COUNTING PROCESSES AND MARTINGALES†

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ABSTRACT. Counting Processes (CP), which are stochastic processes having right-continuous sample paths except for randomly located positive jumps of size one, are examined here in the light of a new notion resulting from the Doob-Meyer decomposition for supermartingales: the Integrated Conditional Rate (ICR). It is shown elsewhere ([4], [15]) that this ICR is particularly pertinent to the solution of the problems of filtering and detection for CP's.

The terminology ICR is motivated by the fact that when a CP(N_t) satisfies some sufficient conditions its ICR takes on the form $(\int_0^t \lambda_s ds)$ where (λ_t) is a nonnegative process called the conditional rate, satisfying $\lambda_t = \lim_{h \to 0} E[h^{-1}(N_{t+h} - N_t) | \mathcal{F}_t]$. Our approach, however, requires only the weak assumption that N_t is a. s finite for each t; there always exists an ICR while in general a conditional rate cannot be defined. Sufficient conditions for the existence of a conditional rate are presented.

Based on the character (e.g., totally inaccessible) of the stopping times defined by its jumps any CP is shown to be uniquely decomposable into the sum of a regular CP and an accessible CP. It is also demonstrated that each class is completely characterized by continuity properties of the ICR. CP's with independent increments are uniquely distinguished by a property of their ICR's: they are deterministic and given by the mean of the CP.

1.0 INTRODUCTION AND SUMMARY

1.1 INTRODUCTION. We are interested here in a description of Counting Processes which is appropriate to study the problems of detection and filtering. By a Counting Process we mean

Definition 1.1.1: A Counting Process (N_t) (hereafter abbreviated CP) is a stochastic process having sample paths which are zero at the time origin, right-continuous step functions with positive jumps of size one.

CP's have been examined in terms of counting properties (i.e. properties related to the number of jumps falling within specified subsets or more generally in terms of random measures) or in terms of interval properties (i.e., relative spacing between points, see e.g. [2]); but none of these approaches are specifically tailored to treat the problems of detection and filtering; hence we propose here a new approach.

The solution to these problems, which involves the computation of conditional expectations, is very much dependent on the information available to the observer or, in mathematical terms, on an increasing family of σ-algebras: the family of observation σ-algebras $(\mathcal{A}_t)$. Let (N_t) be a CP adapted to $(\mathcal{A}_t)$ and with the sole assumption that

(i) The random variable N_t is a.s. finite for each t. Then the Doob-Meyer decomposition for supermartingales associates to (N_t) a unique natural increasing process (A_t) dependent on the family $(\mathcal{A}_t)$. This process plays a central part in solving the detection and filtering problems (see [5], [15]). We call it the Integrated Conditional Rate (ICR) because under sufficient conditions (given in section 5.0) it takes on the form $(\int_0^t \lambda_s ds)$ where (λ_t) is a nonnegative process called the rate, satisfying $\lambda_t = \lim_{h \to 0} E[(N_{t+h} - N_t)h^{-1} | \mathcal{A}_t]$. This paper is concerned with a study of this notion of ICR. It should be strongly emphasized that an ICR always exists, while in general a conditional rate cannot be defined. Hence this study also generalizes and puts in the proper mathematical context previous works on modelling CP's by conditional rates (see e.g. [12], [13], [14], [3], [11], [1]).

1.2 SUMMARY. Let (N_t) be a CP satisfying assumption (i) and adapted to an increasing right-continuous family of σ-algebras $(\mathcal{F}_t)$.

It is shown in Section 3.0 (Section 2.0 is concerned with preliminaries) that the Doob-Meyer decomposition for supermartingales associates to the CP (N_t) a unique natural increasing process (A_t) which makes the process $(M_t \triangleq N_t - A_t)$ a local martingale with respect to $(\mathcal{F}_t)$. This decomposition $(N_t = M_t + A_t)$ is intuitively a decomposition into the part (M_t) which is not predictable and the ICR (A_t) which can be perfectly predicted. We refer to that as the separating property of the Doob-Meyer decomposition for CP's (see Section 4.3). Properties and examples of ICR's are exhibited.

In Section 4.0 three classes, regular, accessible and predictable CP's, are defined, these latter constituting a subclass of accessible CP's. We show that any counting process can be uniquely decomposed into the sum of two counting processes which are respectively regular and accessible. Regular counting processes have, loosely speaking, totally unexpected times of jump. Poisson processes are of this type. On the contrary, the times of jump of an accessible counting process can be predicted with some chance of success. A counting process which jumps with some positive probability at given fixed times is an example of this kind of processes. It is also demonstrated that each class is completely characterized by continuity properties of the ICR.

Finally in Section 5.0 we give sufficient conditions for the existence of conditional rates.

2.0 PRELIMINARIES. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. By (X_t) we denote a real valued stochastic process defined on $\mathbb{R}_+$, the positive real line. By a <u>right-(resp. left-) continuous process</u> we mean a process with right- (resp. left-) continuous sample paths. The process (X_t) is a <u>modification</u> of the process (Y_t) if $X_t = Y_t$ a. s. for each $t \in \mathbb{R}_+$. We do not distinguish here between modifications of the same process. If two right- (resp. left-) continuous processes (X_t) and (Y_t) are modifications of each other then we also have $P\{X_t = Y_t, t \in \mathbb{R}_+\} = 1$ so that we can safely use the simplified notation $X_t = Y_t$ a. s.

Let $(\mathcal{F}_t)$ be an increasing, right-continuous[1] family of σ-subalgebras of $\mathcal{F}$, with $\mathcal{F}_0$ containing all the P negligible sets. In particular we often consider the family $(\sigma(N_u, 0 \leq u \leq t))$, <u>denoted $(\mathcal{N}_t)$</u>, generated by a CP(N_t) up to and at time t. The notions of stopping times and martingales are assumed known. The basic references for this material are [9] and [7]. We recall simply some definitions and facts useful in the context of this study.

- (a) The family $(\mathcal{F}_t)$ is said to be <u>free of times of discontinuity</u> if for every increasing sequence (S_n) of $(\mathcal{F}_t)$ stopping times

$$\mathcal{F}_{\lim_n S_n} = \bigvee_n \mathcal{F}_{S_n}$$

- (b) With respect to the family $(\mathcal{F}_t)$, a stopping time T is said to be

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[1] If the family $(\mathcal{F}_t)$ is not right-continuous, we consider then the family $(\mathcal{F}_{t+} \triangleq \bigcap_{s>t} \mathcal{F}_s)$ (see [8]).

totally inaccessible if T is not a. s. infinite and if for every increasing sequence (S_n) of stopping times majorized by T we have

$P\{\lim_n S_n = T,\ S_n < T < \infty \text{ for every } n\} = 0$

- inaccessible if there exists a totally inaccessible $(\mathcal{F}_t)$ stopping time S such that $P\{T = S < \infty\} > 0$

- accessible if it is not inaccessible

- predictable if there exists an increasing sequence (S_n) of stopping times which converges a. s. to T and such that for every n one has $S_n < T$ on the set $\{T > 0\}$

- (c) An increasing process (A_t) is a stochastic process adapted to the family $(\mathcal{F}_t)$ with (1) sample paths which are a. s. zero at t=0, increasing and right-continuous with (2) A_t integrable for each t. The increasing process (A_t) is integrable if $\sup_t E\, A_t < \infty$.

- (d) A $(\mathcal{F}_t)$ supermartingale (X_t) admits a Doob-Meyer decomposition

$$X_t = M_t - A_t$$

where (M_t) is a $(\mathcal{F}_t)$ martingale and (A_t) an increasing process if and only if (X_t) belongs to the class (DL).

- (e) This decomposition is unique if the increasing process (A_t) is natural .

- (f) The natural increasing process (A_t) is continuous if and only if the supermartingale (X_t) is regular.

- (g) If the supermartingale (X_t) is of class (D) (hence uniformly integrable) with a unique Doob-Meyer decomposition $(X_t = M_t - A_t)$ then $M_t = E(A_\infty + X_\infty | \mathcal{A}_t)$ and (A_t) is also the unique natural increasing process which generates the potential $P_t = X_t - E(X_\infty | \mathcal{A}_t)$.

- (h) If the supermartingale (X_t) is bounded by a constant c (i.e., $|X_t| \leq c$ a.s.) then $\sup_t E M_t^2 < \infty$ (for necessary and sufficient conditions - see [5], Lemma 2.2.2).

- (i) A $(\mathcal{A}_t)$ martingale with $\sup_t E M_t^2 < \infty$ is called a <u>square integrable martingale</u> (see [9], Chapter VIII, Section 3).

- (j) A <u>$(\mathcal{A}_t)$ local martingale</u> (X_t) is a process such that there exists a sequence of $(\mathcal{A}_t)$ stopping times (T_n) increasing a.s. to ∞ which makes each process $(X_{t \wedge T_n})$ a uniformly integrable martingale. If furthermore $(X_{t \wedge T_n})$ is square integrable then (X_t) is a <u>square integrable local martingale</u>.

- (k) We denote by $\mathcal{L}(\mathcal{A}_t)$ the space of $(\mathcal{A}_t)$ local martingale which are zero a.s. at the time origin.

- (l) A sequence of stopping times (T_n) <u>reduces</u> the local martingale (X_t) if $(X_{t \wedge T_n})$ is a uniformly integrable martingale.

- (m) As a consequence of the Doob-Mayer decomposition we can associate to every square integrable $(\mathcal{A}_t)$ local martingale (X_t) a unique natural increasing process $(\langle X \rangle_t)$ such that $(X_t^2 - \langle X \rangle_t) \in \mathcal{L}(\mathcal{A}_t)$ (see [9], Chapter VIII Section 3 [7]).

- (n) For a local martingale (X_t) (not necessarily square integrable) with sample paths of bounded variation on every finite interval, the quadratic variation process $[X]_t$ is defined by[2]

$$[X]_t = \sum_{s \leq t} \Delta_s^2$$

and the process $(X_t^2 - [X]_t) \in \mathcal{L}$ (see [4]).

3.0 INTEGRATED CONDITIONAL RATE. The points in time at which a CP (N_t) jumps are basic to this study:

Definition 3.0.1: The stopping time:

$$J_n = \begin{cases} \inf \{t : N_t \geq n\} \\ \infty \text{ if the above set is empty} \end{cases}$$

is called the time of the n^{th} jump of the CP (N_t).

The fact that J_n is a stopping time with respect to any family $(\mathcal{F}_t)$ to which the CP (N_t) is adapted can be easily verified: the set $\{J_n \leq t\}$ $= \{N_t \geq n\}$ belongs to $\mathcal{F}_t$ for every t.

3.1 DOOB-MEYER DECOMPOSITION FOR COUNTING PROCESSES. As a direct application to CP's of the Doob-Meyer decomposition of supermartingales into the sum of a martingale and an increasing process we have (see [9], Theorem 31-VII; [6])

[2] For a right-continuous function f with left-hand limits Δf_t denotes the jump $f_t - f_{t-}$.

THEOREM 3.1.1: (Doob-Mayer Decomposition for CP's). Let (N_t) be a CP adapted to an increasing family $(\mathcal{F}_t)$.

(a) If for each $t \in \mathbb{R}_+$, N_t is a.s. finite then there exists a unique natural increasing process (A_t) such that the process $(M_t \triangleq N_t - A_t)$ is a square integrable $(\mathcal{F}_t)$ local martingale. The unique decomposition $(N_t = M_t + A_t)$ is called the Doob-Meyer decomposition for the CP (N_t) with respect to the family $(\mathcal{F}_t)$.

(b) If furthermore EN_t is finite for each t then the process $(M_t = N_t - A_t)$ is a $(\mathcal{F}_t)$ martingale.

Proof: (a) Let J_n be the time of the n^{th} jump of the CP (N_t) and define $(N_t^n \triangleq N_{t\wedge J_n})$. By assumption N_t is a.s. finite for each t. Hence the sequence of stopping times (J_n) increases a.s. to infinity. Also by construction the stopped process (N_t^n) is bounded by n. For $t \geq s$ we obviously have $E(-N_t^n|\mathcal{F}_s) \leq -N_s^n$. Thus $(-N_t^n)$ is a bounded $(\mathcal{F}_t^n)$ supermartingale and by the Doob-Meyer decomposition we can obtain the unique decomposition:

$$N_t^n = M_t^n + A_t^n \tag{3.1}$$

where (M_t^n) is a square integrable $(\mathcal{F}_t^n)$ martingale (see section 2.0, (h)) and (A_t^n) a natural integrable increasing process. Now for $n \leq m$ the unique Doob-Meyer decomposition of (N_t^n) with respect to $(\mathcal{F}_t^n)$ is also given by

$$N_t^n = M_{t\wedge J_n}^m + A_{t\wedge J_n}^m \tag{3.2}$$

Therefore comparing (3.1) and (3.2) we get

$$M_{t\wedge J_n}^m = M_t^n \text{ a.s. and } A_{t\wedge J_n}^m = A_t^n \text{ a.s.}$$

Hence we can uniquely define for all t an increasing natural process (A_t) and a square integrable local martingale (M_t) by $A_t \triangleq A_t^n$ and $M_t \triangleq M_t^n$ for $t \leq J_n$ and we clearly have $N_t = M_t + A_t$ a. s.; this proves part (a).

(b) If EN_t is finite for each t then the process $(-N_t)$ is a right-continuous negative supermartingale. By Theorem 19-VI of [9], this supermartingale belongs to the class (DL). Then result (b) follows directly from the Doob-Meyer decomposition (Theorem 31-VII of [9]).

Remark: If the random variable N_t is not a. s. finite for each t then the sequence (J_n) of times of jump of (N_t) does not converge a. s. to infinity. Define $J \triangleq \lim_n J_n$. By Theorem 42-IV of [9], J is a stopping time. For $t \geq J$, $N_t = \infty$ and the best we can do in this case is to consider what is happening on the stochastic interval $[0, J)$ only. If now a local martingale (X_t) is redefined as being a process such that there exists a sequence of stopping times (R_n) increasing to J a. s. (instead of ∞) which makes the stopped process $(X_{t \wedge R_n})$ a uniformly integrable martingale, then as above we can associate a unique Doob-Meyer decomposition to the CP(N_t) on the stochastic interval $[0, J)$.

When speaking of a CP (N_t) we always assume that the random variable (N_t) is a. s. finite for each t since this is clearly the weakest condition under which the unique Doob-Meyer decomposition of (N_t) is defined on the entire positive real line. Note that this assumption is very weak as it is violated only if the times of jump of the CP (N_t) considered converge with some positive probability to a finite time, or, in other words, that the point process associated with the CP (N_t) contains with some positive probability a point of accumulation, an unlikely situation in practice.

3.2 INTEGRATED CONDITIONAL RATE: DEFINITION. For every CP(N_t) with N_t a.s. finite for each t and adapted to a family $(\mathcal{A}_t)$, the uniqueness of the Doob-Meyer decomposition for this CP (N_t) allows us to propose:

Definition 3.2.1: We will call Integrated Conditional Rate (hereafter abbreviated ICR) with respect to the family $(\mathcal{A}_t)$ the unique natural increasing process which appears in the Doob-Meyer decomposition of (N_t) with respect to the family $(\mathcal{A}_t)$.

The terminology Integrated Conditional Rate is motivated by the following: when (N_t) satisfies some sufficiency conditions given in section 5.0 the ICR takes on the form $(\int_0^t \lambda_s \, ds)$ where (λ_t) is a nonnegative process called the conditional rate as it satisfies

$$\lambda_t = \lim_{h \to 0} E\left(\frac{N_{t+h} - N_t}{h} \,\Big|\, \mathcal{A}_t\right).$$

The terminology ICR will be used even when, as may be the case, a conditional rate does not exist. Note that if (N_t) is a nonhomogeneous Poisson process then the notion of conditional rate with respect to the family of σ-algebras $(\mathcal{A}_t)$ generated by the process itself reduces to the usual notion of rate.

Let (N_t) be a CP and denote by $(\mathcal{N}_t)$ the family of σ-algebras generated by (N_t). Let J_n be the time of n^{th} jump. Clearly for each n the stopped process $(N_{t \wedge J_n})$ is a submartingale with respect to any family of $(\mathcal{A}_t)$ such that $\mathcal{A}_t \supset \mathcal{N}_t$. Hence we can define an ICR, say (A_t), with respect to any such family $(\mathcal{A}_t)$. The process $(N_t - A_t)$, that we will systematically denote by (M_t), is in the general case a square integrable $(\mathcal{A}_t)$ local martingale, and a $(\mathcal{A}_t)$ martingale when the mean EN_t is finite for each t. By definition the ICR is a natural process. This last property is dependent on the family $(\mathcal{F}_t)$ chosen so that the ICR (A_t) is varies according to the family $(\mathcal{F}_t)$ considered. For emphasis we therefore speak of a "$(\mathcal{F}_t)$ ICR."

Given a CP (N_t) and its ICR's with respect to two distinct families $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ such that $\mathcal{F}_t \supset \mathcal{G}_t \supset \mathcal{N}_t$, it is natural to ask how these two ICR's are related. This is what we examine now. Assume that the CP (N_t) has a finite mean; even in this case there is no simple useful answer to this problem. Denote respectively by $(A_t^{\mathcal{F}})$ and $(A_t^{\mathcal{G}})$ the ICR's of (N_t) with respect to the families $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$, and by (C_t) a right-continuous modification of $(E(A_t^{\mathcal{F}} \mid \mathcal{G}_t))$. It is easy to check that

(a) The process $(X_t \triangleq N_t - C_t)$ is a $(\mathcal{G}_t)$ martingale;

(b) The process (C_t) is not necessarily increasing or natural so that (C_t) is generally not the $(\mathcal{G}_t)$ ICR of (N_t);

(c) The process (C_t) is a $(\mathcal{G}_t)$ submartingale of class (DL) which has a Doob-Meyer decomposition $(C_t = Y_t + B_t)$ where (Y_t) is a $(\mathcal{G}_t)$ martingale and (B_t) a natural increasing process.

(d) The relation between $(A_t^{\mathcal{G}})$ and $(A_t^{\mathcal{F}})$ is then

$$A_t^{\mathcal{G}} = B_t = E(A_t^{\mathcal{F}} \mid \mathcal{G}_t) - Y_t$$

It is also clear that if $A_t^{\mathcal{F}}$)) is in fact adapted to the family $(\mathcal{G}_t)$ then

$$A_t^{\mathcal{F}} = A_t^{\mathcal{G}}$$

In conclusion there is no simple way to relate the two ICR's $(A_t^{\mathcal{G}})$ and $(A_t^{\mathcal{F}})$ in the general case. But when conditional rates with respect to the two families $(\mathcal{G}_t)$ and $(\mathcal{F}_t)$ exist then these two conditional rates are simply related (see Theorem 5.0.2).

We now demonstrate two simple propositions. The first one shows the intuitive result that a.s. no jump occurs in an interval on which the ICR is a.s. a constant as a function of time.

THEOREM 3.2.2: Suppose (N_t) is a CP adapted to a family $(\mathcal{G}_t)$ which has an ICR with respect to this family that is a.s. constant as a function of time on the stochastic interval $[T, S]$ (T and S are stopping times, finite or not such that $T < S$ a.s). Then (N_t) is a.s. constant as a function of time for $t \in [T, S]$.

Proof: Let (R_n) be a sequence of stopping times reducing the local martingale $(M_t \triangleq N_t - A_t)$ where (A_t) is the ICR of (N_t). We have $E(N_{S \wedge R_n} - N_{T \wedge R_n}) = 0$. But the random variable $N_{S \wedge R_n} - N_{T \wedge R_n}$ is a.s. nonnegative so that $N_{S \wedge R_n} = N_{T \wedge R_n}$ a.s. and the result follows by taking the limit on n.

THEOREM 3.2.3: Let (A_t) be the $(\mathcal{G}_t)$ ICR of a CP (N_t). Then $EN_t < \infty$ if and only if $EA_t < \infty$ and $EN_t = EA_t$.

Proof: If $EN_t < \infty$ then by Theorem 3.1.1 (b) the process $(M_t \triangleq N_t - A_t)$ is a zero mean $(\mathcal{G}_t)$ martingale so that $EA_t = EN_t < \infty$. Conversely if J_n is the time of the n^{th} jump of (N_t) then the process $(N_{t \wedge J_n} - A_{t \wedge J_n})$ is a zero mean martingale so that $EN_{t \wedge J_n} = EA_{t \wedge J_n}$ and the result follows by the monotone convergence theorem.

We close this section with identification of a special class of ICR:

THEOREM 3.2.4: Let (N_t) be a CP of independent increments with a finite mean m_t for each t. Then the $(\mathcal{N}_t)$ ICR (A_t) is given by $A_t = m_t$.

Proof: It is easy to show that the process $(N_t - m_t)$ is a $(\mathcal{N}_t)$ martingale. Furthermore the increasing process m_t is natural because it is deterministic so that (N_t) has the unique Doob-Meyer decomposition $(N_t = (N_t - m_t) + m_t)$. Finally the uniqueness requires m_t to be the $(\mathcal{N}_t)$ ICR.

We will reexamine CP's of independent increments in a future paper and prove in particular a converse result to the above proposition: namely that if a P (N_t) has a deterministic $(\mathcal{N}_t)$ ICR then it is a process of independent increments. Hence CP's of independent increments are uniquely characterized by the fact that their ICR's are deterministic (see [5], Theorem 2.6.1).

It can be shown (see the Remark following Corollary 4.3.3) that the ICR (A_t) of a CP (N_t) with respect to the family of σ-algebras $(\mathcal{G}_t = \mathcal{N}_\infty)$ is given by $A_t = N_t$. Hence for a CP of independent increments the $(\mathcal{N}_t)$ ICR is given by the mean m_t and the $(\mathcal{G}_t = \mathcal{N}_\infty)$ ICR by (N_t). This illustrates the dependence of ICR's on the choice of family of conditioning σ-algebras.

4.0 REGULAR AND ACCESSIBLE COUNTING PROCESSES

4.1 DEFINITION AND DECOMPOSITION. Let (N_t) be a CP adapted to a family $(\mathcal{G}_t)$. Denote by J_n the time of n^{th} jump. It is natural to classify CP's in terms of the properties of their stopping times J_n.

Definition 4.1.1: A CP (N_t) is called respectively regular, accessible or predictable with respect to the family $(\mathcal{G}_t)$ in accordance with the total inaccessibility, accessibility or predictability of its times of jump J_n with respect to this same faculty (see Section 0.2(b)).

While a process can be none of these, the next theorem will show that any CP (N_t) can be decomposed uniquely into the sum of a regular CP and an accessible CP. Here again these definitions are dependent on the particular family $(\mathcal{A}_t)$ chosen. We will see later on (below Theorem 4.2.2) that a CP can be regular with respect to one family and predictable with respect to another.

The term regular was originally used ([9], Definition 33-VII) to characterize a supermartingale (or submartingale) (X_t) such that for any sequence of stopping times (S_n) increasing to a bounded stopping time S we have $\lim_n EX_{S_n} = EX_S$. The next proposition shows that our terminology is consistent.[3]

THEOREM 4.1.2: Let (N_t) be a CP. Then the three following statements are equivalent:

(a) The CP (N_t) is regular in the sense of Definition 4.1.1.

(b) For any stopping time S such that $EN_S < \infty$ the process $(N_{t \wedge S})$ is a regular submartingale in the sense of Definition 33-VII of [9].)

(c) $\lim_n EN_{R_n} = EN_R$ for any sequence of stopping times increasing a.s. to R and such that $EN_R < \infty$.

<u>Proof:</u> Let S be a stopping time such that $EN_S < \infty$ and (T_n) any sequence of stopping times increasing to T a.s. If the relation

(4.1) $$\lim_n N_{T_n \wedge S} = N_{T \wedge S} \text{ a.s.}$$

[3] However, Rubin [11] uses the term in a different sense: it loosely denotes a CP with a random rate which must possess numerous technical properties.

holds then by the monotone convergence theorem we have

$$(4.2) \qquad \lim_n EN_{T_n \wedge S} = EN_{T \wedge S}.$$

Conversely if relation (4.2) holds we have $E(N_{T\wedge S} - \lim_n N_{T_n \wedge S}) = 0$ by the monotone convergence theorem. As the random variable $N_{T\wedge S} - \lim_n N_{T_n \wedge S}$ is positive, relation (4.1) must be valid. Hence conditions (4.1) and (4.2) are equivalent. We show now that (a) is equivalent to (b). If (a) is true then the times of jump of the submartingale $(N_{t \wedge S})$ are totally inaccessible (the time of n^{th} jump of $(N_{t\wedge S})$ is equal to J_n on the set $\{J_n < S\}$ and to ∞ otherwise) so that relation (4.1) is valid and, being equivalent to (4.2), (b) follows. Conversely if (b) is true, relation (4.2) is satisfied. Then (4.1) holds which implies that the times of jump of $(N_{t\wedge S})$ are totally inaccessible (otherwise we reach a contradiction). By taking $S = J_n$, the time of n^{th} jump of (N_t), we get that J_n is a totally inaccessible stopping time. This is true for each n so that (a) follows. The equivalence of (b) and (c) follows easily from the definition of a regular supermartingale (Definition 33-VII of [9]).

Now the announced decomposition result:

THEOREM 4.1.3: Let (N_t) be a CP adapted to a family $(\mathcal{A}_t)$. Then there exists two CP's, (N^r_t) and (N^a_t) which are respectively regular and accessible with respect to the above family and such that

$$N_t = N^r_t + N^a_t \text{ for every } t$$

This decomposition is unique.

Remark 4.1.4: The $(\mathcal{A}_t)$ ICR of (N_t) is given by

$$A_t = A^r_t + A^a_t$$

where (A^r_t) and (A^a_t) are respectively the $(\mathcal{A}_t)$ ICR's of (N^r_t) and (N^a_t).

Proof: As usual, denote by J_n the time of n^{th} jump. By J_n^A we mean the stopping time

$$J_n^A = \begin{cases} J_n & \text{if } \omega \in A \\ \infty & \text{otherwise} \end{cases}$$

for $A \in \mathcal{F}_{J_n}$. By Theorem 44-VII of [9] there exists for each n an essentially unique partition of the set $\{J_n < \infty\}$ into two sets of $\mathcal{F}_{J_n}$, A and R, such that the stopping times J_n^A and J_n^R are respectively accessible and totally inaccessible. The two CP's $N_t^a \triangleq \sum_n I_{\{t \geq J_n^A\}}$ and $N_t^r \triangleq \sum_n I_{\{t \geq J_n^R\}}$ clearly satisfy the conditions of the theorem. The uniqueness of this decomposition follows from the essential uniqueness of the partition of each set $\{J_n < \infty\}$.

Example 4.1.5: Take $\Omega = [0, 1]$ and P the Lebesgue measure defined on the Lebesgue side of $[0, 1]$. Let J_1 be a random variable uniformly distributed on Ω. Define the random variables $J_{n+1} = J_1 + n$, for $n \geq 1$. Let (N_t) be the CP having J_n as time of n^{th} jump, i.e., $N_t = \sum_n I_{\{t \geq J_n\}}$. One can show that (see [5]) for any CP (N_t) the time of the first jump J_1 is totally inaccessible with respect to $(\mathcal{N}_t)$ if and only if $P\{J_1 = a\} = 0$ for any non-negative constant a. Hence the time of jump J_1, being here uniformly distributed on Ω, is totally inaccessible. It is easy to show that for $n \geq 2$, the times of of jump J_n are predictable. Thus the decomposition $N_t = N_t^r + N_t^a$ with respect to the family $(\mathcal{N}_t)$ is given by $N_t^r = I_{\{t \geq J_1\}}$ and $N_t^a = \sum_{n \geq 2} I_{\{t \geq J_n\}}$.

In this very simple example the CP (N_t^a) is in fact predictable. This is not always the case. If we assume in the above example that jumps may be skipped independently of each other with a positive probability, then (N_t^r) and

(N_t^a) are still given as above but the CP (N_t^a) is no longer predictable (see Example 4.3.5).

For clarity we outline now some of the results we are going to investigate. First regular, then accessible CP's are studied in detail. In particular we will see that a CP is regular with respect to a family $(\mathcal{F}_t)$ if and only if its $(\mathcal{F}_t)$ ICR is continuous (Theorem 4.2.2); when the family $(\mathcal{F}_t)$ is free of times of discontinuity then accessible CP's are predictable (Theorem 4.3.1). Predictable CP's are uniquely characterized by the fact that their ICR is given by the CP itself (Corollary 4.3.3). In other words predictable CP's are natural processes. Combining these facts with the above decomposition for CP's (Theorem 4.1.3) gives, when the family $(\mathcal{F}_t)$ is free of times of discontinuity, the separating property of the unique Doob-Meyer decomposition for CP's (see Corollary 4.3.4). The case where the family $(\mathcal{F}_t)$ does contain times of discontinuity is more complex. Most of these results are obtained by studying the different terms in the equation $\Delta N_T = \Delta M_T + \Delta A_T$ in relation to the appropriate property of the stopping time T (Theorem 4.3.2).

4.2 REGULAR COUNTING PROCESSES. Let (N_t) be a regular CP with respect to a family $(\mathcal{F}_t)$. By definition the times of jump J_n of (N_t) are totally inaccessible. This has the immediate consequence that the probability a jump occurs at time t is zero. Also, if T is a $(\mathcal{F}_t)$ stopping time we cannot make with a positive probability a prediction of any time of jump after T, the prediction being based on the information available up to and at time T. More precisely we have:

THEOREM 4.2.1: Let (N_t) be a regular CP with respect to a family $(\mathcal{F}_t)$ and T a $(\mathcal{F}_t)$ stopping time. Assume W is a strictly positive $\mathcal{F}_T$ measurable random variable. Then for each n

$$P\{T + W = J_n\} = 0$$

where J_n is the time of n^{th} jump of (N_t).

Proof: By contradiction. Assume that for $n = n_o$ there exists $W = W_o$, a strictly positive $(\mathcal{F}_T)$ measurable random variable, with $P\{T + W_o = J_{n_o}\} = p > 0$. The sequence of $(\mathcal{F}_t)$ stopping times (see [9], Theorems 37 and 38-IV) $(T_i \triangleq T + (1-1/i)W_o)$ is increasing and $P\{\lim_i T_i = J_{n_o}\} = p > 0$, i.e. the time of n_o^{th} jump is not totally inaccessible, a contradiction.

The next theorem is a direct consequence of Theorem 4.1.2 and a result on the Doob-Meyer decomposition of regular supermartingales ([9], Theorem 37-VII).

THEOREM 4.2.2: Let (N_t) be a CP adapted to a family $(\mathcal{F}_t)$. Then the $(\mathcal{F}_t)$ ICR (A_t) of (N_t) is continuous if and only if the CP (N_t) is regular with respect to this family.

Proof: Let J_n be the n^{th} time jump and define $(N^n_t \triangleq N_{t\wedge J_n})$, $(A^n_t \triangleq A_{t\wedge J_n})$, $(P^n_t \triangleq \mathcal{F}_{t\wedge J_n})$. Note that by the uniqueness of the Doob-Meyer decomposition (A^n_t) is the $(\mathcal{F}^n_t)$ ICR of (N^n_t). By Theorem 37-VII of [9] the process (A^n_t) is continuous for each n if and only if the CP (N_t) is regular. The result follows then by taking the limit as the sequence (J_n) increases to ∞. One uses here the fact that on any interval $[0, t_o]$, $A_t = A^n_t$ for sufficiently large n (depending on ω).

Examples of regular CP's with respect to the family $(\mathcal{F}_t)$ are, by Proposition 3.3.2 and the above theorem, any CP's of independent increments with continuous mean, in particular Poisson processes. Note that these processes of independent increments with continuous mean are not regular but predictable if we take the family $(\mathcal{F}_t = \mathcal{N}_\infty)$ (see Proposition 3.3.1).

For a regular CP (N_t) with ICR (A_t) we have just proved that all the jumps are contained in the local martingale $(M_t = N_t - A_t)$. But these jumps completely determined the CP (N_t). This suggests that there is a direct relation between (M_t) and the ICR (A_t). This point is made clear in the following theorem. Recall that if $(N_t = M_t + A_t)$ is the unique Doob-Meyer decomposition of (N_t) then (M_t) is a square integrable local martingale (Theorem 3.1.1) to which a unique natural increasing process $(\langle M \rangle_t)$ can be associated (section 2.0, (m)).

THEOREM 4.2.3: Let (N_t) be a regular CP with respect to a family $(\mathcal{F}_t)$. Denote by (A_t) its $(\mathcal{F}_t)$ ICR and by (M_t) the square integrable local martingale $(N_t - A_t)$. [See section 2.0 (h)]. We have

(a) $A_t = \langle M \rangle_t$

(b) If EN_t is finite then so is EM_t^2, with $EM_t^2 \leq EN_t$.

<u>Proof:</u> (a) One has

$$N_t = M_t + A_t \tag{4.3}$$

This shows that (M_t) is a martingale of bounded variation so that (see section 2.0, (n)) the quadratic variation process of (M_t) is given by

$$[M]_t = \sum_{s \leq t} (\Delta M_s)^2 \tag{4.4}$$

But (N_t) is a regular CP and by Theorem 4.2.2 its ICR (A_t) is continuous so that $\Delta M_s = \Delta N_s$. Now ΔN_s is either 0 or 1. Hence $(\Delta M_s)^2 = (\Delta N_s)^2 = \Delta N_s$ which implies by (4.4)

$$(4.5) \qquad [M]_t = N_t \, .$$

The two processes $(M_t^2 - \langle M \rangle_t)$ and $(M_t^2 - [M]_t)$ are local martingales (see section 2.0 (m) and (n)); thus so is their difference $(X_t \triangleq [M]_t - \langle M \rangle_t)$ and by (4.5) we get

$$(4.6) \qquad N_t = X_t + \langle M \rangle_t$$

where $X_t \in \mathcal{L}$. The increasing process $(\langle M \rangle_t)$ is natural so that by uniqueness of the Doob-Meyer decomposition one must have, comparing (4.3) and (4.6), $A_t = \langle M \rangle_t$ and $X_t = M_t$.

(b) We have seen above that the process $(M_t^2 - [M]_t) \in \mathcal{L}$ or by (4.5) $(M_t^2 - N_t) \in \mathcal{L}$. Let (T_n) be a sequence of stopping times reducing this local martingale i.e., the process $(M^2_{t\wedge T_n} - N_{t\wedge T_n})$ is a uniformly integrable martingale. In particular

$$E(M^2_{t\wedge T_n} - N_{t\wedge T_n}) = E(M_o^2 - N_o) = 0$$

Hence

$$(4.7) \qquad E\,M^2_{t\wedge T_n} = E\,N_{t\wedge T_n}$$

Since $M_{t\wedge T_n}$ converges to M_t, Fatou's lemma implies

$$E\,M_t^2 \leq \lim_n \,(\inf E\,M^2_{t\wedge T_n})$$

and by (4.7) and the monotone convergence theorem ($N_{t\wedge T_n}$ increases to N_t) we get

$$E\,M_t^2 \leq \lim_n \,\inf\,(E\,M^2_{t\wedge T_n}) = \lim_n\,(E\,N_{t\wedge T_n}) = E\,N_t.$$

4.3 ACCESSIBLE COUNTING PROCESSES. Theorem 4.2.2, which says that the ICR of a CP is continuous if and only if this CP is regular, implies that the ICR (A_t) of an accessible CP (N_t) is discontinuous. We could conjecture that the times of jump of the ICR (A_t) are the same as those of the accessible CP (N_t). As we will see this would be true, and we would have in fact $(A_t \equiv N_t)$ but for the possible presence of times of discontinuity for the family $(\mathscr{A}_t)$ considered (see Definitions 39 and 40-VII, [9]). Recall that an accessible $(\mathscr{A}_t)$ stopping time which is not a time of discontinuity for the family $(\mathscr{A}_t)$ is $(\mathscr{A}_t)$ predictable (see Theorem 45-VII of [9]). This immediately gives us:

THEOREM 4.3.1: An accessible CP (N_t) with respect to a family $(\mathscr{A}_t)$ which is free of times of discontinuity is predictable.

Let (N_t) be any CP with ICR (A_t). We examine now the jump ΔA_T in relation to the property of the stopping time T. We already know that for a regular CP $\Delta A_T = 0$ for any stopping time T (Theorem 4.2.2). The next result will lead to a unique characterization of predictable CP's (Corollary 4.3.3) and the separating property of the unique Doob-Meyer decomposition for CP's (Corollary 4.3.4).

THEOREM 4.3.2: Suppose (N_t) is any CP adapted to a family $(\mathscr{A}_t)$. Denote by (A_t) its $(\mathscr{A}_t)$ ICR.

(a) If T is $(\mathscr{A}_t)$ predictable then

$$\Delta A_T = E(\Delta N_T | \vee_n \mathscr{A}_{T_n})$$

where (T_n) is any sequence of stopping times increasing to T. In particular $0 \leq \Delta A_T \leq 1$, and $\Delta A_T = 1$ (or 0) a.s. if and only if $\Delta N_T = 1$ (or 0) a.s.

$\mathcal{A}$ (b) If T is $(\mathcal{A}_t)$ accessible but not a time of discontinuity for $(\mathcal{A}_t)$ then

$$\Delta A_T = \Delta N_T .$$

(c) If T is $(\mathcal{A}_t)$ totally inaccessible then $\Delta A_T = 0$.

(d) Let J_n be the n^{th} time of jump of (N_t). Then

$$\Delta A_{J_n} = 1$$

if and only if J_n is a predictable $(\mathcal{A}_t)$ stopping time. In particular $\Delta A_{J_n} = 1$ if J_n is accessible but not a time of discontinuity for the family $(\mathcal{A}_t)$.

Proof: (a) (see [9], section 51-VII) Let J_n be the n^{th} time of jump of (N_t), and define $(N_t^n \triangleq N_{t\wedge J_n})$, $(A_t^n \triangleq A_{t\wedge J_n})$. We know (Theorem 3. 1. 1) that the process $(M_t^n \triangleq N_t^n - A_t^n)$ is a square integrable $(\mathcal{A}_t)$ martingale. Thus for $i \geq m$ and any set $H \in \mathcal{A}_{T_m}$ where (T_m) is a sequence of stopping times increasing to T we have

$$\int_H (M_T^n - M_{T_i}^n)dP = \int_H E(M_T^n - M_{T_i}^n \mid \mathcal{A}_{T_m})dP = 0$$

so that using the relation $(M_t^n = N_t^n - A_t^n)$ one gets

$$\int_H (A_T^n - A_{T_i}^n)dP = \int_H (N_T^n - N_{T_i}^n)dP$$

Letting i increase to infinity one obtains, by the monotone convergence theorem

$$\int_H \Delta A_T^n dP = \int_H \Delta N_T^n dP \qquad \forall H \in \mathcal{A}_{T_m}$$

This implies

$$E(\Delta A_T^n \mid \mathcal{A}_{T_m}) = E(\Delta N_T^n \mid \mathcal{A}_{T_m}) \text{ a. s.}$$

and taking the limit with respect to m, by the Lemma of [10]

$$E(\Delta A_T^n | \bigvee_m \mathcal{F}_{T_m}) = E(\Delta N_T^n | \bigvee_m \mathcal{F}_{T_m})$$

The process (A_t^n) is natural with respect to the family $(\mathcal{F}_t)$ so that, by Theorem 49-VII of [9], the random variable ΔA_T^n is $(\bigvee_m \mathcal{F}_{T_m})$ measurable. Thus the above relation gives

$$\Delta A_T^n = E(\Delta N_T^n | \bigvee_m \mathcal{F}_{T_m})$$

and by the bounded convergence theorem we get the desired result letting n go to ∞.

(b) By Theorem 45-VII of [9], T is predictable so that part (a) is applicable. Furthermore $\mathcal{F}_T = \bigvee_m \mathcal{F}_{T_m}$ (T is not a time of discontinuity of $(\mathcal{F}_t)$). Hence

$$\Delta A_T = E(\Delta N_T | \bigvee_m \mathcal{F}_{T_m}) = E(\Delta N_T | \mathcal{F}_T) = \Delta N_T$$

Part (c) is just a restatement of condition (b) of Theorem 49-VII of [9] and is given here for completeness.

(d) $(\Longleftarrow)$ J_n is predictable and $\Delta N_{J_n} = 1$ so that by part (a) $\Delta A_{J_n} = 1$

$(\Longrightarrow)$ Assume $\Delta A_{J_n} = 1$. Let $C_t^n = \Delta A_{J_n} I_{\{t \geq J_n\}} = I_{\{t \geq J_n\}}$.
The process (C_t^n) is natural because it satisfies the necessary and sufficient conditions (a) and (b) of Theorem 69-VII of [9] (if not then the natural process (A_t) would not satisfy these two conditions, a contradiction). By Theorem 52-VII of [9] J_n is then a predictable stopping time.

COROLLARY 4.3.3: A CP (N_t) with ICR (A_t) with respect to $(\mathcal{F}_t)$ is predictable with respect to this family if and only if $(A_t = N_t)$.

Proof: ($\Longrightarrow$) (N_t) is predictable so by (d) of Theorem 4.3.2

$$\Delta A_{J_n} = 1 \quad \text{for each } n$$

where J_n is the time of n^{th} jump of (N_t). This implies $A_t \geq N_t$ a.s. But for each n we also have $E(N_{t\wedge J_n} - A_{t\wedge J_n}) = 0$ and the relation $A_t = N_t$ a.s. holds.

($\Longleftarrow$) If $(N_t = A_t)$ then $\Delta A_{J_n} = 1$ for each n and by (d) of Theorem 4.3.2 J_n is a predictable stopping time for each n, i.e. (N_t) is predictable.

Remark: It is clear that any $(\mathcal{A}_t)$ stopping time is predictable with respect to the family $(\mathcal{A}_t = \mathcal{N}_\infty)$. Hence the $(\mathcal{A}_t = \mathcal{N}_\infty)$ ICR of a CP (N_t) is given by (N_t).

COROLLARY 4.3.4: Let (N_t) be an CP with $(\mathcal{A}_t)$ ICR (A_t) and define $(M_t \triangleq N_t - A_t)$. Then if the family $(\mathcal{A}_t)$ is free of times of discontinuity

(a) The local martingale (M_t) has jumps of size one taking place only at $(\mathcal{A}_t)$ totally inaccessible stopping times.

(b) The $(\mathcal{A}_t)$ ICR (A_t) has jumps of size one only at $(\mathcal{A}_t)$ predictable stopping times.

Remarks: In other words, (M_t) represents the part of (N_t) which is unexpected and the ICR (A_t) the one which can be perfectly predicted. This is what we have called the separating property of the Doob-Meyer decomposition for CP's.

Proof: Let

$$N_t \triangleq N_t^r + N_t^a \tag{4.9}$$

denote the unique decomposition of Theorem 4.1.3 where (N_t^r) is a regular CP and (N_t^a) an accessible CP. and (N_t^a) an accessible CP. Let respectively (A_t^r) and (A_t^a) be the $(\mathcal{A}_t)$ ICR of (N_t^r) and (N_t^a). By Theorem 4.2.2, (A_t^r) is continuous so that the local martingale

(4. 10) $$M_t^r \triangleq N_t^r - A_t^r$$

has only jumps of size one taking place at totally inaccessible stopping times (namely the times of jump of (N_t^r)). By assumption the family $(\mathcal{A}_t)$ is free of times of discontinuity so that by Theorem 4. 3. 1 (N_t^a) is a predictable CP and by Corollary 4. 3. 3

(4. 11) $$A_t^a = N_t^a$$

Introducing (4. 10) in (4. 9) one gets

$$N_t = M_t^r + (A_t^r + N_t^a)$$

which is a unique Doob-Meyer decomposition of (N_t) as, by (4. 11), $(A_t^r + N_t^a = A_t^r + A_t^a)$ is a natural increasing process. But $(N_t = M_t + A_t)$ is also such a unique decomposition so that one must have

$$M_t = M_t^r$$

$$A_t = A_t^r + N_t^a$$

and the result follows.

Let $(N_t = M_t + A_t)$ denote the unique Doob-Meyer decomposition of the CP (N_t) with respect to the family $(\mathcal{A}_t)$. When this family $(\mathcal{A}_t)$ is free of times of discontinuity the above Corollary 4. 3. 4 completely describes the discontinuities of the local martingale (M_t) and of the $(\mathcal{A}_t)$ ICR (A_t): either (M_t) or (A_t) (but not both) have a discontinuity which is of size one and can only take place at a time of jump of (N_t). When the family $(\mathcal{A}_t)$ does have times of discontinuity the above statement is no longer necessarily true. Because it is likely for a $(\mathcal{A}_t)$ local martingale to have a jump at a time of discontinuity for the family $(\mathcal{A}_t)$, the new following situations may now take place:

(a) If T is a stopping time which is a time of discontinuity for the family $(\mathcal{A}_t)$ and such that $\Delta N_T = 0$ a. s. then it may happen that both ΔM_T and $\Delta A_T = - \Delta M_T$ will be different from zero. In fact this can happen only if T has an accessible part which is not predictable.

(b) Let J_n be the time of n^{th} jump of (N_t) which is supposed to be accessible (but not predictable) and also a time of discontinuity for the family $(\mathcal{A}_t)$. Then Theorems 4.2.2 and 4.3.2(d) imply that both (A_t) and (M_t) have a discontinuity at J_n.

The following example illustrates these two points.

Example 4.3.5: Take $\Omega = \{\omega_1, \omega_2\}$ with $\{\omega_1\} = p$ where $0 < p < 1$. Define the following CP (N_t):

$$N_t(\omega_1) = \begin{cases} 0 & t < 1 \\ 1 & t \geq 1 \end{cases}$$

$$N_t(\omega_2) = 0 \qquad t \geq 0$$

The family $(\mathcal{N}_t)$ is then given by

$$\mathcal{N}_t = \begin{cases} \{\phi, \Omega\} & \text{for } t < 1 \\ \{\phi, \{\omega_1\}, \{\omega_2\}, \Omega\} & \text{for } t \geq 1 \end{cases}$$

and the unique time of jump of (N_t) by

$$J(\omega) = \begin{cases} 1 & \omega = \omega_1 \\ \infty & \omega = \omega_2 \end{cases}$$

This stopping time J is obviously accessible. Observe also that J is a time of discontinuity for the family $(\mathcal{N}_t)$: (see Definition 40-VII of [9]). Here $(S_n \triangleq 1 - 1/n)$ is an increasing sequence of stopping times, $S_n < J$ for each n and the set

$$\{\omega : \lim_n S_n = J\} = \{\omega_1\} \notin \bigvee_n \mathcal{N}_{S_n} = \{\phi, \Omega\}$$

Denote a (not necessarily unique) Doob-Meyer decomposition of (N_t) by

$$N_t = M_t + A_t \tag{4.12}$$

where (M_t) is a uniformly integrable martingale ((N_t) is bounded) and (A_t) an increasing (not necessarily natural) process. It is easy to see that the martingale (M_t) is given by

$$M_t(\omega) = \begin{cases} 0 & \forall\ \omega,\ t < 1 \\ \begin{cases} a & \omega = \omega_1,\ t \geq 1 \\ b & \omega = \omega_2,\ t \geq 1 \end{cases} \end{cases}$$

where a and b are two constants such that

$$\frac{a}{b} = -\left(\frac{1-p}{p}\right). \tag{4.13}$$

Then by (4.12) we must have

$$A_t(\omega) = \begin{cases} 0 & \forall\ \omega,\ t < 1 \\ \begin{cases} 1-a & \omega = \omega_1, & t \geq 1 \\ -b & \omega = \omega_2, & t \geq 1 \end{cases} \end{cases}$$

By the uniqueness theorem only one set of values a and b makes the increasing process (A_t) natural. These values are $a = 1 - p$ and $b = -p$ (this choice obviously satisfies (4.13)), as in this case

$$A_t = pI_{[1,\infty)}(t)$$

is a deterministic hence natural process. Thus the ICR of (N_t) with respect to the family $(\mathcal{N}_t)$ is $pI_{[1,\infty)}(t)$ and the martingale $(M_t = N_t - pI_{[1,\infty)}(t))$ is given by

$$M_t = \begin{cases} 0 \quad \forall\ \omega, \quad t < 1 \\ \begin{cases} 1 - p \quad \omega = \omega_1, \quad t \geq 1 \\ -p \quad \omega = \omega_2, \quad t \geq 1 \end{cases} \end{cases}$$

Therefore both the ICR $pI_{[1,\infty)}(t)$ and the above martingale have a discontinuity at the time of jump J of (N_t). This illustrates case (b). As stated above, this is a consequence of the fact that the accessible stopping time J is not predictable and is also a time of discontinuity for $(\mathcal{N}_t)$. Also if we define the stopping time T

$$T = \begin{cases} \infty & \omega = \omega_1 \\ 1 & \omega = \omega_2 \end{cases}$$

then

$$\Delta A_T = \begin{cases} 0 & \omega = \omega_1 \\ p & \omega = \omega_2 \end{cases}$$

even though $\Delta N_T = 0$ for any ω. This illustrates case (a). It is easy to check that T is a time of discontinuity for $(\mathcal{N}_t)$ which is accessible but not predictable.

5.0 CONDITIONAL RATE. In the previous section we have seen that we can decompose uniquely any CP (N_t) adapted to a family $(\mathcal{A}_t)$ into a sum of two CP's which are respectively regular and accessible with respect to this family $(\mathcal{A}_t)$ (Theorem 4.1.3). Regular CP's relatively to a family $(\mathcal{A}_t)$ are precisely those which have a continuous $(\mathcal{A}_t)$ ICR (Theorem 4.2.2). But a continuous ICR may not have absolutely continuous sample paths. For example, consider a CP of independent increments with a continuous, but not absolutely continuous mean.

In the next theorem we give sufficient conditions under which the ICR (A_t) of a CP (N_t) with respect to a family $(\mathcal{A}_t)$ is absolutely continuous; in other words when does a random process (λ_t) adapted to $(\mathcal{A}_t)$ exist such that we can express the ICR (A_t) as

(5.1) $$A_t = \int_0^t \lambda_s ds \ ?$$

Under these conditions, we also have

(5.2) $$\lambda_t = \lim_{h \to 0} E\left(\frac{N_{t+h} - N_t}{h} \,\Big|\, \mathcal{A}_t \right)$$

and because of this relation we call the process (λ_t) the "conditional rate" of the CP (N_t) with respect to the family $(\mathcal{A}_t)$. Expression (5.1) is then a justification for the terminology "Integrated Conditional Rate" (ICR) introduced in section 3.2, terminology used even though a conditional rate does not generally exist.

Although there is great emphasis in the literature ([3], [1], [11], [12], [13], [14]) on CP's which admit a conditional rate, the problem of existence of these CP's has been treated only lately by Breimand ([1]) where a partial answer to this problem is given: the existence of CP's which possess a bounded random rate with respect to the family of σ-algebras generated by the CP itself is demonstrated by the use of absolutely continuous changes of measures. This technique is discussed and extended in [5]. We now give sufficient conditions under which a CP with finite mean does possess a conditional rate:

THEOREM 5.0.1: If for a CP (N_t) with finite men and adapted to a family $(\mathcal{A}_t)$

(i) for each t the following limit exists a.s.

$$\lim_{h\to 0} \frac{1}{h} Q_m(t,h,\omega) \triangleq \lambda_m(t,\omega) \quad m = 1, 2, \ldots$$

where $Q_m(t,h,\omega) \triangleq P\{N_{t+h} - N_t \geq m | \mathcal{A}_t\}$

(ii) for almost all ω there exists $h_0(\omega) > 0$ such that the series $\sum_m \frac{1}{h} Q_m(t,h,\omega)$ converges uniformly for $h \in (0, h_0(\omega)]$ and which is bounded by a function $a(t,\omega)$ such that $\int_0^t a(s,\omega)ds < \infty$ for each t. Then

(a) The series $\sum_m \lambda_m$ is convergent. Define the process $(\lambda_t \triangleq \sum_m \lambda_m)$. We have the relation:

$$\lambda_t = \lim_{h\to 0} E\left(\frac{N_{t+h} - N_t}{h} \middle| \mathcal{F}_t\right) \text{ a.s. for every } t$$

(b) The $(\mathcal{F}_t)$ ICR of (N_t) is given by

$$A_t = \int_0^t \lambda_s ds$$

Proof: By (i) and (ii)

$$(5.3) \qquad \lim_{h\to 0} \frac{1}{h} \sum_m Q_m(t,h,\omega) = \sum_m \lim_{h\to 0} \frac{1}{h} Q_m(t,h,\omega) = \sum_m \lambda_m(t,\omega) \triangleq \lambda_t(\omega)$$

where the first equality follows by the uniform convergence on $(0, h_o(\omega)]$. Assumption (ii) also implies for almost all ω and $h \leq h_o(\omega)$

$$\sum_m Q_m(t,h,\omega) \leq a(t,\omega)h_o(\omega) < \infty$$

and this is enough to justify the equality

$$\sum_m m(Q_m - Q_{m+1}) = \sum_m Q_m .$$

But

$$Q_m - Q_{m+1} = \{N_{t+h} - N_t = m | \mathcal{F}_t\}$$

so that the above relation gives for $h \leq h_o(\ )$

$$(5.4) \qquad E(N_{t+h} - N_t | \mathcal{F}_t) = \sum_m Q_m(t,h,\ )$$

and by (5.3)

$$(5.5) \qquad \lambda_t = \lim_{h\to 0} \frac{1}{h} Q_m(t,h,\omega) = \lim_{h\to 0} E\left(\frac{N_{t+h} - N_t}{h} \middle| \mathcal{F}_t\right)$$

(b) The CP (N_t) is right-continuous;

there exists a right-continuous modification for the submartingale $(E(N_{t+h}|\mathcal{F}_t))$ (see Definition 27-VII of [9]) and we denote by $(p_h\lambda_t)$ a right-continuous modification of the process $(E(\frac{N_{t+h} - N_t}{h}|\mathcal{F}_t))$. We have seen above that $\lim_{h\to 0} p_h\lambda_t = \lambda_t$ a.s. By (ii) and (5.4)

$$0 \leq p_h\lambda_t = \sum_m \frac{1}{h} Q_m(t,h,\omega) \leq a(t,\omega)$$

for $h \leq h_o(\ )$. Hence the integral

$$\int_0^t p_h\lambda_s ds$$

is well defined for almost all ω and by the dominated convergence theorem

$$\lim_{h\to 0} \int_0^t p_h\lambda_s ds = \int_0^t \lambda_s ds \quad \text{a.s.} \tag{5.6}$$

Denote by (A_t) the $(\mathcal{F}_t)$ ICR of (N_t) and define as usual the martingale $(M_t \triangleq N_t - A_t)$. Let c be any positive constant and define

$$P_t^c \triangleq E(A_c|\mathcal{F}_t) - A_{t\wedge c} \tag{5.7}$$

It is easy to check that (P_t^c) is a potential and by Theorem 29-VII of [9] we know that for each t

$$\int_0^t \frac{1}{h} E(P_s^c - P_{s+h}^c|\mathcal{F}_s)ds \xrightarrow[h\to 0]{(L_1, L_\infty)} A_{t\wedge c} \tag{5.8}$$

where convergence is in the weak sense in L^1. Now $(A_{t\wedge c} = N_{t\wedge c} - M_{t\wedge c})$ so that by (5.7)

$$P_t^c = [E(A_c|\mathcal{F}_t) + M_{t\wedge c}] - N_{t\wedge c}$$

where $(M_{t\wedge c})$ is not only a $(\mathcal{F}_{t\wedge c})$ but also a $(\mathcal{F}_t)$ martingale. Hence for $s \leq t$ and if we choose $c > t + h$

$$E(P_s^c - P_{s+h}^c | \mathcal{F}_s) = E(N_{s+h} - N_s | \mathcal{F}_s)$$

Thus on the one hand by (5.8)

$$\int_0^t \frac{1}{h} E(N_{s+h} - N_s | \mathcal{F}_s) ds \xrightarrow[h \to 0]{(L_1, L_\infty)} A_t$$

and on the other by (5.6)

$$\int_0^t \frac{1}{h} E(N_{s+h} - N_s | \mathcal{F}_s) ds \xrightarrow[h \to 0]{a.s} \int_0^t \lambda_s ds$$

so that we must have a. s. for each t

$$A_t = \int_0^t \lambda_s ds$$

Our last result shows that the two conditional rates of a same CP (N_t) but with respect to two families $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ such that $\mathcal{F}_t \supset \mathcal{G}_t \supset \mathcal{N}_t$ are related by a simple expression.

THEOREM 5.0.2: Let (N_t) be a CP with finite mean. Denote its conditional rate with respect to the family $(\mathcal{F}_t)$ by (λ_t). Let $(\mathcal{G}_t)$ be another family such that $\mathcal{N}_t \subset \mathcal{G}_t \subset \mathcal{F}_t$. Then the conditional rate $(\hat{\lambda}_t)$ of (N_t) with respect to $(\mathcal{G}_t)$ exists and is given by

$$\hat{\lambda}_t = E(\lambda_t | \mathcal{G}_t)$$

Remark: Note that this result makes good intuitive sense, the conditional rate $(\hat{\lambda}_t)$ being the best mean square estimate of the conditional rate (λ_t).

Proof: Part of this proof is a consequence of the innovation theorem ([1], Theorem 1.1), i.e., the process $(N_t - \int_0^t \hat{\lambda}_s ds)$ is a $(\mathcal{G}_t)$ martingale. Now the process $(\int_0^t \hat{\lambda}_s ds)$ is increasing, continuous hence natural and consequently is the $(\mathcal{G}_t)$ ICR of (N_t) by the uniqueness of the Doob-Meyer decomposition.

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