On the Elementary Interactions for the Quasilinear Wave Equation $\frac{\partial\gamma}{\partial t}-\frac{\partial v}{\partial x}=0$ *and* $\frac{\partial v}{\partial t}-\frac{\partial\sigma(\gamma)}{\partial x}=0$

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1. Introduction

In this paper we shall restrict ourselves to the quasilinear wave equation

$$\frac{\partial\gamma}{\partial t}-\frac{\partial v}{\partial x}=0 \quad\text{and}\quad \frac{\partial v}{\partial t}-\frac{\partial\sigma(\gamma)}{\partial x}=0, \tag{E}$$

where $\sigma(\cdot)$ is an odd C^2 function which satisfies

$$\sigma'(\gamma)>0, \qquad -\infty<\gamma<\infty, \tag{1}$$

$$\sigma''(\gamma)<0, \qquad 0<\gamma, \tag{2*}$$

and

$$\lim_{\gamma\to\infty}\int_0^\gamma\sqrt{\sigma'(s)}\,ds=+\infty. \tag{3}$$

The system (E) with functions $\sigma(\cdot)$ satisfying (1)–(3) may be used to describe one dimensional shearing motions of an elastic medium.

Our interest is in the initial value problem (E) together with the initial condition

$$(\gamma,v)(x,0)=\begin{cases}(\gamma_0,v_0), & x<0,\\ (\gamma_1,v_1), & 0<x<1,\\ (\gamma_2,v_2), & 1<x.\end{cases} \tag{IC}$$

The constant states defining the data are not arbitrary. We insist that (γ_0, v_0) is connected to (γ_1, v_1) by a *forward wave* and that (γ_1, v_1) is connected to (γ_2, v_2) by either a *forward* or a *backward wave*. (The notion of a wave will be defined in Section 2.)

We shall limit ourselves to those data where not all the numbers γ_i are of the same sign. The reason for this is as follows. If all the numbers γ_i are of the same

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$\star$ The situation when (2) is replaced by $\sigma''(\gamma)>0$ on $\gamma>0$ is similar.

sign, say positive, then *a priori* considerations guarantee that the solution to (E) and (IC) will satisfy $\gamma(x, t)>0$ for all x and $t \geqq 0$. Since the system (E) behaves in a genuinely nonlinear way for γ's in the interval $(0, \infty)$, we see that for positive γ_i's (E) and (IC) reduce to one of the elementary interactions for a genuinely nonlinear system. These problems have been treated successfully elsewhere; for example, see [2]–[5].

Our goal is to obtain an existence theorem for (E) and (IC). The solutions (γ, v) will in general not be smooth. They will be in $L_\infty\{t \geqq 0\}$ and satisfy (E) in the following sense:

$$\iint_{\{t \geqq 0\}} (\gamma \varphi_t - v \varphi_x)\, dx\, dt + \int_{-\infty}^{\infty} \gamma^0(x)\, \varphi(x, 0)\, dx = 0 \tag{E-1}$$

and

$$\iint_{\{t \geqq 0\}} \big(v \varphi_t - \sigma(\gamma)\, \varphi_x\big)\, dx\, dt + \int_{-\infty}^{\infty} v^0(x)\, \varphi(x, 0)\, dx = 0 . \tag{E-2}$$

Here φ is any smooth function with compact support in $\{t \geqq 0\}$.

The technique we shall use to establish the theorem is as follows. For the given data (γ^0, v^0) we shall construct approximate solutions to (E) and (IC) via a difference scheme used by GLIMM [6] and GLIMM & LAX [7] (see Section 3). Making use of properties of the forward and backward wave curves through a given point in the $\gamma - v$ plane (see Section 2-B), we shall show that the x variation of the approximate solutions at any time t is bounded by a constant independent of the particular approximate (see Lemmas 3.3 and 3.4). This, together with pointwise estimates for the approximates and a convergence theorem due to GLIMM [6; pp. 711–715], guarantees the existence of a solution of the problem (E) and (IC).

2. Shocks, Simple Waves, and Riemann Problems

A. Shocks. Suppose that

$$\mathscr{C} = \{(x, t) \mid x = s(t),\ t_L < t < t_U\}$$

is a smooth curve in the $x - t$ plane, that γ and v is a solution of (E) (in the sense of integration by parts) which is smooth to the left and right of $\mathscr{C}$ and that γ and v suffer a jump discontinuity across $\mathscr{C}$. Then, it is well known that to the left and right of $\mathscr{C}$ γ and v satisfy (E) pointwise, and the following compatibility conditions must hold across $\mathscr{C}$:

$$\frac{ds}{dt}[\gamma] = -[v] \quad \text{and} \quad \frac{ds}{dt}[v] = -[\sigma(\gamma)] . \tag{2.1}$$

For any function $f(x, t)$,

$$[f](t) = (f_r - f_l)(t),\ f_r(t) = \lim_{\substack{x \to s(t) \\ x > s(t)}} f(x, t), \quad \text{and} \quad f_l(t) = \lim_{\substack{x \to s(t) \\ x < s(t)}} f(x, t) . \tag{2.2}^\star$$

* Occasionally we shall use the notation $f_+(t)$ and $f_-(t)$ where $f_+(t) = \lim_{\substack{x \to s(t) \\ x > s(t)}} f(x, t)$ and $f_-(t) = \lim_{\substack{x \to s(t) \\ x < s(t)}} f(x, t)$.

Such curves $\mathscr{C}$ are called shock waves, and equation (2.1) is referred to as the *Rankine-Hugoniot* condition for the system (E). *Forward shocks* are those where $\dot{s}>0$, and *backward shocks* are those where $\dot{s}<0$. For *forward shocks* (2.1) implies that

$$\dot{s}=C(\gamma_r,\gamma_l)\overset{\text{def}}{=}\sqrt{\frac{\sigma(\gamma_r)-\sigma(\gamma_l)}{\gamma_r-\gamma_l}} \tag{2.3}$$

and

$$v_l-v_r=-(\gamma_l-\gamma_r)\,C(\gamma_r,\gamma_l), \tag{2.4}$$

while for *backward shocks* (2.1) yields

$$\dot{s}=-C(\gamma_r,\gamma_l) \tag{2.5}$$

and

$$v_r-v_l=(\gamma_r-\gamma_l)\,C(\gamma_r,\gamma_l). \tag{2.6}$$

Conversely, we see that given constants (γ_l, v_l) and (γ_r, v_r) which satisfy (2.4), the function

$$(\gamma,v)(x,t)=\begin{cases}(\gamma_l,v_l), & x<C(\gamma_r,\gamma_l)\,t\\ (\gamma_r,v_r) & C(\gamma_r,\gamma_l)\,t<x,\ t\geqq 0\end{cases} \tag{2.7}$$

is a solution of (E) taking on the data

$$(\gamma,v)(x,0)=\begin{cases}(\gamma_l,v_l), & x<0,\\ (\gamma_r,v_r), & x>0.\end{cases}$$

A similar statement holds for constants satisfying (2.6). Not every solution of the type just constructed will be *admissible*. To be admissible we shall insist that the solution is obtainable as the zero viscosity limit of progressive wave solutions (with wave speed $C(\gamma_r, \gamma_l)$ for *forward shocks* and $-C(\gamma_r, \gamma_l)$ for *backward shocks*) of the *augmented viscous system*:

$$\frac{\partial\gamma}{\partial t}-\frac{\partial v}{\partial x}=0 \quad\text{and}\quad \frac{\partial v}{\partial t}-\frac{\partial\sigma(\gamma)}{\partial x}=\lambda v_{xx}, \qquad \lambda>0. \tag{2.8}$$

Theorem 2.1. (a) *A forward shock is admissible iff*

$$\begin{Bmatrix}\Gamma(\gamma_r)\leqq\gamma_l<\gamma_r\\ \gamma_l=0\\ \gamma_r<\gamma_l\leqq\Gamma(\gamma_r)\end{Bmatrix} \quad \textit{when} \quad \begin{Bmatrix}\gamma_r>0\\ \gamma_r=0\\ \gamma_r<0\end{Bmatrix}. \tag{2.9}$$

(b) *A backward shock is admissible iff*

$$\begin{Bmatrix}\Gamma(\gamma_l)\leqq\gamma_r<\gamma_l\\ \gamma_r=0\\ \gamma_l<\gamma_r\leqq\Gamma(\gamma_l)\end{Bmatrix} \quad \textit{when} \quad \begin{Bmatrix}\gamma_l>0\\ \gamma_l=0\\ \gamma_l<0\end{Bmatrix}. \tag{2.10}$$

For each $a>0$, $\Gamma(a)$ is the unique number less than zero such that

$$\sigma'(\Gamma)=\frac{\sigma(\Gamma)-\sigma(a)}{\Gamma-a}; \tag{2.11}$$

for $a<0$,

$$\Gamma(a)\stackrel{\text{def}}{=}-\Gamma(-a), \tag{2.12}$$

and

$$\Gamma(0)\stackrel{\text{def}}{=}0. \tag{2.13}$$

It is easily proved that the mapping $a\to\Gamma(a)$ *is* C^1 *and decreasing.*

We point out that for the system (E) *being considered here the strength of admissible shocks (the difference* $|\gamma_r-\gamma_l|$*) is not arbitrary. This should be contrasted with the behavior of genuinely nonlinear systems. There, shocks of arbitrary strength are admissible.*

Proof of Theorem 2.1. We shall only prove (a).

We prove necessity first. Suppose that the pair $(\gamma, v)(x,t)$ given by (2.7) is an admissible forward shock; that is

$$\left.\begin{cases}(\gamma_l, v_l), & x<C(\gamma_r,\gamma_l)\,t\\ (\gamma_r, v_r), & C(\gamma_r,\gamma_l)\,t<x,\ t\geqq 0\end{cases}\right\}=\lim_{\lambda\to 0^+}(\gamma^\lambda, v^\lambda)(\xi),$$

where $\xi=x-C(\gamma_r,\gamma_l)\,t$ and $(\gamma^\lambda, v^\lambda)(\xi)$ satisfies

$$C(\gamma_r,\gamma_l)\,\gamma_\xi+v_\xi=0, \qquad \lambda v_{\xi\xi}=-C(\gamma_r,\gamma_l)\,v_\xi-\sigma(\gamma)_\xi,$$

$$\lim_{\xi\to-\infty}(\gamma,v)(\xi)=(\gamma_l,v_l), \quad \text{and} \quad \lim_{\xi\to+\infty}(\gamma,v)(\xi)=(\gamma_r,v_r).$$

The last set of equations is the progressive wave form of (2.8) for solutions which are functions of $\xi=x-C(\gamma_r,\gamma_l)\,t$ alone. These equations are equivalent to

$$v(\xi)-v_r=-\big(\gamma(\xi)-\gamma_r\big)\,C(\gamma_r,\gamma_l), \tag{2.14}$$

$$\lambda\,C(\gamma_r,\gamma_l)\,\gamma_\xi=\sigma(\gamma)-\sigma(\gamma_r)-\frac{\gamma-\gamma_r}{\gamma_l-\gamma_r}\big(\sigma(\gamma_l)-\sigma(\gamma_r)\big)\stackrel{\text{def}}{=}F(\gamma), \tag{2.15}$$

and

$$\lim_{\xi\to-\infty}\gamma(\xi)=\gamma_l \quad \text{and} \quad \lim_{\xi\to\infty}\gamma(\xi)=\gamma_r. \tag{2.16}$$

To complete the proof of necessity, it suffices to show that (2.15) and (2.16) have no solution when (2.8) is violated. We shall only treat the case where $\gamma_r>0$.

Suppose that $0<\gamma_r<\gamma_l$. Then, $F(\gamma)$ is positive for γ's in (γ_r,γ_l), and hence no solutions of (2.15) can satisfy (2.16). If, on the other hand, $\gamma_l<\Gamma(\gamma_r)$, then $F(\gamma)$ has a zero in (γ_l,γ_r). This latter fact implies that no solution of (2.15) can satisfy (2.16).

We shall now establish sufficiency. We restrict ourselves to the case $\gamma_r>0$. Let $\gamma_l\in[\Gamma(\gamma_r),\gamma_r)$, let a be any number in (γ_l,γ_r), and let $\gamma^{\lambda,a}(\xi)$ be the unique solution of

$$\lambda\,C(\gamma_r,\gamma_l)\,\gamma_\xi=F(\gamma), \qquad \gamma(0)=a.$$

$v^{\lambda,a}(\xi)$ is defined by (2.14). The identity

$$\frac{\xi}{\lambda}=\int_a^{\gamma^{\lambda,a}(\xi)}\frac{d\gamma}{F(\gamma)}$$

implies that

$$\lim_{\xi \to +\infty} \gamma^{\lambda, a}(\xi) = \lim_{\substack{\lambda \to 0^+ \\ \xi > 0}} \gamma^{\lambda, a}(\xi) = \gamma_r$$

and

$$\lim_{\xi \to -\infty} \gamma^{\lambda, a}(\xi) = \lim_{\substack{\lambda \to 0^+ \\ \xi < 0}} \gamma^{\lambda, a}(\xi) = \gamma_l,$$

and this together with the identification $\xi = x - C(\gamma_r, \gamma_l)t$ completes the proof. □

B. Simple Waves and Wave Curves. The Riemann Invariants for the system (E) are the functions α and β defined by

$$\alpha = v - \int_0^\gamma C(s)\,ds \quad \text{and} \quad \beta = v + \int_0^\gamma C(s)\,ds. \tag{2.17}$$

Here,

$$C(\gamma) \stackrel{\text{def}}{=} \sqrt{\sigma'(\gamma)} = \lim_{\gamma_r \to \gamma} C(\gamma_r, \gamma), \tag{2.18}$$

and again $C(\gamma_r, \gamma)$ is defined in (2.3). α is constant on the *forward characteristics* $\frac{dx}{dt} = C(\gamma)$, and β is constant on the *backward characteristics* $\frac{dx}{dt} = -C(\gamma)$.

A *simple wave* is a solution (γ, v) of (E) in which one of the Riemann Invariants is identically constant in some region on the $x-t$ plane. When β is constant, the forward characteristics are straight lines. Such solutions are called *forward facing simple waves*. Similarily, *backward facing simple waves* are solutions where α is constant. Here, backward characteristics are straight.

It should be observed that in a *forward (backward) simple wave* no forward (backward) characteristics cross (as t increases) if $x \to C(\gamma(x, t))$ is an increasing (decreasing) function. We shall deal exclusively with such simple waves.

We are now in a position to define the forward and backward wave curves through a given point in the $\gamma-v$ plane. The forward wave curve will be denoted by $v_F(\cdot\,;\cdot,\cdot)$ and the backward curve by $v_B(\cdot\,;\cdot,\cdot)$. Their significance is as follows.

(a) *Forward Wave Curves.* Let (γ_r, v_r) and (γ_l, v_l) be two points in the $\gamma-v$ plane satisfying

$$v_l = v_F(\gamma_l; \gamma_r, v_r). \tag{2.19}$$

Then the Riemann Problem

$$\frac{\partial \gamma}{\partial t} - \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial t} - \frac{\partial \sigma(\gamma)}{\partial x} = 0, \qquad -\infty < x < \infty,\ t > 0, \tag{2.20}$$

and

$$(\gamma, v)(x, 0) = \begin{cases} (\gamma_l, v_l), & x < 0, \\ (\gamma_r, v_r), & x > 0, \end{cases} \tag{2.21}$$

is solvable, in the sense of integration by parts, by a forward facing wave. That is, the solution $(\gamma, v)(x, t)$ is a function of $\frac{x}{t}$, and for every x and $t \geqq 0$

$$v\left(\frac{x}{t}\right) = v_F\left(\gamma\left(\frac{x}{t}\right); \gamma_r, v_r\right). \tag{2.22}$$

Shocks present in the solution are *admissible*, and in any simple wave

$$\frac{\partial}{\partial x} C\left(\gamma\left(\frac{x}{t}\right)\right) > 0. \tag{2.23}$$

(b) *Backward Wave Curves.* A similar remark holds for states on the backward wave curve. The difference is that we insist that

$$v_r = v_B(\gamma_r;\, \gamma_l,\, v_l). \tag{2.24}$$

Solutions to the Riemann Problem (2.20) and (2.21) now satisfy

$$v\left(\frac{x}{t}\right) = v_B\left(\gamma\left(\frac{x}{t}\right), \gamma_l, v_l\right), \tag{2.25}$$

and (2.23) is replaced by $\frac{\partial}{\partial x} C\left(\gamma\left(\frac{x}{t}\right)\right) < 0$.

Definition 2.1. We let

$$v_F(\gamma;\, \gamma_r,\, v_r) = \begin{cases} v_r - (\Gamma(\gamma_r) - \gamma_r)\, C(\Gamma(\gamma_r)) - \int\limits_{\Gamma(\gamma_r)}^{\gamma} C(s)\, ds, & \gamma \leqq \Gamma(\gamma_r) \\ v_r - (\gamma - \gamma_r)\, C(\gamma_r, \gamma), & \Gamma(\gamma_r) \leqq \gamma \leqq \gamma_r \\ v_r - \int\limits_{\gamma_r}^{\gamma} C(s)\, ds, & \gamma_r \leqq \gamma \end{cases}$$

if $\gamma_r > 0$;

$$v_F(\gamma;\, 0,\, \gamma_r) = v_r - \int\limits_0^{\gamma} C(s)\, ds, \qquad -\infty < \gamma < \infty \quad \text{if} \quad \gamma_r = 0;$$

and

$$v_F(\gamma;\, \gamma_r,\, v_r) = \begin{cases} v_r - \int\limits_{\gamma_r}^{\gamma} C(s)\, ds, & \gamma \leqq \gamma_r \\ v_r - (\gamma - \gamma_r)\, C(\gamma_r, \gamma), & \gamma_r \leqq \gamma \leqq \Gamma(\gamma_r) \\ v_r - (\Gamma(\gamma_r) - \gamma_r)\, C(\Gamma(\gamma_r)) - \int\limits_{\Gamma(\gamma_r)}^{\gamma} C(s)\, ds, & \Gamma(\gamma_r) \leqq \gamma \end{cases}$$

if $\gamma_r < 0$.*

Definition 2.2. We let

$$v_B(\gamma;\, \gamma_l,\, v_l) = \begin{cases} v_l + (\Gamma(\gamma_l) - \gamma_l)\, C(\Gamma(\gamma_l)) + \int\limits_{\Gamma(\gamma_l)}^{\gamma} C(s)\, ds, & \gamma \leqq \Gamma(\gamma_l) \\ v_l + (\gamma - \gamma_l)\, C(\gamma, \gamma_l), & \Gamma(\gamma_l) \leqq \gamma \leqq \gamma_l \\ v_l + \int\limits_{\gamma_l}^{\gamma} C(s)\, ds, & \gamma_l \leqq \gamma \end{cases}$$

if $\gamma_l > 0$;

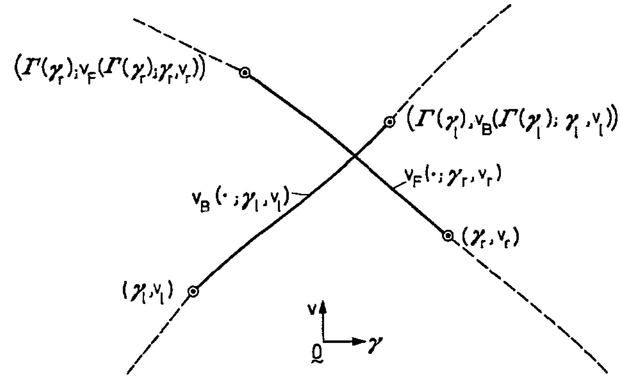
$$v_B(\gamma;\, 0,\, v_l) = v_l + \int\limits_0^{\gamma} C(s)\, ds, \qquad -\infty < \gamma < \infty \quad \text{if} \quad \gamma_l = 0;$$

* Recall that for any numbers a and b $C(a) = \sqrt{\sigma'(a)}$, $C(a, b) = \sqrt{\frac{\sigma(a) - \sigma(b)}{a - b}}$, and $C(\Gamma(a)) = C(\Gamma(a), a)$ (see (2.11)).

and

$$v_B(\gamma;\gamma_l,v_l)=\begin{cases} v_l+\int\limits_{\gamma_l}^{\gamma} C(s)\,ds, & \gamma\leqq\gamma_l \\ v_l+(\gamma-\gamma_l)\,C(\gamma,\gamma_l), & \gamma_l\leqq\gamma\leqq\Gamma(\gamma_l) \\ v_l+(\Gamma(\gamma_l)-\gamma_l)\,C(\Gamma(\gamma_l))+\int\limits_{\Gamma(\gamma_l)}^{\gamma} C(s)\,ds, & \Gamma(\gamma_l)\leqq\gamma \end{cases}$$

if $\gamma_l<0$.



Dashed lines are curves where the Riemann Invariants are constant.

Fig. 2.1

We emphasize that points on either of the wave curves may generate solutions to (2.20) and (2.21) which are combinations of shocks and simple waves. Such solutions are called *contact discontinuities*. To see this, we consider the case where

$$\gamma_r>0, \quad \gamma_l<\Gamma(\gamma_r) \quad \text{and} \quad v_l=v_F(\gamma_l;\gamma_r,v_r).$$

The solution to (2.20) and (2.21) is

$$(\gamma,v)(x,t)=\begin{cases} (\gamma_l,v_l), & x<C(\gamma_l)\,t \\ \left(\bar{\gamma}\left(\frac{x}{t}\right),\bar{v}\left(\frac{x}{t}\right)\right), & C(\gamma_l)\,t\leqq x\leqq C(\Gamma(\gamma_r))\,t \\ (\gamma_r,v_r), & C(\Gamma(\gamma_r))\,t<x,\ t\geqq 0. \end{cases}$$

The number $\bar{\gamma}\left(\frac{x}{t}\right)$ satisfies

$$\bar{\gamma}\left(\frac{x}{t}\right)\in[\gamma_l,\Gamma(\gamma_r)],$$

$$C\left(\bar{\gamma}\left(\frac{x}{t}\right)\right)=\frac{x}{t}, \qquad \frac{x}{t}\in[C(\gamma_l),C(\Gamma(\gamma_r))]^{*},$$

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* Recall, $C(\Gamma(\gamma_r))=C(\Gamma(\gamma_r),\gamma_r)$.

and

$$\bar{v}\left(\frac{x}{t}\right)=v_F\left(\bar{\gamma}\left(\frac{x}{t}\right);\gamma_r,v_r\right).$$

A simple computation shows that for $C(\gamma_l)\leqq\frac{x}{t}\leqq C(\Gamma(\gamma_r))$,

$$\beta\left(\bar{\gamma}\left(\frac{x}{t}\right),\bar{v}\left(\frac{x}{t}\right)\right)=\beta(\Gamma(\gamma_r),v_F(\Gamma(\gamma_r);\gamma_r,v_r)).$$

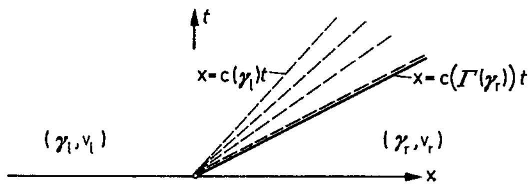

Fig. 2.2

We shall now list some properties of these curves which will be of some use in the sequel.

Theorem 2.2. (a) *The mapping* $(\gamma,\gamma_r,v_r)\to v_F(\gamma;\gamma_r,v_r)$ *is continuous. Moreover, for fixed* (γ_r,v_r), *the maps*

$$\gamma\to v_F(\gamma;\gamma_r,v_r)\quad and\quad \gamma\to\alpha(\gamma,v_F(\gamma;\gamma_r,v_r))$$

are C^1 *and decreasing.*

(b) *For fixed* (γ_r,v_r) *the map* $\gamma\to\beta(\gamma,v_F(\gamma;\gamma_r,v_r))$ *is* C^1. *Moreover,*

(1) *if* $\gamma_r>0$, *then*

$$\beta(\gamma,v_F(\gamma;\gamma_r,v_r))=\beta(\Gamma(\gamma_r),v_F(\Gamma(\gamma_r);\gamma_r,v_r)),\qquad \gamma\leqq\Gamma(\gamma_r),$$

$$\frac{d}{d\gamma}\beta(\gamma,v_F(\gamma;\gamma_r,v_r))=-\frac{(C(\gamma)-C(\gamma_r,\gamma))^2}{2C(\gamma_r,\gamma)}<0,\qquad \Gamma(\gamma_r)<\gamma<\gamma_r,$$

$$\beta(\gamma,v_F(\gamma;\gamma_r,v_r))=\beta(\gamma_r,v_r),\qquad \gamma_r\leqq\gamma;$$

(2) *if* $\gamma_r=0$, *then*

$$\beta(\gamma,v_F(\gamma;\gamma_r,v_r))=\beta(0,v_r),\qquad -\infty<\gamma<\infty;$$

(3) *if* $\gamma_r<0$, *then*

$$\beta(\gamma,v_F(\gamma;\gamma_r,v_r))=\beta(\gamma_r,v_r),\qquad \gamma\leqq\gamma_r,$$

$$\frac{d}{d\gamma}\beta(\gamma;v_F(\gamma;\gamma_r,v_r))=-\frac{(C(\gamma)-C(\gamma_r,\gamma))^2}{2C(\gamma_r,\gamma)},\qquad \gamma_r<\gamma<\Gamma(\gamma_r),$$

$$\beta(\gamma,v_F(\gamma;\gamma_r,v_r))=\beta(\Gamma(\gamma_r),v_F(\Gamma(\gamma_r);\gamma_r,v_r)),\qquad \Gamma(\gamma_r)\leqq\gamma.$$

(c) *If* $\begin{Bmatrix}\gamma_r\geqq 0\\ \gamma_r\leqq 0\end{Bmatrix}$ *and* $\begin{Bmatrix}\bar{\gamma}\leqq\gamma_r\\ \gamma_r\leqq\bar{\gamma}\end{Bmatrix}$, *then*

$$\begin{Bmatrix}v_F(\gamma;\bar{\gamma},v_F(\bar{\gamma};\gamma_r,v_r))\leqq v_F(\gamma;\gamma_r,v_r)\\ v_F(\gamma;\bar{\gamma},v_F(\bar{\gamma};\gamma_r,v_r))\geqq v_F(\gamma;\gamma_r,v_r)\end{Bmatrix}\ for\ all\ \begin{Bmatrix}\gamma\leqq\bar{\gamma}\\ \bar{\gamma}\leqq\gamma\end{Bmatrix}.\qquad\square$$

A similar proposition holds for $v_B(\cdot;\cdot,\cdot)$.

Theorem 2.3. (a) *The mapping* $(\gamma, \gamma_l, v_l) \to v_B(\gamma; \gamma_l, v_l)$ *is continuous. Moreover, for fixed* (γ_l, v_l), *the maps*

$$\gamma \to v_B(\gamma; \gamma_l, v_l) \quad \textit{and} \quad \gamma \to \beta(\gamma, v_B(\gamma; \gamma_l, v_l))$$

are C^1 *and increasing.*

(b) *For fixed* (γ_l, v_l) *the map* $\gamma \to \alpha(\gamma, v_B(\gamma; \gamma_l, v_l))$ *is* C^1. *Moreover.*

(1) *if* $\gamma_l > 0$, *then*

$$\alpha(\gamma, v_B(\gamma; \gamma_l, v_l)) = \alpha(\Gamma(\gamma_l); \gamma_l, v_l)), \qquad \gamma \leqq \Gamma(\gamma_l),$$

$$\frac{d}{d\gamma}\alpha(\gamma, v_B(\gamma; \gamma_l, v_l)) = \frac{(C(\gamma) - C(\gamma, \gamma_l))^2}{2C(\gamma, \gamma_l)} > 0, \qquad \Gamma(\gamma_l) < \gamma < \gamma_l,$$

$$\alpha(\gamma, v_B(\gamma; \gamma_l, v_l)) = \alpha(\gamma_l, v_l), \qquad \gamma_l \leqq \gamma;$$

(2) *if* $\gamma_l = 0$, *then*

$$\alpha(\gamma, v_B(\gamma; \gamma_l, v_l)) = \alpha(0, v_l), \qquad -\infty < \gamma < \infty;$$

(3) *if* $\gamma_l < 0$, *then*

$$\alpha(\gamma, v_B(\gamma; \gamma_l, v_l)) = \alpha(\gamma_l, v_l), \qquad \gamma \leqq \gamma_l,$$

$$\frac{d}{d\gamma}\alpha(\gamma, v_B(\gamma; \gamma_l, v_l)) = \frac{(C(\gamma) - C(\gamma, \gamma_l))^2}{2C(\gamma, \gamma_l)} > 0, \qquad \gamma_l < \gamma < \Gamma(\gamma_l),$$

and

$$\alpha(\gamma, v_B(\gamma; \gamma_l, v_l)) = \alpha(\Gamma(\gamma_l), v_B(\Gamma(\gamma_l); \gamma_l, v_l)), \qquad \Gamma(\gamma_l) \leqq \gamma.$$

(c) *if* $\begin{Bmatrix} \gamma_l \geqq 0 \\ \gamma_l \leqq 0 \end{Bmatrix}$ *and* $\begin{Bmatrix} \bar{\gamma} \leqq \gamma_l \\ \gamma_l \leqq \bar{\gamma} \end{Bmatrix}$, *then*

$$\begin{Bmatrix} v_B(\gamma; \bar{\gamma}, v_B(\bar{\gamma}; \gamma_l, v_l)) \geqq v_B(\gamma; \gamma_l, v_l) \\ v_B(\gamma; \bar{\gamma}, v_B(\bar{\gamma}; \gamma_l, v_l)) \leqq v_B(\gamma; \gamma_l, v_l) \end{Bmatrix} \textit{ for all } \begin{Bmatrix} \gamma \leqq \bar{\gamma} \\ \bar{\gamma} \leqq \gamma \end{Bmatrix}. \qquad \square$$

We shall prove Theorem 2.2 (see Figure 2.3).

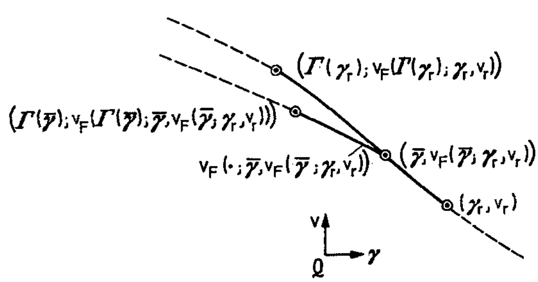

Fig. 2.3

Proof of Theorem 2.2. Parts (a) and (b) of the theorem are routine. We shall limit ourselves to establishing part (c). We shall show that if $\gamma_r > 0$ and $\bar{\gamma} < \gamma_r$, then

$$v_F(\gamma; \bar{\gamma}, v_F(\bar{\gamma}; \gamma_r, v_r) \leqq v_F(\gamma; \gamma_r, v_r), \qquad \gamma \leqq \bar{\gamma}.$$

The proof of the remaining case is left to the reader.

The following situations must be considered separately:

(1) $0 \leqq \bar{\gamma} < \gamma_r$ and either

(A) $\Gamma(\bar{\gamma}) < \gamma \leqq \bar{\gamma}$ or

(B) $\Gamma(\gamma_r) < \gamma \leqq \Gamma(\bar{\gamma})$ or

(C) $\gamma \leqq \Gamma(\gamma_r)$;

(2) $\Gamma(\gamma_r) \leqq \bar{\gamma} < 0$ and either

(A) $\Gamma(\gamma_r) < \gamma \leqq \bar{\gamma}$ or

(B) $\gamma \leqq \Gamma(\gamma_r)$;

(3) $\gamma \leqq \bar{\gamma} < \Gamma(\gamma_r)$.

If (1-A) holds, then

$$v_F(\gamma; \gamma_r, v_r) - v_F(\gamma; \bar{\gamma}, v_F(\bar{\gamma}; \gamma_r, v_r))$$
$$= (\gamma_r - \gamma)\left[\sqrt{\frac{\sigma(\gamma_r)-\sigma(\gamma)}{\gamma_r - \gamma}} - \frac{(\gamma_r - \bar{\gamma})}{(\gamma_r - \gamma)}\sqrt{\frac{\sigma(\gamma_r)-\sigma(\bar{\gamma})}{\gamma_r - \bar{\gamma}}} - \frac{(\bar{\gamma}-\gamma)}{(\gamma_r-\gamma)}\sqrt{\frac{\sigma(\bar{\gamma})-\sigma(\gamma)}{\bar{\gamma}-\gamma}}\right].$$

Since $(\gamma_r - \gamma) > 0$, it suffices to show that the term in brackets is non-negative. But this follows from the identities

$$\frac{(\gamma_r - \bar{\gamma})}{(\gamma_r - \gamma)} > 0, \quad \frac{(\bar{\gamma}-\gamma)}{(\gamma_r - \gamma)} > 0, \quad \frac{(\gamma_r - \bar{\gamma})}{(\gamma_r - \gamma)} + \frac{(\bar{\gamma}-\gamma)}{(\gamma_r - \gamma)} = 1,$$

and

$$\frac{\sigma(\gamma_r)-\sigma(\gamma)}{\gamma_r - \gamma} = \frac{(\gamma_r - \bar{\gamma})}{(\gamma_r - \gamma)}\,\frac{\sigma(\gamma_r)-\sigma(\bar{\gamma})}{\gamma_r - \bar{\gamma}} + \frac{(\bar{\gamma}-\gamma)}{(\gamma_r - \gamma)}\,\frac{\sigma(\bar{\gamma})-\sigma(\gamma)}{\bar{\gamma}-\gamma}$$

and the inequality

$$\alpha\sqrt{x} + (1-\alpha)\sqrt{y} \leqq \sqrt{\alpha x + (1-\alpha) y}$$

for any $0 \leqq \alpha \leqq 1$ and any non-negative x and y.

If (1-B) holds, then

$$v_F(\gamma; \gamma_r, v_r) - v_F(\gamma; \bar{\gamma}, v_F(\bar{\gamma}; \gamma_r, v_r))$$
$$= (\gamma_r - \gamma)\sqrt{\frac{\sigma(\gamma_r)-\sigma(\gamma)}{\gamma_r - \gamma}} - (\gamma_r - \bar{\gamma})\sqrt{\frac{\sigma(\gamma_r)-\sigma(\bar{\gamma})}{\gamma_r - \bar{\gamma}}} - (\bar{\gamma} - \Gamma(\bar{\gamma}))\sqrt{\frac{\sigma(\bar{\gamma})-\sigma(\Gamma(\bar{\gamma}))}{\bar{\gamma}-\Gamma(\bar{\gamma})}}$$
$$+ \int_{\Gamma(\bar{\gamma})}^{\gamma} \sqrt{\sigma'(s)}\,ds, \qquad \Gamma(\gamma_r) < \gamma \leqq \Gamma(\bar{\gamma}).$$

The preceding calculations imply that

$$v_F(\Gamma(\bar{\gamma}); \gamma_r, v_r) - v_F(\Gamma(\bar{\gamma}); \bar{\gamma}, v_F(\bar{\gamma}; \gamma_r, v_r)) > 0.$$

This, together with

$$\frac{d}{d\gamma}\left(v_F(\gamma; \gamma_r, v_r) - v_F(\gamma; \bar{\gamma}, v_F(\bar{\gamma}; \gamma_r, v_r))\right) = -\frac{(C(\gamma) - C(\gamma_r, \gamma))^2}{2C(\gamma_r, \gamma)} < 0,$$
$$\Gamma(\gamma_r) < \gamma \leqq \Gamma(\bar{\gamma}),$$

now yields the result.

If (1-C) is true, similiar computations show that

$$v_F(\gamma;\gamma_r,v_r)-v_F(\gamma;\bar\gamma,v_F(\bar\gamma;\gamma_r,v_r))=v_F(\Gamma(\gamma_r);\gamma_r,v_r)-v_F(\Gamma(\gamma_r);\bar\gamma,v_F(\bar\gamma;\gamma_r,v_r))$$

for $\gamma\leqq\Gamma(\gamma_r)$, and hence (1-B) implies the result.

We now turn our attention to case (2-A). We have

$$\begin{aligned} &v_F(\gamma;\gamma_r,v_r)-v_F(\gamma;\bar\gamma,v_F(\bar\gamma;\gamma_r,v_r))\\ &\quad=(\gamma_r-\gamma)\sqrt{\frac{\sigma(\gamma_r)-\sigma(\gamma)}{\gamma_r-\gamma}}-(\gamma_r-\bar\gamma)\sqrt{\frac{\sigma(\gamma_r)-\sigma(\bar\gamma)}{\gamma_r-\bar\gamma}}\\ &\quad\quad+\int_{\bar\gamma}^{\gamma}\sqrt{\sigma'(s)}\,ds;\qquad \Gamma(\gamma_r)<\gamma\leqq\bar\gamma<0.\end{aligned}$$

The inequality now follows from the observation that

$$v_F(\bar\gamma;\gamma_r,v_r)-v_F(\bar\gamma;\bar\gamma,v_F(\bar\gamma;\gamma_r,v_r))=0$$

and the identity

$$\frac{d}{d\gamma}\left(v_F(\gamma;\gamma_r,v_r)-v_F(\gamma;\bar\gamma,v_F(\bar\gamma;\gamma_r,v_r))\right)=-\frac{(C(\gamma)-C(\gamma_r,\gamma))^2}{2C(\gamma_r,\gamma)}<0.$$

If (2-B) holds, then

$$v_F(\gamma;\gamma_r,v_r)-v_F(\gamma;\bar\gamma,v_F(\bar\gamma;\gamma_r,v_r))=v_F(\Gamma(\gamma_r);\gamma_r,v_r)-v_F(\bar\gamma;\gamma_r,v_r))>0$$

for $\gamma\leqq\Gamma(\gamma_r)\leqq\gamma$; while if (3) holds, then

$$v_F(\gamma;\gamma_r,v_r)-v_F(\gamma;\bar\gamma,v_F(\bar\gamma;\gamma_r,v_r))=0,\qquad \gamma\leqq\bar\gamma<\Gamma(\gamma_r).\quad\square$$

C. Riemann Problems. The following theorems deal with the Riemann Problem:

$$\frac{\partial\gamma}{\partial t}-\frac{\partial v}{\partial x}=0\quad\text{and}\quad\frac{\partial v}{\partial t}-\frac{\partial\sigma(\gamma)}{\partial x}=0,\qquad -\infty<x<\infty,\ t>0,\tag{E}$$

and

$$(\gamma,v)(x,0)=\begin{cases}(\gamma_1,v_1), & x<0,\\ (\gamma_2,v_2), & x>0.\end{cases}\tag{2.26}$$

Theorems 2.4–2.6 are essentially proved in [4] and [5]. They are stated merely for the readers' convenience. Theorems 2.7 and 2.8 are new. These results will be needed when we construct the approximate solutions described in the Introduction.

We introduce the following notation*. Given (γ_1,v_1) with $\gamma_1>0$, we let

$$R(\gamma_1,v_1)=\left\{(\gamma,v)\,|\,\gamma_1\leqq\gamma\quad\text{and}\quad-(\gamma-\gamma_1)\,C(\gamma_1,\gamma)\leqq v-v_1\leqq\int_{\gamma_1}^{\gamma}C(s)\,ds\right\}\tag{2.27}$$

and

$$\tilde R(\gamma_1,v_1)=\left\{(\gamma,v)\,|\,0\leqq\gamma\leqq\gamma_1\quad\text{and}\quad(\gamma-\gamma_1)\,C(\gamma_1,\gamma)\leqq v-v_1\leqq-\int_{\gamma_1}^{\gamma}C(s)\,ds\right\}.\tag{2.28}$$

* These regions are the same as those considered in [5].

Similarly, given (γ_2, v_2) with $\gamma_2 < 0$, we let

$$L(\gamma_2, v_2) = \left\{(\gamma, v) \mid \gamma \leqq \gamma_2 \quad \text{and} \quad (\gamma - \gamma_2)\, C(\gamma, \gamma_2) \leqq v - v_2 \leqq -\int_{\gamma_2}^{\gamma} C(s)\, ds\right\} \tag{2.29}$$

and

$$\begin{aligned} U(\gamma_2, v_2) &= \{(\gamma, v) \mid \gamma \leqq 0, \alpha(\gamma, v) \geqq \alpha(\gamma_2, v_2) \quad \text{and} \quad \beta(\gamma, v) \geqq \beta(\gamma_2, v_2)\} \\ &= \left\{(\gamma, v) \mid \gamma \leqq 0 \quad \text{and} \quad \left|\int_{\gamma_2}^{\gamma} C(s)\, ds\right| \leqq v - v_2\right\}. \end{aligned} \tag{2.30}$$

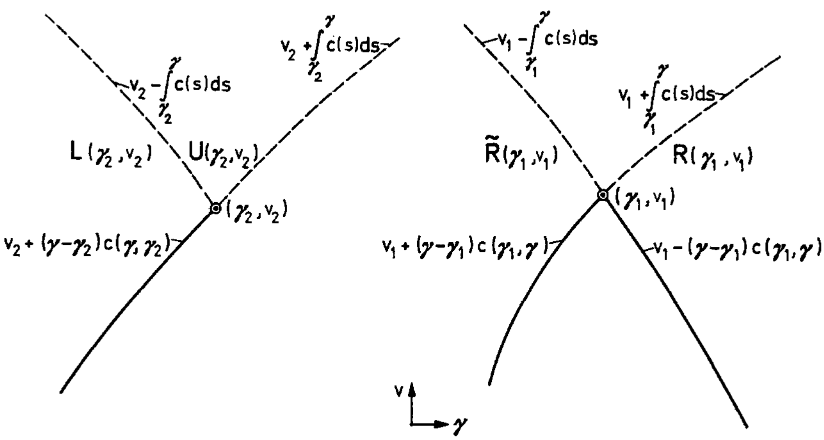

Fig. 2.4

Theorem 2.4. *Let (γ_1, v_1), $\gamma_1 > 0$, be an arbitrary point in the $\gamma - v$ plane, and let*

$$(\gamma_2, v_2) \in \begin{Bmatrix} R(\gamma_1, v_1) \\ \tilde{R}(\gamma_1, v_1) \end{Bmatrix}.$$

Then, $\begin{Bmatrix} R(\gamma_2, v_2) \\ \tilde{R}(\gamma_2, v_2) \end{Bmatrix} \subset \begin{Bmatrix} R(\gamma_1, v_1) \\ \tilde{R}(\gamma_1, v_1) \end{Bmatrix}$, *and the Riemann Problem* (E) *and* (2.26) *is solvable with a* $\begin{Bmatrix} \textit{forward} \\ \textit{backward} \end{Bmatrix}$ *shock and* $\begin{Bmatrix} \textit{backward} \\ \textit{forward} \end{Bmatrix}$ *simple wave. Moreover,*

(1) *for each* $t \in 0$, $\begin{Bmatrix} \gamma(\cdot, t) \text{ is increasing and } \alpha(\gamma, v)(\cdot, t) \text{ is decreasing} \\ \gamma(\cdot, t) \text{ and } \beta(\gamma, v)(\cdot, t) \text{ are decreasing} \end{Bmatrix}$ *in* x,

(2) *for any* $x_1 < x_2$, $(\gamma, v)(x_2, t) \in \begin{Bmatrix} R(\gamma, v)(x_1, t) \\ \tilde{R}(\gamma, v)(x_1, t) \end{Bmatrix}$, *and*

(3) $v(x, t) \begin{Bmatrix} \geqq \min(v_1, v_2) \\ \leqq \max(v_1, v_2) \end{Bmatrix}$. □

Theorem 2.5. *Let (γ_2, v_2), $\gamma_2 < 0$, be an arbitrary point in the $\gamma - v$ plane, and let $(\gamma_1, v_1) \in L(\gamma_2, v_2)$. Then $L(\gamma_1, v_1) \subset L(\gamma_2, v_2)$ and the Riemann Problem* (E) *and* (2.26) *is solvable with a backward shock and a forward simple wave.*

Moreover,

(1) *for each* $t>0$, $\gamma(\cdot,t)$ *and* $\beta(\gamma,v)(\cdot,t)$ *are increasing on* x,

(2) *for any* $x_1<x_2$, $(\gamma,v)(x_1,t)\in L(\gamma,v)(x_2,t)$, *and*

(3) $v(x,t)\geqq\min(v_1,v_2)$. □

Theorem 2.6. *Let* (γ_2,v_2), $\gamma_2<0$, *be an arbitrary point in the* $\gamma-v$ *plane and let* $(\gamma_1,v_1)\in U(\gamma_2,v_2)$. *Then* $U(\gamma_1,v_1)\subset U(\gamma_1,v_1)$ *and the Riemann Problem* (E) *and* (2.26) *is solvable with a backward and a forward simple wave. In addition*

(1) *for each* t, $\alpha(\gamma,v)(\cdot,t)$ *and* $\beta(\gamma,v)(\cdot,t)$ *are decreasing in* x,

(2) *for any* $x_1<x_2$, $(\gamma,v)(x_1,t)\in U(\gamma,v)(x_2,t)$, *and*

(3) $\gamma(x,t)\leqq\max(\gamma_1,\gamma_2)$ *and* $v_2\leqq v(x,t)\leqq v_1$. □

Proof of Theorem 2.6. The existence of the solution just described depends on the unique solvability of the equation

$$v_F(\gamma_I;\gamma_2,v_2)=v_B(\gamma_I;\gamma_1,v_1).$$

That this equation is solvable follows from the fact that

$$\lim_{\gamma\to\infty}\int_0^\gamma C(s)\,ds=+\infty.$$

We conclude this section with a discussion of the Riemann Problem (E) and (2.26) when the points (γ_1,v_1) and (γ_2,v_2) satisfy one of the following inequalities:

$$\gamma_1<0<\gamma_2,\quad v_1\leqq v_F(\gamma_1;\gamma_2,v_2)\quad\text{and}\quad v_2\leqq v_B(\gamma_2;\gamma_1,v_1) \tag{2.31}$$

or

$$\gamma_1<0<\gamma_2\quad\text{and}\quad v_1\geqq v_F(\gamma_1;\gamma_2,v_2). \tag{2.32}$$

We shall treat the situation described by (2.31) first. Condition (2.31), together with the fact that v_F is decreasing and v_B is increasing in γ, implies that the equation

$$v_F(\gamma^*;\gamma_2,v_2)=v_B(\gamma^*;\gamma_1,v_1) \tag{2.33}$$

has a unique solution in $[\gamma_1,\gamma_2]$. Throughout γ^* will denote this solution and

$$v^*\overset{\text{def}}{=}v_F(\gamma^*;\gamma_2,v_2). \tag{2.34}$$

The solution to the Riemann Problem will depend on whether $\gamma^*\leqq 0$ or $\gamma^*>0$. We consider the case $\gamma_1\leqq\gamma^*\leqq 0$ first. The solution to the left of $x=0$ is a back shock

$$(\gamma,v)(x,t)=\begin{cases}(\gamma_1,v_1), & -\infty<x\leqq -C(\gamma_1,\gamma^*)t,\\ (\gamma^*,v^*), & -C(\gamma_1,\gamma^*)t<x\leqq 0,\quad t\geqq 0.\end{cases}$$

The solution to the right of $x=0$ depends on whether $\gamma^*\leqq\Gamma(\gamma_2)$ or $\Gamma(\gamma_2)<\gamma^*<0$. When $\gamma^*\leqq\Gamma(\gamma_2)$, the solution is a back facing contact discontinuity

$$(\gamma,v)(x,t)=\begin{cases}(\gamma^*,v^*), & 0\leqq x\leqq C(\gamma^*)t,\\ \left(\eta\left(\frac{x}{t}\right),v_F\left(\eta\left(\frac{x}{t}\right);\gamma_2,v_2\right)\right), & C(\gamma^*)t<x\leqq C(\Gamma(\gamma_2))t,\\ (\gamma_2,v_2), & C(\Gamma(\gamma_2))t<x<\infty,\quad t\geqq 0.\end{cases}$$

Here $\eta\left(\frac{x}{t}\right)$ is the solution of

$$C(\eta)=\frac{x}{t}, \frac{x}{t}\in(C(\gamma^*), \quad c(\Gamma(\gamma_2))]$$

in the interval $(\gamma^*, \Gamma(\gamma_2)]$. When $\Gamma(\gamma_2)<\gamma^*\leqq 0$, the solution is a back shock

$$(\gamma, v)(x, t)=\begin{cases}(\gamma^*, v^*), & 0\leqq x\leqq C(\gamma^*, \gamma_2)t, \\ (\gamma_2, v_2), & C(\gamma^*, \gamma_2)t<x<\infty, \ t\geqq 0.\end{cases}$$

The construction of the solution when $0<\gamma^*\leqq\gamma_2$ is handled in like manner.

The following general statements about the solutions just constructed are easily verified:

(1) $\gamma_1\leqq\gamma(x, t)\leqq\gamma_2$ and $\min(v_1, v_2)\leqq v(x, t)\leqq v^*$ (see (2.33) and (2.34)), and

(2) for each $t>0$, $\gamma(\cdot, t)$ is an increasing function of x and

$$\operatorname*{var}_{(-\infty,\infty)} v(\cdot, t)\leqq 2(v^*-\min(v_1, v_2)).$$

More important we have

Theorem 2.7. *Let* (γ_1, v_1) *and* (γ_2, v_2) *satisfy* (2.31), *and let* $(\tilde{\gamma}, \tilde{v})$ *denote the solution of the Riemann Problem* (E) *and* (2.26). *Then, for any points* $(\bar{\gamma}_1, \bar{v}_1)\in\Gamma(\gamma_1, v_1)$ (*see* (2.29)) *and* $(\bar{\gamma}_2, \bar{v}_2)\in R(\gamma_2, v_2)$ (*see* (2.27)) *and any* (y, t) *with* $t\geqq 0$

(a) $(\bar{\gamma}_1, \bar{v}_1)\in L(\tilde{\gamma}(y, t), \tilde{v}(y, t)), \quad \tilde{v}(y, t)\leqq v_F(\tilde{\gamma}(y, t); \bar{\gamma}_2, \bar{v}_2)$ *and*
$\bar{v}_2\leqq v_B(\bar{\gamma}_2; \tilde{\gamma}(y, t), \tilde{v}(y, t))$ *if* $\tilde{\gamma}(y, t)\leqq 0$;

(b) $(\bar{\gamma}_2, \bar{v}_2)\in R(\tilde{\gamma}(y, t), \tilde{v}(y, t)), \quad \bar{v}_1\leqq v_F(\bar{\gamma}_1; \tilde{\gamma}(y, t), \tilde{v}(y, t))$ *and*
$\tilde{v}(y, t)\leqq v_B(\tilde{\gamma}(y, t); \bar{\gamma}_1, \bar{v}_1)$ *if* $\tilde{\gamma}(y, t)>0$;

(c) $\bar{v}_1\leqq v_F(\bar{\gamma}_1; \bar{\gamma}_2, \bar{v}_2), \bar{v}_2\leqq v_B(\bar{\gamma}_2; \bar{\gamma}_1, \bar{v}_1)$ *and*
$\bar{v}^*\overset{\text{def}}{=}v_F(\bar{\gamma}^*; \bar{\gamma}_2, \bar{v}_2)\geqq v^*=\max\limits_{(y,t)}\tilde{v}(y, t)$. Here $\bar{\gamma}^*\in[\bar{\gamma}_1, \bar{\gamma}_2]$
is the unique solution of

$$v_F(\bar{\gamma}^*; \bar{\gamma}_2, \bar{v}_2)=v_B(\bar{\gamma}^*; \bar{\gamma}_1, \bar{v}_1) \tag{2.35}$$

and v^* *is defined in* (2.34) (*see Figure* 2.5). □

Proof of Theorem 2.7. We shall establish assertion (a); the proofs of (b) and (c) are similar.

We must consider the various possibilities for the intermediate value γ^* (see (2.33)):

Case 1 $\gamma_1\leqq\gamma^*\leqq 0$ and either (A) $\gamma^*\leqq\Gamma(\gamma_2)$ or (B) $\Gamma(\gamma_2)<\gamma^*\leqq 0$;

Case 2 $0<\gamma^*\leqq\gamma_2$ and either (A) $\gamma^*\geqq\Gamma(\gamma_1)$ or (B) $0<\gamma^*<\Gamma(\gamma_1)$.

If (1-A) holds, then the hypothesis $\tilde{\gamma}(y, t)\leqq 0$ implies

$$(\tilde{\gamma}, \tilde{v})(y, t)=\begin{cases}(\gamma_1, v_1) & \text{or} \\ (\eta, v_F(\eta; \gamma_2, v_2)) & \text{where} \quad \gamma^*\leqq\eta\leqq\Gamma(\gamma_2).\end{cases}$$

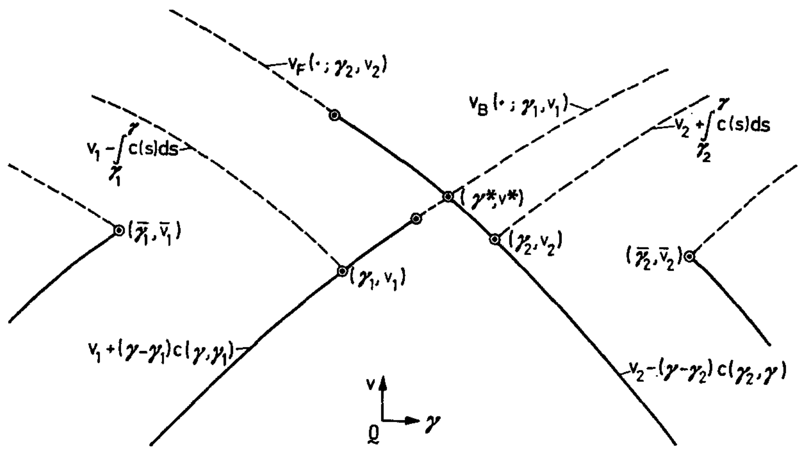

Fig. 2.5

When $(\tilde{\gamma}(y,t),\tilde{v}(y,t))=(\gamma_1,v_1)$, assertion (a) is immediate. Now suppose $(\tilde{\gamma}(y,t),\tilde{v}(y,t))=(\eta,v_F(\eta;\gamma_2,v_2))$ and $\gamma^*\leqq\eta\leqq\Gamma(\gamma_2)$. To show that $(\bar\gamma_1,\bar v_1)\in L(\eta,v_F(\eta;\gamma_2,v_2))$, it suffices, by Theorem 2.5, to show that

$$(\gamma_1,v_1)\in L\big(\eta,v_F(\eta;\gamma_2,v_2)\big).$$

That $v_1\leqq v_F(\eta;\gamma_2,v_2)-\int_\eta^{\gamma_1}C(s)\,ds$ is a consequence of

$$\begin{aligned}v_F(\eta;\gamma_2,v_2)-\int_\eta^{\gamma_1}C(s)\,ds&=v_F(\eta;\gamma_2,v_2)-\int_\eta^{\gamma^*}C(s)\,ds-\int_{\gamma^*}^{\gamma_1}C(s)\,ds\\&=v_F(\gamma^*;\gamma_2,v_2)+\int_{\gamma_1}^{\gamma^*}C(s)\,ds\\&=v_B(\gamma^*;\gamma_1,v_1)+\int_{\gamma_1}^{\gamma^*}C(s)\,ds\\&=v_1+(\gamma^*-\gamma_1)\,C(\gamma^*,\gamma_1)+\int_{\gamma_1}^{\gamma^*}C(s)\,ds\end{aligned}$$

and the inequality $\gamma^*\geqq\gamma_1$. The inequality $v_1\geqq v_F(\eta;\gamma_2,v_2)+(\gamma_1-\eta)\,C(\gamma_1,\eta)$ follows from

$$\begin{aligned}v_1&=v_F(\eta;\gamma_2,v_2)+\int_{\gamma^*}^{\eta}C(s)\,ds+(\gamma_1-\gamma^*)\,C(\gamma_1,\gamma^*)\\&\geqq v_F(\eta;\gamma_2,v_2)+(\gamma_1-\gamma^*)\,C(\gamma_1,\gamma^*)\end{aligned}$$

and from

$$(\gamma_1-\gamma^*)\,C(\gamma_1,\gamma^*)\geqq(\gamma_1-\eta)\,C(\gamma_1,\eta).$$

Our next task is to show that

$$v_F(\eta;\gamma_2,v_2)\leqq v_F(\eta;\bar\gamma_2,\bar v_2)\quad\text{and}\quad\bar v_2\leqq v_B\big(\bar\gamma_2;\eta,v_F(\eta;\gamma_2,v_2)\big)$$

when $\gamma^* \leqq \eta \leqq \Gamma(\gamma_2)$. The fact that $\bar{v}_2 \geqq v_2 - (\bar{\gamma}_2 - \gamma_2)\, C(\bar{\gamma}_2, \gamma_2)$ when $\gamma_2 \leqq \bar{\gamma}_2$ implies that for any $\gamma \leqq \bar{\gamma}_2$

$$v_F(\gamma; \bar{\gamma}_2, \bar{v}_2) \geqq v_F(\gamma; \bar{\gamma}_2, v_2 - (\bar{\gamma}_2 - \gamma_2)\, C(\bar{\gamma}_2, \gamma_2)).$$

The observation $v_2 = v_F(\gamma_2; \bar{\gamma}_2, v_2 - (\bar{\gamma}_2 - \gamma_2)\, C(\bar{\gamma}_2, \gamma_2))$, part (c) of Theorem 2.2, and the preceding inequality imply the first result.

We shall now show that $\bar{v}_2 \leqq v_B(\bar{\gamma}_2; \eta, v_F(\eta; \gamma_2, v_2))$. The hypothesis

$$\bar{v}_2 \leqq v_2 + \int_{\gamma_2}^{\bar{\gamma}_2} C(s)\, ds$$

and the fact that $v_F(\eta; \gamma_2, v_2) \geqq v_2$ when $\eta \leqq \gamma_2$ imply that

$$v_2 \leqq v_F(\eta; \gamma_2, v_2) \leqq v_B(\gamma_2; \eta, v_F(\eta; \gamma_2, v_2))$$

and hence that

$$\begin{aligned} \bar{v}_2 &\leqq v_B(\gamma_2; \eta, v_F(\eta; \gamma_2, v_2)) + \int_{\gamma_2}^{\bar{\gamma}_2} C(s)\, ds \\ &= v_B(\bar{\gamma}_2; \gamma_2, v_B(\gamma_2; \eta, v_F(\eta; \gamma_2, v_2))). \end{aligned}$$

But Theorem 2.3-(c) implies that

$$v_B(\bar{\gamma}_2; \gamma_2, v_B(\gamma_2; \eta, v_F(\eta; \gamma_2, v_2))) \leqq v_B(\bar{\gamma}_2; \eta, v_F(\eta; \gamma_2, v_2))$$

which together the preceding inequality yields the desired result.

The proof of (a) in the remaining cases is similar. □

We now turn our attention to the Riemann Problem (E) and (2.26) when the data satisfies (2.32). The following facts about the solution to this problem will

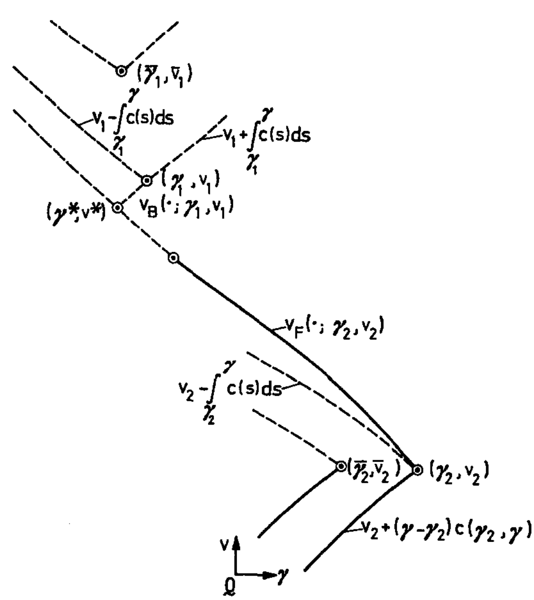

Fig. 2.6

now be summarized. For each t, $\alpha(\gamma, v)(\cdot, t)$ are decreasing in x. Moreover, the following estimates prevail:

$$\alpha(\gamma_2, v_2) \leqq \alpha(\gamma, v)(x, t) \leqq \alpha(\gamma_1, v_1), \tag{2.36}$$

$$\beta(\gamma_2, v_2) \leqq \beta(\gamma, v)(x, t) \leqq \beta(\gamma_1, v_1), \tag{2.37}$$

and

$$v_2 \leqq v(x, t) \leqq v_1. \tag{2.38}$$

We also have the following comparison theorem. Throughout, $(\gamma, v)(x, t)$ will denote the solution to the Riemann Problem just discussed, $(\bar{\gamma}_1, \bar{v}_1) \in U(\gamma_1, v_1)$ and $(\bar{\gamma}_2, \bar{v}_2) \in \tilde{R}(\gamma_2, v_2)$ (see equations (2.28) and (2.30)).

Theorem 2.8. *Let γ^* be the unique solution of*

$$v_B(\gamma^*; \gamma_1, v_1) = v_F(\gamma^*; \gamma_2, v_2). \tag{2.39}$$

If $\gamma^* \leqq \Gamma(\gamma_2)$, then

(a) $(\bar{\gamma}_1, \bar{v}_1) \in U(\gamma(y, t), v(y, t))$ *and* $v_F(\gamma(y, t); \bar{\gamma}_2, \bar{v}_2) \leqq v(y, t)$ *provided* $\gamma(y, t) \leqq 0$, *and*

(b) $v_F(\bar{\gamma}_1; \gamma(y, t), v(y, t)) \leqq \bar{v}_1$ *and* $(\bar{\gamma}_2, \bar{v}_2) \in \tilde{R}(\gamma(y, t), v(y, t))$ *provided* $\gamma(y, t) > 0$. *Moreover,* $v_F(\bar{\gamma}_1; \bar{\gamma}_2, \bar{v}_2) \leqq \bar{v}_1$.

If $\Gamma(\gamma_2) < \gamma^* < \gamma_1$ and $\alpha(\bar{\gamma}_1, \bar{v}_1) = \alpha(\gamma_1, v_1)$, then

(c) $(\bar{\gamma}_1, \bar{v}_1) \in U(\gamma(y, t), v(y, t))$ *and* $v_F(\gamma(y, t); \bar{\gamma}_2, \bar{v}_2) \leqq v(y, t)$ *provided* $\gamma(y, t) \leqq 0$, *and*

(d) $v_F(\bar{\gamma}_1; \gamma(y, t), v(y, t)) \leqq \bar{v}_1$ *and* $(\bar{\gamma}_2, \bar{v}_2) \in \tilde{R}(\gamma(y, t), v(y, t))$ *provided* $\gamma(y, t) > 0$. *Moreover,* $v_F(\bar{\gamma}_1; \bar{\gamma}_2, \bar{v}_2) \leqq \bar{v}_1$.

Proof. The techniques used to establish this theorem are similar to those employed in proving Theorem 2.7. We leave the details to the reader (see Figure 2.6).

3. Approximate Solutions and A Priori Estimates

We now turn to the initial value problem

$$\frac{\partial \gamma}{\partial t} - \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial t} - \frac{\partial \sigma(\gamma)}{\partial x} = 0, \qquad -\infty < x < \infty \quad \text{and} \quad t > 0, \tag{E}$$

$$(\gamma, v)(x, 0) = \begin{Bmatrix} (\gamma_0, v_0), & x < 0 \\ (\gamma_1, v_1), & 0 < x < 1 \\ (\gamma_2, v_2), & 1 < x \end{Bmatrix}. \tag{IC}$$

Again, (γ_0, v_0) is connected to (γ_1, v_1) by a forward wave and (γ_1, v_1) to (γ_2, v_2) by either a forward or a backward wave. Moreover, not all the numbers γ_i are of the same sign. For definiteness we shall assume that $\gamma_0 \leqq 0$.

A number of configurations of the data are possible, and we shall enumerate them. For each configuration we shall describe the solution to the Riemann Problems at $x=0$ and $x=1$. This serves to describe the type of interaction generated when the forward wave from $x=0$ overtakes the wave from $x=1$. Throughout

$$v_0 = v_F(\gamma_0; \gamma_1, v_1), \quad \text{and} \quad \gamma_0 \leqq 0. \tag{3.1}$$

Case 1 ($\gamma_1=0$, $\gamma_2\geqq 0$, and $v_2=v_B(\gamma_2;\gamma_1,v_1)$). The solution to the Riemann Problem at $x=0$ is a forward facing simple wave and the solution to the Riemann Problem at $x=1$ is a backward facing simple wave.

Case 2 ($\gamma_1>0$).

(A) ($\Gamma(\gamma_1)\leqq\gamma_0\leqq 0$). The solution to the Riemann Problem at $x=0$ is a forward shock.

(B) ($\gamma_0\leqq\Gamma(\gamma_1)$). The solution to the Riemann Problem at $x=0$ is a forward facing contact discontinuity.

(a) ($\gamma_1<\gamma_2$ and $v_2=v_B(\gamma_2;\gamma_1,v_1)$). The solution to the Riemann Problem at $x=1$ is a backward facing simple wave.

(b) ($\Gamma(\gamma_1)<\gamma_2<\gamma_1$ and $v_2=v_B(\gamma_2;\gamma_1,v_1)$). The solution to the Riemann Problem at $x=1$ is a backward shock.

(c) ($\gamma_2\leqq\Gamma(\gamma_1)$ and $v_2=v_B(\gamma_2;\gamma_1,v_1)$). The solution to the Riemann Problem at $x=1$ is a backward facing contact discontinuity.

(d) ($\gamma_1<\gamma_2$ and $v_1=v_F(\gamma_1;\gamma_2,v_2)$). The solution to the Riemann Problem at $x=1$ is a forward shock.

(e) ($0\leqq\gamma_2<\gamma_1$ and $v_1=v_F(\gamma_1;\gamma_2,v_2)$). The solution to the Riemann Problem at $x=1$ is a forward facing simple wave.

(f) ($\gamma_2<0$, $0<\Gamma(\gamma_2)\leqq\gamma_1$ and $v_1=v_F(\gamma_1;\gamma_2,v_2)$). The solution to the Riemann Problem at $x=1$ is a forward facing contact discontinuity.

(g) ($\gamma_2<0$, $0<\gamma_1<\Gamma(\gamma_2)$, and $v_1=v_F(\gamma_1;\gamma_2,v_2)$). The solution to the Riemann Problem at $x=1$ is a forward facing shock (compare with (d)).

The relevant problems to be considered are (E) together with the data 2-A-a through g and 2-B-a through g.

Case 3 ($\gamma_1<0$). Since we normalized the problem by insisting that $\gamma_0\leqq 0$, and since we require that not all the numbers γ_i have the same sign, we must take $\gamma_2\geqq 0$.

(A) ($\gamma_0<\gamma_1$). The solution to the Riemann Problem at $x=1$ is a forward facing simple wave.

(B) ($\gamma_1\leqq\gamma_0\leqq 0$). The solution to the Riemann Problem at $x=0$ is a forward facing shock.

(a) ($0\leqq\gamma_2<\Gamma(\gamma_1)$ and $v_2=v_B(\gamma_2;\gamma_1,v_1)$). The solution to the Riemann Problem at $x=1$ is a backward shock.

(b) ($0<\Gamma(\gamma_1)\leqq\gamma_2$ and $v_2=v_B(\gamma_2;\gamma_1,v_1)$). The solution to the Riemann Problem at $x=1$ is a backward facing contact discontinuity.

(c) ($\gamma_1\leqq\Gamma(\gamma_2)$ and $v_1=v_F(\gamma_1;\gamma_2,v_2)$). The solution to the Riemann Problem at $x=1$ is a forward facing contact discontinuity.

(d) ($\Gamma(\gamma_2)<\gamma_1\leqq 0$ and $v_1=v_F(\gamma_1;\gamma_2,v_2)$). The solution to the Riemann Problem at $x=1$ is a forward shock.

The relevant problems to be considered are (E) together with the data 3-A-a through d and 3-B-a through d.

Throughout the remainder of this section we shall adopt the following point of view. We assume that given the data for one of the elementary cases we have already solved the Riemann Problems at $x=0$ and $x=1$ at time $t=0$. These solutions are valid until the forward wave from $x=0$ collides with the wave from $x=1$. At that time, $t=t_c>0$, we are presented with the following continuation problem:

Find the functions γ and v on $t>t_c$ such that

$$\frac{\partial \gamma}{\partial t}-\frac{\partial v}{\partial x}=0 \quad \text{and} \quad \frac{\partial v}{\partial t}-\frac{\partial \sigma(\gamma)}{\partial x}=0, \qquad -\infty<x<\infty \quad \text{and} \quad t>t_c, \tag{E}$$

$$(\gamma, v)(x, t_c)=(\bar{\gamma}, \bar{v})(x, t_c), \qquad -\infty<x<\infty. \tag{CC}$$

Here $(\bar{\gamma}, \bar{v})(x, t_c)$ is the solution of the elementary problem at the collision time $t=t_c$.

Our goal is to show that the solutions to the elementary problems are continuable to all of $t\geqq 0$, that is, to show that the continuation problem (E) and (CC) has a solution (in the integrated by parts sense of (E-1) and (E-2)).

We first note that in the cases where the elementary waves from $x=0$ and $x=1$ are both shocks the solution is trivially continuable. The reason for this is that (E) and (CC) constitute a Riemann Problem. The assumption

$$\lim_{\gamma\to\infty} \int_0^{\gamma} C(s)\,ds=+\infty$$

guarantees that all such problems are solvable. This observation allows us to dismiss cases 2-A-b, 2-A-d, 2-A-g, 3-B-a, 3-B-d immediately.

We now turn to the remaining cases. We first introduce two classes of data G_1 and G_2. A routine calculation then shows that in cases 1, 2-A-a, 2-B-a, 2-B-d, and 3-A-a through 3-A-d the data for (E) and (CC) is in G_1 while in cases 2-A-c, 2-A-e, 2-B-b, 2-B-c, and 2-B-e the data for (E) and (CC) is in G_2. We shall then show that under the Glimm difference scheme data from G_1 (respectively G_2) gets mapped into itself. This fact implies a set of *a priori* estimates for the approximate solutions which are sufficient to establish the existence of a solution of (E) and (CC).

The data for 3-B-b is the mirror image of the data for 2-B-b and hence requires no special treatment. Cases 2-A-f, 2-B-f, 2-B-g, and 3-B-c must be handled separately.

Definition 3.1. We say that a pair (γ^0, v^0) is in G_1 if they are bounded regulated functions and there is a y^0 such that

(1) $\gamma^0(x)\leqq 0$ for $x<y^0$ and $\gamma^0(x)\geqq 0$ for $x>y^0$,

(2) for any $x_1<x_2<y^0$, $(\gamma^0, v^0)(x_1)\in L(\gamma^0(x_2), v^0(x_2))$ (see (2.29)),

(3) for any $y^0<x_1<x_2$, $(\gamma^0, v^0)(x_2)\in R(\gamma^0(x_1), v^0(x_1))$ (see (2.27)), and

(4) the numbers $(\gamma^0_-, v^0_-)(y^0)$ and $(\gamma^0_+, v^0_+)(y^0)$ satisfy

$$v^0_-(y^0)\leqq v_F(\gamma^0_-(y^0); \gamma^0_+(y^0), v^0_+(y^0)) \quad \text{and} \quad v^0_+(y^0)\leqq v_B(\gamma^0_+(y^0); \gamma^0_-(y^0), v^0_-(y^0))$$

(see Figure 2.5 and make the identification $(\gamma_2, v_2)=(\gamma_+^0, v_+^0)(y^0)$ and $(\gamma_1, v_1)=(\gamma_-^0, v_-^0)(y^0))$. □*

Definition 3.2. We say that a pair (γ^0, v^0) is in G_2 if they are bounded regulated functions and there is a y^0 such that

(1) $\gamma^0(x)\leqq 0$ for $x<y^0$ and $\gamma^0(x)\geqq 0$ for $x>y^0$,

(2) for any $x_2<x_1<y^0$, $(\gamma^0, v^0)(x_2)\in U(\gamma^0(x_1), v^0(x_1))$ (see (2.30)),

(3) for any $y^0<x_1<x_2$, $(\gamma^0, v^0)(x_2)\in \tilde{R}(\gamma^0(x_1), v^0(x_1))$ (see (2.28)), and

(4) the numbers $(\gamma_-^0, v_-^0)(y^0)$ and $(\gamma_+^0, v_+^0)(y^0)$ satisfy

$$v_F(\gamma_-^0(y^0);\, \gamma_+^0(y^0), v_+^0(y^0))\leqq v_-^0(y^0).$$

Moreover, if the number γ_0^*, defined as the unique solution of

$$v_F(\gamma_0^*;\, \gamma_+^0(y^0), v_+^0(y^0))=v_B(\gamma_0^*;\, \gamma_-^0(y^0), v_-^0(y^0)), \tag{3.2}$$

satisfies

$$\Gamma(\gamma_+^0(y^0))<\gamma_0^*,$$

then we insist that (γ^0, v^0) satisfy

$$\alpha(\gamma^0, v^0)(x)=\alpha(\gamma_-^0, v_-^0)(y^0)$$

for all $x<y^0$ (see Figure 2.6 and make the identification $\gamma_+^0, v_+^0(y^0)=(\gamma_2, v_2)$ and $(\gamma_-^0, v_-^0)(y^0)=(\gamma_1, v_1))$. □

We admit the possibility that y^0 is plus or minus infinity in either definition.

It is easily checked that if $(\gamma^0, v^0)\in G_1$ (respectively G_2), then they are of bounded variation on $(-\infty, \infty)$. Specifically, we have the following estimates.

Lemma 3.1. *Let* $(\gamma^0, v^0)\in G_1$. *Then*

(1) γ^0 *is increasing*;

(2) $\underline{v}\leqq v^0(x)\leqq \bar{v}$ *where*

$$\underline{v}\stackrel{\text{def}}{=}\min\begin{cases} v_-^0(y^0)+(\gamma^0(-\infty)-\gamma_-^0(y^0))\, C(\gamma^0(-\infty), \gamma_-^0(y^0)),\\ v_+^0(y^0)-(\gamma^0(\infty)-\gamma_+^0(y^0))\, C(\gamma_+^0(y^0), \gamma^0(\infty)),\end{cases}$$

$$\bar{v}\stackrel{\text{def}}{=}v_F\left(\bar{\gamma};\, \gamma^0(\infty), v_+^0(y^0)+\int_{\gamma_+^0(y^0)}^{\gamma^0(\infty)} C(s)\,ds\right), \quad \textit{and}$$

$\bar{\gamma}$ *is the unique solution*

$$v_B\left(\bar{\gamma};\, \gamma^0(-\infty), v_-^0(y^0)-\int_{\gamma_-^0(y^0)}^{\gamma^0(-\infty)} C(s)\,ds\right) = v_F\left(\bar{\gamma};\, \gamma^0(\infty), v_+^0(y^0)+\int_{\gamma_+^0(y^0)}^{\gamma^0(\infty)} C(s)\,ds\right);$$

* Recall, a function f is regulated if for every x the limits $f_+(x)=\lim_{\substack{y\to x\\ y>x}} f(y)$ and $f_-(x)=\lim_{\substack{y\to x\\ y<x}} f(y)$ exist. If $f_+(x)\neq f_-(x)$, we shall agree that $f(x)=f_-(x)$.

and

(3) $$\operatorname*{var}_{(-\infty,\infty)} v^0(\cdot) \leqq 4(\bar{v}-\underline{v}) + 2\int_{\gamma^0(-\infty)}^{\gamma^0(+\infty)} C(s)\,ds.$$

For functions in G_1 we also have the following comparison result.

Lemma 3.2. *Let* $(\gamma^0, v^0) \in G_1$, *let* $\{x_i\}_{i=0,\ldots}$ *be a countable collection of points satisfying* $x_i < x_{i+1}$, *and let*

$$(\bar{\gamma}^0, \bar{v}^0)(x) = \begin{cases} (\gamma^0, v^0)(x), & -\infty < x \leqq x_0 \\ (\gamma^0, v^0)(\xi_i), & x_i < x \leqq x_{i+1}, \quad i=0,1,\ldots, \\ (\gamma^0, v^0)(x), & (x_\infty \overset{\text{def}}{=} \lim\limits_{i\to\infty} x_i) < x < \infty \end{cases}$$

where the numbers ξ_i *are in* $(x_i, x_{i+1}]$. *Then* $(\bar{\gamma}^0, \bar{v}^0) \in G_1$ (i.e., $\exists$ *a number* $\bar{y}^0 \ni$ *conditions* 1–4 *of Definition* 3.1 *holds*) *and in addition* $(\bar{\gamma}^0, \bar{v}^0)$ *satisfies the same bounds as* (γ^0, v^0).

Similar statements hold for functions in G_2.

Lemma 3.3. *Let* $(\gamma^0, v^0) \in G_2$. *Then*

(1) $\beta(\gamma^0, v^0)(\cdot)$ *is decreasing*;

(2) $\underline{\underline{v}} \leqq v^0(x) \leqq v^0(\infty)$ *where*

$$\underline{\underline{v}} = v^0_+(y^0) + \big(\gamma^0(\infty) - \gamma^0_+(y^0)\big)\, C\big(\gamma^0(\infty), \gamma^0_+(y^0)\big);$$

(3) $\gamma^0(x) \leqq \gamma^0_+(y^0)$; *and*

(4) $$\operatorname*{var}_{(-\infty,\infty)} v^0(\cdot) \leqq v^0(-\infty) - \underline{\underline{v}} + \beta(\gamma^0_+, v^0_+)(y^0) - \beta(\gamma^0, v^0)(\infty) + \int_{\gamma^0(\infty)}^{\gamma^0(y^0)} C(s)\,ds$$
$$\leqq 2\left[v^0(-\infty) - \underline{\underline{v}} + \int_{\gamma^0(\infty)}^{\gamma^0_+(y^0)} C(s)\,ds\right].\text{*}$$

Lemma 3.4. *Let* $(\gamma^0, v^0) \in G_2$, *let* $\{x_i\}_{i=0\ldots}$ *be a countable collection of points satisfying* $x_i < x_{i+1}$, *and let*

$$(\bar{\gamma}^0, \bar{v}^0)(x) = \begin{cases} (\gamma^0, v^0)(x), & -\infty < x \leqq x^0, \\ (\gamma^0, v^0)(\xi_i), & x_i < x \leqq x_{i+1}, \quad i=0,1,\ldots \\ (\gamma^0, v^0)(x), & (x_\infty \overset{\text{def}}{=} \lim\limits_{i\to\infty} x_i) < x < \infty \end{cases}$$

where the numbers ξ_i *are in* $(x_i, x_{i+1}]$. *Then* $(\bar{\gamma}^0, \bar{v}^0) \in G_2$ *and satisfies the same bounds as* (γ^0, v^0).

We shall now describe our procedure for constructing the approximate solutions to (E) and (CC) when the data $(\bar{\gamma}, \bar{v})(\cdot, t_c)$ is in G_1 or G_2. Throughout $\bar{c}$ will be a number satisfying

$$\bar{c} > C(0) = \max_{\gamma} C(\gamma),$$

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* Estimates (1)–(4) of Lemma 3.3 together with (2.17) imply an estimate for $\operatorname*{var}_{(-\infty,\infty)} \gamma^0(\cdot)$.

$h>0$ an arbitrary positive number, and $\Delta = \frac{h}{\bar{c}}$. h will be the distance between spacial mesh points and Δ the time step in our difference approximation.

We assume that our approximate solution, (γ^h, v^h), has been defined on the strip $t_c \leqq t \leqq t_c + n\Delta$ and that at time $t = t_c + n\Delta$, γ^h and v^h are piecewise constant functions, specifically

$$(\gamma^h, v^h)(x, t_c + n\Delta) = (\gamma^h_{k,n}, v^h_{k,n})$$

for $kh < x \leqq (k+2)h$ and $n+k$ an even integer. We let $(\Gamma^h_{k,n}, V^h_{k,n})$ be the solution of the Riemann Problem (E) with data

$$(\Gamma^h_{k,n}, V^h_{k,n})(x, t_c + n\Delta) = \begin{cases} (\gamma^h_{k-2,n}, v^h_{k-2,n}), & x \leqq kh \\ (\gamma^h_{k,n}, v^h_{k,n}), & kh < x. \end{cases}$$

The condition $\bar{c} > C(0)$ implies that for $t_c + n\Delta \leqq t \leqq t_c + (n+1)\Delta$ and k such that $n+k$ is an even integer

$$(\Gamma^h_{k,n}, V^h_{k,n})\big((k+1)h, t\big) = (\Gamma^h_{k+2,n}, V^h_{k+2,n})\big((k+1)h, t\big) = (\gamma^h_{k,n}, v^h_{k,n}).$$

Thus, the pair

$$(\Gamma^h, V^h)(x, t) \overset{\text{def}}{=} (\Gamma^h_{k,n}, V^h_{k,n})(x, t), \qquad (k-1)h < x \leqq (k+1)h$$

and

$$t_c + n\Delta \leqq t \leqq t_c + (n+1)\Delta$$

solves the initial value problem (E) with initial conditions

$$(\Gamma^h, V^h)(x, t_c + n\Delta) = (\gamma^h, v^h)(x, t_c + n\Delta), \qquad -\infty < x < \infty.$$

If x is such that $(\Gamma^h_-, V^h_-)(x, t) \neq (\Gamma^h_+, V^h_+)(x, t)$, we shall agree that $(\Gamma^h, V^h)(x, t) = (\Gamma^h_-, V^h_-)(x, t)$.

We define (γ^h, v^h) on $t_c + n\Delta \leqq t < t_c + (n+1)\Delta$ by $(\gamma^h, v^h) = (\Gamma^h, V^h)$. At time $t = t_c + (n+1)\Delta$, (γ^h, v^h) is defined according to the following rule: Pick a number α^{n+1} at random from $(-1, 1]$, and set

$$(\gamma^h, v^h)\big(x, t_c + (n+1)\Delta\big) = (\Gamma^h, V^h)\big((k + \alpha^{n+1})h, t_c + (n+1)\Delta\big)$$

for $(k-1)h < x \leqq (k+1)h$ and $n+k$ even (see Figure 3.1). Observe that by construction $(\gamma^h, v^h)(x, t) = (\gamma^h_-, v^h_-)(x, t)$ for all t.

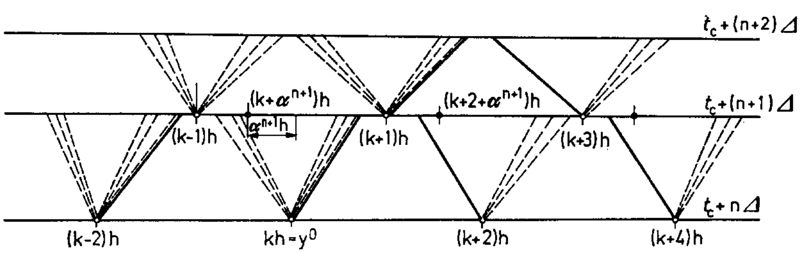

Fig. 3.1

We shall now obtain estimates for the approximate solutions of the problem

$$\frac{\partial \gamma}{\partial t}-\frac{\partial v}{\partial x}=0 \quad \text{and} \quad \frac{\partial v}{\partial t}-\frac{\partial \sigma(\gamma)}{\partial x}=0, \qquad -\infty<x<\infty \quad \text{and} \quad t>t_c \tag{E}$$

$$(\gamma, v)(x, t_c)=(\bar{\gamma}, \bar{v})(x, t_c) \tag{CC}$$

when the data $(\bar{\gamma}, \bar{v})(\cdot, t_c)$ is in G_1 or G_2.

The results of Lemma 3.3 tell us that if $(\bar{\gamma}, \bar{v})(\cdot, t_c)$ is in G_1, then the piecewise constant function

$$(\bar{\gamma}^h, \bar{v}^h)(x)=(\bar{\gamma}, \bar{v})\big((k+1+\alpha^0)\,h, t_c\big), \qquad kh<x\leqq(k+2)\,h,$$

where k is an even integer and α^0 a number chosen at random from $(-1, 1]$, is in G_1 and satisfies the same estimates as $(\bar{\gamma}, \bar{v})(\cdot, t_c)$. Moreover, the results of Theorems 2.4, 2.5, and 2.7 imply that the approximate solution $(\gamma^h, v^h)(\cdot, t)$ is in G_1 and satisfies the same estimates as $(\bar{\gamma}, \bar{v})(\cdot, t_c)$ for every t in $[t_c, t_c+\Delta)$. It then follows from an induction argument that for every $t>t_c$, $(\gamma^h, v^h)(\cdot, t)$ is in G_1 and satisfies the same bounds as the data $(\bar{\gamma}, \bar{v})(\cdot, t_c)$.

A similar statement applies when the data $(\bar{\gamma}, \bar{v})(\cdot, t_c)$ is in G_2. Here, the desired estimates are those of Lemma 3.4. They are obtained by appeal to Theorems 2.4, 2.6, and 2.8.

These estimates, together with the GLIMM convergence proof [6, pp. 711–715], guarantee that (E) and (CC) are solvable in Cases 1, 2-A-a, c, and e, 2-B-a through e, and 3-A-a through d.

We now turn to the remaining cases 2-A-f, 2-B-f, 2-B-g, and 3-B-c. All of these interactions are generated when the forward wave from $x=0$ overtakes the forward wave from $x=1$ (at (x_c, t_c)). In general we cannot be assured that the continuation problem (E) and (CC) is solvable on all of $t\geqq t_c$ yet certain affirmative statements can be made.

In particular, we can conclude that in cases 2-A-f and 2-B-f there is a time $T>t_c$ such that (E) and (CC) are solvable on the strip $t_c\leqq t<T$. T is that time where the forward wave from (x_c, t_c) intersects the straight line $1+C(\Gamma(\gamma_2))t$. T is bounded from below by $t_c+\dfrac{1+C\big(\Gamma(\gamma_2)\big)\,t_c-x_c}{C(0)-C\big(\Gamma(\gamma_2)\big)}$.

The validity of these statements follows from the observation that, for any $t_c\leqq t<T$ and any $x<1+C(\Gamma(\gamma_2))t$, the solution $(\gamma, v)(x, t)$ to (E) and (CC) agrees with the solution $(\tilde{\gamma}, \tilde{v})$ of (E) with the truncated data:

$$(\tilde{\gamma}, \tilde{v})(x, t_c)=\begin{cases}(\bar{\gamma}, \bar{v})(x, t_c), & x<1+C\big(\Gamma(\gamma_2)\big)\,t_c,\\ \big(\Gamma(\gamma_2), v_F(\Gamma(\gamma_2); \gamma_2, v_2)\big), & t+C\big(\Gamma(\gamma_2)\big)\,t_c<x.\end{cases}$$

That this latter problem is solvable follows from the fact that $(\tilde{\gamma}, \tilde{v})(\cdot, t_c)$ is in G_2 (see Figures 3.2 and 3.3).

At time $t=T$, the solution $(\gamma, v)(\cdot, T)$ has the following properties:

(1) $\gamma(x, T)\leqq 0$, $-\infty<x<\infty$,

(2) $(\gamma, v)(x, T)=(\gamma_2, v_2)$, $x>1+C(\Gamma(\gamma_2))T$,

(3) for every $x_1<x_2<1+C(\Gamma(\gamma_2))T$, $(\gamma, v)(x_1, T)\in U(\gamma, v)(x_2, T)$, and

(4) $(\gamma_-, v_-)(1+C(\Gamma(\gamma_2))T, T)\notin U(\gamma_2, v_2)$.

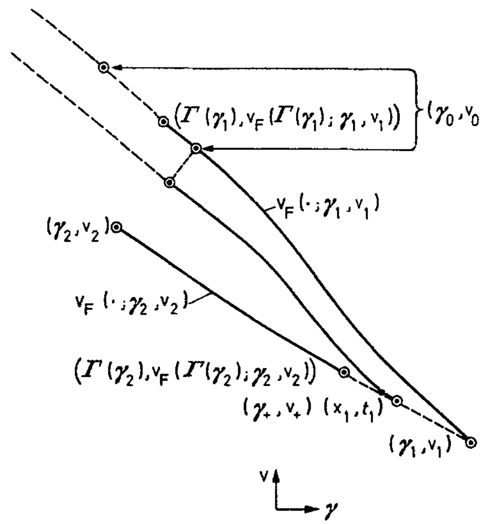

Fig. 3.2

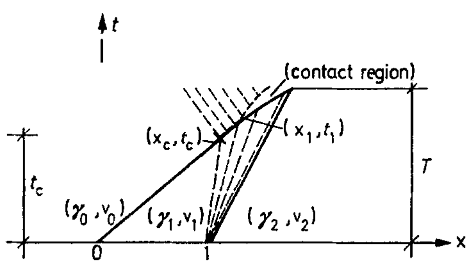

Fig. 3.3

If one now applies the Glimm difference scheme to the Cauchy Problem (E) with data $(\gamma, v)(\cdot, T)$, one obtains the preliminary estimate that the approximate solution (γ^h, v^h) satisfies $\gamma^h \leqq 0$ on $t>T$. Pointwise lower bounds for γ^h and upper and lower bounds for v^h are also obtainable. What is lacking is an estimate for the x variation of γ^h and v^h along the lines $t=\text{const}$. Without these estimates we cannot conclude that the approximate solutions converge to a generalized solution on $t \geqq T$.

With regard to the case 2-B-g, we may conclude that the problem (E) and (CC) is solvable on $t \geqq t_c$ provided the following additional conditions are met:

(A) $v_F(\Gamma(\gamma_1); \gamma_1, v_1) \leqq v_F(\Gamma(\gamma_1); \gamma_2, v_2)$ and

(B) $\gamma^* \leqq 0$ where γ^* is the unique solution of the equation

$$v_B(\gamma^*; \Gamma(\gamma_1), v_F(\Gamma(\gamma_1); \gamma_1, v_1)) = v_F(\gamma^*; \gamma_2, v_2).$$

If these last two conditions hold, then the data $(\bar{\gamma}, \bar{v})(\cdot, t_c)$ for the continuation problem is in G_3.

Definition 3.3. We say that a pair of functions (γ^0, v^0) is in G_3 if they are regulated, $\gamma^0(x) \leqq 0$ for all x, and there is a number y^0 such that

(1) for all $x > y^0$, $(\gamma^0, v^0)(x)$ is a constant state, say $(\gamma_+^0, v_+^0)(y^0)$,

(2) for any $x_1 < x_2 < y^0$, $(\gamma^0, v^0)(x_1) \in L(\gamma^0, v^0)(x_2)$, and

(3) for any $x < y^0$, $v^0(x) \leqq v_F(\gamma^0(x); \gamma_+^0(y^0), v_+^0(y^0))$

and $\gamma^*(x, y^0) \leqq 0$ where $\gamma^*(x, y^0)$ is defined as the unique solution of

$$v_B(\gamma^*; \gamma^0(x), v^0(x)) = v_F(\gamma^*; \gamma_+^0(y^0), v_+^0(y^0)).$$

It then follows from [4, pp. 150–3] and [5, pp. 313–21] that if $(\bar{\gamma}, \bar{v})(\cdot, t_c)$ is in G_3, then the approximate solutions (γ^h, v^h) to (E) and (CC) defined by the Glimm difference scheme are in G_3 for each $t \geqq t_c$. This observation yields a sufficient set of estimates to enable us to conclude that the approximates (γ^h, v^h) converge to a generalized solution of (E) and (CC) on all of $t \geqq t_c$.

If either (A) or (B) fails, we have not succeeded in obtaining estimates for the x variation of γ^h and v^h at each $t \geqq t_c$.

The analysis of 3-B-c is similar to that of 2-B-g.

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