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Maximal Nonparabolic Subgroups of the Modular Group

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0. Introduction

The *modular group* $M = \mathrm{PSL}(2, \mathbb{Z})$ may be defined as the group of all transformations of the set $\mathbb{C} \cup \{\infty\}$ of the form $z \mapsto \dfrac{az+c}{bz+d}$ for $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. It is well known that M has a presentation

$$(0.1) \qquad M = \langle A, B : A^2 = B^3 = 1 \rangle ,$$

where A and B are the transformations

$$A : z \mapsto -1/z, \qquad B : z \mapsto \frac{z-1}{z}.$$

The *elliptic* elements of M, each with two conjugate complex fixed points, are precisely the conjugates of nontrivial powers of A and B. The *parabolic* elements, each with a single real fixed point, are precisely the conjugates of nontrivial powers of $C = AB : z \mapsto z+1$. The remaining nontrivial elements of M are *hyperbolic*, each with two real fixed points.

A subgroup S of M is torsionfree (and therefore by the Kurosh Subgroup Theorem a free group) if and only if it contains no elliptic elements. A subgroup S of M is called *nonparabolic* if it contains no parabolic elements. Neumann [9] showed that, if T is the infinite cyclic group generated by a conjugate of C, and S is a complement to T in M in the sense that $S \cap T = 1$ and $ST = M$, then S is a maximal nonparabolic subgroup of M. We call such subgroups *Neumann subgroups*. Magnus [8] raised the question whether all maximal nonparabolic subgroups of the modular group are Neumann subgroups. We show that this is not the case.

We show that the kernel N of the obvious map from M onto the planar crystallographic group

$$Q = \langle A_1, B_1 : A_1^2 = B_1^3 = (A_1 B_1 A_1 B_1^{-1})^3 = 1 \rangle$$

is a nonparabolic subgroup, indeed maximal in the class of normal nonparabolic subgroups. We show further that none of the maximal nonparabolic subgroups S containing N is a Neumann subgroup. Among the infinitely many such maximal nonparabolic subgroups S containing N, the simplest is perhaps the group S generated by the involutions $A_n = C^n BAB^{-1} C^{-n}$; this group is the free product $S = *\langle A_n \rangle$ of the countably many groups $\langle A_n \rangle$ of order 2.

Our method is to study subgroups S of M by reference to their coset graphs. This method has been employed by many workers, notably in the recent study by Conder [4, 5] of quotient groups of the triangle groups $(2, 3, k)$. It has been applied by Stothers [11–17] and by the present authors [1–3] to the study of nonparabolic subgroups of the modular group. In particular, the observations in Sect. 1 below are formulated by Stothers [13] and, especially, [14].

1. Coset Graphs

If S is any subgroup of M, the *coset graph* $\Gamma(S)$ of S, relative to the set of generators A and B, is defined as follows. The set $V(\Gamma(S))$ of vertices of $\Gamma(S)$ is the set of cosets Sg, $g \in M$. There are exactly three (directed) edges out of each vertex $V = Sg$: an *A-edge* from v to $vA = SgA$, a *B-edge* from v to vB, and a *B^{-1}-edge* from v to vB^{-1}. The inverses of these three edges are the A-edge from vA to v, the B^{-1}-edge from vB to v, and the B-edge from vB^{-1} to v. All graphs considered here will be embedded in the plane or in some other manifold; the edges will be represented by directed arcs, the inverse of an edge being represented by the same arc with opposite orientation.

We define a *(2, 3)-graph* to be connected graph Γ whose set of directed edges is divided into three disjoint sets, of A-edges, B-edges, and B^{-1}-edges, subject to the following conditions.

(i) At each vertex there is exactly one A-edge, one B-edge, and one B^{-1}-edge.

(ii) The inverse of an A-edge is an A-edge, the inverse of a $B^{\pm 1}$-edge is a $B^{\mp 1}$-edge.

(iii) At each vertex v, the B-edge at v is either a loop ending at v, or is one in a cycle of three B-edges.

Evidently every coset graph $\Gamma(S)$ of a subgroup S of M is a (2, 3)-graph. For the converse, let Γ be a (2, 3)-graph. Define a permutation A_1 of $V = V(\Gamma)$ to carry each vertex v to the vertex at the other end of the A-edge beginning at v, and define B_1 analogously. Then $A_1^2 = B_1^3 = 1$. In view of the presentation (0.1) of M, the map $A \mapsto A_1$, $B \mapsto B_1$ defines an action of M on V. Let v_0 be any vertex $v_0 \in V$, and let S be the stabilizer of v_0 under the action of M on V. If $g, h \in M$, then $v_0 g = v_0 h$ is equivalent to $gh^{-1} \in S$, hence to $Sg = Sh$. Thus we can define a bijection $\phi : v_0 g \mapsto Sg$ from V to $V(\Gamma(S))$. Since $(vgA)\phi = SgA = (vg)\phi A$ and $(vgB)\phi = (vg)\phi B$, the map ϕ is an isomorphism from the (2, 3)-graph Γ to the (2, 3)-graph $\Gamma(S)$. If the base point v_0 is changed, then S is replaced by a subgroup conjugate to it in M. This establishes the following.

(1.1) **Proposition.** *There is a bijective correspondence ϕ between the isomorphism classes of* (2, 3)*-graphs and the conjugacy classes of subgroups of M.*

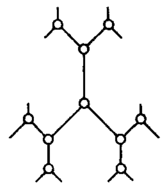

Fig. 1. The graph $\Gamma(1)$

Let $S_0 \leqq S \leqq M$. Then the inclusion map $S_0 g \mapsto Sg$ evidently defines a homomorphism from the (2, 3)-graph $\Gamma(S_0)$ onto the (2, 3)-graph $\Gamma(S)$. This proves the following.

(1.2) **Proposition.** *Let S_0 be a subgroup of M. Then ϕ gives a bijective correspondence between the homomorphic images of $\Gamma(S_0)$ and the conjugacy classes of subgroups of M that contain S_0.*

(1.3) *Remark.* It follows from (1.2) that the (2, 3)-graphs are precisely the homomorphic images of the coset graph $\Gamma(1)$ of the trivial subgroup 1 of M. This graph $\Gamma(1)$, which is a Cayley graph for M, may be described as the truncation of the (infinite) cubic tree; it is shown in Fig. 1.

An *A-loop* [or *B-loop*] in Γ is an A-edge [or B-edge] beginning and ending at the same point v. A *C-orbit* of Γ is an orbit of C under the action of M on the vertex set V of Γ.

(1.4) **Proposition.** *Let S be a subgroup of M. Then the S-conjugacy classes of subgroups of S of order 2 correspond bijectively to the A-loops in $\Gamma(S)$, and the S-conjugacy classes of subgroups of order 3 to the B-loops in $\Gamma(S)$. The S-conjugacy classes of parabolic subgroups of S correspond bijectively to the finite C-orbits of $\Gamma(S)$.*

Proof. A subgroup of order 2 in S is generated by an element of the form gAg^{-1}, $g \in M$. This determines in $\Gamma(S)$ a closed path π at the vertex $v_0 = S \cdot 1$ of the form $\pi = \gamma_1 \gamma \gamma_1^{-1}$, where γ_1 is a path from v_0 to $v_0 g$ and γ is an A-loop at $v_0 g$. The same path corresponds to another subgroup of order 2 in S if and only if the two subgroups are conjugate in S.

A subgroup of order 3, with generater gBg^{-1}, corresponds in the same way to a closed path π in $\Gamma(S)$, where γ is now a B-loop. A parabolic subgroup, with a unique generator of the form $gC^k g^{-1}$, $k \geqq 1$, corresponds to a closed path π in $\Gamma(S)$ in which γ is now a closed path with label $C^k = (AB)^k$, corresponding to a C-orbit of length k. □

(1.5) **Corollary.** *A subgroup S of M is a free group if and only if $\Gamma(S)$ contains no A-loops and no B-loops.*

(1.6) **Corollary.** *A subgroup S of M is nonparabolic if and only if all C-orbits of $\Gamma(S)$ are infinite.*

(1.7) **Corollary.** *A subgroup S of M is a Neumann subgroup if and only if $\Gamma(S)$ contains only a single C-orbit, which is infinite, that is, if and only if $\Gamma(S)$ is infinite and C acts transitively on the vertices of $\Gamma(S)$.*

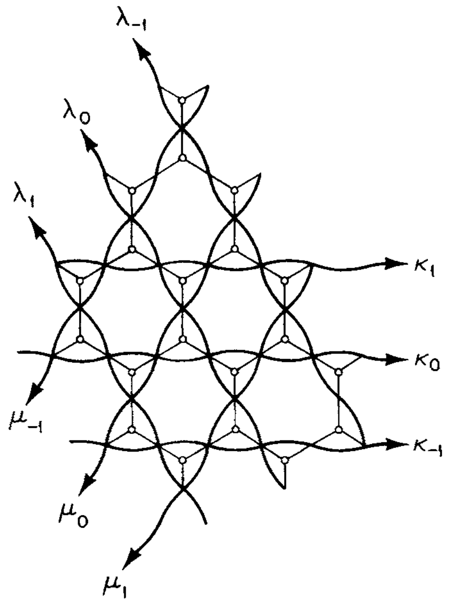

Fig. 2. The coset graph $\Gamma = \Gamma(N)$

2. The Hexagonal Graph

Let H be the 1-skeleton of the regular tessellation of the Euclidean plane by hexagons. The symmetry group of H contains, with index 2, a crystallographic group Q with a presentation

$$Q = \langle A_1, B_1 : A_1^2 = B_1^3 = (A_1 B_1 A_1 B_1^{-1})^3 = 1 \rangle .$$

(See Coxeter-Moser [7, pp. 48–49], especially formula (4.5131); see also Coxeter [6] and Sinkov [10].)

Let N be the kernel of the map from M onto Q that carries A to A_1 and B to B_1. The coset graph $\Gamma = \Gamma(N)$, relative to the generators A and B, is also a Cayley graph for Q. In fact, Γ can be obtained by "truncating" H as follows. Let ε be a positive real number less than half the common length of the edges of H. About each vertex p of H we draw a circle c_p of radius ε. We now obtain Γ by deleting the parts of H interior to the circles c_p. The straight segments remaining from the edges joining points p and q of H now become the A-edges of Γ, joining points on c_p and c_q. Since A_1 represents a reflection of H, we are led to orient the circles c_p alternately positively and negatively, so that circles joined by an A-edge are oriented oppositely. The B-edges of Γ are now the arcs on all the circles c_p, three on each circle. They are oriented in the sense of the c_p.

The graph Γ is indicated in Fig. 2. In this figure the C-orbits are indicated by wavy lines running roughly parallel to the true paths. These true paths consist of successive edges that are alternately (straight) A-edges and (curved) B-edges. The C-orbits are all infinite, and fall into three families K, Λ, M, where each family consists of infinitely many orbits that are "parallel", that is, congruent under

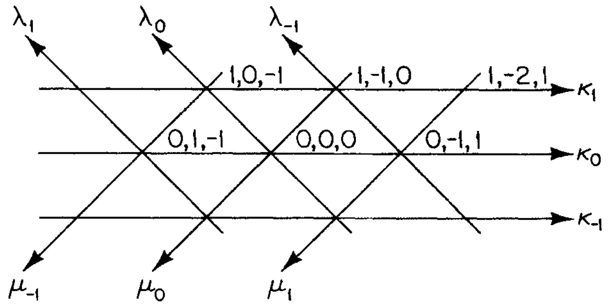

Fig. 3. The graph $\bar{\Gamma}$

translation. The figure also indicates a systematic way of indexing the elements κ_k, λ_l, μ_m of the families K, Λ, M by integers k, l, m. The indexing is arranged so that the three points on any circle c_p lie on three orbits κ_k, λ_l, μ_m, one from each family. Moreover, if c_p is any positively oriented circle, then $k+l+m=0$, and, conversely, if k, l, m are three integers such that $k+l+m=0$, then there is a unique positively oriented circle c_p such that the three points on c_p belong to κ_k, λ_l, and μ_m. We can thus describe the positively oriented circles c_p by the corresponding triple of coordinates (k, l, m). This coordinatization depends of course on the choice of direction of the families K, Λ, M and on the choice of a base point v_0.

For some purposes it is convenient to replace the graph Γ by a less complicated graph $\bar{\Gamma}$. In $\bar{\Gamma}$ each positively oriented circle c_p is replaced by a point $p=p(k,l,m)$, and the negatively oriented circles are omitted. Two points p and q in the graph $\bar{\Gamma}$ are joined by an edge in $\bar{\Gamma}$ if and only if the corresponding points p and q in the hexagonal graph H are at distance 2. The graph $\bar{\Gamma}$ is in fact a regular tessellation of the plane by triangles, and the infinite straight lines in $\bar{\Gamma}$ represent the C-orbits. The graph $\bar{\Gamma}$ is shown in Fig. 3.

We now consider a homomorphism ϕ of Γ onto a (2,3)-graph Γ^*. If two vertices v_1 and v_2 in Γ have the same image $v_1\phi=v_2\phi$, and if g is any element of M, then one sees that also $v_1 g\phi=v_2 g\phi$. It follows that the automorphism of Γ defined by $v_1 g \mapsto v_2 g$, for all $g \in M$, is a rigid motion of the plane. It follows further that if G is the group of all rigid motions γ such that $v\gamma\phi=v\phi$ for all v, then Γ^* is within isomorphism the quotient Γ/G of Γ by the action of G. Thus, to study homomorphic images Γ^* of Γ, and ultimately conjugacy classes of subgroups S that contain N, it suffices to study groups G of rigid motions of Γ. We shall, for brevity, call G and $\Gamma^*=\Gamma/G$ *nonparabolic* if and only if the corresponding subgroups S are nonparabolic.

We first consider rigid motions that preserve Γ and also preserve the orientation of the plane. Such a nontrivial rigid motion is either a translation or a rotation. A translation τ must carry each positively oriented circle $c_p=(k,l,m)$ to a second positively oriented circle $c_q=(k+a, l+b, m+c)$, where necessarily $a+b+c=0$. We write $\tau=\tau(a,b,c)$. Now τ carries some point of an orbit into a different point of the same orbit if and only if exactly one of a, b, c is 0. Thus a nonparabolic group G cannot contain any nontrivial translation $\tau=\tau(a,b,c)$ one of whose coordinates is 0.

Any nontrivial rotation σ that preserves Γ, as a (2, 3)-graph, must likewise carry every positively oriented circle c_p to another positively oriented circle c_q. Such a rotation σ must be of order 3, and can be of one of two kinds: it can carry some positively or negatively oriented circle c_p into itself, or it can have its center at the center of one of the truncated hexagons in Γ, permuting cyclically the three positively oriented circles c_p incident with that hexagon. In either case, σ permutes the three families K, Λ, M of C-orbits cyclically, hence does not carry any point of an orbit to a point on the same orbit.

We next consider a rigid motion ϱ that preserves Γ but reverses the orientation of the plane. Then ϱ must be either a reflection or a "glide reflection". Since ϱ reverses orientation, it must interchange two of the families of C-orbits while leaving the third invariant. We may suppose that ϱ leaves K invariant. Then the axis L of ϱ must be parallel to the C-orbits in the family K. Since ϱ must interchange positively oriented with negatively oriented circles, L must bisect perpendicularly the family of vertical Λ-edges in Fig. 2 that join each positively oriented circle c_p containing a vertex in a certain C-orbit $\kappa_k \in K$ with the negatively oriented circle c_q directly below it, which contains a vertex in the C-orbit κ_{k-1}. Thus ϱ, which reverses the linear order of the family K of orbits $(\ldots, \kappa_{-1}, \kappa_0, \kappa_1, \ldots)$, must interchange κ_k and κ_{k-1}.

Now ϱ^2, which preserves orientation, must leave invariant each orbit of the family K. If ϱ is contained in a nonparabolic group G, this can happen only if $\varrho^2=1$, that is, if ϱ is a reflection.

From the above we conclude that a group G of rigid motions preserving Γ is nonparabolic if and only if G contains no nontrivial translation $\tau=\tau(a,b,c)$ for which any of a, b, c is 0. In particular, although G may contain rotations and reflections, it can contain no proper glide reflection.

We are now prepared to enumerate the nonparabolic groups G of rigid motions that preserve Γ. We begin with those groups G that preserve orientation. Suppose first that $\tau_1=\tau(a_1,b_1,c_1)$ and $\tau_2=\tau(a_2,b_2,c_2)$ are two translations in G. By applying the Euclidean algorithm to the two vectors (a_i,b_i,c_i) in the usual way, we can replace the two generators τ_1 and τ_2 of the group $\langle\tau_1,\tau_2\rangle$ that they generate by two generators τ_1' and τ_2' such that $a_2'=0$. Since G is nonparabolic, this implies that $\tau_2'=1$. We conclude that τ_1 and τ_2 are integer powers of the common element τ_1'; in particular, τ_1 and τ_2 are at least as long as τ_1'. It follows that the translation subgroup of G is either trivial or infinite cyclic, generated by one of the two translations in G of minimal length.

Suppose now that G contains a nontrivial rotation σ. If G also contains a translation τ, it contains another translation $\sigma\tau\sigma^{-1}$, in a different direction. Since τ and $\sigma\tau\sigma^{-1}$ are not powers of a common translation, this is not possible in the nonparabolic group G. Thus G contains no translation. Further, if G contained a nontrivial rotation σ_1 other than σ and σ^{-1}, then one of $\sigma_1\sigma$, $\sigma_1\sigma^{-1}$ would be a nontrivial translation. Thus G contains no nontrivial rotation other than σ and σ^{-1}.

We see that the orientation preserving groups G that preserve Γ are exactly as follows:

(i) $G_0 = 1$, the trivial group;

(ii) $G_\tau = \langle\tau\rangle$, infinite cyclic, generated by a translation τ not in the direction of any C-orbit;

(iii) $G_\sigma = \langle\sigma\rangle$, cyclic of order 3, generated by a rotation σ. Here there are two geometrically distinct cases according as: (iiia), σ preserves some circle c_p, or, (iiib), σ preserves no circle c_p.

Now suppose that G contains elements that reverse orientation. Then G contains a reflection ϱ, and G is generated by ϱ together possibly with some translations and rotations. Suppose that G contains a non trivial translation τ. Then it also contains the translation $\varrho\tau\varrho$, which must be parallel to τ. This is possible only if the direction of τ is either parallel or perpendicular to the axis L of ϱ. Since we have assumed that the axis L of ϱ is parallel to the orbits of the family K, the case that τ is parallel to L is excluded. Thus τ must be a translation in a direction perpendicular to L, that is, vertical in Fig. 2. Because τ must carry the positively oriented circle $c_p = (0,0,0)$ to another positively oriented circle c_q directly above or below c_p, inspection of Fig. 1 shows that τ must carry κ_0 to κ_n for some even $n = 2n_0$. If we choose τ in G of minimal length, then all other translations in G are powers of τ. Moreover, since G contains translations, it can contain no rotation. We conclude that G is generated by ϱ and τ, and is fully described by its action on K, on which it acts as an infinite dihedral group.

Suppose finally that G contains the reflection ϱ and also a nontrivial rotation σ. Then G contains no translation, and no nontrivial rotation other than σ and σ^{-1}. Since $\varrho\sigma\varrho$ is a rotation in a sense opposite to that of σ, it can only be σ^{-1}. The center of σ must therefore be on the axis L of ϱ, which contains no center of a circle c_p, and we conclude that the rotation subgroup of G must be of type (iiib).

From the above we see that the classes of nonparabolic groups G that preserve Γ but do not preserve orientation are exactly as follows:

(iv) $G_\varrho = \langle\varrho\rangle$, of order 2, generated by a reflection ϱ;

(v) $G_{\varrho,\tau} = \langle\varrho,\tau\rangle$, infinite dihedral, generated by a reflection ϱ together with a translation τ of even length $2n_0$ in a direction perpendicular to the axis L of ϱ;

(vi) $G_{\varrho,\sigma} = \langle\varrho,\sigma\rangle$, dihedral of order 6, generated by a reflection ϱ together with a rotation σ, of type (iiib), with center on the axis L of ϱ.

With this catalog of nonparabolic groups G in hand, we can proceed to verify that none of these groups G gives rise to a graph $\Gamma^* = \Gamma/G$ corresponding to a Neumann subgroup S of M, that is, that every case Γ^* has more than one C-orbit. This is obvious if $G = 1$. If $G = G_\tau$, a translation group, then it is clear that the images K^*, Λ^*, M^* of K, Λ, M are disjoint, and it is not difficult to check that K^*, Λ^*, M^* have the finite cardinalities $|a|$, $|b|$, $|c|$. Thus Γ^* has $n = |a| + |b| + |c|$ C-orbits, and every even integer $n \geqq 4$ occurs thus.

If $G = G_\sigma$, a rotation group, then G permutes K, Λ, M cyclically, and their common image $K^* = \Lambda^* = M^*$ in Γ^* is a single infinite family of C-orbits. If $G = G_\varrho$, we see similarly that Γ^* has two infinite families of C-orbits, K^* and $\Lambda^* = M^*$. The case $G = G_{\varrho,\tau}$, in which there are two finite families K^* and $\Lambda^* = M^*$ of cardinalities $|n|$ and $|n| - 1$, and the case $G = G_{\varrho,\sigma}$, in which there are infinitely many C-orbits, are more difficult to describe. These two cases are illustrated by Fig. 4 and 5.

We remark that the conjugacy class of groups S associated with $\Gamma^* = \Gamma/G$ will reduce to a single normal subgroup S of M if and only if G is a normal subgroup of

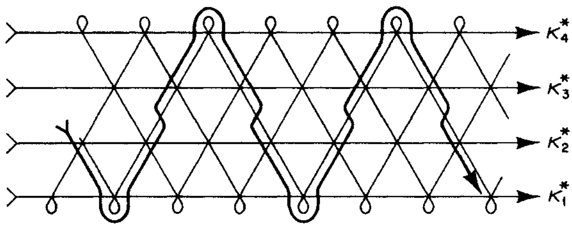

Fig. 4. The graph $\Gamma^* = \Gamma/G$ for $G = G_{\varrho,\tau}$ with $n_0 = 2$

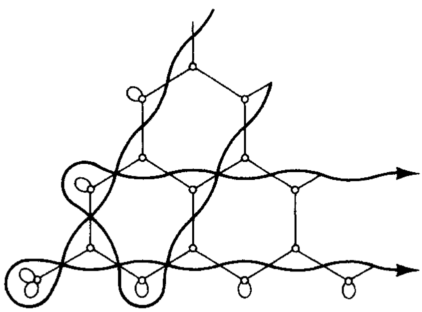

Fig. 5. The graph $\Gamma^* = \Gamma/G$ for $G = G_{\varrho,\sigma}$

the full symmetry group of Γ. Apart from the trivial group $G = 1$, none of the groups G listed above is normal in this full symmetry group. Therefore none of the nonparabolic subgroups S properly containing N is normal.

(2.1) **Theorem.** *N is maximal in the class of normal nonparabolic subgroups of M.*

We have also seen that none of the nonparabolic groups G listed above gives rise to a conjugacy class of Neumann subgroups. This proves the following.

(2.2) **Theorem.** *The nonparabolic subgroup N of M is contained in no Neumann subgroup.*

From this it follows that M contains maximal nonparabolic subgroups that are not Neumann subgroups. To find those that contain N, we determine the maximal groups among the nonparabolic groups G listed above. Clearly $G_0 = 1$ is not maximal. Let $G = G_\tau$ for $\tau = \tau(a, b, c)$ where $a + b + c = 0$ and $a, b, c \neq 0$. Evidently G_τ is not contained in a larger group of the same type if and only if a, b, and c are relatively prime, that is, if $(a, b) = 1$. However, even in this case, G may be contained in a group $G_{\varrho,\tau}$. If ϱ is chosen as before, with axis L parallel to the family K, then, in $G_{\varrho,\tau}$, the triple (a, b, c) has the form $(2n_0, -n_0, -n_0)$. In general, for G_τ not to be contained in some $G_{\varrho,\tau}$, we must further exclude the case that two of a, b, c are equal, and hence, assuming $(a, b) = 1$, have a common value $+1$ or -1.

Evidently a group $G = G_\sigma$ is maximal if and only if it is not contained in a group $G_{\varrho,\sigma}$. Thus G_σ is maximal if and only if it is of type (iiia).

Clearly no group $G = G_\varrho$ is maximal. Likewise, a group $G_{\varrho,\tau}$ is contained in a larger group $G_{\varrho,\tau'}$ if and only if the length $2n_0$ of τ is divisible by the length $2n'_0$ of

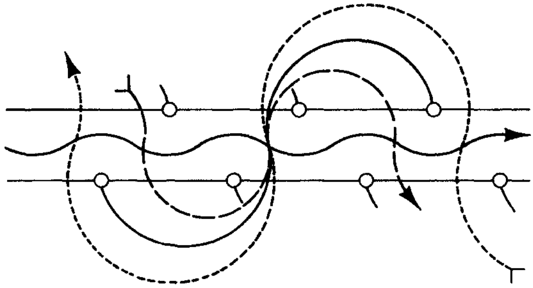

Fig. 6. The graph $\Gamma^* = \Gamma/G$ for $G = G_\tau$, with $\tau = \tau(1, -3, 2)$

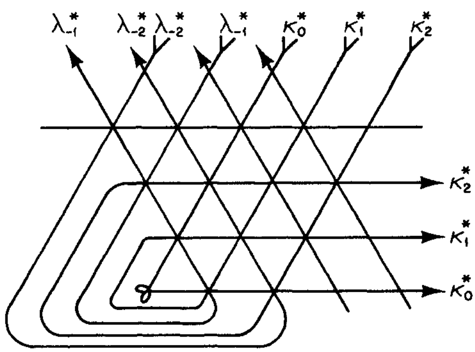

Fig. 7. The graph $\Gamma^* = \Gamma/G$ for $G = G_\sigma$

τ', and τ and τ' are parallel. Thus, among the $G_{\varrho,\tau}$, only those with τ of length $2n_0 = 2$ are maximal. Finally, the groups $G = G_{\varrho,\sigma}$ are obviously maximal. We summarize.

(2.3) **Proposition.** *The maximal nonparabolic subgroups S of M that contain N correspond to the graphs $\Gamma^* = \Gamma/G$, where G is of one of the following types:*

(i) G_τ, *where $\tau = \tau(a, b, c)$, a translation, such that $a + b + c = 0$, that $a, b, c \neq 0$, that a and be are relatively prime, and that no two of a, b, c are $+1$ and no two are -1.*

(ii) G_σ, *where σ is a rotation of order 3 leaving invariant some B-orbit;*

(iii) $G_{\varrho,\tau}$, *where ϱ is a reflection with axis L parallel to a family of C-orbits and τ is a translation of length $2n_0 = 2$ in a direction perpendicular to L.*

(iv) $G_{\varrho,\sigma}$, *dihedral of order 6.*

We note that, in case (i), any permutation of a, b, c, or the replacement of all three by their negatives, yields an isomorphic graph Γ^*, and hence the same conjugacy class of maximal nonparabolic subgroups S of M. Nonetheless, infinitely many conjugacy classes are obtained in this case for various choices of the parameters a, b, c. Each of the remaining cases (ii), (iii), (iv) yields a single conjugacy class of maximal nonparabolic subgroups.

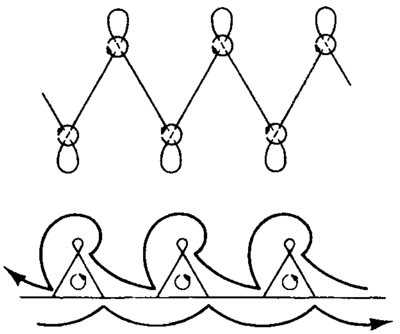

Fig. 8. Two representations of $\Gamma^* = \Gamma/G$ for $G = G_{\varrho,\tau}$ with $\tau = \tau(2, -1, -1)$, of length 2

In Fig. 6 we illustrate case (i) for the group G_τ given by $\tau = \tau(1, -3, 2)$. The resulting graph Γ^* has six C-orbits, the smallest number obtainable in case (i). Figs. 7 and 5 show the cases (ii) and (iv); these are admittedly less perspicuous.

Finally, in Fig. 8, we show Γ^* for the case (iii), first in the form in which it arises naturally, and again, redrawn in a simpler form. From the redrawn form it is clear that Γ^* has exactly two C-orbits, one running to the right and the other to the left, the second containing twice as many points of each B-orbit as the first. One can verify by direct inspection that every proper homomorphic image of Γ^* is finite, and indeed has either three vertices or a single vertex. It is also evident. upon choosing a suitable base point v_0, that the associated class of nonparabolic subgroups contains the subgroup S generated by the set of elements $A_n = C^n B A B^{-1} C^{-n}$ of order 2, for all $n \in \mathbb{Z}$, corresponding to the A-loops in Γ^*. The subgroup S is therefore the free product of the countably infinite set of subgroups of order 2 generated by the A_n. This graph Γ^* is in a reasonable sense the simplest graph yielding a nonparabolic subgroup S of M that is not a Neumann subgroup.

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