CONDITIONAL PROBABILITY GENERATING FUNCTIONS OF COUNTING PROCESSES†

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ABSTRACT

Conditional probability generating functions (CPGF) of counting processes (CP) are studied; these determine expressions for the probabilities of numbers of counts in an interval for special cases, and are in general required for applications to prediction, estimation and detection. Martingale theory–in particular the Meyer-Doob decomposition and the Doléans-Dade integral equation–leads to the desired CPGF in terms of the integrated conditional rate (i.e., natural increasing process) appearing in the Doob-Meyer decomposition.

It is shown that the integrated conditional rate is nonrandom iff the CP has independent increments; the CP then generalizes the Poisson process in the sense that the mean of the CP need not be continuous. For a CP mean with discontinuities, the CPGF involves coefficients furnished by a specified infinite product.

The other special case requires a conditional independence condition between the count and the rate. Here the CPGF is used to derive the probability of the number of counts in an interval. The resulting formula looks like the analogous expression for a Poisson process, but is actually a generalization in which the rate (known for a Poisson process) is replaced by the conditional expectation of the rate, given the past of the CP.

1. INTRODUCTION

Recently martingale theory has been used to study Counting Processes (hereafter abbreviated CP). This has given rise to the notion of Integrated Conditional Rate (ICR)([5]). We examine here an efficient martingale technique to compute some conditional probability generating functions for CP's.

First we obtain a general expression (involving the ICR) of this conditional probability generating function. For CP's with independent increments this expression gives actually an integral equation for this conditional probability generating function; furthermore, due to a result of Doléans-Dade [4], the unique solution of this equation, i.e. the conditional probability generating function, can be obtained. As a first consequence we derive a unique characterization of CP's with independent increments. Then we compute the probability of having n jumps in an interval $(s, t]$. This result is well known when the mean of the process is continuous, but our derivation extends to the general case where no conditional rate is assumed to exist.

Finally we show how similar results can be obtained for CP's having a conditional rate satisfying some kind of conditional independence property.

2. PRELIMINARIES

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space. By (X_t) we denote a real valued stochastic process defined on $\mathbb{R}_+$ (the positive real line) and by a Counting Process (CP) we mean

Definition 2.1:

A CP is a stochastic process having sample paths zero at the time origin, and consisting of right-continuous step functions with positive jumps of size one.

Let $(\mathcal{F}_t)$ be a right-continuous increasing family of σ-subalgebras of $\mathcal{F}$ with $\mathcal{F}_o$ containing all the P negligible sets. Now if (N_t) is a CP adapted to $\mathcal{F}_t$, with the sole assumption (i) the random variable N_t is a.s. finite for each t, then as a consequence of the Doob-Meyer decomposition for supermartingales we can associate to (N_t) a unique natural increasing process (A_t), dependent on the family $(\mathcal{F}_t)$, which makes the process $(M_t \triangleq N_t - A_t)$ a square integrable $(\mathcal{F}_t)$ local martingale [3]. When the mean $m_t = EN_t$ of the CP is finite then the process (M_t) is actually a martingale. This decomposition $(N_t = M_t + A_t)$ is intuitively a decomposition into the part (M_t) which is not predictable and (A_t) which can be perfectly predicted. This unique process (A_t) is called the Integrated Conditional Rate (ICR) of (N_t) with respect to $(\mathcal{F}_t)$ ("the $(\mathcal{F}_t)$ ICR of (N_t)") and has been studied in [5]. The terminology ICR is motivated by the fact that when (N_t) satisfies some sufficiency conditions its ICR takes on the form $(\int_0^t \lambda_s ds)$ where (λ_t) is a nonnegative process called the conditional rate (with respect to $(\mathcal{F}_t)$) satisfying $\lambda_t = \lim_{h \to 0} E[h^{-1}(N_{t+h} - N_t) | \mathcal{F}_t]$ ([5], Section 2.5). The existence of CP's which possess a bounded conditional rate with respect to the family of σ-algebras generated by the process itself has been first shown in [1] and in [5]. Sufficiency conditions for the existence of a conditional rate have been given in [5].

We assume here that modern martingale theory ([3], [7]) is known. Recall that a semimartingale (X_t) is a process which can be written as a sum $(X_t = X_o + L_t + A_t)$ where X_o is $\mathcal{F}_o$-measurable, (L_t) is a $(\mathcal{F}_t)$ local martingale, and (A_t) is a right-continuous process adapted to $(\mathcal{F}_t)$ with $A_o = 0$ a. s. and having sample paths of bounded variation on every finite interval (see [3]). A result basic to this study and due to Doléans-Dade [4] is the following: the stochastic integral equation

$$Z_t = 1 + \int_0^t Z_{s-} \, dX_s$$

with (X_t) a semimartingale has a unique solution, which is a semimartingale given by†

$$Z_t = \exp(X_t - \frac{1}{2} \langle X^c \rangle_t) \prod_{s \leq t} (1 + \Delta X_s) \exp(-\Delta X_s)$$

where the product in the right hand side converges a. s. for each t. Here we define $(\langle X^c \rangle_t)$ as the unique natural increasing process associated to the continuous part of the local martingale (L_t); $(\langle X^c \rangle_t)$ is identically zero when (X_t) is a semimartingale with sample paths of bounded variation on every finite interval (see [3]).

3. A FORMULA FOR THE CONDITIONAL PROBABILITY GENERATING FUNCTION

Let (N_t) be a CP with N_t a. s. finite for each t and adapted to a family $(\mathcal{F}_t)$. Its conditional probability generating function $\psi(z, t, s)$ is defined for

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† When f_t is a right-continuous function with left-hand limits Δf_t denotes the jump $f_t - f_{t-}$.

$t \geq s$ by:

(3.1) $$\psi(z,t,s) \triangleq E[z^{(N_t - N_s)} \mid \mathcal{F}_s] = \sum_n z^n \; P\{N_t - N_s = n \mid \mathcal{F}_s\}$$

where z is a complex number with $|z| \leq 1$. Denote by (A_t) the $(\mathcal{F}_t)$ ICR of (N_t). Recall that the process $(M_t \triangleq N_t - A_t)$ is a $(\mathcal{F}_t)$ local martingale

Lemma 3.1: The conditional probability generating function is given by

(3.2) $$\psi(z,t,s) = 1 + (z-1)E\left(\int_s^t z^{(N_{v-} - N_s)} dA_v \,\Big|\, \mathcal{F}_s\right)$$

Proof: The CP (N_t) is a right-continuous step process with ΔN_t being either zero or one so that for $t \geq s$

$$z^{N_t} - z^{N_s} = \sum_{s<v\leq t} \Delta z^{N_v} = \sum_{s<v\leq t} (z^{N_v} - z^{N_{v-}})$$

$$= \sum_{s<v\leq t} (z^{\Delta N_v} - 1) z^{N_{v-}}$$

$$= \sum_{s<v\leq t} (z-1) z^{N_{v-}} \Delta N_v$$

$$= (z-1)\int_s^t z^{N_{v-}} dN_v$$

where $\sum$ is the sum over the discontinuities of (N_t) in $(s,t]$. Using the expression $(N_t \triangleq M_t + A_t)$ in the above gives

$$z^{N_t} = z^{N_s}$$

(3.3) $$+ (z-1)\left(\int_s^t z^{N_{v-}} dM_v + \int_s^t z^{N_{v-}} dA_v\right)$$

Let (T_n) be a sequence of stopping times reducing the local martingale (M_t) i.e. the stopped process $(M_{t \wedge T_n})$ is a uniformly integrable martingale for each n (see [3]). The sample paths of $(M_{t \wedge T_n})$ are of bounded variation on every finite interval and $|z| \leq 1$ so by Proposition 2 of [3] the process $(\int_s^t z^{N_{v-}} dM_{v \wedge T_n})$ is a martingale. In particular $E(\int_s^t z^{N_{v-}} dM_{v \wedge T_n} | \mathcal{F}_s) = 0$. Now

$$\int_s^t z^{N_{v-}} dM_{v \wedge T_n} = \int_s^t z^{N_{v-}} I_{\{v \leq T_n\}} dM_v$$

and it follows from the bounded convergence theorem that

$$E(\int_s^t z^{N_{v-}} dM_v | \mathcal{F}_s) = 0.$$

Substituting the above relation in (3.3) and multiplying both sides by z^{-N_s} we get the desired result (3.2). ||

Formula (3.2) can be generalized to the case where the jumps of the process (N_t) are of random size. This formula would then contain, in place of the term (z-1), a random quantity which is a function of the random size jumps ΔN_t and z. This additional randomness makes this formula difficult to manipulate and perhaps of less value. Accordingly, we shall limit our future considerations to CP's.

4. APPLICATION TO PROCESSES OF INDEPENDENT INCREMENTS

Suppose now that (N_t) is a CP of independent increments with finite mean m_t and consider the family of σ-algebras $(\mathcal{N}_t \overset{\Delta}{=} \sigma(N_u, 0 \leq u \leq t))$ generated by the process itself up to and at time t. We will show that CP's of independent increments are uniquely distinguished by the fact that their

$(\mathcal{N}_t)$ ICR is deterministic and given by the mean m_t. Also, the probability generating function $\psi(z,t,s)$ and the probability $P\{N_t - N_s = n\}$ will be computed. The method used to devise these formulas is appealing as it does not require the mean m_t to be continuous, and hence generalizes currently known formulas [8].

Theorem 4.1: Let (N_t) be a CP with finite mean m_t for each t. Denote its (N_t) ICR by (A_t). Then

(a) (N_t) is a CP of independent increments if and only if the ICR (A_t) is deterministic.

(b) If the ICR (A_t) is deterministic then

$$A_t = m_t .$$

(c) The probability generating function of a CP with independent increments is given by

(4.1) $$\psi(z,t,s) = \exp[(z-1)(m_t - m_s)] \cdot \prod_{s<v\le t} [1+(z-1)\Delta m_v]\exp[(1-z)\Delta m_v]$$

(d) Denote by t_i the (at most countable) times of jump of m_t on the interval $(s,t]$. Define

(4.2) $$\delta_s^t \triangleq m_t - m_s - \sum_{s<v\le t} \Delta m_v .$$

(1) If the number of jumps of m_t in $(s,t]$ is infinite then the product $\prod_i [1+(z-1)\Delta m_{t_i}]$ is uniformly convergent in the region $|z| \le 1$ to an analytic function and we denote by a_k the coefficients of the Taylor expansion of the above infinite product.

(2) If the number j of jumps of m_t in (s, t] is finite the coefficients a_k of the Taylor expansion can be computed by formulas (4.9) to (4.11) below. In particular, if m_t is continuous (j=0) then $a_o = 1$, $a_k = 0$, $k \geq 1$ and $\delta_s^t = m_t - m_s$.

(3) We have

$$P\{N_t - N_s = n\} = \exp\{-\delta_s^t\} \sum_{k=0}^{n} \frac{a_k}{(n-k)!} (\delta_s^t)^{n-k}. \tag{4.3}$$

Remark:

Observe that when m_t is continuous we get the well known formulas

$\psi(z,t,s) = \exp[(z-1)(m_t - m_s)]$ and

$$P\{N_t - N_s = n\} = \frac{1}{n!} (m_t - m_s)^n \exp[-(m_t - m_s)] \tag{4.4}$$

for the Poisson counting process with variable rate.

Proof: (a) ($\Longrightarrow$) It is easy to show that the process $(N_t - m_t)$ is a $(\mathcal{N}_t)$ martingale.† Furthermore the increasing process m_t is natural because it is deterministic, so (N_t) has the unique Doob-Meyer decomposition $(N_t = (N_t - m_t) + m_t)$, and by definition m_t is the $(\mathcal{N}_t)$ ICR of (N_t).

($\Longleftarrow$) From formula (3.2)

$$\psi(z,t,s) = 1 + (z-1)\, E\left(\int_s^t z^{(N_{v-} - N_s)} dA_v \,\middle|\, \mathcal{N}_s\right)$$

The process (A_t) is deterministic and by Fubini's Theorem we can obtain

$$E\left(\int_s^t z^{(N_{v-} - N_s)} dA_v \,\middle|\, \mathcal{N}_s\right) = \int_s^t E\left(z^{N_{v-} - N_s} \,\middle|\, \mathcal{N}_s\right) dA_v$$

† Here $\mathcal{N}_t$ is the completion of the σ-algebra generated by $\{N_s, 0 \leq s \leq t\}$.

so that $\psi(z, t, s)$ satisfies the following integral equation:

$$\psi(z,t,s) = 1 + (z-1)\int_s^t \psi(z,v-,s)dA_v \tag{4.5}$$

By the Doléans-Dade result, Theorem 1 of [4], the above equation has the unique solution

$$\psi(z,t,s) = \exp[(z-1)(A_t - A_s)] \prod_{s<v\leq t} [1+(z-1)\Delta A_v]\exp[(1-z)\Delta A_v] \tag{4.6}$$

The right hand side of this relation is a deterministic function and it follows that (N_t) is a process with independent increments.

(b) By definition of the $(\mathcal{H}_t)$ ICR (A_t) the process $(M_t \overset{\Delta}{=} N_t - A_t)$ is a local martingale. This process is in fact a martingale since $m_t = EN_t$ is finite for each t (see [5], Theorem 2.3.1) and the result follows from $EM_t = 0 = EN_t - A_t$.

Part (c) is a restatement of (4.6) where we have used the fact that $A_t = m_t$.

(d) For a process of independent increments we clearly have $P\{N_t - N_s = n|\mathcal{H}_s\} = P\{N_t - N_s = n\}$. Now define

$$\delta_s^t = m_t - m_s - \sum_i \Delta m_{t_i}$$

Formula (4.1) can be rewritten with the above relation in the form

$$\psi(z,t,s) = \exp\{-\delta_s^t\}\exp\{z\delta_s^t\} \prod_i [1 + (z-1)\Delta_{m t_i}] \tag{4.7}$$

We examine now the infinite product

$$\prod_i [1 + (z-1)\Delta m_{t_i}] \tag{4.8}$$

Observe that: (a) for each n the partial product

$$f_n(z) = \prod_{i=1}^{n} [1 + (z-1)\Delta m_{t_i}]$$

is analytic in the complex plane and (b) the series

$$\sum_i |(z-1)\Delta m_{t_i}|$$

is uniformly convergent in the region $|z| \leq 1$. This last point follows from the Weierstrass test:

$$|(z-1)\Delta m_{t_i}| \leq 2\Delta m_{t_i}$$

and because the mean m_t is finite for each t the series

$$\sum_i \Delta m_{t_i} < m_t < \infty$$

is convergent. Conditions (a) and (b) above imply that the infinite product (4.8) converges uniformly to a function f(z) which is analytic in the region $|z| < 1$ (see [6], Corollary to Theorem 8.6.3; or [2], Theorem 5.4.8). Hence we can get a Taylor series expansion for $f(z) = \prod_i [1 + (z-1)\Delta m_{t_i}]$ in the region $|z| \leq 1$.

$$f(z) = \sum_{\ell} a_{\ell} z^{\ell} = \prod_i [1 + (z-1)\Delta m_{t_i}]$$

This power series can also be differentiated term by term in the region $|z| < 1$. It is then easy to compute from (4.7)

$$P\{N_t - N_s = n\} = \frac{1}{n!} \frac{d^{(n)}}{dz^n} \psi(z,t,s)\Big|_{z=0}$$

$$= \frac{1}{n!} \exp(-\delta_s^t) \frac{d^{(n)}}{dz^n} (\exp z\delta_s^t \sum_{\ell} a_\ell z^\ell)\Big|_{z=0}$$

$$= \exp(-\delta_s^t) \sum_{k=0}^{n} \frac{a_k}{(n-k)!} (\delta_s^t)^{n-k}$$

Now if the mean m_t has only a finite number of jumps $j \geq 1$ in the interval $(s,t]$ then the coefficients a_k are such that

$$\sum_{\ell=1}^{j} a_\ell z^\ell = \prod_{\ell=1}^{j} [1 + (z-1)\Delta m_{t_\ell}];$$

they can then be computed by

(4.9) $$a_o = \sum_{i=1}^{j} (1 - \Delta m_{t_i}) .$$

For $0 < k < j$

(4.10) $$a_k = \frac{1}{k!} \sum_{\substack{\text{all permutations} \\ \{\ell_q,\ q=1,\dots,j\} \\ \text{of} \\ \{1,\dots\dots,j\}}} \left(\prod_{q=1}^{k} \Delta m_{t_{\ell_q}} \right) \left[\prod_{q=k+1}^{j} (1 - \Delta m_{t_{\ell_q}}) \right]$$

(4.11) $$a_k = \prod_{i=1}^{k} \Delta m_{t_i} \qquad \text{for } k=j,$$

and finally for $k > j$, $a_k = 0$.

If $j = 0$ (continuous case) then $\prod_i [1 + (z-1)\Delta m_{t_i}] = 1$ so that $a_o = 1$ and $a_k = 0$, $k \geq 1$, and result (4.3) reduces to (4.4) ($\delta_s^t = m_t - m_s$ in this case). ||

If we define a non-homogeneous Poisson process (N_t) as being a CP of independent increments with a characteristic function $E\, z^{N_t}$ given by $\exp\{(z-1)\int_s^t \lambda_s ds\}$ where λ_t is a nonnegative function called the rate, then we have:

Corollary 4.2: A CP (N_t) of independent increments with finite mean m_t for each t is a non-homogeneous Poisson process if and only if the mean m_t is absolutely continuous. The rate λ_t is then given by the Radon-Nikodym derivative $\frac{dm_t}{dt}$.

Proof: By Theorem 4.1 it is easy to see that

$$E\, z^{N_t} = \exp\{(z-1)\int_0^t \lambda_s ds\}$$

if and only if

$$m_t = \int_0^t \lambda_s ds. \; ||$$

5. APPLICATION TO COUNTING PROCESSES WITH A CONDITIONAL RATE

Assume now that (N_t) is a CP with finite mean for each t and for which a conditional rate (λ_t) with respect to a family $(\mathcal{F}_t)$ exists and satisfies the condition

$$E(z^{N_{v-}} \lambda_v | \mathcal{F}_s) = E(z^{N_{v-}} | \mathcal{F}_s) E(\lambda_v | \mathcal{F}_s) \tag{5.1}$$

for all $v \geq s$. This condition will be discussed later on. From (3.2) we get

$$\psi(z,t,s) = 1 + (z-1)E\left(\int_s^t z^{(N_{v-}-N_s)} \lambda_v dv \,|\, \mathcal{F}_s\right) \tag{5.2}$$

Now the CP has a finite mean so that $E\int_0^t \lambda_s ds$ is finite which implies

$$E\int_s^t |z^{(N_{v-}-N_s)}|\lambda_v dv < \infty \qquad \text{for } |z| < 1.$$

Then by Fubini's Theorem

$$E[\int_s^t z^{(N_{v-}-N_s)}\lambda_v dv|\mathcal{F}_s] = \int_s^t E[z^{(N_{v-}-N_s)}\lambda_v|\mathcal{F}_s]dv$$

Hence by the above relations (5. 1) and (5. 2) one has

$$(5.3) \qquad \psi(z,t,s) = 1 + (z-1)\int_s^t \psi(z,v^-,s)\hat{\lambda}_v^s dv$$

where $\hat{\lambda}_v^s \triangleq E(\lambda_v|\mathcal{F}_s)$ i.e., $(\hat{\lambda}_v^s)$ is the minimum mean square error prediction of (λ_v) based on past information up to and at time s. As before, this equation has a unique solution which is a semimartingale (Theorem 1, [4])

$$(5.4) \qquad \psi(z,t,s) = \exp\{(z-1)\int_s^t \hat{\lambda}_v^s dv\}$$

and

$$(5.5) \qquad \{N_t-N_s = n|\mathcal{F}_s\} = \frac{1}{n!}(\int_s^t \hat{\lambda}_v^s dv)^n \exp\{-\int_s^t \hat{\lambda}_v^s dv\}$$

It is interesting to note that both these formulas generalize the corresponding expressions for Poisson processes directly, with the best estimate $\hat{\lambda}_v^s$ of λ_v replacing the latter, which is deterministic for a Poisson process.

All this is very appealing but is true only if condition (5. 1) is satisfied. This condition which can be rewritten (by adding and subtracting terms) as

$$(5.6) \quad E[z^{(N_{v-}-N_s)}(\lambda_v-\lambda_s)|\mathcal{F}_s] = E[z^{(N_{v-}-N_s)}|\mathcal{F}_s]E[(\lambda_v-\lambda_s|\mathcal{F}_s]$$

is difficult to interpret. But in the particular case where s = 0 and $\mathcal{F}_o = \{\phi, \Omega\}$ (this is the case for $\mathcal{F}_t = \mathcal{N}_t$) the above condition (5.1) becomes

$$(5.7) \qquad E(z^{N_{v-}}\lambda_v) = E(z^{N_{v-}})E(\lambda_v) \quad v \geq 0$$

and is satisfied if for each t the two random variables N_{t-} and λ_t are independent. This seems a reasonable assumption if we suppose the value of N_{t-} does not influence the rate at time t. Then under this condition (5.7) relation (5.5) gives

$$P\{N_t = n\} = \frac{1}{n!}\left(\int_0^t (E\lambda_v)dv\right)^n \exp\{-\int_0^t (E\lambda_v)dv\} .$$

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