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Classification of recurrent domains for some holomorphic maps

John Erik Fornæss, Nessim Sibony
[1] Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1003, USA
[2] Mathématiques, Université de Paris-Sud, Bâtiment 425, F-91405 Orsay Cedex, France

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0 Introduction

Let $f: P^2 \to P^2$ be a holomorphic map of degree $d \geqq 2$ on the two dimensional projective space. Then f is given in homogeneous coordinates by $[f_0 : f_1 : f_2]$ where each f_j is a homogeneous polynomial of degree d and the f_j have no common zero except the origin. Observe that f is a d^2 to one map. We denote by H_d the family of such self maps. In analogy with the one variable theory the Fatou set of f is the maximal open set where the family (f^n) is equicontinuous. A Fatou component is a connected component of the Fatou set. The higher dimensional analog of the Fatou Julia theory has been studied in [FS1] [FS2]. We will always assume that $d \geqq 2$.

Our purpose in this paper is to study periodic Fatou components Ω for f, so $f^k(\Omega) = \Omega$ for some $k \geqq 1$. Without lack of generality we can assume that $f(\Omega) = \Omega$ replacing f by an iterate if necessary.

Definition 0.1. *A Fatou component Ω is recurrent if for some $p_0 \in \Omega$ the ω-limit set of p_0 intersects Ω. More precisely there exists $p_0 \in \Omega$ such that $f^{n_i}(p_0)$ is relatively compact in Ω for some subsequence n_i.*

In P^1, the recurrent Fatou components are attractive basins, Siegel discs and Herman rings. For the one variable theory we refer to [CG] or [Be].

We describe in Theorem 1.2 the analogous components for holomorphic maps in $\mathbb{P}^2$.

In a second paragraph we study the recurrent Fatou components for Hénon mappings in $\mathbb{P}^2$. Compare with [BS].

Hénon mappings are holomorphic polynomial automorphisms of $\mathbb{C}^2$, they are the dynamically interesting ones, see [FM], and they do not extend as holomorphic maps from $\mathbb{P}^2$ to $\mathbb{P}^2$.

1 Holomorphic maps in $\mathbb{P}^2$

Let $f: \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic map, $f \in H_d$. Assume $\Omega = f(\Omega)$ is a recurrent Fatou component. It is a priori possible that there exists $p_0 \in \Omega$ such that the orbit $f^n(p_0)$ clusters both at some interior point and at some boundary point of Ω. In $\mathbb{P}^1$ this cannot happen, we will show that this does not happen in $\mathbb{P}^2$ either. This is a trivial consequence of the classification of recurrent domains in Theorem 1.2.

If f is a rational map in $\mathbb{P}^1$ there are only finitely many recurrent domains. A recent theorem by E. Gavosto [G] shows that holomorphic maps on $\mathbb{P}^2$ can have infinitely many recurrent domains.

It is shown in [FS1] that any Fatou component is a domain of holomorphy. A recent result of Ueda [U] shows that every Fatou component is Kobayashi hyperbolic.

Definition 1.1. *A Fatou component Ω is a Siegel domain if there exists a subsequence (f^{n_i}) converging uniformly on compact sets of Ω to identity.*

We have the following result.

Theorem 1.2. *Let $f \in H_d$, $d \geqq 2$. Let Ω be a recurrent Fatou component such that $f(\Omega) = \Omega$. Then one of the following happens:*

(i) *There is a fixed attractive point $p \in \Omega$, the eigenvalues λ_1, λ_2 of f' at p satisfy $|\lambda_1| < 1$, $|\lambda_2| < 1$.*

(ii) *There exists a Riemann surface $\tilde{\Sigma}$ which is a closed complex submanifold of Ω and $f|\tilde{\Sigma} \to \tilde{\Sigma}$ is an automorphism, moreover $d(f^n(K), \tilde{\Sigma}) \to 0$ for any compact set K in Ω. The Riemann surface $\tilde{\Sigma}$ is biholomorphic to a disc, a punctured disc or an annulus and $f|\tilde{\Sigma}$ is conjugate to a rotation. The limit h of any convergent subsequence, f^{n_i}, has the same image. Any two limits h_1, h_2 differ only by a rotation in $\tilde{\Sigma}$.*

(iii) *The domain Ω is a Siegel domain. Any limit of a convergent subsequence of (f^n) is an automorphism of Ω.*

Proof. Since Ω is recurrent, there exists $p_0 \in \Omega$ and $n_i \to \infty$ such that $f^{n_i}(p_0)$ is relatively compact in Ω. In this case we prove that we are in one of the situations described in i), ii), iii).

Assume $f^{n_i}(p_0) \to p$, $n_{i+1} - n_i \to \infty$. Taking a subsequence $\{i = i(j)\}$ and recalling that we are in the Fatou set, we can suppose that the sequence $\{f^{n_{i+1}-n_i}\}_i$ converges uniformly on compact sets in Ω to a holomorphic map $h: \Omega \to \bar{\Omega}$. Let $p_i = f^{n_i}(p_0)$. Then $f^{n_{i+1}-n_i}(p_i) = f^{n_{i+1}}(p_0) = p_{i+1}$. Hence $f^{n_{i+1}-n_i}(p) = p_{i+1} + O(|p_i - p|)$ so converges to p. Therefore, necessarily $h(p) = p$.

Consider all maps $h: \Omega \to \bar{\Omega}$, with $h(p) = p$ for some $p \in \Omega$ and $h = \lim f^{k_j}$ for some subsequence k_j.

Let $Fix(h)$ denote the collection of fixed points of h. Since h commutes with f, it follows that f maps $Fix(h)$ to itself.

If, for some h the rank of h is 0, then $h(\Omega) = p$ and necessarily $f(p) = p$, hence p is also a fixed point for f. Also both eigenvalues of f' at p must have modulus strictly less than one since some iterates of f converge to the constant map. Hence this leads to case i).

Assume for some h the rank of h is two. Then for some sequence $f^{k_{i+1}-k_i} \to Id$ and hence Ω is a Siegel domain. The restriction of f to Ω is clearly an automorphism of Ω. We are then in case (iii). We want to show that if (f^{n_j}) converge to a map g then g is an automorphism of Ω.

Let $Aut(\Omega)$ denote the holomorphic automorphisms of Ω. We know from Ueda's, [U], result that Ω is Kobayashi hyperbolic, so $Aut(\Omega)$ has the structure of a Lie group [Ko]. Let $\tilde{G}$:= closed subgroup generated in $Aut(\Omega)$ by f. Since $\tilde{G}$ is a closed subgroup of a Lie group it is a Lie group. Let G^0 be the connected component of Id in $\tilde{G}$, it is also a Lie group. Since (f^{n_i}) converges to identity, then G^0 is not reduced to identity. But G^0 is clearly commutative, hence it is isomorphic to $\mathbb{T}^k \times \mathbb{R}^l$. Let Φ denote the isomorphism from $\mathbb{T}^k \times \mathbb{R}^l$ to G^0. For some $(a, b) \in \mathbb{T}^k \times \mathbb{R}^l$, we have $\Phi(a, b) = f$. If $b \neq 0$ we cannot have (f^{n_i}) converging to Id. So $b = 0$ and hence G^0 is isomorphic to $\mathbb{T}^k$, consequently G^0 is compact. It follows that each convergent subsequence of (f^n) tend to an element of $Aut(\Omega)$.

We now assume that for all h, the maximal rank of h is one. Fix an h and let $\Sigma := h(\Omega)$. Then $\Sigma \subset \bar{\Omega}$. For $p \in \Omega$, there is an irreducible piece of a Riemann surface with singularities $\Sigma_p \subset \bar{\Omega}$ and a neighborhood $U(p)$ so that $h(U(p)) = \Sigma_p$. We define an abstract Riemann surface R as the union $\bigcup \Sigma_{p_i}$ for a covering $U(p_i)$ of Ω, with the identifications at $q \in \Sigma_{p_i} \cap \Sigma_{p_j}$ if the two pieces agree as germs. Then R is Hausdorff by the identity theorem.

The map $h: \Omega \to \Sigma$ factors naturally as a map $h = \pi \circ \bar{h}$ where $\bar{h}: \Omega \mapsto R$ and $\pi: R \mapsto \Sigma$.

We show first that f is a surjective self map of Σ.

If $x \in \Sigma$ then $x = h(y)$ for some $y \in \Omega$ and $f(x) = f(h(y)) = h(f(y)) \in \Sigma$. So $f(\Sigma) \subset \Sigma$. We show next that the restriction of f to Σ is surjective on Σ.

Let $x = h(y), y \in \Omega$. Choose $y_- \in \Omega$ such that $f(y_-) = y$. Define $x_- = h(y_-)$. Then $x_- \in \Sigma$ and $f(x_-) = f(h(y_-)) = hf(y_-) = h(y) = x$.

Define $\Sigma^0 := \Sigma \cap \Omega$. Since $f(\Omega) \subset \Omega$, then $f(\Sigma^0) \subset \Sigma^0$. Since f is an open map ([FS1]), f maps the boundary of Ω to itself and hence $f(\Sigma^0) = \Sigma^0$.

We first show that h is not constant on the irreducible component Σ_p of $\Sigma \cap \Omega$ which contains p. Assume not. Since $h(p) = p$, then $h|\Sigma_p \equiv p \in \Sigma$. But $f^{2k_i} \to p$ so we are in case i).

Since $f^{k_{i+1}-k_i}(f^{k_i}) = f^{k_{i+1}}$ we can assume, using a diagonal process, that for a subsequence m_i, (f^{m_i}) converges to a new map h and $h = Id$ on Σ_p. Since $f^l \circ h = h \circ f^l$ it follows that $h = Id$ on each $f^l(\Sigma_p)$, $l \geqq 1$. We use this new h from now on.

We want to show that $\bigcup_{l \geqq 0} f^l(\Sigma_p)$ is closed in Ω. We know that $\bigcup_{l \geqq 0} f^l(\Sigma_p) \subset S := \{q; q \in \Omega, h(q) = q\}$. Since $Id - h'$ has at least rank one, S is a countable union of disjoint irreducible components each of which is a point or a smooth complex curve. It follows that Σ_p is a component of S and since f is a proper self map of Ω, $\bigcup_{l \geqq 0} f^l(\Sigma_p)$ is a closed countable union of irreducible curves in S.

Suppose Σ is a torus. Then $f: \Sigma \to \Sigma$ is an l to 1 map, $l \geqq 2$ by Proposition 7.5 ([FS1]). Hence repelling points for $f|\Sigma$ are dense in Σ, which contradicts normality in Ω.

We next show that if Σ is a $\mathbb{P}^1, \mathbb{C}$ or $\mathbb{C}^*$ then $f|\Sigma$ is an automorphism. Suppose not, then f is an l to 1 surjective map $f: \Sigma \to \Sigma$ with $l \geqq 2$. From the Fatou–Julia theory in one variable repelling periodic points for $f|\Sigma$ are dense in the Julia set of $f|\Sigma$. Choose q a repelling periodic point for $f|\Sigma$ say $f^s(q) = q$ and $h(z_0) = q$ with $z_0 \in \Omega$. Recall that $h = \lim f^{m_i}$. We can assume $h'(z_0) \neq 0$ in some direction.

Fix $0 < \delta \ll 1$. For each $l \in$ choose $r_l > 0$ such that $f^{ls}(B(q, r_l)) \subset B(q, \delta)$. Choose $m_{i(l)}$ such that $f^{m_{i(l)}}(z_0) \in B(q, r_l)$. Then $f^{ls+m_{i(l)}}(z_0) \in B(q, \delta)$. The sequence $(f^{ls+m_{i(l)}})_l$ is equicontinuous, hence we can even assume that in a ball B_1 containing z_0, we have $f^{ls+m_{i(l)}}(B_1) \subset B(q, \delta)$. We can always increase $m_{i(l)}$ so that $f^{ls+m_{i(l)}}$ is as close as we wish to $f^{ls} \circ h$. Then the derivative of $f^{ls+m_{i(l)}}$ at z_0 is not bounded in all directions, a contradiction. Hence we have shown that if Σ is a $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{C}^*$ then $f|\Sigma$ is an automorphism. As a consequence Σ cannot be a P^1 since by Theorem 7.5 in [FS1] $f|\Sigma$ cannot be an automorphism.

So if Σ is a $\mathbb{C}$ or $\mathbb{C}^*$, since $f^{m_i}|_{\Sigma_p} \to Id$, and $\bigcup f^l(\Sigma_p)$ is closed, then necessarily f (or f^2) is conjugate to an irrational rotation. This proves our claim.

If Σ is hyperbolic, we use the classification of holomorphic maps $g: \Sigma \to \Sigma$ in Theorem 3.3 [M]. Since $f^{n_i} \to Id$ on the open set Σ_p of Σ, we know that not all orbits converge to an attracting fixed point, nor do all orbits diverge to infinity. From Proposition 7.5 in [FS1], we know also that f is not of finite order, hence Theorem 3.3 in [M] implies that Σ is isomorphic to the unit disc D, D^* or an annulus and f is conjugate to an irrational rotation.

We want to prove next that Σ is independent of h.

Assume $f^{m_i} \to k$ uniformly on compact sets of Ω. Let $\Sigma' = k(\Omega)$. We have $f^{m_i}|\Sigma^0 \to k|\Sigma^0$ but since f is conjugate to a rotation on Σ^0, $k(\Sigma^0) \subset \Sigma^0$ and Σ' is an extension of Σ^0 and one can prove similarly that f is conjugate to rotation on Σ'. Similarly Σ^0 is an extension of $\Sigma' \cap \Omega$, so $\Sigma' \cap \Omega = \Sigma^0$. Let $\tilde{\Sigma}$ be the maximal extension of Σ in $\bar{\Omega}$ such that f is conjugate to a rotation on $\tilde{\Sigma}$. We then get that (f^n) converges u.c.c. on Ω to $\tilde{\Sigma}$, i.e. $d(f^n, \tilde{\Sigma}) \to 0$. We would like to show next that $\Sigma = \Sigma'$. Pick a point p in Σ^0. Then we can find a holomorphic coordinate system in a neighborhood of p such that in that neighborhood $\Sigma = \{w = 0;\ a < |z| < b\}$ and

$$f(z, w) = (e^{i\theta}z + wg_1(z, w), wa_1(z) + w^2k_1(z, w)) .$$

Then

$$f^n(z, w) = \left(e^{ni\theta}z + O(w), w\prod_{j=0}^{n-1} a_1(e^{ij\theta}z) + O(w^2)\right) .$$

Write $a_n(z) = \prod_{j=0}^{n-1} a_1(e^{ij\theta}z)$. Since we are in a Fatou component, the functions $a_n(z)$ are necessarily uniformly bounded on any smaller set

$a' < |z| < b'$, $a < a', b' < b$. We must even have that $a_n \to 0$ uniformly, since all limits have rank 1. It follows that a neighborhood of Σ^0 is attracted to Σ^0.

An easy estimate gives that if we start with small enough w, and consider tangent vectors $v = (1, \alpha)$ based at (z, w), $|\alpha|$ small enough, then $(f^n)'(v) = c_n(1, \alpha_n)$, $\alpha_n \to 0$, $||c_n| - 1|$ as small as we want.

For each n, let $\mathscr{F}_n$ be the "vertical" foliation consisting of leaves L with $f^n(L) \subset \{z = const\}$. We show next that $\mathscr{F}_n \to \mathscr{F}$, a foliation with leaves of the form $z = g(w)$.

The above observation shows that in order to compute the horizontal distance between leaves of $\mathscr{F}_n$ and corresponding leaves of $\mathscr{F}_{n+1}$, it is enough to compute the distance after applying f^n. If one considers the leaves of $\mathscr{F}_n$ as almost vertical discs of radius C then after applying the map f, n times, the discs have radius at most cCr^n for some $r < 1$.

Since the discs of $\mathscr{F}_1 = f^n(\mathscr{F}_{n+1})$ and $(z_1 = constant) = f^n(\mathscr{F}_n)$ start at the same point, they can be at most at cCr^n away from each other. So the horizontal distance between $\mathscr{F}_n$ and $\mathscr{F}_{n+1}$ is at most Cr^n, so $\mathscr{F}_n$ converge to a foliation $\mathscr{F}$ with leaves of the form $z = g(w)$.

Moreover f maps leaves of $\mathscr{F}$ to leaves of $\mathscr{F}$. It follows that these leaves are in level sets of h. In particular, in a neighborhood of $\Sigma \cap \Omega$, the level sets of h are independent of h.

Next, pick any two limits h_1, h_2, say $h_i(\Omega) = \Sigma_i$. By connectivity reasons one must contain the other, say $\Sigma_1 \subset \Sigma_2$. Let λ_θ denote rotation by θ in Σ_2. Then near one component of $\Sigma_1 \cap \Omega$, there must exist a θ so that $h_1 \equiv \lambda_\theta \circ h_2$. But then this must hold everywhere. So $\Sigma_1 = \Sigma_2$. In particular, the level sets of h_1 and h_2 are the same (even globally) and f maps level sets to level sets (globally). It follows that if $p_0 \in \Omega$, then either $\{f^n(p_0)\}$ converges to the boundary or is a relatively compact set in Ω.

Note that using the local coordinates above, it follows from the maximum principle that $\Sigma \cap \Omega$ cannot have more than one component. Indeed, let A be a subannulus of Σ whose boundary with respect to Σ is in Ω. Assume A intersects $\partial\Omega$. Then A has a Stein neighborhood isomorphic to $A \times disc$, and we can apply the maximum principle there. Since (f^n) converges towards Σ near the boundary of A we still have convergence in a neighborhood of A, so $A \subset \Omega$.

Next we prove that actually Σ is a closed complex manifold of Ω. Namely, let us assume not. We consider a circle in Σ so that one side is in Ω and the other side is in the boundary. We then choose a local coordinate system as above. Consider the coefficient $a_1(z)$. For each radius r let $A(r)$ denote the average of $log|a_1(z)|$ over the circle of radius r. Similarly let $A_n(r)$ denote the average of $log|a_n(z)|$. Then $A_n(r)$ and $A(r)$ have the same sign always and they are continuous and monotonic. Also note that the rotation by θ on the circle is ergodic. Hence it follows that $\frac{1}{n} log|a_n(z)| \to A(r)$ in L^2 on the circle $|z| = r$. Note that the functions $\frac{1}{n} log|a_n|$ are equicontinuous so they converge uniformly to $A(r)$ except near circles where a_1 has a zero.

In particular it follows that if $A(r) < 0$, then the circle with radius r is in the Fatou component. Since $A(|z|)$ is subharmonic, it follows that $A(r) > 0$ on the side which belongs to the boundary. But then it follows from ergodicity that for large n, $|a_n(z)| > 1$ uniformly, on circles $|z| = r$. But this implies that these points repell points from Ω. Hence there can be no points in Ω converging to them. So $\Sigma = \Sigma^0 \subset \Omega$.

From Ueda's Theorem ([U]) Ω is Kobayashi hyperbolic so Σ which is contained in Ω cannot be $\mathbb{C}$ or $\mathbb{C}^*$. This finishes the proof.

We describe more precisely the structure of the closure of iterates in the case of a Siegel domain.

Proposition 1.3. *Let $f: \mathbb{P}^2 \to \mathbb{P}^2, f \in H_d$. Let $\Omega = f(\Omega)$ be a Fatou component which is a Siegel domain. Let G denote the closure of $(f^n)_{n \geq 0}$ in the topology of uniform convergence on compact sets. Then G is a sub-Lie group of $Aut(\Omega)$ and G is isomorphic to $\mathbb{T}^k \times F$ where F is a finite group and $k = 1$ or 2.*

Proof. We have shown in the proof of Theorem 1.2 that $\tilde{G} :=$ {closed subgroup generated by f} $\cap\, Aut\,\Omega$ is a compact Lie group which is isomorphic to $\mathbb{T}^k \times F$ where F is a finite group.

Since Ω is Kobayashi hyperbolic and of complex dimension two, it follows from a theorem of Kruzhilin [Kr] that the maximal real dimension of a commutative group in $Aut(\Omega)$ is two. Indeed Kruzhilin shows that, fields that commute in the Lie algebra are real linearly independent iff they are complex linearly independent; so $0 \leqq k \leqq 2$. We cannot have $k = 0$ since an iterate of f cannot be the identity, see [FS1]. Assume $k = 2$. Then we have an effective $\mathbb{T}^2$ action on Ω. It follows from a theorem of Barrett, Bedford, Dadok [BBD] that there exists a hyperbolic Reinhardt domain U in $\mathbb{C}^2$ and a biholomorphic map $\Phi: \Omega \to U$, such that for some l, $\Phi \circ f^l = R \circ \Phi$ where $R(z, w) = (e^{i\alpha} z, e^{i\beta} z)$. So f is conjugate to $R_1(z, w) = (e^{i\alpha/l} z, e^{i\beta/l} w)$, hence $\tilde{G} = \mathbb{T}^2 \times F$ and also $G = \tilde{G}$.

When $k = 1$, then $\tilde{G} \simeq \mathbb{T} \times F$ where A is a finite group, it is also clear in that case that $G = \tilde{G}$.

Examples.
1. Let $f[z:w:t] = [\lambda zt + z^2 : \lambda^p wt + w^2 : t^2]$ where $\lambda = e^{2i\pi\theta}$, $p \in \mathbb{Z}^+$ and θ is a diophantine number. Then f has a Siegel component Ω with $[0:0:1] \in \Omega$. Since in a neighborhood of $(0, 0)$ in $\mathbb{C}^2$, $(z, w) \to (\lambda z + z^2, \lambda^p w + w^2)$ is conjugate to $(\lambda z, \lambda^p w)$, G is isomorphic to $\mathbb{T}$, which can be seen as the subgroup in $\mathbb{T}^2$ generated by (λ, λ^p).

2. Let $g[z:w:t] = [\lambda zt + z^2 : \mu wt + w^2 : t^2]$ with $\lambda = e^{2i\pi\theta}$, $\mu = e^{2i\pi\psi}$ with λ, μ satisfying the Brjuno condition, see [H], in order that f be linearizable near $[0:0:1]$ then G is isomorphic to $\mathbb{T}^2$.

2 Hénon mappings

In this paragraph we want to study recurrent periodic domains for Hénon mappings.

Recall that $f: \mathbb{C}^2 \to \mathbb{C}^2$ is a Hénon map if it is a finite composition of maps of the following type

$$f_j(z, w) = (p_j(z) - aw, z)$$

where p_j is a polynomial of degree $d_j \geqq 2$. Hénon mappings are the dynamically interesting polynomial automorphisms of $\mathbb{C}^2$, see [FM], [BS], [FS3]. The article [FS4] is a survey.

We just recall the following facts. Let $K^+ = \{(z, w), f^n(z, w), n \geqq 0$, is bounded$\}$, $K^- = \{(z, w) \setminus f^{-n}(z, w), n \geqq 0$ is bounded$\}$ and $K = K^+ \cap K^-$. Then $U^+ := \mathbb{C}^2 \setminus K^+$ is the basin of attraction of infinity, more precisely if we identify $\mathbb{C}^2$ with the open set $t \neq 0$ in $\mathbb{P}^2$, then points in $U^+ \subset \mathbb{P}^2$ converge towards $p^+ := [1:0:0]$ in $\mathbb{P}^2$. The sets K^+ and K^- are closed. The sequence (f^n) converge uniformly on compact sets of K^+ towards K, which is compact.

We describe the Fatou components in the interior of K^+ which are recurrent. The question was considered in [BS] and [FS3]. We however want to give more details for the case ii) in the following theorem.

Theorem 2.1. *Let $f \mathbb{C}^2 : \to \mathbb{C}^2$ be a Hénon mapping. Let Ω be a recurrent Fatou component in the interior of K^+. Assume $f(\Omega) = \Omega$. Then Ω is of one of the following types:*

(i) *There is a fixed attracting point $p \in \Omega$ and Ω is biholomorphic to $\mathbb{C}^2$.*

(ii) *There exists a Riemann surface $\tilde{\Sigma}$ which is a closed complex submanifold in Ω such that $d(f^n(X), \tilde{\Sigma}) \to 0$ for any compact X in Ω. The Riemann surface $\tilde{\Sigma}$ is biholomorphic to a disc, a punctured disc or an annulus and $f|\tilde{\Sigma}$ is conjugate to an irrational rotation.*

(iii) *The domain Ω is a Siegel domain and all convergent subsequence of (f^n) converge to an automorphism of Ω.*

Proof. Since Ω is recurrent, there is $p_0 \in \Omega$ and (n_i) such that $f^{n_i}(p_0) \to p_0$. Taking a subsequence of $f^{n_{i+1}-n_i}$, we can assume that $f^{n_{i+1}-n_i}$ u.c.c. to a holomorphic map $h: \Omega \to \bar{\Omega}$. We have that $h(p) = p$. If for some h, *rank* $h = 0$, then $h(\Omega) = p$ and $f(p) = p$. Consequently p should be attracting and we are in case i). The last assertion is classical, see [RR] for example.

Assume, for some h, the rank of h is two. Then if a denotes the constant, jacobian determinant of f, we necessarily have $|a| = 1$ and so f is volume preserving. Let $G = \overline{\{f^n\}}_{n \in \mathbb{Z}}$ where the closure is taken for the topology of u.c.c.. It follows from results of Cartan, see [N], that G is a compact Lie group, isomorphic to $\mathbb{T}^2 \times F$ or $\mathbb{T} \times F$ where F is a finite commutative group. The argument is even simpler than the one given in the first paragraph since we have the information that f is volume preserving.

We can assume that all limits h have rank one and consequently $|a| < 1$. Let $\Sigma = h(\Omega)$. We have $\Sigma \subset \bar{\Omega} \cap K$, so Σ is a Riemann surface which is hyperbolic and $f: \Sigma \to \Sigma$ is an automorphism. Let Σ_p be the irreducible component of $\Sigma \cap \Omega$ containing p. The difficulty here, as in Theorem 1.2, in order to say that f is conjugate to a rotation on Σ is that possibly other components of $\Sigma \cap \Omega$ might cluster on Σ_p. So the fact that say (f^{n_i}) converge to identity on Σ_p does not imply the convergence on Σ, considered as an abstract Riemann surface. As in Theorem 1.2 we show first that f is not constant on any component $f^l(\Sigma_p), l \geqq 0$. Hence we can assume using a diagonal process that for a subsequence $(f^{m_i}), f^{m_i} \to h$ and $h = Id$ on $\bigcup_{l \geqq 0} f^l(\Sigma_p) \subset \{q \in \Omega;\ h(q) = q\}$. Consequently $\bigcup_{l \geqq 0} f^l(\Sigma_p)$ is closed in Ω. Since $f^{m_i} \to Id$ on Σ_p and since f is not of finite order, see [FM], then f is conjugate to an irrational rotation on the Riemann surface Σ. We then show as in Theorem 2.3 of [FS3] that Σ is contained in Ω, so Σ is closed in Ω. The rest of the argument is as in Theorem 1.2 or as in [FS3] Theorem 2.3.

References

[B] Beardon, A.: Iteration of rational functions. Springer Verlag (1991)

[BBD] Barrett, D., Bedford, E., Dadok, J.: T^n-actions on holomorphically separable complex manifolds. Math. Z. **202** (1989), 65–82

[BS] Bedford, E., Smillie, J.: Polynomial diffeomorphisms of $\mathbb{C}^2$. II. Amer. J. Math. **4** (1991), 657–679

[CG] Carleson, L., Gamelin, T.: Complex dynamics. Springer Verlag (1993)

[FM] Friedland, S., Milnor, J.: Dynamical properties of plane automorphisms. Ergodic theory and dynamical systems **9** (1989), 67–99

[FS1] Fornæss, J.E., Sibony, N.: Complex dynamics in higher dimension. I. Asterisque **222** (1994), 201–231

[FS2] Fornæss, J.E., Sibony, N.: Complex dynamics in higher dimension. II. To appear in Ann. Math. Studies

[FS3] Fornæss, J.E., Sibony, N.: Complex Henon mappings in ^{2} and Fatou-Bieberbach domains. Duke Math. J. **65** (1992), 345–380

[FS4] Fornæss, J.E., Sibony, N.: Holomorphic dynamical systems. (To appear)

[G] Gavosto, E.: To appear

[H] Herman, M.: Recent results on some open questions on Siegel's linearization

[Ko] Kobayashi, S.: Hyperbolic manifolds and holomorphic mappings. New York, Marcel Dekker, 1970

[Kr] Kruzhilin, N.G.: Holomorphic automorphisms of hyperbolic Reinhardt domains. Math. USSR Izvestya **32** (1989), 15–38

[M] Milnor, J.: Dynamics in one complex variable: Introductory Lectures. SUNY Stony Brook. Institute for Mathematical Sciences. Preprint #1990/5

[N] Narashiman, R.: Several complex variables. Univ. Chicago Press, 1971

[RR] Rosay, J.-P., Rudin, W.: Holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$. TAMS **310** (1988), 47–86

[U] UEDA, T.: Fatou set in complex dynamics in projective spaces. Preprint