{\alpha d}\,\mathrm{sh}\,\alpha d\,A_1 \\
H_{zII} &= e^{\alpha d}\,\mathrm{sh}\,\alpha d\,\cot\psi\quad A_1
\end{aligned}\right\}\; e^{-\alpha y}\,e^{i(\omega t - kx - \beta z)} \qquad (13')$$

with

$$\begin{aligned}
\alpha &= \left(\frac{s}{p\,\sin\psi} - \cot\psi\right) K \\
\beta &= \left(\frac{1}{\sin\psi} - \frac{s}{p}\cot\psi\right) K \\
k &= \frac{s}{p}\,K\,.
\end{aligned} \qquad (14')$$

<u>Remarks</u>: We see that for ψ near to $\frac{\pi}{2}$, the expression for the field on the interior are approximately the same as those given by relations (4), which would be obtained beginning from a simple theory which neglects the influence of the vector potentials.

Calculating the flux of the Poynting vector

$$P = 1/2\left(\vec{E} \times \vec{H}^{*}\right)$$

across the right-hand cross section, we obtain the power transported, W, for a helix length, h,

$$W = A_1^2\sqrt{\frac{\mu_o}{\varepsilon_o}}\,\frac{\cot\psi}{\sin\psi}\,\frac{h}{4\alpha}\left[\mathrm{sh}2\alpha d + \mathrm{ch}2\alpha d - 1\right] ,$$

and the resistance of coupling R_x can be deduced from the relation

$$W = \frac{E_x E_x^*}{2R_x} ,$$

which allows us to write

$$R_x = \frac{\alpha}{h} \sqrt{\frac{\mu_o}{\varepsilon_o}} \frac{\sin\psi}{\cot\psi} \frac{e^{-\alpha d}}{\mathrm{sh}\,\alpha d} \mathrm{sh}^2\, \alpha y . \tag{15'}$$

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PART IV

THE TRAVELLING-WAVE TUBE WITH A MAGNETIC FIELD

Extension of the Linear Theory, The Effects of Nonlinearities and the Efficiency

by O. Doehler, J. Brossart, and G. Mourier

Annales de Radioélectricité
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Summary

The authors study again the linear theory of travelling-wave tubes with constant magnetic field without taking into consideration one of the assumptions made in the preceding issue.

Two additional waves are found in that case, which are neither amplified nor attenuated. By computing nonlinear effects, they point out that the essential fact is the absorption by the anode. The efficiency is then evaluated. When the electron beam is sent with a velocity small with respect to the anode voltage, its value is larger, and a simple expression can be found for it. (U.D.C. 621.385.1.029.)

Introduction

In a previous publication, Warnecke and Guénard have pointed out, along general lines, the properties of a new type of amplifier for UHF, the travelling-wave tube in a magnetic field. The first experimental results obtained on these tubes have been mentioned in reference 2.

One of the forms arising from this, the plane magnetron travelling-wave tube, is represented very schematically in Fig. 1, in a plane perpendicular to the direction of the magnetic field, B. An electron gun produces a beam of electrons, F, which is caused to be displaced between a delay line for electromagnetic propagation, Ln, and a plate, P_2. Ln is maintained at the high voltage of the tube, the plate, P_2, being at a potential near to that of the cathode. The space in which the beam is displaced is also the seat of an electric field and of a crossed magnetic field, uniform and constant; they form with the direction of propagation a "trihedral trirectangle" (three mutually perpendicular vectors).

We analyze generally the motion of the electrons in this case in two parts:

a) A uniform rectilinear drift motion of which the velocity is perpendicular to the electric field, E, and magnetic field, B; the velocity, v_o, is given by the ratio E/B. It is characterized by equilibrium between the force exerted by the electric field and the force due to the magnetic field. The equilibrium velocity v_o is independent of the potential of the plane in which the electron is moved.

b) With respect to a system of reference endowed with a drift motion, a relative motion of free oscillation in two dimensions whose pulsation, ω_c, is defined uniquely by the magnetic field and which can be avoided by proper initial conditions for the electrons. We will call "optically ideal" an electron gun which produces a rectilinear and laminar beam, that is to say, where the relative motion is eliminated, and we will limit ourselves to this case.

In Fig. 1, the beam is extracted from the cathode, C, by a positive plate, P_1, held at a potential lower than that of Ln. The electrons move with a velocity very near to that of a wave guided by the delay line, in such a manner that they remain practically in the same phase during all of their transit in the tube; the HF field produces a progressive bunch in the midst of the beam and borrows from the energy of the electrons. These latter should be retarded longitudinally as in the travelling-wave tube if the presence of the magnetic field were not converting this longitudinal slowing-up into a transverse motion which grows continually toward the line where the potential is higher, allowing all to have practically the same velocity. Thus, the electrons take the energy from the continuous electric field, which they do not keep but which they give immediately to the HF field.

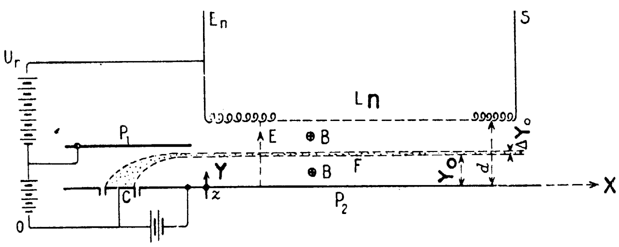

Fig. 1 Essential Elements of a Plane Magnetron Travelling-Wave Tube

Ln, "slow-wave" line (anode); En, input of HF circuit; S, output of HF circuit E, continued electric field; B, continuous magnetic field; F; static trajectory of electronic beam.

One deduces immediately an estimate of the maximum theoretical efficiency of the tube if the beam is very small. If it enters into the condenser made up of the plate P_2 and the line Ln with a potential U_o and if Ln is at potential U_p with respect to the cathode, the energy given to the electrons in the space of the discharge is $e(U_p - U_o)$. Let us assume that all the electrons reach the line with the initial velocity v_o (if the signal is very small, the velocity imported to the HF field is negligible); the field, therefore, receives all the energy $e(U_p - U_o)$, and the electronic efficiency is

$$\eta_{el} = \frac{e(U_p - U_o)}{eU_p} = 1 - \frac{U_o}{U_p} .$$

In the plane condenser, the potential grows linearly with the distance to the electrodes, and one has accordingly

$$\eta_{el} = 1 - \frac{y_o}{d} ,$$

y_o being the initial distance of the beam to the plate P_2, and d the distance between Ln and P_2, provided that P_2 is at potential of the cathode. The efficiency is therefore bound in a very simple fashion to the geometrical characteristics of the tube.

Contrary to the situation prevailing in the tube with linear progressive waves, one can thus transform into kinetic energy almost all of the energy derived from the source of voltage, provided that the beam passes sufficiently close to the plate P_2 in the static condition.

In addition, experiment has shown that one could easily cause sufficiently large currents to pass in the space of the discharge. One has therefore in the magnetron travelling-wave tube an amplifier with large power output and high efficiency.

The nature of the known quantities concerning the operation are such that if one bends a plane tube to resemble the one we have described, giving it a diameter of the order of one-half wavelength, the centrifugal force to which each of the electrons is subjected is very small compared to the two other forces which would be exerted on it primarily in the static operation. One can therefore construct from plane tubes circular magnetron travelling-wave tubes which will show only minimum differences.

A first theoretical study on the magnetron travelling-wave tube has already appeared in this journal (3). We plan to complete this now:

a) Up to now we have limited ourselves to the case where

$$\left|\frac{eB}{m}\right| = \omega_c \gg \left| i\omega + \Gamma v_o \right| , \qquad (1)$$

e and m being, respectively, the charge and mass of the electron, B the magnetic field, ω the angular frequency of the amplified signal, v_0 the electron velocity, and $\Gamma = \bar{\gamma} - i\bar{k}$ the propagation constant of the wave in the direction of the beam. This relation indicates that the angular velocity of relative motion of the electrons is large compared to the modulus of the angular frequency of the wave seen by an electron according to a linear theory; let us remember that this apparent angular frequency is a complex quantity if the applied signal varies exponentially along the electron trajectory.

This hypothesis is certainly always very well verified. In fact, ω_c is of the same order as ω, or even larger; $\bar{\gamma}$ is small compared with $\bar{k}$ (if one has $\bar{\gamma} = \bar{k}$, the gain would be 55 decibels per wavelength in the line, much smaller than the wavelength in vacuo!), and $j\omega + \Gamma v_0$ is little different from $\bar{\omega} - \bar{k}v_0$; $\omega - \bar{k}v_0$ is zero if the electrons are in absolute synchronism with the wave ($v_0 = \omega/\bar{k}$, velocity of the wave); even if the synchronism is not absolute, this quantity is smaller than ω and therefore smaller than ω_c by at least a factor of ten.

Nevertheless, we have been led to abandon this hypothesis for the following purely theoretical reason: in reference (3), we found two waves; that is, there are four initial conditions: one for each of the two components of HF velocity, and one for the current and one for the field:

$$\Sigma \text{ fields} = \text{injected field} \tag{2}$$

$$\Sigma \text{ currents} = \text{injected current;} \tag{3}$$

as in (3); but, on the other hand,

$$\delta \dot{y}_0 = 0 \quad , \quad \delta \dot{x}_0 = 0 \ .$$

We will find here two supplementary waves and we will have four unknown quantities to determine in the four equations. This will permit us to explain a remark in reference (3) relative to this matter. We will see that the supplementary waves involve a motion with an angular velocity very near to

ω_c, which characterizes the relative motion of the electrons in the purely static case. The small difference from the static case is due to the mechanism by which these waves are coupled by the fields of the amplified end attenuated waves. We can compare these last to the forced oscillations of an oscillating system and the two supplementary waves to the free oscillations.

b) In (3), we wrote (see Notations) that the power given by the beam along the element dx is

$$- dP = \frac{E_x(y_0) i_x^*}{2} dx.$$

The field E_x varies with y, and the electrons do not keep the coordinate y_0 but withdraw by δy; one must therefore adopt $E_x(y_0 + \delta y)$ instead of $E_x(y_0)$. This effect, that is studied for the travelling-wave tube in (4) and (5) leads to an electronic gain in decibels $\sqrt{2}$ times larger.

c) In the Appendix of (3), we have given a calculation of the coefficient of coupling of the beam and the wave, defined by

$$R_x = \frac{E_x(y_0) E_x^*(y_0)}{2P} .$$

We will give here a calculation corresponding to the conditions nearer to those which are effectively realized, in using a method given by Pierce[6].

d) But the most important part of this article treats the nonlinear effects in the tube and leads to an evaluation of the efficiency which shows that the method given in (2) is justified. It is remembered that, in the first theoretical study (3) we limited ourselves to small signals, assuming that the HF motions of the electrons would be sufficiently small so that they remain almost in the same phase of the field. This is only for comparison with this hypothesis which leads us to the linear equation. As has been done for the linear travelling-wave tube[7], we shall use here the method of successive approximations and we shall limit ourselves to the third. Actual practice differs very much from this approximation; however, we can imagine that some other effects, of which these initial hypotheses do not take account, influence appreciably the power and the efficiency (for example, the absorption of electrons by the electrodes); the gain calculated up to the third approximation must be sufficient to isolate a certain number of essential parameters and their influence.

This method has the advantage of being more general and valid for all types of magnetron travelling-wave tubes than that of Nordsieck, which is more precise.

Hypotheses

The initial hypotheses are the following:

1. We restrict ourselves to the plane magnetron, the case which is approached sufficiently by the practical realizations.

2. We neglect the effects of space charge, continuous as well as alternating.

3. We neglect the boundary effects, that is to say, we treat the magnetron travelling-wave tube having two dimensions.

4. We have an ideal optical system: the static trajectories are rectilinear.

5. The beam is thin, that is, $|\Gamma \Delta y_0| \ll 1$ (Δy_0, being the thinness of the beam.)

6. The propagation constant $\Gamma = \bar{\gamma} - i\bar{k}$ is such that $\bar{\gamma}$ is clearly smaller than $\bar{k}$.

7. For the calculations of large signals, we will assume

$$\left|\frac{eB}{m}\right| = \omega_c \gg |i\omega + \Gamma v_0| \ .$$

8. The velocity of the electrons, v_0, is small compared with the velocity of light.

9. The resistance of coupling is zero for the harmonics of the fundamental wave, 2ω, 3ω, etc.

10. We do not take account of the perturbations applied to the HF fields by the absorption of the beam along the line.

11. We assume that the HF line possesses a continuous structure; actually it may be either a helix or a line with waves; in this case, the free wave allows a large number of components tightly bound together; our hypothesis considers only the largest and most rapid among these, although the others undoubtedly are effective if the behavior of the tube is nonlinear. This approximation is less valid when the "opening" of the circuit is larger (case of large accelerating voltage).

Notations

v_o - velocity of electrons

v - velocity of free waves

$\bar{v}$ - velocity of forced wave

$\rho = 1 - \frac{v_o}{v}$

$\Gamma_o = \gamma - ik$ - propagation constant of free wave

$\Gamma = \bar{\gamma} - i\bar{k}$ - propagation constant of forced wave

B = magnetic field

U_p - voltage between anode and the interior cylinder

U_c - voltage between anode and cathode

$\omega_c = \frac{eB}{m}$

ω - high frequency angular frequency

η - efficiency

R_x - resistance of coupling Ω/cm^2

E_x - electric field in the direction of the beam

E_y - electric field in the direction of the continuous electric field

ρ_o - density of the continuous space charge

ρ_1, ρ_2 - density of the alternating space charge in first and second approximation

$\left.\begin{matrix} v_{x1}, & v_{x2} \\ v_{y1}, & v_{y2} \end{matrix}\right\}$ ..., alternating electron velocities in first and second approximation

d - distance between anode and interior cylinder

$\phi = \omega t_o + \omega \rho \tau$

$\Phi = \omega t - \bar{k}x$

$\tau = t - t_o$ = transit time

$$a = \frac{\rho}{\rho^2 + \frac{\bar{\gamma}^2}{k_o^2}} , \qquad b = \frac{\bar{\gamma}/k_o}{\rho^2 + \bar{\gamma}^2/k_o^2} ,$$

$$c = \frac{\rho}{\rho^2 + 9\bar{\gamma}^2/k_o^2} \qquad g = \frac{3\bar{\gamma}/k_o}{\rho^2 + 9\bar{\gamma}^2/k_o^2} .$$

4. Linear Theory

(a) Determination of the Electron Trajectories. The system is represented in Fig. 1 with the three axes of reference. A constant electric field U_p/d is directed along Oy and a constant magnetic field along + z.

The electrons enter into the system parallel to the + x direction. In the absence of an HF field their trajectories are therefore defined by

$$y = y_o , \quad x = v_o \tau , \tag{4}$$

with $\tau = t - t_o$, t_o being the instant of passage of the electron in the plane x = o, and t the instant when this electron is found at x. The velocity v_o is given by

$$v_o = \frac{U_p}{dB} . \tag{5}$$

A HF wave is propagated in the x direction with a propagation constant $\Gamma = \bar{\gamma} - i\bar{k}$, with $\bar{\gamma} \ll k$. If the velocity of this wave is small compared with the velocity of light we can derive the fields from a scalar potential and we have

$$E_y = -\bar{k}\,\Delta U_p \frac{\cosh(ky)}{\sinh(kd)} e^{i\omega t + \Gamma x}, \qquad (6)$$

$$E_x = +i\bar{k}\,\Delta U_p \frac{\sinh(ky)}{\sinh(kd)} e^{i\omega t + \Gamma x}.$$

Let us consider the trajectories in the presence of the HF wave as the perturbed static trajectories

$$\begin{aligned} y &= y_o + \delta_1 y + \delta_2 y + \ldots, \\ x &= v_o \tau + \delta_1 x + \delta_2 x + \ldots, \end{aligned} \qquad (7)$$

and we obtain for the first approximation in δx and δy the equations:

$$\begin{aligned} \delta\ddot{y} + \omega_c \delta\dot{x} &= \eta k\,\Delta U_p \frac{\cosh(ky_o)}{\sinh(kd)} e^{i\omega t_o + i\xi\tau}, \\ \delta\ddot{x} - \omega_c \delta\dot{y} &= -i\eta k\,\Delta U_p \frac{\sinh(ky_o)}{\sinh(kd)} e^{i\omega t_o + i\xi\tau}, \end{aligned} \qquad (8)$$

with

$$\xi = \omega\rho - i v_o \bar{\gamma}. \qquad (9)$$

There follows from this that:

$$\delta y = \frac{i\eta\Gamma\Delta U_p}{\omega_c \xi} \frac{\sinh(i\Gamma y_o)}{\sinh(i\Gamma d)} \times \frac{1 + \frac{\xi}{\omega_c}\coth(i\Gamma y_o)}{1 - \left(\frac{\xi}{\omega_c}\right)^2} e^{i\omega t_o + i\xi\tau},$$

$$(10)$$

$$\delta x = \eta \frac{\Gamma\Delta U_p}{\xi\omega_c} \frac{\cosh(i\Gamma y_o)}{\sinh(i\Gamma d)} \frac{1 + \frac{\xi}{\omega_c}\tanh(i\Gamma y_o)}{1 - \left(\frac{\xi}{\omega_c}\right)^2} e^{i\omega t_o + i\xi\tau},$$

and

$$\delta v_y = i\xi\delta y, \quad \delta v_x = i\xi\delta x. \qquad (11)$$

(b) Calculation of the Current. The unmodulated beam is assumed to occupy the interval y_0, $y_0 + \Delta y_0$ and to carry constant current I_0. The alternating current is then given by

$$\tilde{i} \quad = \quad \rho(v_0 + \delta\dot{x})\left(1 + \frac{\partial}{\partial y}\,\delta y\right)\Delta y_0 - I_0. \tag{12}$$

The term in $\frac{\partial}{\partial y}$ (δy) represents the variation of the cross section of the beam due to the action of the HF field.

The density ρ_1 is given by the equation of conservation of charge

$$\nabla \cdot \rho v \quad = \quad - \frac{\partial \rho}{\partial t}\,, \tag{13}$$

which gives as first approximation

$$v_0 \frac{\partial \rho 1}{\partial x} + \frac{\partial \rho 1}{\partial t} \quad = \quad - \rho_0\left(\frac{\partial}{\partial y}\delta v_y + \frac{\partial}{\partial x}\delta v_x\right)\,. \tag{14}$$

The introduction of equations (11) into (14) gives

$$\rho 1 \quad = \quad 0, \tag{15}$$

that is to say, the alternating charge density is zero in first approximation and we have, for the current,

$$i_x \quad = \quad i I \Gamma d \frac{\Delta U_p}{U_p} \frac{\omega}{\xi} \frac{\cosh (i \Gamma y_0)}{\sinh (i \Gamma d)} \frac{1 + \frac{\xi}{\omega_c} \tanh (i \Gamma y_0)}{1 - \left(\frac{\xi}{\omega_c}\right)^2} e^{i\omega t_c} e^{i\xi\tau}\,. \tag{16}$$

(c) Energy Balance. In reference (3), we determined the power given up by the beam along the element dx by

$$- dP \quad = \quad \frac{E_x(y_0)\, i_x^*(y_0)}{2}\, dx. \tag{17}$$

In reality, in writing - dP in this form, we have ignored the fact that the beam is displaced along y in a field E_x as a function of y, and it is necessary to write

$$- dP = \frac{dx}{2}\int_{y_o}^{y_o+\Delta y_o} E_x\left[y_o + \delta y(y_o)\right]\left[1 + \frac{\delta v_x(y_o)}{v_o}\right]\left[1 + \frac{\partial}{\partial y}\,\delta y(y_o)\right]\rho_o v_o\, dy_o . \quad (18)$$

We obtain therefore, in studying the balance of energy as given in (1),

$$\Gamma - \Gamma_o = -\frac{idR_x}{Z_o}\,\frac{\coth(i\Gamma y_o)}{1-(\xi/\omega_c)^2}\times\frac{\omega_c}{\xi}\left[1 + \frac{\xi}{\omega_c}\,\frac{\tanh(i\Gamma y_o) + \coth(i\Gamma y_o)}{2}\right]$$

$$-\left[\frac{1}{2} + \frac{\xi}{\omega_c}\,\frac{\coth(i\Gamma y_o)}{2}\right], \quad (19)$$

where $\Gamma_o = \gamma - ik$, i.e., propagation constant of the free wave. In replacing ξ by its value (9), we obtain an equation of fourth degree in Γ, which involves the existence of four waves.

We determine the solution of (19) by approximation in assuming that we have a small current to deal with, i.e., $Z_o = \frac{U_p}{I}$ is large. We have therefore two groups of solutions:

1. $|\xi|$ small; we neglect $\left(\frac{\xi}{\omega_c}\right)^2$ compared with unity, and (19) is written

$$(\bar{\gamma}-\gamma)\left(\bar{\gamma}^2 + \frac{\omega^2}{v_o^2}\,\rho^2\right) = \frac{R_x \coth(ky_o)}{Z_o}\,\frac{\omega}{v_o}\,\bar{\gamma}, \quad (20)$$

$$(\bar{k}-k)\left(\bar{\gamma}^2 + \frac{\omega^2}{v_o^2}\,\rho^2\right) = \frac{R_x \coth(ky_o)}{Z_o}\,\frac{\omega^2}{v_o^2}\,\rho . \quad (21)$$

Equations (20) and (21) are identical with a factor of nearly two to relations (16) and (17) of reference (3). The coefficient of amplification $\bar{\gamma}$ is therefore $\sqrt{2}$ times the value given in reference (3).

2. $\left|1 - \frac{\xi^2}{\omega_c^2}\right|$ is small. We will obtain Γ by writing first $\left|1 - \frac{\xi^2}{\omega_c^2}\right| = 0$, and in carrying the value, we find

$$\Gamma = -i\omega\,\frac{1 \pm \omega_c/\omega}{v_o} \quad (22)$$

in the second member, which finally gives:

$$\Gamma_{3,4} = -i\omega\frac{1 \pm \omega_c/\omega}{v_o} + i\frac{d R_x \coth (ky_o)}{2 Z_o}\left\{\mp\frac{\omega}{\omega_c}\left[1 \mp \frac{\tanh (ky_o) + \coth (ky_o)}{2}\right]\right.$$

$$\left.-\left[\frac{1}{2} \mp \frac{\coth (ky_o)}{2}\right]\right\}. \qquad (23)$$

We see therefore that the two supplementary waves are neither attenuated nor amplified but that they have a large dispersion of velocity. These resemble two waves found in the linear travelling-wave tube when we take account of the radial field.[5]

5. Study of Large Signals

In order to define the efficiency of an electronic tube, we can assume different points of view. In the arrangement where the HF energy of the current is collected by a single circuit (klystrons, amplifiers with grids), this energy is written $\frac{Ui^*}{2}$, U being the HF voltage and i the fundamental of the electronic current. U_oI_o being the applied power, the efficiency, in the case where $U = U_o$ and where i is in phase with U, is written

$$\eta = \frac{|i|}{2I_o}. \qquad (24)$$

The maximum of i is therefore $2I_o$, which would correspond to the bunches of electrons infinitely small and infinitely dense; and in this case the efficiency is equal to 100%. Now, in the travelling-wave tube, we have seen that i is of the order 1.2 I_o; if we therefore take (24) to define the efficiency, this can attain 60 per cent. But we know that in the travelling-wave tube, the electrons lose only that amount of kinetic energy which corresponds to the difference between this velocity and that of the forced wave; as the result, finally, the HF power given to the forced wave is very small compared with the applied power.

We will consider, on the other hand, a current which is of the form represented in Fig. 2.

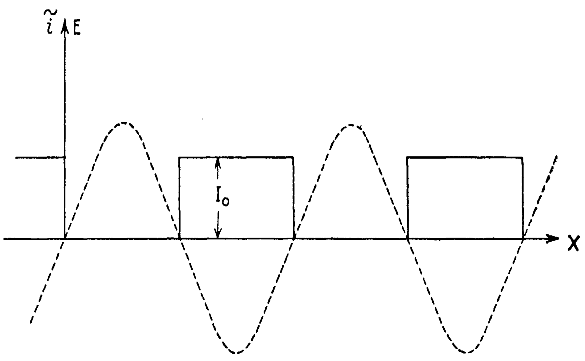

Fig. 2

The amplitude of the fundamental term is $\frac{4}{\pi} I_0$. However, this electron current is moving constantly in a retarding sinusoidal HF field (shown in dotted lines in Fig. 2). The field and the beam having always the same velocity, the electrons will finish by giving up all of their kinetic energy, and an efficiency of 100 per cent can be attained. The following, for example, takes place in the magnetron travelling-wave tube for the electrons of favorable phase: These electrons will form bunches, each located in a retarding field and displaced with the same velocity as that of the field; they will, therefore, have a tendency to be slowed down, which will bring them closer to the anode. In this process, they will, on the one hand, have given up high-frequency energy and, on the other, have taken energy from the continuous field in such a way as to overtake their retarded velocity; this mechanism will be repeated until such a time that they will be absorbed by the anode. Those which will be in an intense retarding field will arrive quickly at the anode and will have a transit time shorter than those which will be in a weaker retarding field. As a consequence, in order to determine the efficiency of the magnetron travelling-wave tube, it is necessary to determine the percentage of electrons captured by the anode as well as their velocity of arrival at the anode; this allows, then, the following course:

1. Determine the perturbed trajectories of the electrons. This will be done up to the third approximation.

2. Determine the velocity of the electrons as a function of y.

3. Calculate the density with the aid of equation (13).

4. Determine the influence of the nonlinearity on the gain.

5. Determine the velocities of the electrons which are absorbed by the anode and take account of the nonlinearities on the gain.

6. Determine the percentage of electrons captured by the anode.

7. Finally, calculate the amplitude of the alternating current. As we have said, this quantity does not play an essential role in the determination of the efficiency; it gives only an estimate of the amplitudes.

6. The Perturbed Trajectories

We have the equations of motion

$$\delta\ddot{y} + \omega_c \delta\dot{x} = -i\eta\Gamma\Delta U_p \frac{\cosh(i\Gamma y)}{\sinh(i\Gamma d)} e^{i\omega t_o} e^{i\omega\tau+\Gamma x}, \tag{25}$$

$$\delta\ddot{x} - \omega_c \delta\dot{y} = -\Gamma\eta\Delta U_p \frac{\sinh(i\Gamma y)}{\sinh(i\Gamma d)} e^{i\omega t_o} e^{i\omega\tau+\Gamma x}. \tag{26}$$

Except in the exponentials $e^{\Gamma x}$ we shall replace everywhere in (25) and (26) Γ by $\overline{ik}$.

t_o = time of passage of the electrons into the plane x = o;

τ = transit time.

Equations (25) and (26) are solved only by successive approximations. We develop the solution in growing powers of ΔU_p and we put, according to (3),

$$y = y_o + \delta_1 y + \delta_2 y + \delta_3 y, \tag{27}$$

$$x = x_o + v_o\tau + \delta_1 x + \delta_2 x + \delta_3 x, \tag{28}$$

which we introduce into the terms on the right.

Equations (25) and (26) are, in this way, made linear and easily integrated, a condition on the basis of which one calculates real quantities, the complex notation being valid only in the linear equations.

In this calculation, we are concerned only with the amplified wave, and we neglect all of the terms which are small compared to the terms in $\frac{1}{\xi}$. We obtain:

$$\delta_1 y = d\frac{\Delta U_p}{U_p}\frac{\sinh(ky_o)}{\sinh(kd)} \epsilon^{v_o\bar{\gamma}\tau}(a\cos\phi - b\sin\phi), \tag{29}$$

$$\delta_2 y = d\frac{kd}{4\sinh^2(kd)}\left(\frac{\Delta U_p}{U_p}\right)^2 \frac{\omega b}{v_o\bar{\gamma}}\sinh(2ky_o)e^{2v_o\bar{\gamma}z}, \tag{30}$$

$$\delta_3 y = -i\omega d\left(\frac{\Delta U_p}{U_p}\right)^3 \frac{(kd)^2}{4\sinh^3(kd)} \times \left\{\frac{\sinh(ky_o)}{2}\left[a^2 + 3b^2 - \frac{\omega}{v_o\bar{\gamma}}(b+ia) - 2iab\right]\right.$$

$$\left. + \frac{\sinh(3ky_o)}{2}\left[a^2 - b^2 + i2ab - \frac{\omega}{v_o\bar{\gamma}}(b-ia)\right]\right\} \times \frac{e^{i\omega t}e^{(3v_o\bar{\gamma}+i\omega p)\tau}}{3v_o\bar{\gamma} + i\,p\omega} \tag{31}$$

and

$$\delta_1 x = d\frac{\Delta U_p}{U_p}\frac{\cosh(ky_o)}{\sinh(kd)} e^{v_o\bar{\gamma}\tau}(a\sin\phi + b\cos\phi), \tag{32}$$

$$\delta_2 x = d\frac{kd}{2\sinh^2(kd)}\left(\frac{\Delta U_p}{U_p}\right)^2 \frac{\omega a}{2v_o\bar{\gamma}}\cosh(2ky)e^{2v_o\bar{\gamma}\tau} \tag{33}$$

$$-d\frac{kd}{4\sinh^2(kd)}\left(\frac{\Delta U_p}{U_p}\right)^2 e^{2v_o\bar{\gamma}\tau}\left[(a^2-b^2)\sin 2\phi + 2ab\cos 2\phi\right],$$

$$\delta_3 x = -d\left(\frac{\Delta U_p}{U_p}\right)^3 \frac{(kd)^2}{4\sinh^3(kd)} \frac{e^{3v_o\bar{\gamma}\tau} e^{i\omega t_o + i\omega\rho\tau}}{3v_o\bar{\gamma} + i\omega\rho}$$

$$\times\left\{\frac{\cosh(ky)}{2}\left[3a^2+b^2+i2ab+\frac{\omega}{v_o\bar{\gamma}}(b+ia)\right]\right. \tag{34}$$

$$\left. - \cosh(3ky)\left[\frac{\omega}{v_o\bar{\gamma}}(b-ia) + (a-ib)^2\right]\right\},$$

where

$$a = \frac{\rho}{\rho^2 + \frac{v_o^2\bar{\gamma}^2}{\omega^2}}, \quad b = \frac{\frac{v_o\bar{\gamma}}{\omega}}{\rho^2 + \frac{v_o^2\bar{\gamma}^2}{\omega^2}},$$

$$c = \frac{\rho}{\rho^2 + 9\frac{v_o^2\bar{\gamma}^2}{\omega^2}}, \quad g = \frac{3\frac{v_o\bar{\gamma}}{\omega}}{\rho^2 + 9\frac{v_o^2\bar{\gamma}^2}{\omega^2}}.$$

In equations (29) and (32), we have neglected the terms in $e^{i3\omega t}$. Equations (29) and (34) determine δy and δx as functions of the transit time τ and of y, which is the ordinate of the entrance of the current into the space of the discharge.

As a result of the nonlinearity of equations (25) and (26), $\delta_3 y$ and $\delta_3 x$ are the terms of frequency ω; but they contain the exponential factor $e^{3v_o\bar{\gamma}\tau}$, which entails also the existence of a term in $(\Delta U_p)^3\, e^{3v_o\bar{\gamma}\tau}$ for the current. As a result of the coupling, we should therefore, necessarily have in addition to the field given by (6), a field in $(\Delta U_p)^3\, e^{3\bar{\gamma}x}$, given by

$$E_y = -kA\,\frac{(\Delta U_p)^3}{U_p^2}\,\frac{\cosh(kx)}{\sinh^3(kd)}\, e^{3\bar{\gamma}x}\, e^{i(\omega t - \bar{k}_x)}, \tag{35}$$

$$E_x = ikA\,\frac{(\Delta U_p)^3}{U_p^2}\,\frac{\sinh(kx)}{\sinh^3(kd)}\, e^{3\bar{\gamma}x} e^{i(\omega t - \bar{k}x)}; \tag{36}$$

A is a complex amplitude factor which will be determined in section 8.

The fields (35) and (36) show the perturbations $\delta_3'x$ and $\delta_3'y$ given by

$$\delta_3'y \quad = \quad id(g-ic)\left(\frac{\Delta U_p}{U_p}\right)^3 \frac{\sinh\,(ky_o)}{\sinh^3\,(kd)}\; A\; e^{3\bar{\gamma}\,v_o\tau\,+i\phi} \quad , \tag{37}$$

$$\delta_3'x \quad = \quad d(g-ic)\left(\frac{\Delta U_p}{U_p}\right)^3 \frac{\cosh\,(ky_o)}{\sinh^3\,(kd)}\; A\; e^{3\,v_o\gamma\tau+i\phi} \quad , \tag{38}$$

which are to be added to the perturbations given by (31) and (34). We must make a remark here relative to the subsequent calculations of the charge densities: in order to determine the alternating densities to second and third approximations, we make use of equation (31), which expresses the conservation of charge:

$$\nabla\cdot\rho v \quad = \quad -\frac{\partial\rho}{\partial t}\ .$$

We make this equation linear by developing ρ and v in growing powers of ΔU_p:

$$\rho \quad = \quad \rho_o \;+\; \rho_1 \;+\; \rho_2 \;\ldots.,$$

$$v \quad = \quad v_o \;+\; v_1 \;+\; v_2 + \;\ldots..$$

We know, moreover, that ρ_1 is zero. But we must necessarily be careful to express v as function of x, y, t; we knew it up till now only as function of y_o, t_o, $\tau = t - t_o$.

Letting $\Phi = \omega t - \bar{k}x$, we obtain, according to (27), (29), and (30):

$$y_o \quad = \quad y - d\,\frac{\Delta U_p}{U_p}\;\frac{\sinh\,(ky)}{\sinh\,(kd)}\;e^{\bar{\gamma}x}(a\,\cos\Phi\,-\,b\,\sin\Phi) \tag{39}$$

$$+\;d\left(\frac{\Delta U_p}{U_p}\right)^2 \frac{kd}{4\,\sinh^2\,(kd)}\;\sinh\,(2ky)\quad e^{2\bar{\gamma}\,x}\left(2\,a^2+2\,b^2\,-\,\frac{\omega\,b}{v_o\bar{\gamma}}\right)\ ,$$

and, according to (28), (32), and (33):

$$v_o\tau = x - d\,\frac{\Delta U_p}{U_p}\,\frac{\cosh(ky)}{\sinh(kd)}\,e^{\bar{\gamma}x}(b\cos\Phi + a\sin\Phi) \tag{40}$$

$$- d\left(\frac{\Delta U_p}{U_p}\right)^2 \frac{kd}{4\sinh^2(kd)}\quad e^{2\bar{\gamma}x}\Big(2\,ab\cos 2\Phi + (a^2-b^2)\sin 2\Phi$$

$$+ \frac{a\,\omega}{v_o\bar{\gamma}}\cosh(2ky)\Big)\;.$$

7. The Alternating Velocities and the Alternating Densities

According to equations (29) and (34), we obtain for the alternating velocities as function of y, t_o, and τ :

$$\delta v_y = i v_o \frac{kd}{\sinh(kd)}\,\frac{\Delta U_p}{U_p}\,\sinh(ky_o)\,e^{\bar{\gamma}v_o\tau}e^{i\phi} \tag{41}$$

$$+ v_o\frac{(kd)^2}{2\sinh^2(kd)}\left(\frac{\Delta U_p}{U_p}\right)^2 \sinh(2ky)\,b\,e^{2v_o\bar{\gamma}\tau}$$

$$- \frac{(kd)^3}{8\sinh^3(kd)}\left(\frac{\Delta U_p}{U_p}\right)^3 e^{3\bar{\gamma}v_o\tau}e^{i\phi}\,\sinh(ky_o)\left[i(a-ib)^2 + 4ib^2 + \frac{\omega}{v_o\bar{\gamma}}(a-ib)\right]$$

$$+ \sinh(3ky_o)\left[i(a+ib)^2 - \frac{\omega}{v_o\bar{\gamma}}(a+ib)\right] + i v_o\frac{kd}{\sinh^3(kd)}\left(\frac{\Delta U_p}{U_p}\right)^3 A\,\sinh(ky_o)\,e^{3\bar{\gamma}v_o\tau}e^{i\phi}.$$

and

$$\delta v_x = v_o\frac{kd}{\sinh(kd)}\,\frac{\Delta U_p}{U_p}\cosh(ky_o)\,e^{\bar{\gamma}v_o\tau}e^{i\phi} + v_o\frac{(kd)^2}{2\sinh^2(kd)}\left(\frac{\Delta U_p}{U_p}\right)^2 e^{2\bar{\gamma}\tau} \tag{42}$$

$$\Big\{a\cosh(2ky_o) - a\cos 2\phi + b\sin 2\phi\Big\} - v_o\frac{(kd)^3}{8\sinh^3(kd)}\left(\frac{\Delta U_p}{U_p}\right)^3 e^{3\bar{\gamma}v_o\tau}e^{i\phi}$$

$$\left\{\cosh(ky_o)\left[3a^2 + b^2 + i2ab + \frac{\omega}{v_o\bar{\gamma}}(b+ia)\right] - \cosh(3ky_o)\left[\frac{\omega}{v_o\bar{\gamma}}(b-ia) + (a+ib)^2\right]\right\}$$

$$+ v_o\frac{kd}{\sinh^3(kd)}\left(\frac{\Delta U_p}{U_p}\right)^3\cosh(ky_o)\,A\,e^{3\bar{\gamma}v_o\tau}e^{i\phi}.$$

Replacing in equations (41) and (42), y_o and τ by y, t, and x with the aid of equations(39) and (40), we have the alternating velocities $\delta_2 v_y$, $\delta_3 v_y$, $\delta_2 v_x$, $\delta_3 v_x$ as function of y, t, and x. We find

$$\delta_2 v_y = \delta_2 v_x = o \tag{43}$$

and

$$\begin{aligned} \delta_3 v_y &= i v_o \frac{kd}{\sinh^3 (kd)} \left(\frac{\Delta U_p}{U_p}\right)^3 A \sinh(ky) e^{2\bar{\gamma}x} e^{i\omega t + \Gamma x} , \\ \delta_3 v_x &= v_o \frac{kd}{\sinh^3 (kd)} \frac{\Delta U_p}{U_p}^3 A \cosh(ky) e^{2\bar{\gamma}x} e^{i\omega t + \Gamma x} ; \end{aligned} \tag{44}$$

or for the total velocity,

$$v_y = i v_o \frac{kd}{\sinh (kd)} \frac{\Delta U_p}{U_p} \left[1+ \left(\frac{\Delta U_p}{U_p}\right)^2 \frac{A e^{2\bar{\gamma}x}}{\sinh^2 (kd)}\right] \sinh(ky) e^{i\omega t + \Gamma x} \tag{45}$$

and

$$v_x = v_o + v_o \frac{kd}{\sinh(kd)} \frac{\Delta U_p}{U_p} \left[1+ \left(\frac{\Delta U_p}{U_p}\right)^2 \frac{A e^{2\bar{\gamma}x}}{\sinh^2 (kd)}\right] \cosh(ky) e^{i\omega t + \Gamma x} . \tag{46}$$

We have therefore the following important result: the perturbed velocity of the electrons as function of y and x is proportional to the ratio of the alternating field

$$E_\sim = K \Delta U_p \left[1 + \left(\frac{\Delta U_p}{U_p}\right)^2 \frac{A e^{2\bar{\gamma} x}}{\sinh^2 (kd)}\right] e^{i\omega t + \Gamma x} ,$$

to the continuous field

$$E_c = \frac{U_p}{d} .$$

The alternating velocity $v_{y\sim}$ is proportional to the field E_x, and the alternating velocity $v_{x\sim}$ is proportional to the electric field E_y up to the third approximation. However, we can consider that, in a general fashion, in the plane magnetron or in the plane magnetron travelling-wave tube the velocity is given by

$$\vec{v} = \frac{\vec{E} \times \vec{Z}_o}{B} \tag{47}$$

if the amplitude of relative motion at the entrance is zero. Z_o is the unit vector in the direction of the magnetic field.*

Equations (45) and (46) determine the kinetic energy with which the electrons reach the anode.

The introduction of the alternating velocities from equations (43) and (44) into equation (13) gives the alternating densities ρ_2 and ρ_3. It is found that

$$\rho_2 = \rho_3 = 0 . \tag{48}$$

Therefore in the magnetron travelling-wave tube the alternating densities are zero up to the third approximation, and we can assume more generally that the density remains constant in the beam.

8. Determination of the Amplitude A of the Third Approximation

The value A appearing in equations (35) and (36) is determined by a balance of power. The power -dP given by the current of width Δy is

$$- dP = \rho\, v\, \Delta y h \left[1 + \frac{\partial}{\partial y_o} \delta y \right] E_x(y_o + \delta y) dx, \tag{49}$$

where h is the width of the beam in the z direction. Equation (49) is written in terms of real quantities. The term $\left(1 + \frac{\partial}{\partial y_o} \delta y \right)$ gives the variation of the section Δy due to the action of the HF fields. The term $E_x(y_o + \delta y)$ takes account of the fact that the current is moving in a field variable with y. In taking account of equations (15), (42), (48), and the hypothesis $\bar{\gamma} \ll \bar{k}$, equation (49) when we neglect all small terms, becomes

* We can demonstrate that equation (47) is also valid if the density of the space charge is continuous and constant (case of the plane magnetron).

$$- dP = I_o\, dx \left(1 + \frac{\partial}{\partial y_o}\delta y\right) E_x \sinh k(y_o + \delta y) \quad , \tag{50}$$

I_o being the continuous current.

With equations (29), (37), and (38), we obtain, for the third approximation term of the power, by using complex quantities:

$$\begin{aligned} -dP = {} & \frac{I_o k dx}{64}\, k^3 d^3 \frac{\Delta U_p^4}{U_p^3} \frac{e^{4\bar{\gamma}x}}{\sinh^4(kd)} \left\{ \sinh(4ky_o)(b+ia) \left[-7b^2 - 7a^2 + 6b\frac{\omega}{\bar{\gamma}v_o} \right.\right. \\ & \left.-8ia\frac{\omega}{\bar{\gamma}v_o} + 8(g+ic)\left(b + \frac{\omega}{\bar{\gamma}v_o} + ia\right)\right] + \sinh(2ky_o)\left[(b+ia)\left(4b\frac{\omega}{\bar{\gamma}v_o}\right.\right. \\ & \left.\left.\left. -2b^2 - 2a^2 + 8ia\frac{\omega}{\bar{\gamma}v_o}\right) - 8(g+ic)\left(2b^2 + 2iab + ia\frac{\omega}{\bar{\gamma}v_o}\right)\right]\right\} \\ & + \frac{I_o k dx}{2}\, kd \frac{\Delta U_p^4}{U_p^3} \frac{e^{4\bar{\gamma}x}}{\sinh^4(kd)} \left\{ A(b+ia) + (g+ic)A^* \right\} \sinh(2ky_o) \quad . \end{aligned} \tag{51}$$

The power absorbed by the attenuation along dx is

$$+ dP_1 = - \frac{E_x E_x^*}{R_x} (\bar{\gamma} + ik)\, dx \quad , \tag{52}$$

with

$$E_x = - \Gamma \Delta U_p \left[1 + A \left(\frac{\Delta U_p}{U_p}\right)^2 \frac{e^{2\bar{\gamma}x}}{\sinh^2(kd)} \right] \frac{\sinh(ky)}{\sinh(kd)}\, e^{i\omega t + \Gamma x} \tag{53}$$

which gives, for the third approximation,

$$dP_1 = \frac{k^2 \Delta U_p^4}{U_p^3 R_x} \frac{\sinh^2(ky)}{\sinh^4(kd)} (A + A^*)(\gamma + ik)\, e^{4\bar{\gamma}x}\, dx \quad . \tag{54}$$

The growth along dx of the apparent power dP_2, which is propagated in the direction of the wave, is, for the third approximation

$$dP_2 = \frac{k^2 \Delta U_p^4}{U_p^3 R_x} \frac{\sinh^2(ky)}{\sinh^4(kd)}\, e^{4\bar{\gamma}x} \left[A(\bar{\gamma} + i\bar{k}) + (3\bar{\gamma} + i\bar{k})A^* \right] \quad . \tag{55}$$

According to the principle of conservation of energy we have

$$dP + dP_1 + dP_2 = o \; . \tag{56}$$

The introduction of the relations (51), (54), and (55) into equation (56) gives the equation which determines A.

We limit ourselves to two particular cases:

1. $a = o$, $\gamma \neq o$, that is, the forced wave has the same velocity as the free wave and as the electrons.

2. $a \neq o$, $\gamma = o$, that is, the slow wave line has no attenuation.

1. $a = o, \gamma \neq o$. We obtain according to the equations (51) and (54) to (56)

$$A = \frac{k^2d^2}{128} \frac{\bar{\gamma} - \gamma}{\left(\bar{\gamma} - \frac{\gamma}{4}\right)\bar{\gamma}^2 \; \frac{\omega^2}{v_o^2}} \left[13 \cosh (2ky_o) - 5\right] \; . \tag{57}$$

Therefore A is real and positive ($\gamma < o$), and the gain increases with amplitude. We can attempt to calculate the power at the output $P = \frac{E_x E_x^*}{2R}$, as a function of the power P_o, which is transported from the input by the amplified wave. We will then have:

$$P = P_o \, e^{2\bar{\gamma} x} + A_2 P_o^2 \, e^{4\bar{\gamma} x} + A_3 P_o^3 \, e^{6\bar{\gamma} x} \; .$$

The calculation made previously gave the coefficient A_2, and we have

$$P = P_o \, e^{2\bar{\gamma} x} \left[1 + \frac{(\bar{\gamma} - \gamma)^2}{\bar{\gamma}\left(\bar{\gamma} - \frac{\bar{\gamma}}{4}\right)} \frac{kd}{16} \; \frac{13 \cosh(2ky_o) - 5}{\sinh(2ky_o)} \; \frac{P_o \, e^{2\bar{\gamma} x}}{U_p \; I_o}\right] . \tag{58}$$

Quantitatively, the preceding expression is evidentally valid only if the term $P_o \, e^{4\bar{\gamma} x}$ is small compared with $P_o \, e^{2\bar{\gamma} x}$, which allows us to determine the limit of validity of the third approximation. Let us take an example which corresponds to a practical case,

$$kd = 2 \; , \quad ky_o = 1.25 \; , \quad \gamma = 0,$$

voltage between anode and cathode $= \frac{U_p}{2}$, we then deduce

$$P = P_o e^{2\bar{\gamma} x} \left[1 + 0.78 \; \frac{P_o e^{2\bar{\gamma} x}}{U_c \; I_o} \right] ,$$

where U_c I_o is the applied power.

For an efficiency of 50 per cent the second term, therefore, gives a correction of 25 per cent.

We can note, in passing, an essential difference between the nonlinearities in the magnetron travelling-wave tube and the phenomena of the same nature in the travelling-wave tube, namely, that the factor $\frac{P_o e^{2\bar{\gamma} x}}{U_c \; I_o}$ is generally negative and inversely proportional to the optimum gain $\bar{\gamma}_{opt}$, while here the factor is positive and independent of the optimum gain.

2. $a \neq 0, \gamma = 0$. Starting from equation (51) and (54) to (56), and the relation (20), which gives R_x, we obtain the real part A_1 from $A = A_1 + j B$ (in fact, only the real part occurs in P). If $2ky_o$ is sufficiently large, A_1 is reduced to

$$A_1 = \frac{kd}{128} \; \frac{9 - 8 \; a\rho}{1 - a\rho} \; \frac{3 \; b \; \theta_1 + a \; \theta_2}{9 \; b^2 + a^2} \cosh \left(2ky_o\right) \quad , \tag{59}$$

with

$$3b \; \theta_1 + a \; \theta_2 = (3b^2 + a^2) \left[-7(a^2 + b^2) + \frac{6\omega b}{v_o \bar{\gamma}} + 8g \left(b + \frac{\omega}{v_o \bar{\gamma}} \right) - 8ac \right]$$

$$- 16 \; ab \left[- \frac{\omega \; a}{v_o \bar{\gamma}} + ga + 7c \left(b + \frac{\omega}{v_o \bar{\gamma}} \right) \right] \quad .$$

Fig. 3 gives the relative value $\frac{A_1}{A_{o1}}$ (A_{o1} being the value of A_1 for $a = 0$) as a function of $\bar{u}$. $\bar{u}$ represents the normalized difference between the velocity of the free wave and the velocity of the electrons

$$\bar{u} = \left(1 - \frac{v_o}{v} \right) \frac{k}{\bar{\gamma}_{opt.}}$$

$\bar{\gamma}_{opt}$ is the optimum value of gain, which is that which we have for a tube without attenuation. It is seen that A_1 increases rapidly with $\bar{u}$. It follows that we must expect hysteresis effects if the velocity of the electrons is not equal to the velocity of the free wave, as we have found experimentally in magnetrons for the region of oscillations of resonance.

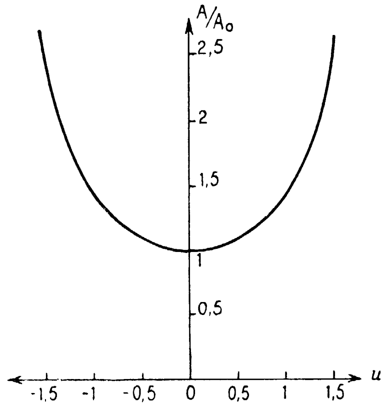

Fig. 3

9. Electron Efficiency

In section 8, we have seen that the third approximation leads to a gain higher than the first and that the nonlinearities give only corrections in the neighborhood of 25 per cent for efficiencies of the order of 50 per cent. It is,therefore, not, as in the travelling-wave tube, the mechanism of energy exchange which determines the amplitude, but, to the contrary, the absorption by the anode. In order to determine the electronic efficiency, it is therefore necessary to determine the velocity with which the electrons are captured by the electrodes.

In [2] , we have evaluated the electronic efficiency by assuming that all of the electrons would be absorbed by the anode with a velocity $v_0 = E/B$. Now, in section 7 we have determined the velocity of the electrons under the influence of the HF fields and we have found [see equations (45) and (46)] that the velocity of the electrons is, following Ox:

$$v_x = v_0 \left(1 + \frac{E_y}{E_0}\right) , \tag{60}$$

(E_y = alternating field at point y, x at time t) and following Oy:

$$v_y = - v_o \frac{E_x}{E_o} \quad . \tag{61}$$

The kinetic energy is found immediately:

$$\frac{m}{2} (v_x^2 + v_y^2) = \frac{m}{2} v_o^2 \left[\left(1 + \frac{E_y}{E_o}\right)^2 + \left(\frac{E_x}{E_o}\right)^2 \right] . \tag{62}$$

We can show that

$$\left|E_y\right| \ll E_o \qquad \left|(E_x)^2\right| \ll E_o^2 \quad \text{for } y = d \quad , \tag{63}$$

and the original evaluation is found to be justified.

According to the definition of R_x, we have

$$E_y = \sqrt{2 R_x P_s} \; \frac{\cosh (ky)}{\sinh (ky)} \; ,$$

$$P_s = \eta \, U_c \, I_o = \text{power at the output.}$$

Replacing R_x by equation (20), we obtain

$$\frac{E_y}{E_o} = \sqrt{2kd\eta \frac{U_c}{U_p} \; \frac{\cosh (ky)}{\sinh (ky)}} \quad \frac{\bar{\gamma}}{k} \quad . \tag{64}$$

In practical cases:

$$kd \approx 2, \quad y = d, \quad \eta < 1 \quad \text{and} \quad \frac{U_c}{U_p} \leq 1 \quad ,$$

we have

$$\left|\frac{E_y}{E_o}\right| \leqslant \frac{2\bar{\gamma}}{k} \ll 1 \quad ,$$

where $\frac{\gamma}{k}$ is of the order of $\frac{1}{50}$. We do not commit appreciable error in writing equation (62)

$$E_{\text{kinetic}} = \frac{m}{2} v_o^2 \quad .$$

It is necessary, therefore, to demonstrate that all the electrons which enter for different phases ωt_o into the space of the discharge reach the anode. We will restrict ourselves to the case where the velocity of the electrons is equal to that of the free wave, that is to say, we will let $a = 0$, and where the attenuation of the slow line is negligible. We calculate the distance y of the electrons as a function of the phase of entrance ωt_o and we obtain, in fourth approximation,

$$
\begin{aligned}
y = y_o &- d\,\frac{\Delta U_p}{\bar{\gamma}\frac{v_o}{\omega}U_p}\,e^{\bar{\gamma}x}\,\frac{\sinh(ky_o)}{\sinh(kd)}\,\sin\omega t_o + d\frac{kd}{4}\left(\frac{\Delta U_p}{\bar{\gamma}\frac{v_o}{\omega}U_p}e^{\bar{\gamma}x}\right)^2\frac{\sinh(2ky_o)}{\sinh^2(kd)} \\
&- d\,\frac{k^2d^2}{12}\left(\frac{\Delta U_p}{\bar{\gamma}\frac{v_o}{\omega}U_p}e^{\gamma x}\right)^3\frac{\sinh(3ky_o)-\sinh(ky_o)}{\sinh^3(kd)}\,\sin\omega t_o \\
&+ d\,\frac{k^3d^3}{48}\left(\frac{\Delta U_p}{\bar{\gamma}\frac{v_o}{\omega}U_p}e^{\bar{\gamma}x}\right)^4\left\{\frac{\sinh(4ky_o)-\sinh(2ky_o)}{\sinh^4(kd)}\right. \\
&\qquad\left. - \cos 2\omega t_o\,\frac{\frac{1}{2}\sinh(4ky_o)-\sinh(2ky_o)}{\sinh^4(kd)}\right\} \qquad (65) \\
&- d\,\frac{k^2d^2}{6\cdot 128}\left(\frac{\Delta U_p e^{\bar{\gamma}x}}{\bar{\gamma}\frac{v_o}{\omega}U_p}\right)^3\frac{13\sinh(3ky_o)-23\sinh(ky_o)}{\sinh^3(kd)}\,\sin\omega t_o \\
&+ d\,\frac{k^3d^3}{12\cdot 128}\left(\frac{\Delta U_p\,e^{\bar{\gamma}x}}{\bar{\gamma}\frac{v_o}{\omega}U_p}\right)^4\frac{13\sinh(4ky_o)-10\sinh(2ky_o)}{\sinh^4(kd)}\,.
\end{aligned}
$$

The last two terms in equation (65) occur because of the nonlinearities of the initial equations. In equation (65), we have neglected the absorption by the anode. This absorption entails a diminution of gain and, therefore, in the last two terms of equation (65), of the other coefficients. We shall neglect these two terms in the following considerations because they do not change appreciably the motions of the electrons in the y direction.

We see that, according to equation (65), in the first, third, etc., approximations, the electrons which enter in a phase $\pi < \omega t_o < 2\pi$ are moved toward the anode whereas the electrons in the phase $0 < \omega t_o < \pi$ are moved toward the negative electrode. In the second and the fourth approximation, all of the electrons have a component toward the anode. If we take account of the two last terms of equation (65), the behavior of the motion in the y direction is the same; only the coefficients change. Replacing ΔU_p by E_x (6) and deducing the coupling resistance of equation (20), we obtain

$$\left(\frac{\Delta U_p \, e^{\bar{\gamma} x}}{\bar{\gamma} \frac{v_o}{\omega} U_p} \right) = \frac{\sinh (kd)}{\sinh (ky_o)} \sqrt{\frac{2 \, P_{H.F.}}{kd \, U_p \, I_o} \, \frac{\sinh (ky_o)}{\cosh (ky_o)}}$$

$$= \frac{\sinh (kd)}{\sinh (ky_o)} \sqrt{\frac{2 \eta_o}{kd} \frac{U_c}{U_p} \tanh (ky_o)} \quad ,$$

where P_{HF} = HF power in first approximation.

$$P_{HF} = \frac{Ex_o \, Ex_o^*}{2 \, R_x} \, e^{2 \bar{\gamma} x} \quad ,$$

$$\eta_o = \frac{P_{HF}}{U_c \, I_o} \quad .$$

In Figs. 4 and 5, we have drawn y/d as a function of the entering phase ωt_o for different values of kd, ky_o, and $\eta_o \, U_c/U_p$, according to equation (65), by neglecting the two last terms (Figs. 4 and 5). We see that the rate at which the electrons are captured by the anode increases as $\eta_o \, U_c/U_p$ and kd increase and ky_o decreases.

Let us take a practical example: $kd = 2$, $U_c/U_p = 0.5$. For $ky_o = 1.25$, η_o cannot exceed about 0.75, the value of the ideal efficiency that we calculated if all of the electrons reach the anode. In this case, $\eta_o \, U_c/U_p$ is of the order of 0.3 or 0.4; therefore, only about 50 per cent of the electrons reach the anode, and nearly 80 per cent of the electrons give

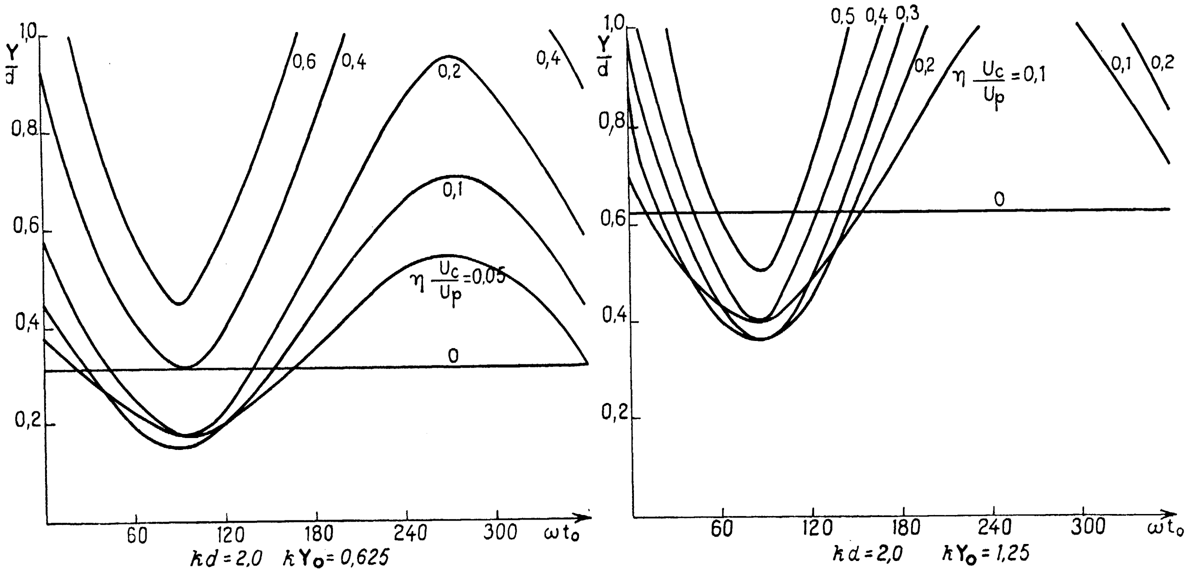
Y/d
ωt_0
$kd = 2,0 \quad kY_0 = 0,625$
$\eta \frac{U_c}{U_p} = 0,05$
$kd = 2,0 \quad kY_0 = 1,25$
$\eta \frac{U_c}{U_p} = 0,1$

Fig. 4.

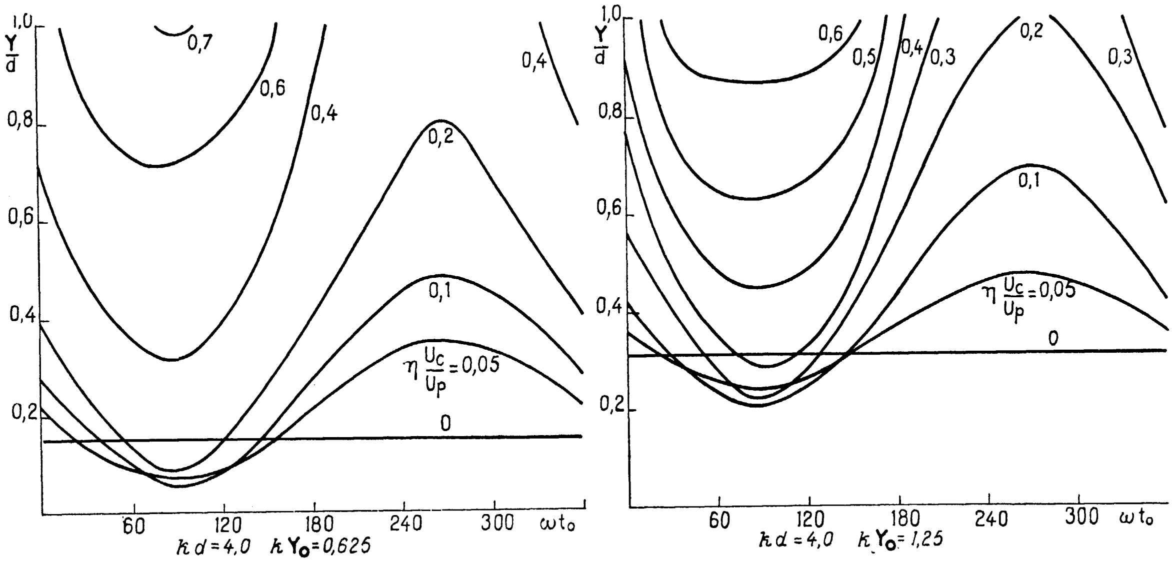
Y/d
ωt_0
$kd = 4,0 \quad kY_0 = 0,625$
$\eta \frac{U_c}{U_p} = 0,05$
$kd = 4,0 \quad kY_0 = 1,25$
$\eta \frac{U_c}{U_p} = 0,05$

Fig. 5.

energy to the HF field, while 20 per cent gain energy. The efficiency calculated according to the method given in [2] is much larger than the practical efficiency. In the case of Fig. 5, $kd = 4$, $ky_o = 1.25$, $U_c/U_p = 1$, the efficiency calculated according to [2] is 69 per cent. Therefore, $\eta_o\, U_c/U_p$ is of the order of 0.6 or 0.7. In this case, almost all the electrons reach the anode. The evaluation of the efficiency, according to [2], is therefore justified. This is the same in the case of Fig. 4, with

$$kd = 2.0, \quad ky_o = 0.625 \quad \text{and} \quad \frac{U_c}{U_p} = 1 .$$

To sum up: If the electrons move sufficiently far from the anode, almost all reach the anode and the efficiency is high. If the electrons move in the vicinity of the anode, the electrons with unfavorable entering phase do not reach the anode. The efficiency is smaller than the result of the calculation according to the method given in [2] . The error which we commit in this case is smaller than 0.5.

This effect is understandable if we consider the physical behavior of the tube. The field E_y gives a focussing action in favorable phase. The field E_x of unfavorable phase guides the electrons toward the negative electrode. The field E_x of favorable phase guides the electrons toward the anode. Therefore, if the electrons move in the vicinity of the negative electrode, that is, in a large field E_y, compared to E_x, there is a focussing action in favorable phase, and the beam is guided slowly toward the anode. If the electrons move in the vicinity of the anode, E_x is of the same order of magnitude as E_y, and the electrons of unfavorable phase move toward the negative electrode. They arrive in a smaller E_y field and the focussing is not sufficient.

For a high efficiency it is therefore favorable to use the same voltage on the cathode and the negative electrode.

10. Calculation of the Fundamental Term of HF Current

The current is given by

$$\tilde{i} + I_o = \rho_o \, (v_o + \delta v_x) \left(1 + \frac{\partial}{\partial y_o} \, \delta y \right) \Delta y_o \quad , \tag{66}$$

which is practically

$$\frac{\tilde{i}}{I_o} = \frac{\partial}{\partial y_o} \, \delta_1 y + \frac{\partial}{\partial y_o} \, \delta_3 y \quad .$$

Letting $\tilde{i} = \tilde{i}_1 + \tilde{i}_3$, we have, assuming, for simplicity, that $\gamma = 0$ and $a = 0$:

$$\frac{\tilde{i}_1}{I_o} = i \frac{kd}{\sinh(kd)} \, \frac{\Delta U_p}{\bar{\gamma} \frac{v_o}{\omega} U_p} \, e^{\bar{\gamma} x} \cosh\,(ky_o) \, e^{i\Phi} \quad , \tag{67}$$

$$\frac{\tilde{i}_3}{I_o} = i \frac{(kd)^3}{\sinh^3\,(kd)} \left(\frac{\Delta U_p}{\bar{\gamma} \frac{v_o}{\omega} U_p} \, e^{\bar{\gamma} x} \right)^3 e^{i\Phi} \left[\frac{37}{6.128} \cosh\,(ky_o) \right.$$

$$\left. + \frac{5}{2.128} \cosh\,(3ky_o) \right] \quad . \tag{68}$$

From which, for the amplitude

$$\left| \frac{\tilde{i}}{I_o} \right| = \left| \frac{\tilde{i}_1}{I_o} \right| - \left| \frac{\tilde{i}_1}{I_o} \right|^3 \frac{30 \cosh\,(2ky_o) + 22}{6.128 \cosh^2\,(ky_o)} \quad . \tag{69}$$

Thus the maximum value of $\left| \frac{\tilde{i}}{I_o} \right|$ is

$$\left| \frac{\tilde{i}}{I_o} \right| = \frac{16\sqrt{2}}{3} \, \frac{\cosh\,(ky_o)}{\sqrt{11 + 15 \cosh\,(2ky_o)}} \quad . \tag{70}$$

For $e^{ky_o} \gg 0$, we obtain

$$\left| \frac{\tilde{i}}{I_o} \right|_{max.} \sim 1.38 \ ,$$

and for $ky_o \ll 1$

$$\left| \frac{\tilde{i}}{I_o} \right|_{max.} \sim 1.48 \ .$$

When we inject the beam in the vicinity of the negative electrode, the modulation of current is greater than if we inject near the anode. This is understandable because the field E_x is zero on the negative electrode, and the electrons therefore are not subjected, at the moment of their entrance into the space of the discharge, to the focussing action of the field E_y; on the contrary, when they enter in the vicinity of the anode, they are subjected at the same time to the action of E_x and E_y, which entails a smaller modulation of current.

11. Conclusions

In this article we have calculated more exactly the gain for small signals, neglecting the hypothesis expressed by equation (1). We have found four waves which fulfill the initial conditions. The gain is increased by a factor $\sqrt{2}$ because the beam moves in a field variable with y. On the other hand, in the appendix, we have calculated the resistance of coupling for a helical plate and we found that the resistance of coupling is smaller by approximately a factor of two. Therefore, the numerical values calculated according to (3) remain almost the same.

The influence of the nonlinearities of the initial equations [equation (8)] entail an increase of the gain with amplitude. The increase of the gain is larger if the velocity of the electrons is not equal to the velocity of the free wave. The attenuating densities are zero up to the third approximation, and the effect of the space charge are accordingly small.

The velocity of the electrons at the place x, y, t, is given by $\frac{\vec{E} \times \vec{B}}{\vec{B}^2}$, where $\vec{E}$ is the electric field at the point x, y, t perpendicular to the velocity. Therefore, the electrons are captured with their continuous velocity, since, on the other hand, the HF field is small compared with the continuous field.

The rate of electron capture by the anode is greater than 50 per cent. If the electrons enter in the vicinity of the negative electrode, the rate is 100 per cent. For a high efficiency, it is therefore favorable to use the same voltage on the cathode and on the negative electrode.

There remain still three problems to be discussed:

1. To calculate the rate of capture of the electrons by the anode, it is necessary to know the influence of the absorption by the anode on the gain.

2. In this article, the calculation is made under the hypothesis of an ideal optical system. In practice, there always exists a relative motion. The influence of this will be studied in a future article.

3. Up to the present we have assumed that the effects of the space charge are small because the electronic density is constant. But it is well known from the work of Brillouin that the continuous density of space charge entails a continuous velocity variable in the section of the beam. This effect must influence the gain of the magnetron travelling-wave tube. Up to the present, we have studied only an electron flow with different velocities without a slow line [8,9] .

On the other hand, the alternating density must influence the behavior of the magnetron travelling-wave tube. We will treat these effects in another article.

APPENDIX

We are going to calculate here the coupling resistance of a helical plate following a method given by Pierce. Since the velocity of the wave which is propagated in the direction of the beam is small compared with the velocity of light, the fields can be derived simply from a scalar potential.

We shall assume that the helix is formed of two plane conductors separated by an interval 2E, the negative electrode being located in the plane y = - D (see Fig. 6).

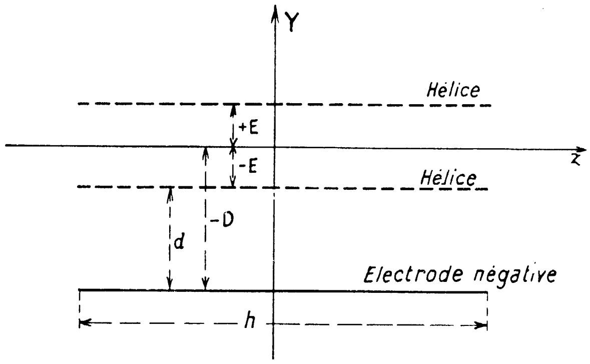

Fig. 6

On the helix we have, in the x direction, a sinusoidal force which is derived from a potential of the form

$$\phi = F(y)\, e^{i(\omega t - \alpha x)} . \qquad (1')$$

We have

for $-D \leqslant y \leqslant -E$:

$$\phi = B \sinh [\alpha(y + D)] \quad e^{i(\omega t - \alpha x)} , \qquad (2')$$

for $-E \leqslant y \leqslant +E$

$$\phi = A \cosh \alpha y \, e^{i(\omega t - \alpha x)} , \qquad (3')$$

for $+E \leqslant y \leqslant \infty$

$$\phi = C e^{-\alpha y} \; e^{i(\omega t - \alpha x)} . \qquad (4')$$

The fields are derived from ϕ by

$$E_y = - \frac{\partial \phi}{\partial y} , \qquad E_x = - \frac{\partial \phi}{\partial x} ,$$

where:

for $-D \leqslant y \leqslant -E$:

$$E_{y1} = -\alpha A \cosh(\alpha E) \frac{\cosh \alpha (y + D)}{\sinh \alpha (D - E)} e^{i(\omega t - \alpha x)} , \tag{5'}$$

$$E_{x1} = +i \alpha A \cosh(\alpha E) \frac{\sinh \alpha (x + D)}{\sinh \alpha (D - E)} e^{i(\omega t - \alpha x)} ;$$

for $-E \leqslant y \leqslant +E$:

$$\begin{aligned} E_{yII} &= -\alpha A \sinh(\alpha y) e^{i(\omega t - \alpha x)} , \\ E_{xII} &= +i \alpha A \cosh(\alpha y) e^{i(\omega t - \alpha x)} ; \end{aligned} \tag{6'}$$

for $+E \leqslant y \leqslant +\infty$:

$$\begin{aligned} E_{yIII} &= \alpha A \cosh(\alpha E) e^{-\alpha(y - E)} e^{i(\omega t - \alpha x)} , \\ E_{xIII} &= +i \alpha A \cosh(\alpha E) e^{-\alpha(y - E)} e^{i(\omega t - \alpha x)} . \end{aligned} \tag{7'}$$

The power W transported by a length, h, of helix is

$$W = Q v_g ,$$

where Q is the electromagnetic energy stored per unit length, and V_g is the group velocity of the wave.

Now, Q is given by

$$Q = \frac{\varepsilon_0 h}{2} \int_{-D}^{\infty} (E_x E_x^* + E_y E_y^*) \, dy , \tag{9'}$$

where

$$W = {}_g \frac{\alpha A^2}{2} \varepsilon_0 h \left\{ \sinh(2 \alpha E) + \cosh^2(\alpha E) \left[1 + \cot \alpha (D-E) \right] \right\} . \tag{10'}$$

The resistance of coupling is therefore

$$R_x = \frac{E_x E_x^*}{2\,W} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\;\frac{c}{v_g}\;\frac{\alpha}{h}\;\frac{\sinh^2[\alpha\,(D+y)]}{\sinh[\alpha(D-E)]\,[2\tanh\alpha E\,\sinh\alpha(D-E)+e^{\alpha(D-E)}]} \qquad (11')$$

The value of R_x given by (15') in [3] is identical to the value found here in (11') when we put $E = 0$.

On the other hand, we have in practice:

$$\alpha E \approx \alpha\,(D-E) \approx 2$$

and in this case, R_x is practically half of R_x $(E = 0)$.

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