RECURSIVE INTEGRAL EQUATIONS FOR THE DETECTION OF COUNTING PROCESSES†

F. B. Dolivo

and

F. J. Beutler

Computer, Information and Control Engineering Program
The University of Michigan, Ann Arbor, Michigan 48104

September 1974

† This research was sponsored by the Air Force Office of Scientific Research, AFSC, USAF, under Grant No. AFOSR-70-1920C, and the National Science Foundation under Grant No. GK-20385.

011

MRC655

ABSTRACT. A recursive stochastic integral equation for the detection of Counting Processes is derived from a previously known formula [5] of the likelihood ratio. This is done quite simply by using a result due to Doléans-Dade [4] on the solution of stochastic integral equations.

1. INTRODUCTION. Recently modern martingale theory has been used to describe Counting Processes (hereafter abbreviated CP) in a way specially appropriate to the problems of detection and filtering. This has given rise to the notion of Integrated Condtional Rate (ICR) [5], which generalizes the notion of random rate.

Expressions for likelihood ratios (involving ICR's) for the detection of CP's have been obtained in [5] using a three-step technique introduced by Kailath [9] and Duncan ([6], [7]) in their works on detection of a stochastic signal in white noise. The three steps are the Likelihood Ratio Representation Theorem ([2], [5], [6]), the Girsanov Theorem ([5], [8], [13]) and the Innovation Theorem ([2], [5], [9]). By this method likelihood ratios for a large class of CP's can be found. These expansions represent a generalization of the formulas given in [1] and [12] in the context of Poisson processes and [2] in the context of CP's which admit a conditional rate.

The purpose of this paper is not to present a proof of the likelihood ratio formula (for that see [5]) but to derive from this formula stochastic integral equations by which the likelihood ratio can be computed recursively. This can be done quite simply using a result due to Doléans-Dade [4] on the solution of stochastic integrals equations involving semimartingales. These recursive equations are most useful in applications as they give a way of implementing the computation of the likelihood ratio continuously in time.

2. PRELIMINARIES. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. By (X_t) we denote a real valued stochastic process defined on $\mathbb{R}_+$, the positive real line and by a Counting Process (CP) we mean

Definition 2.1: A CP is a stochastic process having sample paths which are zero at the time origin and consisting of right-continuous step functions with positive jumps of size one.

The time of n^{th} jump J_n of a CP (N_t) is the stopping time defined by

$$J_n = \begin{cases} \inf \{t: N_t \geq n\} \\ \\ \infty \text{ if the above set is empty.} \end{cases}$$

Let $(\mathcal{F}_t)$ be a right-continuous increasing family of σ-subalgebras of $\mathcal{F}$ with $\mathcal{F}_o$ containing all the P negligible sets, and suppose (N_t) is a CP, adapted to $\mathcal{F}_t$, with the sole assumption that EN_t is finite for each t. Then, as a consequence of the Doob-Meyer decomposition for supermartingales we can associate to (N_t) a unique natural increasing process (A_t), dependent on the family $(\mathcal{F}_t)$, which makes the process $(M_t \triangleq N_t - A_t)$ a martingale (see [11]). This decomposition $(N_t = M_t + A_t)$ is intuitively a decomposition into the part (M_t) which is not predictable and (A_t) which can be perfectly predicted. This unique process (A_t) is called the Integrated Conditional Rate (ICR) of (N_t) with respect to $(\mathcal{F}_t)$ ("the $(\mathcal{F}_t)$ ICR of (N_t)") and has been studied in [5]. The terminology ICR is motivated by the fact that when (N_t) satisfies some sufficiency conditions its ICR takes on the form $(\int_0^t \lambda_s ds)$ where (λ_t) is a nonnegative process called the conditional rate (with respect to $(\mathcal{F}_t)$) satisfying $\lambda_t = \lim_{h \to 0} E[h^{-1}(N_{t+h} - N_t) | \mathcal{F}_t]$ ([5], Section 2.5). The existence of CP's possessing a bounded conditional rate with respect to the family of σ-algebras generated by the process itself has been first shown in [2] and in [5]. Sufficiency conditions for the existence of a conditional rate have been given in [5]. By a change of time we can show similar results (i.e., existence (see [5], Corollary 3.1.3) and sufficiency conditions) for $(\mathcal{F}_t)$ ICR's of the form $(\int_0^t \lambda_s dm_s)$ where (λ_t) is a locally bounded predictable process and m_t a deterministic increasing right-continuous function with $m_0 = 0$. Denote by $\mathcal{H}(\mathcal{F}_t)$ the class of all locally bounded predictable (with respect to $(\mathcal{F}_t)$) processes (see [3], p. 98). For example, processes adapted to $(\mathcal{F}_t)$ and having left-continuous sample paths belong to $\mathcal{H}(\mathcal{F}_t)$.

Remark 2.2: Let the ICR (A_t) be of the form $(\int_0^t \lambda_s dm_s)$ and denote by Λ the union of all intervals of $\mathbb{R}_+$ on which the function m_t is constant. Observe that the ICR (A_t) is not affected by a change of values of (λ_t) for $t \in \Lambda$ and we may well have $\lambda_t = \infty$ for $t \in \Lambda$. To avoid problems due to this indeterminacy we adopt the following convention: for $t \in \Lambda$ we set λ_t equal to unity.

We assume here that modern martingale theory ([11], [3]) is known. Recall that a semimartingale (X_t) is a process which can be written as a sum $(X_t = X_0 + L_t + A_t)$ where X_0 is $\mathcal{F}_0$-measurable, (L_t) is a $(\mathcal{F}_t)$ local martingale and (A_t) is a right-continuous process adapted to $(\mathcal{F}_t)$

having sample paths of bounded variation on every finite interval and with $A_o = 0$ a. s. (see [3]). A result basic to this study and due to Doléans-Dade [4] is the following: the stochastic integral equation

$$Z_t = 1 + \int_0^t Z_{s-} dX_s$$

where (X_t) is a semimartingale has a unique solution, which is a semimartingale given by[†]

$$Z_t = \exp(X_t - \frac{1}{2} \langle X^c \rangle_t) \prod_{s \leq t} (1 + \Delta X_s) \exp(-\Delta X_s)$$

where the product in the right hand side converges a. s. for each t. Here we define $(\langle X^c \rangle_t)$ as the unique natural increasing process (see [3]) associated to the continuous part of the local martingale (L_t); $(\langle X^c \rangle_t)$ is identically zero when (X_t) is a semimartingale with sample paths of bounded variation on every finite interval (see [3]).

3. THE DETECTION PROBLEM. Let P_0 and P_1 be two measures carried on $(\Omega, \mathcal{A})$. Suppose that (N_t) is a CP defined on $(\Omega, \mathcal{A})$ and denote by $\mathcal{N}_t$ the minimal σ-algebra generated by (N_t) up to and at time t. The notation $E_i(\cdot)$ for i=0, 1 is intended for the expectation operator with respect to the measure P_i.

Definition 3. 1: For a $(\mathcal{N}_t)$ stopping time R (possibly infinite) denote by $\bar{P}_i^R$ for i=0, 1 the restriction of the measure P_i to the σ-algebra $\mathcal{N}_R$.

---

† When f_t is a right-continuous function with left-hand limits Δf_t denotes the jump $f_t - f_{t-}$.

We have the inclusion $\mathcal{N}_R \subset \mathcal{F}$ so that if $P_0 << P_1$† then $\bar{P}_0^R << \bar{P}_1^R$ and the Radon-Nikodym derivative $d\bar{P}_0^R/d\bar{P}_1^R$ is well defined. We examine now the meaning of this Radon-Nikodym derivative. In the case where the stopping time R is equal to a constant a then $\mathcal{N}_R = \mathcal{N}_a = \sigma(N_u,\ 0 \leq u \leq a)$ so that $d\bar{P}_0^a/d\bar{P}_1^a$ is the likelihood ratio for testing the two hypotheses H_i for i=0, 1: P_1 is the probability measure on $(\Omega, \mathcal{F})$, by observations on the CP (N_t) for $t \leq a$. The detection scheme then consists in selecting H_0 or H_1. according as $d\bar{P}_0^a/d\bar{P}_1^a$ is above or below a given threshold. Now in the case where R is a stopping time which is not a constant we know that $\mathcal{N}_R \supset \sigma(N_{u \wedge R},\ 0 \leq u)$ (this follows from the fact that $N_{u\wedge R}$ is $(\mathcal{N}_R)$ measurable by Theorem 49-IV of [11]) but the reverse inclusion is not necessarily true. For this reason $d\bar{P}_0^R/d\bar{P}_1^R$ is not the likelihood ratio for our detection problem when the time of observation is the stochastic interval [0, R], as one could have conjectured. But one can interpret $d\bar{P}_0^R/d\bar{P}_1^R$ as a likelihood ratio if we assume that the information accessible to the observer is $\mathcal{N}_R$ and not simply $\sigma(N_{u\wedge R},\ 0 \leq u)$. For i=0, 1 with the measure P_i carried on $(\Omega, \mathcal{F})$ suppose that the CP (N_t) has the process $(\int_0^t \lambda_s^i dm_s)$ for $(\mathcal{F}_t^i)$ ICR, where $(\mathcal{F}_t^i)$ is a family of σ-algebras with $\mathcal{F}_t^i \supset \mathcal{N}_t$, $(\lambda_t^i) \in \mathcal{B}(\mathcal{F}_t^i)$ is a positive process, and m_t is an increasing deterministic function with $m_0 = 0$.

It is known that we can make a change of measure under which (N_t) is a CP of independent increments with mean $m_t = EN_t$ under the new measure P (Theorem 2.6.1 of [5]). Using this fact and the three-step technique

† P_0 << P means that the measure P_0 is absolutely continuous with respect to P while $P_0 \sim P$ indicates that the two measures are equivalent.

of Duncan and Kailath (see Introduction) the likelihood ratio for detecting CP's has been obtained according to

THEOREM 3.2 (Theorem 3.4.4 of [5]): For i=0, 1 let (N_t) be, under the measure P_i, the CP described above. Assume

(a) $P_0 \ll P$ and $P \sim P_1$ and define for i=0, 1 the $(P, \mathcal{N}_t)$ martingale

$$L_t^i = E\left(\frac{d\bar{P}_i^\infty}{d\bar{P}^\infty} \,\middle|\, \mathcal{N}_t\right);$$

(b) For i=0, 1, the stopping times T^i are such that there exists increasing sequences of stopping times (T_n^i) for which $T^i = \lim_n T_n^i$ a.s. and $E(\ln^- L^i_{T_n^i})^2 < \infty$ for each n. Let $T = T^1 \wedge T^0$;

(c) For i=0, 1 $E_i \int_0^t \lambda_s^i dm_s < \infty$.

Then

(1) $$\frac{d\bar{P}_0^{t\wedge T}}{d\bar{P}_1^{t\wedge T}} = \prod_{J_n \le t\wedge T} \left[\frac{\hat{\lambda}^0_{J_n}}{\hat{\lambda}^1_{J_n}}\right] \exp\left[\int_0^{t\wedge T} (\hat{\lambda}_s^1 - \hat{\lambda}_s^0)dm_s\right]$$

where $\hat{\lambda}_t^i \triangleq E_i(\lambda_t^i|\mathcal{N}_t)$ for i=0, 1 and J_n is the time of n^{th} jump of (N_t). By convention the product $\Pi(\cdot) = 1$ for $J_1 > t \wedge T$.

<u>Remark 3.3:</u> (a) The stopping time T^i which is the first time after which the martingale (L_t^i) can behave badly may take the value $+\infty$. It is in fact desirable for T^i to be as large as possible.

(b) By our convention (Remark 2.2) condition (c) above insures that the process $(\hat{\lambda}_t^i)$ is well defined.

4. RECURSIVE INTEGRAL EQUATIONS FOR LIKELIHOOD RATIOS

We show here that the likelihood ratio (1) of our detection problem can be obtained as the unique solution of a stochastic integral equation. This stochastic integral equation can be mechanized by a feedback scheme tantamount to a recursive filter, as shown in Figure 1.

THEOREM 4.1: The likelihood ratio $d\bar{P}_0^{t\wedge T}/d\bar{P}_1^{t\wedge T}$ of Theorem 3.2 is the unique solution of the following stochastic integral equation:

$$Z_t = 1 + \int_0^t Z_{s-}\,dX_{s\wedge T} \tag{2}$$

where

$$X_t = \int_0^t \left\{ \left[\frac{\hat{\lambda}_s^0}{\hat{\lambda}_s^1} \right] - 1 \right\} dN_s + \int_0^t (\hat{\lambda}_s^1 - \hat{\lambda}_s^0)\,dm_s \tag{3}$$

Proof: By assumption (λ_t^i), i=0, 1, is positive a.s. finite for all t (by condition (c) of Theorem 3.2 and Remark 2.2). The process (N_t) has a finite number of jumps in any finite interval so that the process $(\int_0^{t\wedge T} [(\hat{\lambda}_s^0/\hat{\lambda}_s^1)-1]dN_s)$ has sample paths of bounded variation on any finite interval; and so does the process $(\int_0^{t\wedge T} (\hat{\lambda}_s^1-\hat{\lambda}_s^0)dm_s)$ by assumption (c) of Theorem 3.2. Hence $(X_{t\wedge T})$ is a semimartingale with sample paths of bounded variation on any finite interval so that $(\langle X^c\rangle_{t\wedge T} \equiv 0)$ (see the remark, on p. 90, following proposition 3 of [3]). Then by Theorem 1 of [4] the unique solution of (2) is given by

$$Z_t = \exp(X_{t\wedge T}) \prod_{s\leq t} (1 + \Delta X_{s\wedge T})\exp(-\Delta X_{s\wedge T}) \tag{4}$$

Now $\Delta X_{s\wedge T} = ((\hat{\lambda}_s^0/\hat{\lambda}_s^1) - 1)\Delta N_{s\wedge T}$ and hence the product in (4) becomes

$$\Pi_{s\leq t}(\cdot) = \Pi_{s\leq t}\left[1 + \left[\frac{\hat{\lambda}_s^0}{\hat{\lambda}_s^1} - 1\right]\Delta N_{s\wedge T}\right]\exp\left[\sum_{s\leq t\wedge T} - \left[\frac{\hat{\lambda}_s^0}{\hat{\lambda}_s^1} - 1\right]\Delta N_{s\wedge T}\right]$$

$$= \Pi_{J_n\leq t\wedge T}\left[\frac{\hat{\lambda}_{J_n}^0}{\hat{\lambda}_{J_n}^1}\right]\exp\left[-\int_0^{t\wedge T}\left[\frac{\hat{\lambda}_s^0}{\hat{\lambda}_s^1} - 1\right]dN_s\right]$$

Substituting the above relation and expression (3) in (4) gives the desired result (compare with (1))

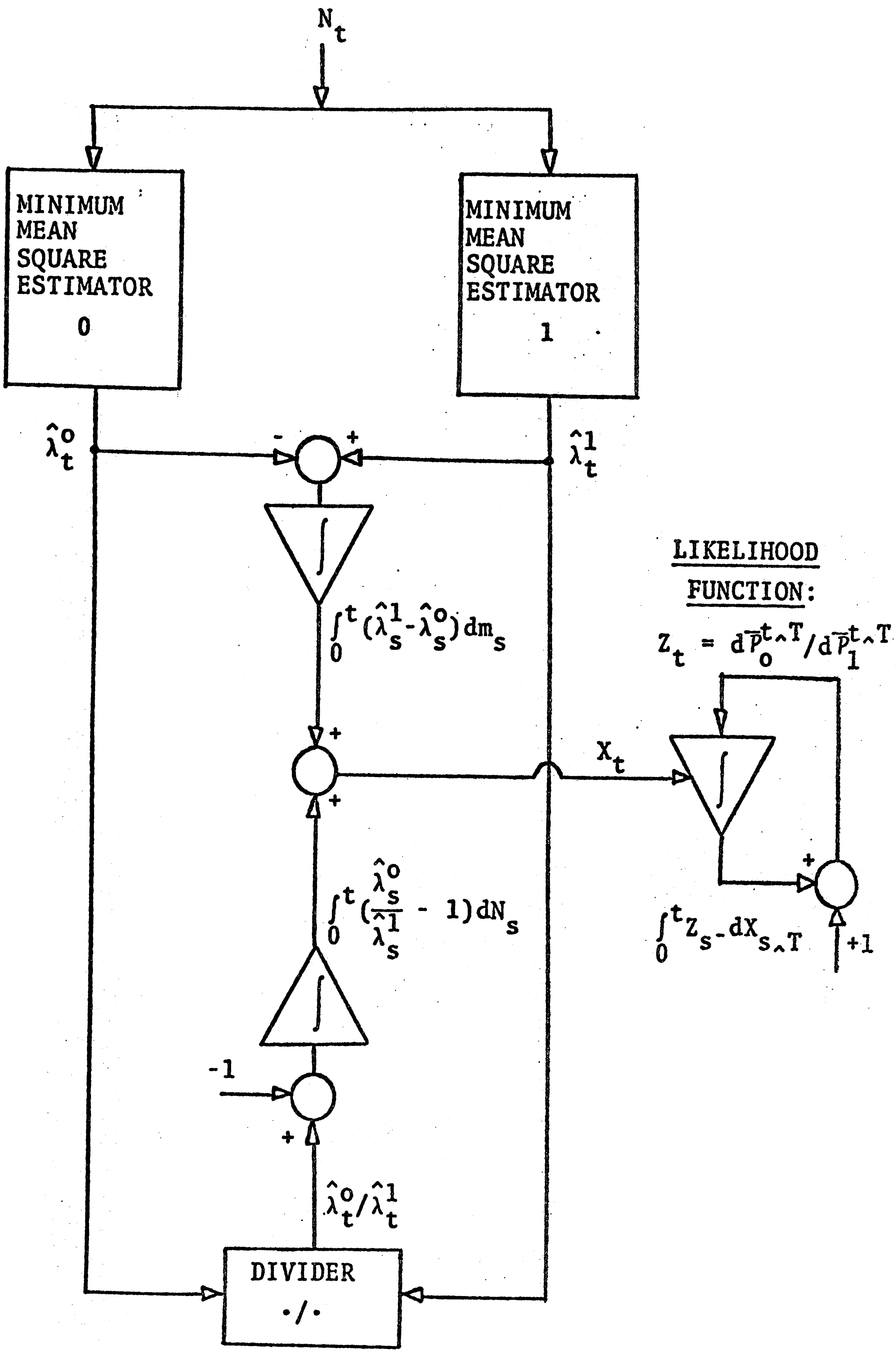
$$Z_t = \Pi_{J_n\leq t\wedge T}\left[\frac{\hat{\lambda}_{J_n}^0}{\hat{\lambda}_{J_n}^1}\right]\exp\left[\int_0^{t\wedge T}(\hat{\lambda}_s^1 - \hat{\lambda}_s^0)dm_s\right] = \frac{d\bar{P}_0^{t\wedge T}}{d\bar{P}_1^{t\wedge T}}$$

Observe that if under the measure P_1 the CP (N_t) is a process of independent increments with mean m_t then $P \equiv P_1$, $\lambda_t^1 = 1$ and Eq. (3) becomes

$$(5) \qquad X_t = \int_0^t (\hat{\lambda}_s^0 - 1)d(N_s - m_s)$$

The process $(M_t \triangleq N_t - m_t)$ is a $(P, \mathcal{N}_t)$ martingale. Hence (5) shows that the process $(X_{t\wedge T})$ is a local martingale. In turn, (2) then implies that the process (Z_t) is a local martingale. In this case we in fact have $Z_t = E_1[(d\bar{P}_0^{\infty}/d\bar{P}_1^{\infty})|\mathcal{N}_{t\wedge T}]$, i.e. the likelihood function is a uniformly integrable martingale.

In applications, Eqs. (2) and (3) give a way of implementing the computation of the likelihood ratio continuously in time. They represent recursive equations if one also obtains the best estimates $(\hat{\lambda}_t^i)$ in a recursive manner. The block diagram of this implementation is given in Figure 1.



Recursive Scheme for Obtaining the Likelihood Function Z_t.

Figure 1

REFERENCES

1. I. Bar David, Communication under the Poisson regime, IEEE Transactions on Information Theory, IT-15, January 1969, pp. 31-37.

2. P. M. Brémaud, A martingale approach to point processes, Memorandum No. ERL-M345, Electronic Research Laboratory, University of California, Berkeley, California, August 1972.

3. C. Doléans-Dade and P. A. Meyer, Intégrales stochastiques par rapport aux martingales locale, Séminaires de Probabilités IV, Lecture Notes in Mathematics No. 124, Springer-Verlag, Berlin, 1970, pp. 77-107.

4. C. Doléans-Dade, Quelques applications de la formule de changement de variables pour les semimartingales, Z. Wahrscheinlichkeitstheorie verw. Geb., 16, 1970, pp. 181-194.

5. F. B. Dolivo, Counting Processes and Integrated Conditional Rates: A Martingale Approach with Application to Detection, Ph. D. Thesis, The University of Michigan, Ann Arbor, Michigan, June 1974.

6. T. E. Duncan, On the absolute continuity of measures, Ann. Math. Stat., 41 (1970), pp. 30-38.

7. T. E. Duncan, Likelihood functions for stochastic signals in white noise, Information and Control, 16 (1970), pp. 303-310.

8. I. V. Girsanov, On transforming a certain class of stochastic processes by absolutely continuous substitution of measures, Theory of Probability and Its Applications, V:3 (1960), pp. 285-301.

9. T. Kailath, A further note on a general likelihood formula for random signals in a Gaussian noise, IEEE Transactions on Information Theory, IT-16, July 1970, pp. 393-396.

10. H. Kunita and S. Watanabe, On square integrable martingales, Nagoya Math. Journal, 30 (1967), pp. 209-245.

11. P. A. Meyer, Probability and Potentials, Blaisdell, Waltham, Massachusetts, 1966.

12. B. Reiffen and H. Sherman, An optimum demodulator for Poisson processes: photon source detectors, Proceedings of the IEEE, 51, October 1963, pp. 1316-1320.

13. J. H. Van Shuppen and E. Wong, Transformation of local martingales under a change of law, Electronic Research Laboratory, Memorandum No. ERL-M385, University of California, Berkeley, California, May 1973.