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Motifs, L-Functions, and the K-Cohomology of Rational Surfaces over Finite Fields

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Let X be a smooth projective variety over a field. Let $\mathscr{K}_j$ denote the Zariski sheaf associated to the presheaf $U \mapsto K_j(\Gamma(U, \mathcal{O}_U))$ of Quillen K-groups. The collection of Zariski cohomology groups $H^i(X, \mathscr{K}_j)$ will be referred to as the K-cohomology of X. The groups contain a great deal of information about the geometry of X. For example, Bloch's formula [11] says that $H^i(X, \mathscr{K}_i)$ is isomorphic to the group $CH^i(X)$ of codimension i cycles on X modulo rational equivalence.

In this paper, the K-cohomology of a rational surface over a finite field will be calculated up to p-torsion, where p is the characteristic. Recall that a surface X over a field F is called rational if $\bar{X} = X \otimes_F \bar{F}$ is birational with $\mathbb{P}^2_{\bar{F}}$ over the algebraic closure $\bar{F}$.

Motifs play a central role in the computation. The idea that motifs could be used effectively to study the K-theory of varieties over finite fields is due to Soulé, whose paper [15] is a fertile source of inspiration. He introduced the K-cohomology of motifs and proved several important finiteness results. In particular, he showed for a wide class of varieties, including rational surfaces, that $K_j(X)$ is torsion for $j > 0$. A second source of ideas is the fundamental work of Merkurjev and Suslin [18], especially as applied in [4] to codimension two cycles and in [17] to the K_2-cohomology of rational surfaces.

The paper is organized as follows. The first section contains a brief review of the theory of motifs. From the motivic point of view, all the relevant data about a rational surface is contained in the action of the Galois group on $\mathrm{Pic}(\bar{X})$. This action is a continuous, finite dimensional, integral Galois representation. The second section contains a computation of the prime-to-p K-theory of such representations when F is finite.

In the final section, the orders of the K-cohomology groups are related to the values at integer points of the L-functions associated to X. Let $\# F = q = p^r$ and let l be a prime different from p. Let ϕ_j denote the endomorphism of the étale cohomology group $H^j(\bar{X}, \mathbb{Q}_l)$ which is induced by the q-power Frobenius. Define $P_j(t) = \det(1 - \phi_j t)$ and $L_j(s) = P_j(q^{-s})$. Also, if $a, b \in \mathbb{Q}$, write $a \sim b$ to mean that a/b is a power of p.

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Theorem. *If X is a rational surface over a finite field and if $n>2$, then*

$$L_{2i}(n) \sim \# H^i(X, \mathcal{K}_{2n-i-1}) \qquad 0 \leqq i \leqq 2$$
$$L_{2i+1}(n) \sim \# H^i(X, \mathcal{K}_{2n-i-2}) \qquad 0 \leqq i \leqq 1 .$$

The restrictions on i in the theorem correspond to the range of possibly nontrivial étale cohomology groups and L-functions. The restriction on n avoids any contribution from $K_0(X)$, hence by [15] lies in a range where the K-groups are torsion. The conclusion also holds for certain other particularly simple varieties. It is compatible with the Quillen conjecture for curves. It is unlikely that such a simple formula holds in general. Beilinson [2] and Lichtenbaum [8] have conjectured a theory of arithmetic cohomology which would yield a more elaborate formula for the values of the zeta function of X at any positive integer n.

1. Motifs and Rational Surfaces

For a fuller discussion of the theory of motifs, see [9, 15]. The category of motifs used here is constructed from the integral Chow theory as follows. Let $V(F)$ be the category of smooth projective varieties over a field F. Let $CV(F)$ denote the category of correspondences between varieties over F. That is, the objects of $CV(F)$ are the same as those of $V(F)$. If $\hat{X}$, $\hat{Y}$ are objects of $CV(F)$ which represent irreducible varieties X, Y, then

$$\mathrm{Hom}_{CV(F)}(\hat{X}, \hat{Y}) = CH^{\dim X}(X \times Y) .$$

A pair of morphisms $f \in CH^{\dim X}(X \times Y)$, $g \in CH^{\dim Y}(Y \times Z)$ is composed by the rule

$$g \circ f = (\mathrm{pr}_{XZ})_*(\mathrm{pr}_{XY}^*(f) \cap \mathrm{pr}_{YZ}^*(g)) ,$$

where the cycle theoretic intersection is computed on $X \times Y \times Z$. These definitions are extended to reducible varieties componentwise.

Define the category $M^+(F)$ of effective motifs over F to be the pseudo-abelian completion of $CV(F)$. Thus, the objects of $M^+(F)$ are pairs (X, c) consisting of a variety X and a correspondence $c \in CH^{\dim X}(X \times X)$ which is a projector; i.e., $c \circ c = c$. The morphisms $(X, c) \to (Y, d)$ are defined as

$$\{f \in CH^{\dim X}(X \times Y) : f \circ c = d \circ f\} / \{f : f \circ c = d \circ f = 0\} .$$

Composition of morphisms is induced from $CV(F)$.

There are natural functors $V(F)^{op} \to CV(F)$, which assigns X to $\hat{X}$ and $f: Y \to X$ to the class of its graph as a cycle on $X \times Y$, and $CV(F) \to M^+(F)$ which embeds $\hat{X}$ as $\tilde{X} = (X, 1_X)$. The category of effective motifs has a number of advantages. It is additive, and every projector has a kernel and an image. It has a direct sum induced from $\tilde{X} \oplus \tilde{Y} = (X \coprod Y)^\sim$ and a tensor product from $\tilde{X} \otimes \tilde{Y} = (X \times Y)^\sim$. Furthermore, let $H: V(F)^{op} \to \Lambda$-algebras be a graded cohomology theory which admits a multiplicative structure, a Λ-linear covariant structure subject to the usual projection formula, and a compatible cycle class map. Then H extends to an additive functor $M^+(F) \to \Lambda$-algebras. This result holds for the Chow theory, for étale cohomology, and for the bigraded K-cohomology.

Let $U=(\operatorname{Spec} F, 1)$ be the motif of a point. Then U is a unit for the tensor product operation. Let $e \in \mathbb{P}^1$ be an F-point. Then $U \approx (\mathbb{P}^1, e \times \mathbb{P}^1)$ and, if $L=(\mathbb{P}^1, \mathbb{P}^1 \times e)$, there is a canonical decomposition

$$\tilde{\mathbb{P}}^1 = U \oplus L .$$

The motif L plays a particularly important role. For instance,

$$\tilde{\mathbb{P}}^n \approx U \oplus L \oplus \ldots \oplus L^n .$$

If $\mathscr{E}$ is a vector bundle on X of rank $n+1$ with associated projective bundle $Z=\mathbb{P}(\mathscr{E})$, then

$$\tilde{Z} \approx \tilde{X} \oplus (\tilde{X} \otimes L) \oplus \ldots \oplus (\tilde{X} \otimes L^n) .$$

If B is the blowup of X along a smooth center Y of codimension $r+1$, then

$$\tilde{B} \approx \tilde{X} \oplus (\tilde{Y} \otimes L) \oplus \ldots \oplus (\tilde{Y} \otimes L^r) .$$

Twisting a motif M by L effects cohomology by

$$H^j(M \otimes L, \mathscr{K}_i) \approx H^{j-1}(M, \mathscr{K}_{i-1})$$
$$H^j(M \otimes L, \mathbb{Z}_l) \approx H^{j-2}(M, \mathbb{Z}_l(-1)) .$$

Finally, the endofunctor $M \mapsto M \otimes L^i$ is fully faithful. One completes the construction of the category $M(F)$ of all motifs by formally adjoining an inverse L^{-1} of L for tensor product.

Now suppose that X is a rational surface over F. A Galois extension E/F is called a splitting field for X provided that the surface X_E obtained by base change is birational to $\mathbb{P}^2_E$ by a sequence of monoidal transformations centered at E-points. Over E, the motif of X becomes

$$\tilde{X}_E \approx U \oplus mL \oplus L^2 .$$

This decomposition follows immediately from the motivic behavior of blowups. The K-theory and the étale cohomology of X_E are completely understood. One would like to proceed by descent theory. A key step is provided by the following consequence of [4].

(1.1) **Theorem.** *Let X, Y be smooth, projective, geometrically connected varieties over a finite field F of characteristic p. Let $G=\mathrm{Gal}(\bar{F}/F)$. Let l be a prime different from p. Assume*

(i) *$\bar{X}$ and $\bar{Y}$ are simply connected;*

(ii) *$H^3(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l(2)) = 0 = H^3(\bar{Y}, \mathbb{Q}_l/\mathbb{Z}_l(2))$;*

(iii) *$H^3(\bar{X} \times \bar{Y}, \mathbb{Z}_l(2))$ is torsion free.*

Then $CH^2(X \times Y)$ has no l-primary torsion.

Proof. It is shown in [4] that the l-primary torsion of $CH^2(X \times Y)$ is a subquotient of $H^3(X \times Y, \mathbb{Q}_l/\mathbb{Z}_l(2))$. It suffices, therefore, to show that the latter group vanishes. Since $\bar{X}$ and $\bar{Y}$ are simply connected with vanishing H^3, the Kunneth formula shows that $H^3(\bar{X} \times \bar{Y}, \mathbb{Q}_l/\mathbb{Z}_l(2)) = 0$. The Serre spectral sequence thus yields an isomorphism

$$\begin{aligned} H^3(X \times Y, \mathbb{Q}_l/\mathbb{Z}_l(2)) &\approx H^1(G, H^2(\bar{X} \times \bar{Y}, \mathbb{Q}_l/\mathbb{Z}_l(2))) \\ &\approx H^2(\bar{X} \times \bar{Y}, \mathbb{Q}_l/\mathbb{Z}_l(2))_G . \end{aligned}$$

By the Weil conjectures [6],

$$H^2(\bar{X} \times \bar{Y}, \mathbb{Q}_l(2))^G = 0 = H^2(\bar{X} \times \bar{Y}, \mathbb{Q}_l(2))_G.$$

Since $H^3(\bar{X} \times Y, \mathbb{Z}_l(2))$ has been assumed torsion free, the long exact cohomology sequence shows that $H^2(\bar{X} \times \bar{Y}, \mathbb{Q}_l/\mathbb{Z}_l(2)) = 0$, as desired.

(1.2) **Corollary.** *Let X and Y be rational surfaces over the finite field F. Then $CH^2(X \times Y)$ has no prime-to-p torsion.*

Proof. Since rational surfaces have no odd-dimensional cohomology, hypotheses (i) and (ii) of the theorem hold. Since $\bar{X} \times \bar{Y}$ can be obtained from $\mathbb{P}^2 \times \mathbb{P}^2$ by a sequence of blowups centered at "point $\times$ surface" or "surface $\times$ point," it also has no odd-dimensional cohomology, and hypothesis (iii) also holds for all primes $l \neq p$.

(1.3) **Theorem.** *Let X be a rational surface over a finite field F. Then the motif of X, and hence its cohomology, is determined (up to p-torsion) by the action of Frobenius on* $\mathrm{Pic}(\bar{X})$.

Proof. Let E/F be a splitting field. Write

$$\tilde{X}_E \approx U \oplus mL \oplus L^2.$$

Let Y be a rational surface over F obtained by blowing up $m-1$ F-points on $\mathbb{P}^2$. Then

$$\tilde{Y} \approx U \oplus mL \oplus L^2$$

and $\tilde{Y}_E$ is E-isomorphic to $\tilde{X}_E$. This isomorphism determines descent data or, equivalently, a cohomology class in

$$H^1(\mathrm{Gal}(E/F), \mathrm{Aut}_{M(E)}(\tilde{Y}_E)).$$

These descent data will suffice to determine the motif of X provided the base change functor $-\otimes_F E$ is faithful on the full subcategory of motifs of rational surfaces. For such surfaces X, Y, the usual norm argument shows that the base change homomorphism

$$\beta: CH^2(X \times Y) \to CH^2(X_E \times Y_E)$$

has torsion kernel. By Theorem 1.1, β is injective up to p-torsion. Hence the base change functor is faithful provided one works in the motivic category localized away from p. So, the descent data determine the motif of X up to p-torsion.

Since X is defined over a finite field, it contains an F-rational zero cycle e of degree 1 [15, Lemma 1, 1.5.3]. The projectors $e \times X$ and $X \times e$ determine a decomposition

$$\tilde{X} \approx U \oplus N(X) \oplus L^2$$

for some motif $N(X)$. Arguing as in the proof of Theorem 1.1, one sees that $CH^2(X) = \mathbb{Z}$. So, this decomposition is canonical. Upon base change to E, $N(X) \otimes L^{-1}$ extends to the motif mU. But

$$\mathrm{End}_{M(E)}(mU) = CH^0(m^2 \text{ points}) = M_m(\mathbb{Z})$$

is the ring of $m \times m$ integral matrices. Hence

$$\mathrm{Aut}_{M(E)}(mU) = \mathrm{GL}(m, \mathbb{Z}).$$

To give descent data which determine $\tilde{X}$ it suffices to give descent data on mU which determine $N(X) \otimes L^{-1}$; that is, a class in

$$H^1(\mathrm{Gal}(E/F), \mathrm{GL}(m, \mathbb{Z})) = \mathrm{Hom}(\mathrm{Gal}(E/F), \mathrm{GL}(m, \mathbb{Z})).$$

Since F is finite, each homomorphism is determined by the image of Frobenius. Finally, the action of Frobenius on $N(X) \otimes L^{-1}$ is faithfully reflected on its étale cohomology

$$H^0(N(\bar{X}) \otimes L^{-1}, \mathbb{Z}_l) = H^2(N(\bar{X}), \mathbb{Z}_l(1)) = \mathrm{Pic}(\bar{X}) \otimes \mathbb{Z}_l.$$

(1.4) *Example.* Let X be the surface in $\mathbb{P}^4$ defined by

$$\begin{cases} a(dR^2 - S^2) = b(T+U)(T+V) \\ c(dR^2 - T^2) = b(S+U)(S-V) \end{cases}$$

for some $a, b, c, d \in F$. If $d \notin F^2$, then X is not birational to $\mathbb{P}^2$ over F, but is so over the splitting field $F(\sqrt{d})$. Such surfaces are degree 4 del Pezzo surfaces of Manin's type IV [10]. If $\{l, e_1, \ldots, e_5\}$ is the standard basis of $\mathrm{Pic}(\bar{X})$, then Frobenius acts through

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 1 & 1 \\ -2 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

A more convenient basis for understanding the representation is $\{l - e_1 - e_5, e_5, e_1 - e_2 - e_3, e_1 - e_4 - e_5, e_4 - e_5, e_2 - e_5\}$ which yields the similar matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

On the one hand, these surfaces give nontrivial examples to which the theorems can be applied.

On the other hand, suppose F is a local or global field. Colliot-Thélène [17] showed that $CH^2(X)_{\mathrm{tor}}$ is finite. Coombes and Muder [5] showed that there exists nontrivial torsion varying with the parameters. Theorem 1.1 fails over such fields, and Theorem 1.3 fails in the sense that the Galois representation on $\mathrm{Pic}(\bar{X})$ does not determine the motif of X.

2. The *K*-Theory of Representations

To each rational surface is associated a continuous, finite dimensional, integral Galois representation. Other motifs also give rise to such representations. For instance, let E/F be a finite Galois extension with group G. Write $m=[E:F]$. Consider the F-motif $P(E)=(\operatorname{Spec} E, 1)$. After base extension, $P(E)$ becomes isomorphic to the motif of m points. As before, descent theory assigns to $P(E)$ a representation of G in $\mathrm{GL}(m,\mathbb{Z})$; this is just the regular representation.

It is interesting to note that not every representation can be realized as a motif of the form (X,c) where X is a union of points. For example, let ϱ be the integral representation with Frobenius acting on $\mathbb{Z}$ by multiplication by -1. The only zero-dimensional candidate to realize ϱ over $F=\mathbb{F}_q$ is as a summand of $\mathbb{P}(\mathbb{F}_{q^2})$. But the representation $P(\mathbb{F}_{q^2})$ is indecomposable; if a is the nontrivial automorphism, then the projector $(1+a)/2$ is only defined rationally, not integrally. Even more interestingly, ϱ is a direct summand of the representation associated to a degree 4 del Pezzo surface of Manin's Type IV, so ϱ is actually an integral motif. In order to pick out the representations within the category of motifs, one has the

Definition. Let M be an F-motif and let ϕ_M be the Frobenius endomorphism of $\bar{M}$. Let l be a prime different from p. Say that M is l-pure of weight w if

(a) $H^i(\bar{M},\mathbb{Q}_l)=0$ for $i\neq w$;

(b) the natural map $\mathbb{Z}[\phi_M]\subset \operatorname{End}(\bar{M})\to \operatorname{End}(H^w(\bar{M},\mathbb{Q}_l))$ is injective.

Every l-pure motif of weight w thus gives an l-adic representation of the Galois group. If this representation is actually integral independently of l, then M is said to be pure of weight w. A motif will be called continuous if ϕ_M has finite order.

The continuous pure motifs of weight 0 form an additive subcategory $M^0(F)$ of $M(F)$, closed under direct sum and tensor product. The functor of étale cohomology faithfully describes $M^0(F)$ as a subcategory of the category $R(F)$ of continuous, finite dimensional, integral Galois representations. Objects of $M^0(F)$ have a K-theory by virtue of being motifs [15]. There is also a natural candidate for the K-theory of objects in $R(F)$.

Let $\varrho\in R(F)$. Let E/F be a Galois extension field over which the restricted representation $\varrho_E\in R(E)$ becomes trivial. Let $m=\operatorname{rank}(\varrho)$. Clearly, $K_i(\varrho_E)\approx\mathbb{Z}^M\otimes K_i(E)$. View this as a tensor product of Galois modules for $\mathrm{Gal}(E/F)$, acting in the usual way on $K_i(E)$ and through the representation ϱ on $\mathbb{Z}^m$. Define

$$K_i(\varrho)=(\mathbb{Z}^m\otimes K_i(E))^{\mathrm{Gal}(E/F)}.$$

This definition has the following desirable properties.

(2.1) *Independence of Splitting Field.* Suppose E_1 and E_2 are two different fields over which ϱ becomes trivial. Let $G_i=\mathrm{Gal}(E_i/F)$. Then

$$(\mathbb{Z}^m\otimes K_iE_1)^{G_1}\approx(\mathbb{Z}^m\otimes K_iE_2)^{G_2}.$$

To prove this, it is enough to assume $E_1\subset E_2$. Then $G_2\twoheadrightarrow G_1$ and the kernel H acts trivially on $\mathbb{Z}^m$. So, it suffices to check that

$$K_iE_1\approx(K_iE_2)^H.$$

The latter follows immediately from Quillen's computation of the K-theory of finite fields [12].

(2.2) *Normalization.* Let λ be the regular representation of $\mathrm{Gal}(E/F)$. Then $K_i(\lambda)=K_i(E)$.

(2.3) *Direct Sums.* If ϱ, $\sigma\in R(F)$, then

$$K_i(\varrho\oplus\sigma)=K_i(\varrho)\oplus K_i(\sigma).$$

(2.4) *Localization.* Let $\mathbb{Z}\subset R\subset\mathbb{Q}$ and let σ be a continuous Galois representation into $\mathrm{GL}(m,R)$. Let

$$K_i(\sigma)=(R^m\otimes K_i(E))^G.$$

Then for every $\varrho\in R(F)$,

$$K_i(\varrho\otimes R)=K_i(\varrho)\otimes R.$$

(2.5) *Functoriality.* The K-groups are covariantly functorial on $R(F)$.

(2.6) *Projection Formula.* Let E/F be any field extension. Let ϱ_E be the representation obtained by base change from $\varrho\in R(F)$. There are natural maps

$$\alpha^*: K_i(\varrho)\to K_i(\varrho_E).$$

If E/F is finite, then there are also transfer maps

$$\alpha_*: K_i(\varrho_E)\to K_i(\varrho).$$

These maps are related by the usual projection formula.

(2.7) **Theorem.** *Let $M\in M^0(F)$ be a continuous pure motif of weight 0 with motivic K-groups $K_i(M)$. Let $\varrho:\mathrm{Gal}(E/F)\to\mathrm{GL}(m,\mathbb{Z})$ be the associated representation, with K-groups $K_i(\varrho)$ defined as above. Then, up to p-torsion,*

$$K_i(M)\approx K_i(\varrho).$$

Proof. The proof will proceed through several lemmas. If X is any variety, let KX denote the K-theory spectrum of X with homotopy groups $\pi_i(KX)=K_i(X)$. If $M=(X,c)$ is any motif, let KM denote the homotopy fibre of $1-c$: $KX\to KX$. The appropriateness of the name arises from

(2.8) **Lemma.** $\pi_i(KM)\approx K_i(M)$.

Proof. There is a homotopy commutative square

$$\begin{array}{ccc} KX & \xrightarrow{c} & KX \\ \downarrow & & \downarrow{\scriptstyle 1-c} \\ * & \longrightarrow & KX. \end{array}$$

The induced map on the homotopy fibres of the vertical arrows is a function $g: KX\to KM$. Since c is a projector, the natural composite

$$KM\longrightarrow KX\xrightarrow{g} KM$$

is homotopic to the identity. So, $\pi_i(KM)$ is a direct summand of $K_i(X)$. Since c on spectra induces the obvious map on K-theory, this summand is precisely

$$\pi_i(KM)=c(K_i(X))=K_i(M).$$

Now let ϕ be the Frobenius endomorphism of the extended motif M_E. Also write ϕ for the induced endomorphism of the spectrum KM_E. Define Φ to be the homotopy fibre of $1-\phi: KM_E \to KM_E$.

(2.9) **Lemma.** *$KM \otimes \mathbb{Z}[1/p]$ is weakly homotopically equivalent to $\Phi \otimes \mathbb{Z}[1/p]$.*

Proof. Here $-\otimes\mathbb{Z}[1/p]$ means to smash with a Moore spectrum which localizes homotopy and cohomology away from p. In the motivic category, M is the kernel of $1-\phi$: $M_E \to M_E$. So, there is a natural map of spectra $f: KM \to \Phi$. By (2.4) and (2.6), f is a weak equivalence after tensoring with $\mathbb{Z}[1/d]$ where $d=[E:F]$. In particular, f is a rational isomorphism. But a straightforward étale cohomology computation reveals that f is an isomorphism on cohomology with finite prime-to-p coefficients. So, f is a $\mathbb{Z}[1/p]$-cohomology isomorphism. Since f is a map of spectra, it is therefore a weak equivalence away from p.

Let BU be the classifying space for complex topological vector bundles. Let ψ^i denote the ith Adams operations. As in [12], KE is the homotopy fibre of

$$1-\psi^{q^d}: BU \to BU,$$

where $q=\#F$ and $d=[E:F]$. Since M_E is a trivial E-motif, it has spectrum

$$KM_E \approx (KE)^m.$$

Let ϱ denote the action of Frobenius on U^m by the given integral representation associated to M. Define $F\varrho$ to be the homotopy fibre of

$$1-\varrho\psi^q: BU \to BU.$$

(2.10) **Lemma.** *Φ is homotopy equivalent to $F\varrho$.*

Proof. Consider the homotopy commutative diagram

$$\begin{tikzcd}
\Phi \arrow[r] \arrow[d, "\alpha"] & (KE)^m \arrow[r, "1-\phi"] \arrow[d] & (KE)^m \arrow[d] \\
F\varrho \arrow[r] \arrow[d, "\beta"] & BU^m \arrow[r, "1-\varrho\psi^q"] \arrow[d, "1-\psi^{q^d}"] & BU^m \arrow[d, "1-\psi^{q^d}"] \\
F\varrho \arrow[r] & BU^m \arrow[r, "1-\varrho\psi^q"] & BU^m.
\end{tikzcd}$$

The final two columns are homotopy fibrations. The first column is obtained as the homotopy fibres of the horizontal maps. In order to show α is a homotopy equivalence, it suffices to show that β is homotopically trivial. But $\varrho^d=1$. So, formally,

$$1-t^d=1-\varrho^d t^d=(1-\varrho t)(1+\varrho t+\ldots+\varrho^{d-1}t^{d-1}).$$

Setting $t=\psi^q$, one sees that $1-\psi^{q^d}$ factors through $1-\varrho\psi^q$. Hence $\beta\simeq 0$ and the lemma follows.

(2.11) **Lemma.** *$\pi_i(F\varrho)=K_i(\varrho)$.*

Proof. First, the vanishing of the even K-groups of E implies $K_{2n}(\varrho)=0$. Next, Deligne's proof of the Weil conjectures [6] implies that $1-\varrho q^n$ is an injective endomorphism of $\mathbb{Z}^m$ for $n\geqq 1$. Since ψ^q acts on $\pi_{2n}(BU)=\mathbb{Z}$ by multiplication by q^n, the long exact homotopy sequence associated to the fibration defining $F\varrho$ shows that $\pi_{2n}(F\varrho)=0$. Finally, take the homotopy groups of the diagram of the previous lemma to obtain an exact commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \mathbb{Z}^m \arrow[r, "1-\varrho q^n"] \arrow[d, "1-q^{dn}"] & \mathbb{Z}^m \arrow[r] \arrow[d, "1-q^{dn}"] & \pi_{2n-1}F\varrho \arrow[r] \arrow[d, "\beta"] & 0 \\
 & 0 \arrow[r] & \mathbb{Z}^m \arrow[r, "1-\varrho q^n"] \arrow[d] & \mathbb{Z}^m \arrow[r] \arrow[d] & \pi_{2n-1}F\varrho \arrow[r] & 0 \\
0 \arrow[r] & K_{2n-1}(\varrho) \arrow[r] & \mathbb{Z}^m\otimes K_{2n-1}E \arrow[r, "1-\phi"] \arrow[d] & \mathbb{Z}^m\otimes K_{2n-1}E \arrow[d] & & \\
 & & 0 & 0. & &
\end{tikzcd}$$

Since $1-\varrho q^n$ divides $1-q^{dn}$, one has $\beta=0$. By the Snake Lemma, $K_{2n-1}(\varrho)\approx\pi_{2n-1}(F\varrho)$ and the lemma follows.

The proof of Theorem 2.7 is now complete since, up to p-torsion

$$K_i(\varrho)\approx\pi_i(F\varrho)\approx\pi_i(\Phi)\approx\pi_i(KM)\approx K_i(M).$$

(2.12) *Remark.* Suppose E/F is a splitting field for X of degree d relatively prime to p. Then only d-torsion is relevant. So, the motif of X is completely determined by the representation ϱ on $\operatorname{Pic}(\bar{X})$ and $K_i(\varrho)\approx K_i(N(X)\otimes L^{-1})$. The restriction to prime-to-p torsion when $p|d$ arises because of the mediation of étale cohomology. This restriction can presumably be removed in general by using the techniques of [4] to handle the p-torsion.

Theorems 1.3 and 2.7 combine to give a complete computation of the K-cohomology of rational surfaces over finite fields. One should note that Colliot-Thélène [17] has computed the K_2-cohomology over any field.

(2.13) *Example.* Let X be a degree 4 del Pezzo surface of Manin's type IV. Then

$$H^0(X,\mathscr{K}_j)=H^2(X,\mathscr{K}_{j+2})=K_j(F)=\begin{cases}\mathbb{Z} & j=0\\ 0 & j=2n\\ \mathbb{Z}/q^n-1 & j=2n-1.\end{cases}$$

$$H^1(X,\mathscr{K}_{j+1})=\begin{cases}\operatorname{Pic}(X)=\mathbb{Z}^2 & j=0\\ 0 & j=2n\\ (\mathbb{Z}/q^{2n}-1)^2\times(\mathbb{Z}/q^n+1)^2 & j=2n-1.\end{cases}$$

3. Connections with L-Functions

Let M be a motif over a finite field F with q elements. Let $\bar{M}$ be the induced motif over the algebraic closure. Let ϕ_j denote the endomorphism induced by Frobenius on $H^j(\bar{M}, \mathbb{Q}_l)$ for some prime l different from the characteristic. Define

$$P_j(M,t) = \det(1-\phi_j t) \qquad L_j(s) = P_j(q^{-s}).$$

By Deligne's proof of the Weil conjectures [6], P_j is a polynomial with integral coefficients independent of l.

(3.1) **Theorem.** *Let X be a rational surface over a finite field. Then for all $n > 2$,*

$$L_{2i}(X,n) \sim \# H^i(X, \mathscr{K}_{2n-i-1}) \qquad 0 \leqq i \leqq 2$$

$$L_{2i+1}(X,n) \sim \# H^i(X, \mathscr{K}_{2n-i-2}) \qquad 0 \leqq i \leqq 1.$$

Proof. For any motifs M, N, one has

$$L_j(M \oplus N, s) = L_j(M,s) L_j(N,s).$$

Since the K-cohomology of motifs is additive, it suffices to prove the result for each of the motifs in the canonical decomposition

$$\tilde{X} \approx U \oplus N(X) \oplus L^2.$$

The effect of twisting by the Tate motif L is

$$L_j(M \otimes L, s) = L_{j-2}(M, s-1)$$

$$H^j(M \otimes L, \mathscr{K}_i) = H^{j-1}(M, \mathscr{K}_{i-1}).$$

The conclusion of the theorem is therefore compatible with shifting dimensions by twisting. So, it is enough to verify the theorem for motifs $M \in M^0(F)$. Let ϱ be the integral representation associated to M. Then, up to p-torsion,

$$H^i(M, \mathscr{K}_j) = \begin{cases} K_j(\varrho) & i=0 \\ 0 & i>0 \end{cases}$$

$$H^i(\bar{M}, \mathbb{Q}_l) = \begin{cases} \varrho \otimes \mathbb{Q}_l & i=0 \\ 0 & i>0. \end{cases}$$

Moreover, ϕ_0 is the action of Frobenius prescribed by ϱ. Thus,

$$|L_0(M,n)| = |\det(1-\varrho q^{-n})| \sim |\det(1-\varrho q^n)|$$
$$= \#(\mathbb{Z}^m/(1-\varrho q^n)) = \# K_{2n-1}(\varrho).$$

(3.2) *Remark.* The part of the theorem concerning the odd L-functions is rather specious, since $L_{2i+1}(X,s) = 1$ and $H^i(X, \mathscr{K}_{2n-i-2}) = 0$ (up to p-torsion). There are several reasons for its inclusion with this precise choice of indices. First is its compatibility with twisting. Second, the K-cohomology associated to the even L-functions for a given n all lie on a common increasing diagonal of the E_2-terms of the Gersten-Quillen spectral sequence [11]. The odd terms are chosen to lie on the parallel diagonal one step higher. Further, let X be a smooth projective F-variety. Consider the property $P(X)$: For all $n > \dim(X)$,

$$L_{2i}(X,n) \sim \# H^i(X, \mathscr{K}_{2n-i-1}) \quad 0 \leqq i \leqq \dim X$$
$$L_{2i+1}(X,n) \sim \# H^i(X, \mathscr{K}_{2n-i-2}) \quad 0 \leqq i \leqq \dim X - 1.$$

There are partial results [7, 16] suggesting that curves have the property $P(X)$.

(3.3) **Theorem.** *Let X be a smooth projective curve over F. Let $n>1$. Then the Quillen conjecture for affine curves and $P(X)$ predict the same prime-to-p value for $\zeta(X,n)$.*

Proof. The Quillen conjecture [13] for an affine curve A is

$$K_{2i-2}(A) \otimes \mathbb{Z}_l \approx H^2(A, \mathbb{Z}_l(i))$$
$$K_{2i-1}(A) \otimes \mathbb{Z}_l \approx H^1(A, \mathbb{Z}_l(i)).$$

The Bayer-Neukirch-Schneider [1, 14] cohomological formula for the value of the zeta function is

$$|\zeta(X,n)|_l^{-1} = \frac{\# H^0(X, \mathbb{Z}_l(n)) \# H^2(X, \mathbb{Z}_l(n))}{\# H^1(X, \mathbb{Z}_l(n)) \# H^3(X, \mathbb{Z}_l(n))}.$$

Without loss of generality, one may assume $A \subset X$ is the open affine complement of an F-point. Compare the Gysin exact sequence in étale cohomology with the K-theoretic localization sequence

$$\begin{array}{l} 0 \to H^1(X, \mathbb{Z}_l(n)) \to H^1(A, \mathbb{Z}_l(n)) \to H^0(F, \mathbb{Z}_l(n-1)) = 0 \\ \ldots \to \quad K_{2n-1}(X) \quad \to \quad K_{2n-1}(A) \quad \to \quad K_{2n-2}(F) \quad = 0 \\ \to H^2(X, \mathbb{Z}_l(n)) \to H^2(A, \mathbb{Z}_l(n)) \to H^1(F, \mathbb{Z}_l(n-1)) \to H^3(X, \mathbb{Z}_l(n)) \to 0 \\ \to \quad K_{2n-2}(X) \quad \to \quad K_{2n-2}(A) \quad \to \quad K_{2n-3}(F) \quad \to \quad K_{2n-3}(X) \quad \to \ldots . \end{array}$$

For curves, the Gersten-Quillen spectral sequence collapses to the short exact sequences

$$0 \to H^1(X, \mathscr{K}_{j+1}) \to K_j(X) \to H^0(X, \mathscr{K}_j) \to 0.$$

Soulé has shown [15] for curves over finite fields that $H^1(X, \mathscr{K}_{j+1}) = K_j(F)$.

Now by Quillen's computation of the K-theory of finite fields [12] and by assuming the Quillen conjecture, the corresponding terms in the long exact sequences involving F and A have the same l-primary orders. Therefore,

$$\begin{aligned} \zeta(X,n) &\sim \frac{1 \cdot \# K_{2n-2}(X)}{\# H^0(X, \mathscr{K}_{2n-1}) \# H^1(X, \mathscr{K}_{2n-2})} \\ &= \frac{\# H^0(X, \mathscr{K}_{2n-2}) \# H^1(X, \mathscr{K}_{2n-1})}{\# H^0(X, \mathscr{K}_{2n-1}) \# H^1(X, \mathscr{K}_{2n-2})} \\ &= \frac{\# H^0(X, \mathscr{K}_{2n-2})}{\# H^0(X, \mathscr{K}_{2n-1}) \# H^1(X, \mathscr{K}_{2n-2})}. \end{aligned}$$

Since $\zeta(X, n) = L_1(n)/L_0(n)L_2(n)$, the latter is precisely the value predicted by $P(X)$.

(3.4) **Proposition.** (a) *$P(\mathbb{P}^m)$ is true.*

(b) *If $P(X)$ is true and if $Z = \mathbb{P}(\mathscr{E})$ is a projective bundle over X, then $P(Z)$ is true.*

(c) *If $P(X)$, $P(Y)$ are true and if B is the blowup of X along Y, then $P(B)$ is true.*

Proof. All three parts follow immediately from the motivic decompositions and the compatibility with twisting by L.

(3.5) *Remark.* Let X be a surface. Then the motif of X decomposes as

$$\tilde{X} \approx U \oplus J \oplus M \oplus (A \otimes L) \oplus L^2,$$

where J is the 1-dimensional motif associated to the Picard variety of X and A is the 1-dimensional motif associated to its Albanese. The truth of $P(J)$ and $P(A)$ should follow from work on curves. Whenever $M \otimes L^{-1}$ can be reduced to a representation, the techniques used here should confirm $P(M)$ and hence $P(X)$. Up to small torsion, this should be the case for Enriques surfaces and hyperelliptic surfaces, since the second étale cohomology group is generated by algebraic cycles. It is considerably less clear if one should expect $P(X)$ to hold for abelian surfaces or $K3$ surfaces.

It seems unlikely that $P(X)$ holds in full generality. Several authors [2, 3, 8] have advanced more elaborate suggestions relating the values of L-functions to objects constructed from algebraic K-theory. The structure of rational surfaces simplifies the cohomology and the Gersten-Quillen spectral sequence sufficiently to replace those elaborate constructions with a direct formula.

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