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The asymptotic expansion of a CR invariant and Grauert tubes

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1 Introduction

CR manifolds, the abstract models of real hypersurfaces in complex manifolds, are $2n+1$ dimensional manifolds M with a codimension one subbundle H of the tangent bundle, which carries a complex structure. The "CR" refers to Cauchy–Riemann because for $M \subset \mathbb{C}^{n+1}$, the subbundle H consists of induced Cauchy–Riemann operators. There is a wealth of geometry and analysis associated with these structures, especially when the CR manifolds are strictly pseudoconvex. For example, two strictly pseudoconvex domains are biholomorphically equivalent if and only if their boundaries are CR equivalent.

A fundamental problem in CR geometry is to find computable invariants associated with the CR structures. The global CR invariant we will consider in this paper is the Chern–Simons type invariant μ discovered by Burns and Epstein [B-E 1]. It is a real-valued global CR invariant of a compact 3-dimensional strictly pseudo-convex CR manifold whose holomorphic tangent bundle is trivial. (Cheng and Lee independently found this invariant, and extend the definition of B-E invariant to a relative invariant on an arbitrary compact 3-dimensional CR manifold, cf. [C-L].) We will evaluate this μ asymptotically on the boundary of small Grauert tubes. Before posing the question in a more precise form we will first say a few workds about Grauert tubes.

Let X be a real analytic manifold. Then every coordinate patch $U \subset \mathbb{R}^n$ can be thickened to obtain an open set $\mathbb{C}U \subset \mathbb{C}^n$. Since the coordinate changes of X are real analytic maps, by taking power series expansions and by shrinking $\mathbb{C}U$ to get convergence, they can be extended holomorphically to such enlarged domains and thus they can be used as holomorphic transition

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functions for a complex manifold $\mathbb{C}X$. This complexification process makes it possible to extend real analytic objects given on X to holomorphic ones on the complexification. Grauert [Gr] used this idea in his famous proof about embeddability of abstract real analytic manifolds. One remarkable byproduct of Grauert's construction is the existence of a neighborhood M of X in $\mathbb{C}X$ and a smooth strictly plurisubharmonic function ρ

$$\rho : M \to [0, 1)$$

such that X is the zero set of ρ. This ρ is not canonically defined, since $c\rho$ keeps this property for any $c \in (0, 1)$. Recently Guillemin and Stenzel (and independently Lempert and Szöke) proved the uniqueness of ρ under two additional hypotheses: the Kähler metric with Kähler form $\frac{\sqrt{-1}}{2}\partial\bar{\partial}\rho$ on M is compatible with the Riemannian metric on X, and $\sqrt{\rho}$ is a solution of the homogeneous complex Monge–Ampère equation on $M - X$. This result can be regarded as defining a canonical complexification of Riemannian manifolds with real analytic metrics. The set $\{\rho < \varepsilon^2\}$ is a certain disk bundle over X. We call it the Grauert tube of radius ε.

We will concentrate on real 4-dimensional Grauert tubes, and find the B-E invariant μ on the boundaries of these tubes. Our motivation for this study of the invariant μ in Grauert tubes comes from the volume formula proved by H. Weyl. He showed in [Wey] that the volume V_r of an n-dimensional tubular neighborhood around a Riemannian submanifold (X^m, g) of $\mathbb{R}^n$ has a Taylor series in the radius r. Specifically,

$$\begin{aligned} V_r =& C_m(\text{volume of } X)\ r^{n-m} + C_{m,2}\int_X (\text{scalar curvature})\ r^{n-m+2}\ \text{dvol} \\ &+ C_{m,e}\sum_{e \text{ even}} \frac{r^{n-m+e}}{(m+2)(m+4)\dots(m+e)} k_e \end{aligned}$$

where k_e are certain integral invariants of X, determined by the intrinsic metric nature of X only.

We address the following questions. On $\{\rho = \varepsilon^2\}$, as ε varies, how do the invariants μ depend on the manifold X and the radius ε? To what extent are they analogous to H. Weyl's volume formula?

One of the main results we prove in this paper is Theorem 6.1 which says that: μ has an asymptotic expansion in ε^2

$$\mu \sim \frac{3}{16\pi\varepsilon^2}\int_X dA - \frac{1}{8\pi}\int_X k(x)dA + \sum_{l=1}^{\infty}\varepsilon^{2l}\int_X F_l(g_{ij}^{(k)})dA\,,$$

where $k(x)$ is the scalar curvature, and $\lambda^{2l+2}F_l(\lambda^2 g_{ij}^{(k)}) = F_l(g_{ij}^{(k)})$ for any nonzero real number λ. The leading term was suggested by the calculation of a Reinhardt example (cf. [B-E 1]). Using this result, together with group representations, we can prove the biholomorphic inequivalence of Grauert tubes with centers of constant sectional curvature and classify these kinds of tubes.

The contents of the various sections are as follows:

Section 2 is a quick review of definitions of a CR structure and the Burns–Epstein invariant.

In Sect. 3 we establish the necessary background information and develop some properties of the Monge–Ampère equation which will play a key role in the sequel.

In Sect. 4 we first show that the invariant μ is well-defined on the boundaries of Grauert tubes, then point out a natural pseudo-hermitian structure.

Section 5 will concentrate on the B-E invariant. We prove in this section, the invariant μ possesses an asymptotic expansion in powers of the radius.

Section 6 is devoting to the calculation of the second term of the asymptotic expansion of the invariant μ. Our main result is Theorem 6.1, which is summarized above, saying that the invariant μ of the boundary of a Grauert tube has an asymptotic expansion in the radius of the tube. The leading and the second-order terms are respectively the volume and the scalar curvature times some dimensional constant.

In Sect. 7 we discuss the location of CR spherical structures and the biholomorphic inequivalence of Grauert tubes by examining the behavior of μ on their boundaries. Though the answer is not clear for general Riemannian manifolds, we do have a definitie result for those Grauert tubes whose centers have constant sectional curvature, the main results are stated as Theorem 7.1 and Theorem 7.2.

2 CR manifolds and the Burns–Epstein invariant

Let M be a smooth manifold of real dimension $2n+1$. A CR structure on M is defined by choosing an n-dimensional subbundle $T_{1,0}M$ of the complexified tangent bundle $\mathbb{C}TM$ of M, such that

(1) $T_{1,0}M \cap \overline{T_{1,0}M} = \{0\}$;

(2) $T_{1,0}M$ is integrable, i.e., if X and Y are two sections of $T_{1,0}M$, so is their Lie bracket $[X, Y]$.

We call M a CR manifold with the given CR structure $T_{1,0}M$. Also $T_{1,0}M$ is called the holomorphic tangent bundle, and $T_{1,0}M \oplus \overline{T_{1,0}M}$ is usually denoted by $\mathbb{C}H$. In fact, $\mathbb{C}H$ carries a natural complex structure given by the map J

$$J : \mathbb{C}H \to \mathbb{C}H\,,$$

$$J(V) = iV, \quad J(\overline{V}) = -i\overline{V} \quad \text{for } V \in T_{1,0}M\,.$$

The most important example of a CR structure is of course that induced by an embedding $M \subset \mathbb{C}^n$, in which we can choose $T_{1,0}M = T_{1,0}\mathbb{C}^{n+1} \cap \mathbb{C}TM$. We call this the embedded CR structure. For three-dimensional CR manifolds, the integrability condition is automatically fulfilled: any complex line bundle V with $V \cap \overline{V} = \{0\}$ defines a CR structure. This property, together with the fact that there are many nondegenerate CR structures on any compact orientable three-manifold, makes the 3-dimensional CR structures strikingly different from higher dimensions.

A contact form θ is a real non-vanishing one-form which annihilates $T_{1,0}$ (hence annihilates $\mathbb{C}H$); it is determined only up to a conformal factor. A CR structure $T_{1,0}$ with a specified choice of contact form θ is called a pseudohermitian structure. The Levi form associated with this θ is a Hermitian form L_θ on $T_{1,0}$:

$$L_\theta(V,W) = -id\theta(V,\overline{W}).$$

The structure is strictly pseudoconvex if the Levi form is definite; thus by changing the sign of θ if necessary, we may assume that it is positive-definite. Let $\{X_1,X_2,\dots,X_n\}$ be a local frame field for $T_{1,0}$, and let $\{\theta_1,\theta_2,\dots,\theta_n\}$ be a dual coframe field. Then

$$d\theta = ig_{\alpha\bar\beta}\theta_\alpha \wedge \theta_{\bar\beta} + \theta\wedge\varphi,$$

where $\theta_{\bar\beta} = \overline{\theta_\beta}$, and φ is a real one-form. Calculations on pseudohermitian manifolds are simplified tremendously if we work with special coframes. With the contact form θ fixed, Webster [Web] chose a coframe $\{\theta_\alpha,\theta_{\bar\alpha}\}$ of $T_{1,0}$ by requiring

$$d\theta = ig_{\alpha\bar\beta}\theta_\alpha \wedge \theta_{\bar\beta},$$

and defined the connection form (ω_β^α) as well as the torsion form (τ^α) via the structure equations

$$d\theta_\alpha = \theta_\beta\wedge\omega_\beta^\alpha + \theta\wedge\tau^\alpha, \quad \omega_\beta^\alpha + \overline{\omega_\alpha^\beta} = 0, \quad \tau^\alpha\wedge\bar\theta_\alpha = 0.$$

In this setting, the curvature matrix (Π_α^β) is

$$\begin{aligned}\Pi_\alpha^\beta &= d\omega_\alpha^\beta - \omega_\alpha^\gamma\wedge\omega_\gamma^\beta \\ &= R_{\alpha\rho\bar\sigma}^\beta\theta_\rho\wedge\bar\theta_\sigma + \omega_{\alpha\rho}^\beta\theta_\rho\wedge\theta - \omega_{\alpha\bar\sigma}^\beta\bar\theta_\sigma\wedge\theta + i\bar\theta_\alpha\wedge\tau^\beta - i\bar\tau^\alpha\wedge\theta_\beta,\end{aligned}$$

and the pseudo-hermitian scalar curvature R is defined by

$$R = R_{\alpha\beta}^{\alpha\beta}.$$

In the sequel, we will only deal with three-dimensional CR manifolds in which the tedious indices of the above forms could be simplified tremendously. First of all, since M is strictly pseudoconvex, the matrix $(g_{\alpha\bar\beta})$ is positive-definite. Therefore, we can normalize $\{\theta_1,\theta_{\bar 1}\}$ so that

$$d\theta = i\theta_1\wedge\theta_{\bar 1}.$$

Since there is only one connection form ω_1^1, and only one torsion form τ^1, we may denote the connection form ω_1^1 by ω and the torsion form τ^1 by τ. The structure equations then become

$$d\theta_1 = \theta_1\wedge\omega + \theta\wedge\tau, \quad \omega+\bar\omega = 0, \quad \tau\wedge\theta_{\bar 1} = 0. \tag{2.1}$$

and the pseudo-hermitian scalar curvature R is obtained from the equation

$$d\omega = R\theta_1\wedge\theta_{\bar 1} + W\theta_1\wedge\theta - \overline{W}\theta_{\bar 1}\wedge\theta.$$

Based on this structure, Burns and Epstein [B-E 1] defined a real-valued global CR invariant μ of Chern–Simons type for a compact, strictly pseudoconvex 3-dimensional CR manifold whose holomorphic tangent bundle is trivial. This μ can be written down explicitly as

$$\mu = \int_M \widetilde{T}C_2(\Pi)\,, \tag{2.2}$$

where

$$\widetilde{T}C_2(\Pi) = \frac{i}{8\pi^2}\left[\frac{-2i}{3} d\omega \wedge \omega + \frac{1}{6} R\theta \wedge d\omega - 2\theta \wedge \tau \wedge \bar{\tau}\right] + \text{exact form}\,. \tag{2.3}$$

Remark. Cheng and Lee [C-L] independently found this invariant and extended the definition as a relative invariant to arbitrary compact three-dimensional CR manifolds.

3 Grauert tubes and Monge–Ampère equations

Let X be an n-dimensional differentiable manifold. Bruhat and Whitney showed that if X is a real-analytic manifold of dimension n, then X can be complexified; i.e., there exists a complex n-dimensional manifold M, and a real-analytic imbedding of X in M, such that X is a totally real submanifold of M, where totally real means: $V \in T_x(X)$ implies $JV \notin T_x(X)$ for the complex structure J on $T_x(X)$, any $x \in X$. In addition, Grauert [Gr] showed that there exists a neighborhood U of X in M, and a nonnegative smooth strictly plurisubharmonic function ρ on U such that X is the zero set of ρ. The fact that ρ is strictly plurisubharmonic implies that the domains

$$M_\varepsilon = \rho^{-1}([0, \varepsilon^2)), \quad \varepsilon > 0$$

are strictly pseudoconvex.

Clearly this ρ is not uniquely defined for a given X, because any positive real number c times ρ still gives a nonnegative strictly plurisubharmonic function. However, Guillemin and Stenzel [G-S] (simultaneously and independently, Lempert and Szöke) imposed additional conditions on ρ to assure its uniqueness; they proved the following theorem.

Theorem (Guillemin–Stenzel) *Let X be a compact, real-analytic, n-dimensional manifold with a real-analytic Riemannian metric ds^2. Then there exists a neighborhood M of X in the ambient complexified space, and a unique real-analytic nonnegative smooth strictly plurisubharmonic function ρ such that*

(1) $X = \rho^{-1}(0)$;

(2) *the metric ds^2_M obtained from the Kähler form $\frac{i}{2}\partial\bar{\partial}\rho$ is compatible with ds^2 (i.e., $ds^2_M|_X = ds^2$);*

(3) $(\partial\bar{\partial}\sqrt{\rho})^n = 0$ *on $M - X$.*

Let us say a few more words about the condition (3). Let $u : M \to \mathbb{R}$ be a plurisubharmonic function on a complex n-dimensional manifold M. The homogenous complex Monge–Ampère equation for u is

$$(**) \qquad (\partial\bar{\partial}u)^n = 0\,,$$

or in local coordinates $z_1, z_2, \ldots, z_n$,

$$det\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) = 0\,.$$

When $n = 1$, this equation reduces to the Laplace equation $\Delta u = 0$, and, indeed, the Monge–Ampère equation is the most natural extension of the Laplace equation to higher-dimensional complex manifolds. The above theorem shows that M and ρ are uniquely determined by X and the metric ds^2. We can also regard the theorem as defining a canonical complexification of a Riemannian manifold with a real-analytic metric. $(M, X, \sqrt{\rho})$ is called a *Monge–Ampère model* (of bounded type, if $\sqrt{\rho}$ is bounded; of unbounded type, otherwise). Let

$$M_\varepsilon = \rho^{-1}[0, \varepsilon^2)\,.$$

Then M_ε is an open, strictly pseudoconvex domain. We refer to this as the Grauert tube of radius ε. One of the main objects of this paper is to find the Burns–Epstein invariant μ on this Grauert tube, and to see how it depends on the geometry of X, which we will call the *middle manifold* or *center*, and the radius ε. Since the invariant μ is defined only on three-dimensional CR manifolds; we will fix $n = 2$ from now on, and point out some properties of the Monge–Ampère solution $\sqrt{\rho}$.

Let ρ be a positive smooth function on a complex manifold M of dimension two. Since

$$(\partial\bar{\partial}\sqrt{\rho})^2 = -\tfrac{1}{4}\rho^{-2}[\partial\rho \wedge \bar{\partial}\rho \wedge \partial\bar{\partial}\rho - \rho(\partial\bar{\partial}\rho \wedge \partial\bar{\partial}\rho)]\,,$$

$\sqrt{\rho}$ is a solution of the Monge–Ampère equation $(**)$ if and only if

$$(3.1) \qquad \partial\rho \wedge \bar{\partial}\rho \wedge \partial\bar{\partial}\rho = \rho(\partial\bar{\partial}\rho) \wedge (\partial\bar{\partial}\rho)\,;$$

or in local coordinates z, w,

$$(3.2) \qquad 2\rho(\rho_{z\bar{z}}\rho_{w\bar{w}} - \rho_{z\bar{w}}\rho_{w\bar{z}}) = \rho_z\rho_{\bar{z}}\rho_{w\bar{w}} - \rho_{\bar{z}}\rho_w\rho_{z\bar{w}} - \rho_z\rho_{\bar{w}}\rho_{w\bar{z}} + \rho_w\rho_{\bar{w}}\rho_{z\bar{z}}\,.$$

These differentials could also be expressed in terms of real coordinates x_1, x_2, y_1, y_2, with $z = x_1 + iy_1$, $w = x_2 + iy_2$. Then the equation (3.2) takes the form

(3.3)
$$\begin{aligned}
&2\rho[(\rho_{x_1x_1} + \rho_{y_1y_1})(\rho_{x_2x_2} + \rho_{y_2y_2}) - (\rho_{x_1x_2} + \rho_{y_1y_2})^2 - (\rho_{x_1y_2} - \rho_{x_2y_1})^2] \\
&\quad = [(\rho_{x_1})^2 + (\rho_{y_1})^2](\rho_{x_2x_2} + \rho_{y_2y_2}) + [(\rho_{x_2})^2 + (\rho_{y_2})^2](\rho_{x_1x_1} + \rho_{y_1y_1}) \\
&\qquad - 2(\rho_{x_1}\rho_{x_2} + \rho_{y_1}\rho_{y_2})(\rho_{x_1x_2} + \rho_{y_1y_2}) \\
&\qquad + 2(\rho_{x_2}\rho_{y_1} - \rho_{x_1}\rho_{y_2})(\rho_{x_1y_2} - \rho_{x_2y_1})\,.
\end{aligned}$$

Let (g_{ij}) denote the Riemannian metric on X. The metric compatibility property (2) of the Guillemin–Stenzel Theorem holds if and only if

$$\left(\frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}\right)\bigg|_X = (g_{ij}),$$

i.e., when we pull back and evaluate the Kähler metric on real vectors tangent to X, it coincides with the Riemannian metric (g_{ij}). Therefore, locally ρ must have the form

$$\begin{aligned}(\#)\quad \rho(z,w) &= \rho(x_1,y_1,x_2,y_2)\\ &= 2g_{11}(x_1,x_2)y_1^2 + 4g_{12}(x_1,x_2)y_1y_2 + 2g_{22}(x_1,x_2)y_2^2\\ &\quad + \text{ higher order terms in } y_1 \text{ and } y_2\,.\end{aligned}$$

On the other hand, $\varrho(z,w) \equiv \rho(\bar{z},\bar{w})$ is a real-analytic, non-negative smooth strictly plurisubharmonic function which satisfies all of three conditions listed in the Guillemin–Stenzel Theorem. By uniqueness

$$\begin{aligned}(\#\#)\quad \rho(z,w) &= \varrho(z,w) \equiv \rho(\bar{z},\bar{w})\\ &= 2g_{11}(x_1,x_2)y_1^2 + 4g_{12}(x_1,x_2)y_1y_2 + 2g_{22}(x_1,x_2)y_2^2\\ &\quad + \text{higher order terms in } (-y_1) \text{ and } (-y_2)\,.\end{aligned}$$

Comparing (#) and (##), we see that it is not possible for odd-order terms in y_1, y_2 to appear. Setting

$$g_{ij}^p = \frac{\partial g_{ij}(x_1,x_2)}{\partial x_p}, \quad g_{ij}^{pq} = \frac{\partial g_{ij}(x_1,x_2)}{\partial x_p \partial x_q},$$

ρ can be expressed more precisely as follows:

Proposition 3.1.

$$\begin{aligned}(3.4)\qquad \rho(z,w) &= 2g_{11}(x_1,x_2)y_1^2 + 4g_{12}(x_1,x_2)y_1y_2 + 2g_{22}(x_1,x_2)y_2^2\\ &\quad + \varphi_1(x_1,x_2)y_1^4 + \varphi_2(x_1,x_2)y_1^3y_2 + \varphi_3(x_1,x_2)y_1^2y_2^2\\ &\quad + \varphi_4(x_1,x_2)y_1y_2^3 + \varphi_5(x_1,x_2)y_2^4\\ &\quad + \textit{higher even-order terms in } y_1 \textit{ and } y_2\,.\end{aligned}$$

The coefficients $\varphi_j(x_1,x_2)$, $0 \leqq j \leqq 5$ are smooth functions of the metric (g_{ij}) and their k-th derivatives $(g_{ij}^{(k)})$, $1 \leqq \|k\| \leqq 2$. More precisely,

$$\varphi_l(x_1,x_2) = \frac{\beta_{labcdefp}\, g_{ab} g_{cd}^p g_{ef}^p + \gamma_{lhijmnrs}\, g_{hi} g_{hj} g_{mn}^{rs}}{g_{11}g_{22} - g_{12}g_{12}}$$

for some real numbers $\beta_{labcdefp}$ and $\gamma_{lhijmnrs}$.

Proof. Because the Taylor expansion of ρ in y_1 and y_2 contains even-order terms only, (3.4) is proved. The idea for proving this proposition is to insert the expression (3.4) into the equation (3.3), and collect those $y_1^l y_2^k$ terms with $l+k=4$, then equate the coefficients of each monomials $y_1^l y_2^k$ on both sides of the equation. We first observe that

$$
\begin{aligned}
(\rho_{x_p})^2 &\sim g_{ij}^p g_{kl}^p y_i y_j y_k y_l + O(|y|^6),\\
(\rho_{y_p})^2 &\sim g_{pi} g_{pj} y_i y_j + O(|y|^4),\\
\rho_{x_p x_q} &\sim g_{ij}^{pq} y_i y_j + O(|y|^4),\\
\rho_{y_p y_q} &\sim g_{pq} + \varphi_j y_k^m y_l^n + O(|y|^4), \quad m+n=2\,.
\end{aligned}
$$

Collecting y_1^4 terms in the first part of (3.3), they are

$$
\begin{aligned}
(3.5)\quad & 32g_{11}[g_{22}g_{11}^{11} + g_{11}g_{11}^{22} - 2g_{12}g_{11}^{12} - 2g_{12}^1 g_{12}^1 - 2g_{11}^2 g_{11}^2 + 4g_{11}^2 g_{12}^1]y_1^4\\
&+ 32g_{11}g_{11}\varphi_3 y_1^4 + (224g_{11}g_{22} - 32g_{12}g_{12})\varphi_1 y_1^4\,.
\end{aligned}
$$

In the second part of (3.3), those y_1^4 terms are

$$
\begin{aligned}
(3.6)\quad & [16g_{11}g_{11}^2 g_{11}^2 + 16g_{22}g_{11}^1 g_{11}^1 - 32g_{12}g_{11}^1 g_{11}^2 + 64g_{12}g_{11}^1 g_{11}^2\\
&- 64g_{11}g_{11}^2 g_{11}^2 - 64g_{11}g_{11}^2 g_{12}^1 - 64g_{12}g_{11}^1 g_{12}^1 + 32g_{11}g_{11}g_{11}^{22}\\
&+ 32g_{12}g_{12}g_{11}^{11} - 64g_{11}g_{12}g_{11}^{12}]y_1^4 + 32g_{11}g_{11}\varphi_3 y_1^4 + 192g_{12}g_{12}\varphi_1 y_1^4\,.
\end{aligned}
$$

Therefore

$$
\varphi_l(x_1,x_2) = \frac{\beta_{labcdefp} g_{ab} g_{cd}^p g_{ef}^p + \gamma_{lhijmnrs}\, g_{hi} g_{hj} g_{mn}^{rs}}{224(g_{11}g_{22} - g_{12}g_{12})}
$$

for some real numbers $\beta_{labcdefp}$ and $\gamma_{lhijmnrs}$, by comparing (3.5) and (3.6). Similarly, we can obtain $\varphi_{i,j} = 2,3,4,5$, which will possess the same kind of expressions.

We pursue Proposition 3.1 a bit more by choosing a specific coordinate system, the geodesic normal coordinates at the origin of X, which will be important to us at various times. Let (x_1,x_2) be the geodesic normal coordinates on X centered at 0, and let (z,w) be the holomorphic extension of (x_1,x_2), $z = x_1 + iy_1$, $w = x_2 + iy_2$. Then the Monge–Ampère solution $\sqrt{\rho}$ is locally

$$
\begin{aligned}
(3.7)\quad \rho(z,w) &= 2y_1^2 + 2y_2^2 + \tfrac{1}{3}kx_1^2 y_2^2 - \tfrac{2}{3}kx_1 x_2 y_1 y_2 + \tfrac{1}{3}kx_2^2 y_1^2\\
&+ \alpha_1 k y_1^4 + \alpha_2 k y_1^3 y_2 + \alpha_3 k y_1^2 y_2^2 + \alpha_4 k y_1 y_2^3 + \alpha_5 k y_2^4\\
&+ \text{higher order terms in } x \text{ and } y
\end{aligned}
$$

where k is the scalar curvature at 0, and α_j are certain constants.

For those $(x_1, y_1, x_2, y_2) \in \partial M_\varepsilon$, we also observe that y_1, y_2 can be solved as the following asymptotic expansions in ε:

(3.8)

$$
\begin{cases}
y_1 = (2\Box)^{\frac{-1}{2}} g_{22}^{\frac{1}{2}} \varepsilon \sin\theta,\\
y_2 = (2g_{22})^{\frac{-1}{2}} \varepsilon\cos\theta - (2\Box)^{\frac{-1}{2}} g_{12} g_{22}^{\frac{-1}{2}} \varepsilon\sin\theta + \sum_{j=1}^{\infty} \eta_j(x_1,x_2,\theta)\varepsilon^{2j+1},\\
\Box = g_{11}g_{22} - g_{12}g_{12},\ \theta\in[0,2\pi) \text{ and } \eta_j \text{ are certain smooth functions}.
\end{cases}
$$

In particular, if we choose the geodesic normal coordinates on X, then y_1 and y_2 can be approximated by

$$y_1 = \tfrac{1}{\sqrt{2}}\varepsilon \sin\theta + \text{h.o.t. in } x \text{ and } \varepsilon, \quad y_2 = \tfrac{1}{\sqrt{2}}\varepsilon\cos\theta + \text{h.o.t. in } x \text{ and } \varepsilon,$$

where h.o.t. stands for "higher order terms".

4 Formulation of the problem and the pseudo-hermitian structures

Let (X, g_{ij}) be a real-analytic, oriented, compact, two-dimensional Riemannian manifold, $M_\varepsilon = \{\rho < \varepsilon^2\}$ be the Grauert tube of radius ε around X, and $\partial M_\varepsilon = \{\rho = \varepsilon^2\}$ be the boundary of the tube. We would like to show that there exists a CR structure on ∂M_ε with trivial holomorphic tangent bundle, and therefore prove that the invariant μ is well-defined on ∂M_ε.

Theorem 4.1. *Let $(M, X, \sqrt{\rho})$ be the Monge–Ampère model of a compact, oriented, real-analytic, two-dimensional manifold X, and $\partial M_\varepsilon = \{\rho = \varepsilon^2\}$ be the boundary of the Grauert tube of radius ε around X. Then the natural CR structure on ∂M_ε has a trivial holomorphic tangent bundle.*

Proof. Let

$$\eta = \partial\rho = \rho_z dz + \rho_w dw\,.$$

We observe that the vector

$$V = \rho_w \frac{\partial}{\partial z} - \rho_z \frac{\partial}{\partial w}$$

is a local choice of a generator of the holomorphic tangent space $T_{1,0}(\partial M_\varepsilon)$. Choose a section of the dual of $T_{1,0}(\partial M_\varepsilon)$ to be dual to V:

$$\varphi = \sqrt{det(g_{ij}(z,w))}\,\frac{\rho_{\bar w}\, dz - \rho_{\bar z}\, dw}{|\rho_z|^2 + |\rho_w|^2}\,. \tag{4.1}$$

Then

$$\varphi \wedge \eta = \sqrt{det(g_{ij}(z,w))}\, dz\, dw\,, \tag{4.2}$$

which is the complexification of the volume element $\sqrt{det(g_{ij}(x_1,x_2))}dx_1 dx_2$ of the oriented Riemann manifold X. Therefore it is globally defined and is non-vanishing in a small neighborhood $\{\rho < \varepsilon\}$ of X. As η is defined globally, we conclude that φ is globally defined and nowhere vanishing, at least as a section of the dual of the holomorphic tangent bundle of ∂M_ε. This proves the natural CR structure on ∂M_ε has a trivial holomorphic tangent bundle. However, the "trivial holomorphic tangent bundle condition" is a homotopy condition. If it is true for $0 < \varepsilon \ll 1$, then it is true for all non-singular levels of ρ.

Remark. If $X = \widetilde{X}/\Gamma$, $|\Gamma| = k$, is not oriented, but its k-th covering $\widetilde{X}$ is oriented, then $\partial\widetilde{M_\varepsilon}$ – the Grauert tube of radius ε around $\tilde{X}$-has trivial holomorphic tangent bundle, which could be viewed as a k-th tensor power of $T_{1,0}(\partial M_\varepsilon)$.

A discussion in [B-E 1] about this situation shows that the definition of the invariant μ can be extended to such ∂M_ε.

The existence of μ leads to the seeking of the pseudo-hermitian structure, i.e., we are trying to find out those dual one-forms $\theta_1, \theta_{\bar{1}}$, connection form w, torsion form τ, and scalar curvature R on ∂M_ε. For our convenience, from now on, we will use the following abbreviations:

$$A = \rho_{\bar{w}}\rho_{z\bar{z}} - \rho_{\bar{z}}\rho_{z\bar{w}}, \quad B = \rho_{\bar{w}}\rho_{w\bar{z}} - \rho_{\bar{z}}\rho_{w\bar{w}}, \quad \Delta = \rho_{z\bar{z}}\rho_{w\bar{w}} - \rho_{z\bar{w}}\rho_{w\bar{z}} > 0\,.$$

Restrict all of the calculations to the level surface $\partial M_\varepsilon = \{\rho = \varepsilon^2\}$. Then (3.2) is equivalent to

$$(4.4) \qquad \begin{aligned} &(1) \quad \partial\rho \wedge \bar{\partial}\rho \wedge \partial\bar{\partial}\rho = 2\varepsilon^2 \Delta \\ &(2) \quad A\rho_w - B\rho_z = 2\varepsilon^2 \Delta\,, \end{aligned}$$

also,

$$(4.5) \qquad A_w - B_z = 2(\rho_{z\bar{z}}\rho_{w\bar{w}} - \rho_{z\bar{w}}\rho_{w\bar{z}}) = 2\Delta\,.$$

By (1) of (4.4), $\partial\rho \neq 0$ and $\bar{\partial} \neq 0$ on $\{\rho = \varepsilon^2\}$, therefore $d\rho \neq 0$ off $\{\rho = 0\}$. So,

$$\operatorname{grad} \rho = (\rho_z, \rho_w, \rho_{\bar{z}}, \rho_{\bar{w}}) \neq 0 \quad \text{on } \partial M_\varepsilon\,.$$

On the other hand, for any tangent vector $X \in T_p(\partial M_\varepsilon)$,

$$0 = d\rho(X) = (\rho_z dz + \rho_w dw + \rho_{\bar{z}} d\bar{z} + \rho_{\bar{w}} d\bar{w})(X)\,.$$

Therefore, when we consider the actions of one-forms on the tangent bundle of ∂M_ε, without loss of generality, we may assume locally $\rho_{\bar{w}} \neq 0$ and

$$(4.6) \qquad d\bar{w} = \frac{\rho_z dz + \rho_w dw + \rho_{\bar{z}} d\bar{z}}{-\rho_{\bar{w}}}\,.$$

Choose the contact form θ by

$$\theta = -i\partial\rho = -i(\rho_z dz + \rho_w dw)\,.$$

By (4.1), a natural globally defined section of the dual of the holomorphic tangent bundle to ∂M_ε will be

$$\varphi = \sqrt{det(g_{ij}(z,w))}\, \frac{\rho_{\bar{w}} dz - \rho_{\bar{z}} dw}{|\rho_z|^2 + |\rho_w|^2}\,.$$

However, to construct a pseudo-hermitian structure, we need to construct another one form θ_1 so that $d\theta = i\theta_1 \wedge \theta_{\bar{1}}$. Consider φ locally as a one-form; it is only well-defined modulo addition of multiples of θ. We let

$$(4.7) \qquad \begin{aligned} \theta_1 &= 2\Delta\varepsilon^2\alpha(|\rho_z|^2 + |\rho_w|^2)\varphi + i\sqrt{det(g_{ij}(z,w))}\,\alpha(B\rho_{\bar{w}} + A\rho_{\bar{z}})\theta \\ &= \beta(Adz + Bdw), \end{aligned}$$

for some complex-valued functions α and β. The structure equation $d\theta = i\theta_1 \wedge \theta_{\bar{1}}$ will determine a unique extension of φ as a globally defined one-form θ_1. We solve

$$\beta = \frac{1}{\sqrt{2}\varepsilon\Delta^{\frac{1}{2}}}, \quad \alpha > 0,$$

then

$$\theta_1 = \frac{Adz + Bdw}{\sqrt{2}\varepsilon\Delta^{\frac{1}{2}}}.$$

Similarly,

$$\omega = ia\theta + b(\theta_1 - \theta_{\bar{1}}), \quad \tau = c\theta_{\bar{1}}$$

where

$$a = \frac{-1}{\varepsilon^2} + \frac{A\Delta_w - B\Delta_z}{4\varepsilon^2\Delta^2} + \frac{\rho_w A_{\bar{w}} - \rho_z B_{\bar{w}}}{2\varepsilon^2\Delta\rho_{\bar{w}}} - \frac{\overline{B}(\rho_{\bar{z}}\Delta_{\bar{w}} - \rho_{\bar{w}}\Delta_{\bar{z}})}{4\varepsilon^2\Delta^2\rho_{\bar{w}}} - \frac{\Delta_{\bar{w}}}{2\Delta\rho_{\bar{w}}},$$

$$b = \frac{1}{2\sqrt{2}\varepsilon\Delta^{\frac{3}{2}}}(-\rho_{\bar{z}}\Delta_{\bar{w}} + \rho_{\bar{w}}\Delta_{\bar{z}}),$$

$$c = \frac{i}{2\varepsilon^2\Delta\rho_w}(-\rho_{\bar{z}}B_{\bar{w}} + \rho_{\bar{w}}B_{\bar{z}}) + \frac{iB}{2\varepsilon^2\Delta\rho_w}(-\rho_{\bar{w}}\Delta_{\bar{z}} + \rho_{\bar{z}}\Delta_{\bar{w}}).$$

Finally, the scalar curvature R is uniquely determined by the equation

$$d\omega = R\theta_1 \wedge \theta_{\bar{1}} + W\theta_1 \wedge \theta - \overline{W}\theta_{\bar{1}} \wedge \theta, \quad \text{where}$$

$$R = -a - 2b^2 - \frac{1}{\sqrt{2}\varepsilon\Delta^{\frac{1}{2}}}(b_{\bar{z}}\rho_{\bar{w}} - b_{\bar{w}}\rho_{\bar{z}} - b_w\rho_z + b_z\rho_w),$$

$$W = iab + b\bar{c} + \frac{i}{\sqrt{2}\varepsilon\Delta^{\frac{1}{2}}}(a_z\rho_w - a_w\rho_z) - \frac{i}{2\varepsilon^2\Delta}(\overline{B}b_{\bar{z}} - Bb_z + Ab_w - \overline{A}b_{\bar{w}}).$$

5 The asymptotic expansion of the invariant μ

The main purpose of this section is to show the invariant μ of the boundaries of Grauert tubes are very much like Weyl's volume formula, which says that the volume of a tubular domain around a Riemannian manifold depends only on the geometric nature of this Riemannian manifold and the radius of the tube; furthermore, it admits a Taylor expansion in powers of the radius. In our case, the invariant μ also possesses an asymptotic expansion in powers of the radius, but this time, we get some singularities as the radius goes to zero. The leading order term of this expansion is suggested by the calculation of a simple Reinhardt example. (cf. [B-E 1]). We start by interpreting μ in terms of the pseudo-hermitian structure $\{\theta, \theta_1, \bar{\theta}_1, \omega, \tau, R\}$ computed above. We now can compute the invariant μ. First,

$$d\omega \wedge \omega = (iaR - b\overline{W} + bW)\theta_1 \wedge \theta_{\bar{1}} \wedge \theta,$$

$$R\theta \wedge d\omega = R^2\theta_1 \wedge \theta_{\bar{1}} \wedge \theta,$$

$$\theta \wedge \tau \wedge \bar{\tau} = -c\bar{c}\theta_1 \wedge \theta_{\bar{1}} \wedge \theta.$$

So, on ∂M_ε,

$$\begin{aligned}\mu &= \int_{\{\partial M_\varepsilon\}} \widetilde{T}C_2(\Pi)\\ &= \frac{i}{8\pi^2}\int_{\{\rho=\varepsilon^2\}}\left[\frac{-2i}{3}(iaR - b\overline{W} + bW) + \frac{1}{6}R^2 + 2c\bar{c}\right]\theta_1\wedge\theta_{\bar{1}}\wedge\theta\,.\end{aligned}$$

In local coordinates $z = x_1 + iy_1$, $w = x_2 + iy_2$,

$$dz\wedge d\bar{z}\wedge dw = 2idx_1\wedge dx_2\wedge dy_1 + 2dx_1\wedge dy_1\wedge dy_2\,.$$

The volume form

$$\begin{aligned}\theta_1\wedge\theta_{\bar{1}}\wedge\theta &= \frac{i}{\rho_{\bar{w}}}[\rho_z\rho_{\bar{w}}\rho_{wz} - \rho_w\rho_{\bar{w}}\rho_{z\bar{z}} - \rho_z\rho_{\bar{z}}\rho_{w\bar{w}} + \rho_{\bar{z}}\rho_w\rho_{zw}]dz\wedge d\bar{z}\wedge dw\\ &= \frac{-i}{\rho_{\bar{w}}}(2\varepsilon^2\Delta)dz\wedge d\bar{z}\wedge dw\,.\end{aligned}$$

Therefore,

$$\begin{aligned}(5.1)\qquad \mu &= \frac{\varepsilon^2}{12\pi^2}\int_{\{\rho=\varepsilon^2\}}\frac{\Delta}{\rho_{\bar{w}}}[-4i(iaR - b\overline{W} + bW) + R^2 + 12c\bar{c}]dx_1dy_1dy_2\\ &+ \frac{i\varepsilon^2}{12\pi^2}\int_{\{\rho=\varepsilon^2\}}\frac{\Delta}{\rho_{\bar{w}}}[-4i(iaR - b\overline{W} + bW) + R^2 + 12c\bar{c}]dx_1dx_2dy_1\,.\end{aligned}$$

Lemma 5.1. *Each integrand of* (5.1) *is the sum of an even-order, real-valued function and an odd-order purely-imaginary-valued function. In other words,*

$$\textit{integrand} = f_e + if_o$$

where f_e is a real-valued, even-ordered function and f_o is a real-valued, odd-order function.

Proof. Since

$$\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x_1} - i\frac{1}{2}\frac{\partial}{\partial y_1},\qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x_1} + i\frac{1}{2}\frac{\partial}{\partial y_1}\,.$$

We obtain the following type of functions, when taking the z derivatives of ρ,

$$(*)\qquad \text{(even-order terms)} + i\text{(odd-order terms)}\,.$$

Similarly, it is also true for the $\bar{z}, w, \overline{w}$ derivatives. Inductively, when taking one more derivative of ρ, one will have

$$\text{(even)} + i[\text{odd} + i\text{(even)}] = \text{even} + i\text{(odd)} = (*)$$

So, any ∂ and $\bar{\partial}$ derivatives of the function ρ reduce ρ to be of the type $(*)$. Also, the product or the quotient of any two functions of this type will still preserve this type. Now, we check the functions in the integrand: $\rho_{\bar{w}}, \Delta a, b, c, R, W$

all have the type $(*)$, so do $R^2, c\bar{c}, aR, bW, \frac{1}{\rho_{\bar{w}}}$. The only one we need to be careful about is

$$i(iaR - b\overline{W} + bW) \in i[i(\text{even} + i(\text{odd})) - b(\overline{W} - W)]$$
$$\in \text{even} + i\,\text{odd} - [\text{even} + i(\text{odd})]i(\text{odd}) \in \text{even} + i(\text{odd})\,.$$

Putting all of the above arguments together gives the result.

Lemma 5.2. *μ has an asymptotic expansion in ε^2.*

Proof. By the above lemma and (5.1)

$$\mu = \frac{\varepsilon^2}{12\pi^2}\int (f_e + if_o)dx_1dy_1dy_2 + \frac{i\varepsilon^2}{12\pi^2}\int (f_e + if_o)dx_1dx_2dy_1$$
$$= \frac{\varepsilon^2}{12\pi^2}\int f_e dx_1dy_1dy_2 - \frac{\varepsilon^2}{12\pi^2}\int f_o dx_1dx_2dy_1$$
$$+ \frac{i\varepsilon^2}{12\pi^2}\left[\int f_o dx_1dy_1dy_2 + \int f_e dx_1dx_2dy_1\right]\,.$$

Since μ, is real, the imaginary part has to vanish. So,

$$\mu = \frac{\varepsilon^2}{12\pi^2}\int f_e dx_1dy_1dy_2 - \frac{\varepsilon^2}{12\pi^2}\int f_o dx_1dx_2dy_1\,. \tag{5.2}$$

We then interpret y_1, y_2 in terms of ε and θ as in (4.1). Both $f_e dy_1dy_2$ and $f_o dy_1$ give even-order terms in ε. Therefore, there is no odd-order terms in ε appearing in the asymptotic expansion of μ.

Once we know μ has an asymptotic expansion in ε^2, it is quite natural to ask where does it start? Is it a Taylor series in ε^2 similar to the H. Weyl's volume formula or do we have some singular terms? This answer was suggested by the computation in [B-E 1] about the Reinhardt domain $(log|z_1|)^2 + (log|z_1|)^2 \leqq r^2$, which has the invariant $\mu = \frac{3}{8\pi r^2}$. We reach the following lemma.

Lemma 5.3. *The asymptotic expansion of μ in ε^2 starts from $\varepsilon^{-2}, \varepsilon^0, \varepsilon^2, \ldots$, and so forth.*

Proof. Collect each leading order term of the integrands in (5.1). Then

$$\text{the integrands} = i\left(\frac{-9\Box}{8g_{22}y_2\varepsilon^4}\right) + \text{h.o.t.},$$

and the orders of μ in ε start from $-5, -4, -3, \ldots$, where the leading term comes from the second part of (5.2),

$$-\frac{\varepsilon^2}{12\pi^2}\int \frac{-9\Box}{8g_{22}y_2\varepsilon^4}dx_1dx_2dy_1 = \frac{3}{32\pi^2\varepsilon^2}\int \frac{\Box}{g_{22}y_2}dx_1dx_2dy_1\,, \tag{5.3}$$

which has order -2. This completes the proof.

In order to obtain some further properties of μ, we replace y_1 and y_2, by ε and θ as in (3.8).

$$dx_1dx_2dy_1 = \frac{-\varepsilon}{\sqrt{2}\Box} g_{22}^{\frac{1}{2}} \sin\theta d\theta dA\,, \tag{5.4}$$

where $dA = \Box^{\frac{1}{2}} dx_1dx_2$ is the volume element of the Riemannian manifold X. The second term of μ in (5.2) is then

$$\frac{-\varepsilon^2}{12\pi^2} \int_{\{\rho=\varepsilon^2\}} f_o dx_1dx_2dy_1 = \frac{\varepsilon^3}{12\sqrt{2}\pi^2} \int_X \Box^{-1} g_{22}^{\frac{1}{2}} \int_\theta f_o \sin\theta\, d\theta dA\,,$$

the order of f_o is $-5, -3, -1, \ldots$

$$\sim \sum_{l=-1}^{\infty} \varepsilon^{2l} \int_X F_l(x_1, x_2) dA\,,$$

for certain smooth functions F_l of the metric g_{ij} and its derivatives $g_{ij}^{(k)}$. Similarly,

$$dx_1dy_1dy_2 \sim \sum_{l=1}^{\infty} \varepsilon^{2l} \eta_l(x_1, x_2, \theta) d\theta dA\,.$$

Substitute this into the first part of (5.2), and also notice that order $(f_e) \geqq -4$:

$$\frac{\varepsilon^2}{12\pi^2} \int_{\{\rho=\varepsilon^2\}} f_e dx_1dy_1dy_2 \sim \frac{\varepsilon^2}{12\pi^2} \sum_{l=1}^{\infty} \varepsilon^{2l} \int_X \int_\theta f_e(x_1, x_2, \varepsilon, \theta) \eta_l(x_1, x_2, \theta) d\theta dA$$

$$\sim \frac{1}{12\pi^2} \sum_{l=0}^{\infty} \varepsilon^{2l} \int_X F_l(x_1, x_2) dA\,.$$

We include this as part of the following proposition.

Proposition 5.4.

$$\mu \sim \sum_{l=-1}^{\infty} \varepsilon^{2l} \int_X F_l(g_{ij}^{(k)}) dA$$

where $\lambda^{2l+2} F_l(\lambda^2 g_{ij}^{(k)}) = F_l(g_{ij}^{(k)})$ for any nonzero real number λ.

Proof. Let $G_{ij} = \lambda^2 g_{ij}$ be a new metric on X. Then

$$\partial M_\varepsilon = \{\rho = \varepsilon^2\}$$

$$= \{\varepsilon^2 = 2g_{11}y_1^2 + 4g_{12}y_1y_2 + 2g_{22}y_2^2 + \varphi_1 y_1^4 + \varphi_2 y_1^3 y_2 + \ldots\} \tag{a}$$

$$= \{(\lambda\varepsilon)^2 = 2G_{11}y_1^2 + 4G_{12}y_1y_2 + 2G_{22}y_2^2 + \varphi_1^G y_1^4 + \varphi_2^G y_1^3 y_2 + \ldots\}\,. \tag{b}$$

Let dA_g and dA_G denote the surface integrals of the metrics g_{ij} and G_{ij}, respectively. By (a)

$$\mu \sim \sum_{l=-1}^{\infty} \varepsilon^{2l} \int_X F_l(g_{ij}^{(k)}) dA_g\,.$$

But, from this point of view of (b),

$$\mu \sim \sum_{l=-1}^{\infty} (\lambda\varepsilon)^{2l} \int_X \lambda^2 F_l(\lambda^2 g_{ij}^{(k)}) dA_g \sim \sum_{l=-1}^{\infty} \varepsilon^{2l} \int_X \lambda^{2l+2} F_l(\lambda^2 g_{ij}^{(k)}) dA_g\,.$$

Comparing these two μ, one proves

$$\lambda^{2l+2}F_l(\lambda^2 g_{ij}^{(k)}) = F_l(g_{ij}^{(k)}), \quad l \geqq -1$$

This proposition implies the next corollary immediately.

Corollary 5.5.

$$F_{-1}(\lambda^2 g_{ij}^{(k)}) = F_{-1}(g_{ij}^{(k)})$$

$$F_0(\lambda^2 g_{ij}^{(k)}) = \lambda^{-2}F_0(g_{ij}^{(k)}), \quad 0 \neq \lambda \in \mathbb{R} .$$

We have proved that the leading term of μ is $\varepsilon^{-2}\int_X F_{-1}(g_{ij}^{(k)})dA$, and any rescaling of this metric (g_{ij}) will not change $F_{-1}(g_{ij}^{(k)})$. From the geometric point of view, it seems quite possible that this function F_{-1} is actually a constant. We will prove this fact here. Notice that each $F_l(g_{ij}^{(k)})$ is a geometric integrand which can be evaluated at the origin, and all of the calculations in previous sections work for any coordinate system. We now choose a specific one, the geodesic normal coordinate.

For the leading order term, it is sufficient to choose the first approximation of ρ, i.e., let

$$\rho = 2y_1^2 + 2y_2^2 + \text{ h.o.t. in } x \text{ and } y$$

Inserting (5.4) into (5.3), then replacing y_2 by the first approximation $\frac{\varepsilon\cos\theta}{\sqrt{2}}$, we obtain

$$\text{(5.5)} \qquad \text{the leading term} = \frac{3}{32\pi^2\varepsilon^2}\int_X\int_\theta \frac{\varepsilon}{\sqrt{2}y_2}\cos\theta d\theta dA = \frac{3}{16\pi\varepsilon^2}\int_X dA .$$

As for the second order term, we have $F_0(\lambda^2 g_{ij}^{(k)}) = \lambda^{-2}F_0(g_{ij}^{(k)})$ which suggests that the scalar curvature might be the best candidate for F_0. Let

$$[-4i(iaR - b\overline{W} + bW) + R^2 + 12c\bar{c}] \equiv [I] .$$

We go back to (5.1) again, and examine the first part of μ, which is

$$\frac{\varepsilon^2}{12\pi^2}\int \frac{\Delta}{\rho_{\bar{w}}}[I]dx_1dy_1dy_2 ,$$

where

(a) $\varepsilon^2\Delta[I]$ is real, and has orders starting from $-2, 0, 2, 4, \ldots$
(b) dy_1dy_2 is a real two-form with even orders no less than 2
(c) $\frac{1}{\rho_{\bar{w}}}$ has orders $-1, 0, 1, 2, \ldots$

So, the only chance we get ε^0 terms is by taking

(a) order of $\varepsilon^2\Delta[I] = -2$, i.e., taking $\varepsilon^2\Delta[I] = \frac{9\square}{4\varepsilon^2}$
(b) order of $dy_1dy_2 = 2$, i.e., taking first approximation of y_1 and y_2
(c) order of $\frac{1}{\rho_{\bar{w}}} = 0$.

We need to make some explanation of (c). Recall the expression of ρ in (3.7),

$$\rho_{\bar w} = 2iy_2 + \frac{i}{3}kx_1^2y_2 - \frac{i}{3}kx_1x_2y_1 - \frac{1}{3}kx_1y_1y_2 + \frac{1}{3}kx_2y_1^2 + \frac{1}{2}\alpha_2ky_1^3 + 2i\alpha_3ky_1^2y_2 + \frac{3i}{2}\alpha_4ky_1y_2^2 + 2\alpha_5ky_2^3 + \text{higher order terms in } x \text{ and } y\,. \tag{5.6}$$

Therefore

$$\frac{1}{\rho_{\bar w}} = \frac{1}{2iy_2}[1 + \text{ h.o.t. in } x \text{ and } y]$$

where the coefficients of those higher order terms are polynomials of the scalar curvature k.

(5.6) together with (a) (b) and (c), shows that those terms which can be left after we integrate out the angular θ term are those curvature terms. We conclude that the ε^0 term coming from this part is $c\int_X k(x)dA$ – some constant times the integration of the scalar curvature over X.

In the second part of (5.2), since dy_2 always has order 1, there are two possible cases to get an ε^0 term.

Case (I): order of $\varepsilon^2\Delta[I] = -2$, i.e., taking $\varepsilon^2\Delta[I] = \frac{9\Box}{4\varepsilon^2}$, and order of $\frac{1}{\rho_{\bar w}} = 1$,

Case (II): order of $\varepsilon^2\Delta[I] = 0$, and order of $\frac{1}{\rho_{\bar w}} = -1$, i.e., $\frac{1}{\rho\bar w} = \frac{1}{2iy_2}$.

We left some crucial points to be checked in either case. In Case (I), it is the $\frac{1}{\rho_{\bar w}}$ term. By (5.5), when restricted to the origin of X, the order-one term of $\frac{1}{\rho\bar w}$ is

$$\frac{-1}{2iy_2}\left[\frac{1}{4}\alpha_2k\frac{y_1^3}{y_2} + \alpha_2ky_1^2 + \frac{3}{4}\alpha_4ky_1y_2 + \alpha_5ky_2^2\right]. \tag{$*$}$$

This tells us again

$$\int \frac{9\Box}{4\varepsilon^2}(*)dx_1dx_2dy_1 = (\text{const})\int_X k(x)dA\,.$$

Case (II) is a little bit complicated. We have to check carefully what happens when the order of $\varepsilon^2\Delta[I]$ is zero. We divide this into two subcases.

(1) order of $[I] = -4$, order of $\Delta = 2$, then $[I] = \frac{9}{4\varepsilon^4}$ and Δ is a polynomial of k,

(2) order of $\Delta = 0$, i.e., $\Delta = \Box$, order of $[I] = -2$, we check functions in $[I]$ for which the curvature k can't appear in the denominator; it is a polynomial of k. Both of these two subcases check the ε^0 term is of the form

$$(\text{constant})\int_X k(x)dA\,.$$

We conclude that:

$$\mu \sim \frac{3}{16\pi\varepsilon^2} \int_X dA + (\text{constant}) \int_X k(x) dA + \text{ h.o.t.} \tag{5.7}$$

6 The constant term

We will devote this section to the calculation of the invariant μ for a one-parameter family of compact, homogeneous CR manifolds which are the boundaries of Grauert tubes. We compute these μ through two different approaches, one is via the formula we got in (5.7) to find the invariant of a Grauert tube, another way is by examining the standard CR structure. Both will give us the same μ. We could therefore determine the coefficient of the second order term and double check the constant $\frac{3}{16\pi}$ of the ε^{-2} term obtained from the previous section. Let Q denote the standard hyperquadric in $\mathbb{C}^3$, defined by the equation,

$$Q = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon z_1^2 + z_2^2 + z_3^2 = 1\},$$

S_r be the 5-sphere of radius $\sqrt{r}$ in $\mathbb{C}^3$,

$$S_r = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon |z_1|^2 + |z_2|^2 + |z_3|^2 = r\},$$

∂M_r be the intersection of Q and S_r. In terms of real coordinates $x_j + \sqrt{-1}\, y_j = z_j$, we could veiw ∂M_r as an embedded submanifold of $\mathbb{R}^6$ defined by the equations

$$\partial M_r : \begin{cases} x_1^2 + x_2^2 + x_3^2 = \frac{r+1}{2}, \\ y_1^2 + y_2^2 + y_3^2 = \frac{r-1}{2}, \\ x_1 y_1 + x_2 y_2 + x_3 y_3 = 0. \end{cases} \tag{6.1}$$

This shows that r has to be greater than or equal to one. ∂M_r is a three dimensional hypersurface when $r > 1$, whereas it degenerates to a totally real unit sphere S^2 in $\mathbb{R}^3$ as r goes to one, $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$.

The first attempt is to find a Monge–Ampère solution u on $Q - S^2$ with the desired properties described in Sect. 3, which was done by G. Patrizio and P-M Wong in [P-W]. Since Q can be sliced by the level surfaces ∂M_r, which are the intersection of Q and S_r, we shall take u as a function of $r = z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 = |z|^2$, with $z_1, z_2, z_3 \in Q$, then

$$\bar{\partial} u = u' \bar{\partial} |z|^2$$

$$\partial\bar{\partial} u = u'' \partial |z|^2 \wedge \bar{\partial} |z|^2 + u' \partial\bar{\partial} |z|^2 .$$

u is a Monge–Ampère solution if and only if $(\partial\bar{\partial} u)^2 = 0$. A solution is

$$u(r) = cosh^{-1} r . \tag{6.2}$$

Thus u is a Monge–Ampère solution on $Q - S^2$, is positive for $r > 1$, equals zero if and only if r is 1, and is a plurisubharmonic function. Notice that cu, for any positive constant c, preserves all of the above properties. So the solution is not uniquely determined. On coordinate neighborhood $U = \{(x_1, x_2, x_3) \in S^2 \,|x_3 > 0\}$, we consider the projection

$$\varphi : U \to \mathbb{R}^2$$
$$(x_1, x_2, x_3) \to (x_1, x_2)$$

which gives a local coordinate system, the metric inherited from the Euclidean space is

$$g_{11} = \frac{1 - x_2^2}{1 - x_1^2 - x_2^2}, \quad g_{12} = \frac{x_1 x_2}{1 - x_1^2 - x_2^2}, \quad g_{22} = \frac{1 - x_1^2}{1 - x_1^2 - x_2^2}.$$

To assure the uniqueness, we need to find c such that the Kähler metric $\left(\frac{\partial^2 u^2}{\partial z_i \partial \bar{z}_j}\right)$, when pulled back to the center, agrees with (g_{ij}) where z_j is the complexification of x_j for $j = 1, 2$. Let $s = r - 1$. We could also write the Taylor expansion of u^2 at $s = 0$.

$$u^2(z) = (cosh^{-1}(s+1))^2 \sim 2s - \tfrac{1}{3}s^2 + \tfrac{4}{45}s^3 + O(s^4). \tag{6.3}$$

Taking derivatives of the expansion (6.3) implies

$$\left.\frac{\partial^2 u^2}{\partial z_i \bar{\partial} z_j}\right|_{S^2} = \left.\frac{\partial^2}{\partial z_i \bar{\partial} z_j}\left(2s - \frac{1}{3}s^2\right)\right|_{S^2}.$$

On ∂M_r:

$$s = r - 1 = |z_1|^2 + |z_2|^2 + \sqrt{(1 - z_1^2 - z_2^2)(1 - \bar{z}_1^2 - \bar{z}_2^2)} - 1,$$

then

$$\begin{cases} \left.\frac{\partial^2 s}{\partial z_1 \partial \bar{z}_1}\right|_{S^2} = \frac{1 - x_2^2}{1 - x_1^2 - x_2^2} = g_{11}, \\ \left.\frac{\partial^2 s}{\partial z_1 \partial \bar{z}_2}\right|_{S^2} = \frac{x_1 x_2}{1 - x_1^2 - x_2^2} = g_{12}, \\ \left.\frac{\partial^2 s}{\partial z_2 \partial \bar{z}_2}\right|_{S^2} = \frac{1 - x_1^2}{1 - x_1^2 - x_2^2} = g_{22}, \\ \left.\frac{\partial^2 s^2}{\partial z_i \partial \bar{z}_j}\right|_{S^2} = \left.\left(\frac{\partial s}{\partial z_i}\,\frac{\partial s}{\partial \bar{z}_j}\right)\right|_{S^2} \\ \qquad = \left.\left\{\frac{\bar{z}_i |z_3|^2 - z_i z_3^2}{2|z_3|^2}\,\frac{z_j |z_3|^2 - \bar{z}_j \bar{z}_3^2}{2|z_3|^2}\right\}\right|_{S^2} = 0. \end{cases}$$

The last equation holds because $z_1 = \bar{z}_1$, $z_2 = \bar{z}_2$, $z_3 = \bar{z}_3$ when restricted to S^2.

This checks

$$\left.\left(\frac{\partial^2 u^2}{\partial z_i \bar{\partial} z_j}\right)\right|_{S^2} = 2(g_{ij}).$$

We rescale u by taking $c = \frac{1}{\sqrt{2}}$. Then the unique Monge–Ampère solution defined by Guillemin–Stenzel for this tubular domain is

$$\rho(z) = \tfrac{1}{2}u^2(z) = \tfrac{1}{2}(cosh^{-1}(z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3))^2 ,$$

and the three-dimensional hypersurface ∂M_r is the boundary of a Grauert tube centered at S^2 of radius ε,

$$\varepsilon = \frac{cosh^{-1}(r)}{\sqrt{2}} .$$

It is therefore possible to compute the invariant μ of ∂M_r. Notice that the center is the unit 2-sphere which has constant scalar curvature 2 and surface area 4π. The expansion (6.3) also shows $\varepsilon^{2n} \sim O(s^{2n})$ on the level surface $\partial M_\varepsilon = \{u^2 = \varepsilon^2\}$. By (5.6),

$$\mu = \frac{3}{16\pi\varepsilon^2}\int_{S^2} dA + c\int_{S^2} k(x)dA + O(\varepsilon^2) = \tfrac{3}{4}s^{-1} + \tfrac{1}{8} + 8c\pi + O(s) . \tag{6.4}$$

On the other hand, since ∂M_r is defined by the equation (6.1), we could also view ∂M_r as the unit tangent bundle of a unit sphere. It is diffeomorphic to $SO(3)$, the special orthogonal group, and the diffeomorphism is given by

$$(x_1, y_1, x_2, y_2, x_3, y_3) \to g = \begin{pmatrix} \sqrt{\frac{2}{r+1}}x_1 & \sqrt{\frac{2}{r-1}}y_1 & \frac{2}{\sqrt{r^2-1}}A_1 \\ \sqrt{\frac{2}{r+1}}x_2 & \sqrt{\frac{2}{r-1}}y_2 & \frac{2}{\sqrt{r^2-1}}A_2 \\ \sqrt{\frac{2}{r+1}}x_3 & \sqrt{\frac{2}{r-1}}y_3 & \frac{2}{\sqrt{r^2-1}}A_3 \end{pmatrix} \in SO(3)$$

where

$$(A_1, A_2, A_3) = ((x_1, x_2, x_3) \times (y_1, y_2, y_3)) .$$

To simplify the notation, we use the abbreviation

$$a \equiv \sqrt{\frac{2}{r+1}}, \quad b \equiv \sqrt{\frac{2}{r-1}}, \quad c \equiv \frac{2}{\sqrt{r^2-1}} .$$

And the Cartan connection form on this group is

$$\Omega = g^{-1}dg = \begin{pmatrix} 0 & -\alpha & \gamma \\ \alpha & 0 & -\beta \\ -\gamma & \beta & 0 \end{pmatrix} .$$

Where

$$\alpha = ab(y_1dx_1 + y_2dx_2 + y_3dx_3) ,$$
$$\beta = bc(A_1dy_1 + A_2dy_2 + A_3dy_3) ,$$
$$\gamma = -ac(A_1dx_1 + A_2dx_2 + A_3dx_3) ,$$

are three independent left-invariant one-forms on $SO(3)$. The fundamental property of this Cartan connection is that

$$d\Omega = -\Omega \wedge \Omega$$

which shows

$$d\alpha = -\beta \wedge \gamma, \quad d\beta = -\gamma \wedge \alpha, \quad d\gamma = -\alpha \wedge \beta .$$

Choosing

$$\theta = -\alpha, \quad \theta_1 = \tfrac{1}{\sqrt{2}}(\beta + i\gamma),$$

then

$$d\theta = i\theta_1 \wedge \theta_{\bar{1}}, \quad d\theta_1 = i\theta \wedge \theta_1 .$$

This choice of $\{\theta, \theta_1, \theta_{\bar{1}}\}$ provides a CR structure on ∂M_r, but it is not necessarily the embedded structure, the one inherited from the complex structure of $\mathbb{C}^3$. However, since all $SO(3)$ invariant CR structures are obtained from the perturbation of this $\{\theta_1, \theta_{\bar{1}}\}$, there exists $t \in (-1, 1)$, such that

$$\theta_1^t = (1 - t^2)^{\frac{-1}{2}}(\theta_1 + t\theta_{\bar{1}})$$

gives the embedded CR structure. In other words, in terms of the local coordinates z_1, z_2, z_3, none of the $d\bar{z}_1, d\bar{z}_2, d\bar{z}_3$ terms is contained in the one form θ_1^t, when written in terms of dz_j and $d\bar{z}_j$, $j = 1, 2, 3$. We collect those $d\bar{z}_j$ terms in $\theta_1 + t\theta_{\bar{1}}$. They are

$$\left[\frac{-i}{2\sqrt{2}}(ac - bc) + t\frac{i}{2\sqrt{2}}(bc + ac)\right](A_1 d\bar{z}_1 + A_2 d\bar{z}_2 + A_3 d\bar{z}_3) .$$

So, we choose

$$t = \frac{a - b}{a + b} = \sqrt{r^2 - 1} - r, \quad t \in (-1, 0) . \tag{6.5}$$

This set of $\{\theta, \theta_1^t, \theta_{\bar{1}}^t\}$ is the embedded CR structure of ∂M_r, with

$$d\theta = i\theta_1^t \wedge \theta_{\bar{1}}^t ,$$
$$d\theta_1^t = \theta_1^t \wedge (-ih)\theta + \theta \wedge (-ik)\theta_{\bar{1}}^t ,$$
$$\text{where } h = \frac{1 + t^2}{1 - t^2}, \quad k = \frac{2t}{1 - t^2} .$$

Therefore,

$$\omega = -ih\theta, \quad \tau = -ik\theta_{\bar{1}}^t, \quad R = h .$$

The local form defining μ is

$$\widetilde{T}C_2(\Pi) = \frac{-1}{16\pi^2}(1 - 3k^2)\theta \wedge d\theta ,$$

and the invariant

$$\mu = \int_{\partial M_r} \widetilde{T}C_2(\Pi) = \frac{-1}{16\pi^2}(1 - 3k^2) \int_{\partial M_r} \theta \wedge d\theta . \tag{6.6}$$

We use the substitutions $s = r - 1$, and write k^2 as the asymptotic expansion

$$k^2 \sim \tfrac{1}{2}s^{-1} - \tfrac{1}{4} + O(s) .$$

It is left to calculate $\int_{\partial M_r} \theta \wedge d\theta$, where

$$(6.7) \qquad \theta \wedge d\theta = a^2b^2(y_1dx_1dy_2dx_2 + y_1dx_1dy_3dx_3 + y_2dx_2dy_1dx_1 + y_2dx_2dy_3dx_3 + y_3dx_3dy_1dx_1 + y_3dx_3dy_2dx_2)$$

To compute the surface area, we introduce two independent spherical coordinates on ∂M_r. Let

$$(6.8) \qquad \begin{array}{ll} x_1 = a^{-1}sin(\varphi - \pi)\, cos\, \theta, & y_1 = b^{-1}sin\, \eta\, cos\, \zeta\,, \\ x_2 = a^{-1}sin(\varphi - \pi)sin\, \theta, & y_2 = b^{-1}sin\, \eta\, sin\, \zeta\,, \\ x_3 = a^{-1}\, cos\,(\varphi - \pi), & y_3 = b^{-1}\, cos\, \eta, \\ 0 \leqq \varphi, \eta \leqq \pi, & 0 \leqq \theta, \zeta \leqq 2\pi\,. \end{array}$$

The condition $x_1y_1 + x_2y_2 + x_3y_3 = 0$ makes it possible to interprate η in terms of φ, θ, ζ, with

$$\eta = tan^{-1}(-cot\varphi sec(\theta - \zeta))\,.$$

Integrating the first term of (6.7) over ∂M_r, with help from Maple, we have

$$-\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi}\left[\frac{sin^3(2\varphi)\, sin\, 2(\theta-\zeta)\, sin\, 2\zeta}{32(\, sin^2\varphi\, cos^2(\theta-\zeta) + \, cos^2\varphi)^2} - \frac{cos^3\varphi\, sin\, \varphi\, cos\, ^2\zeta}{sin^2\varphi\, cos^2(\theta-\zeta) + \, cos^2\varphi}\right] d\varphi\, d\theta d\zeta = \frac{4}{3}\pi^2\,.$$

By symmetry, integrating out each one of the rest of the terms in (6.7) will have the same value $\frac{4}{3}\pi^2$. So, $\int_{\partial M_r} \theta \wedge d\theta = 8\pi^2$, $\forall r > 0$. Therefore, by (6.6)

$$(6.9) \quad \mu = \tfrac{-1}{2}(1 - 3k^2) = \tfrac{-1}{2}\left[1 - 3(\tfrac{1}{2}s^{-1} - \tfrac{1}{4} + O(s))\right] = \tfrac{3}{4}s^{-1} - \tfrac{7}{8} + O(s)\,.$$

Equating this μ with that in (6.4) proves

$$c = \frac{-1}{8\pi}\,.$$

We have thus arrived at the decisive theorem of this paper.

Theorem 6.1. *Let X be a two-dimensional compact, real-analytic oriented manifold with a real-analytic metric (g_{ij}), and let $(M, X, \sqrt{\rho})$ be the Monge–Ampère model of X. Then*

(1) *the invariant μ is well-defined on the level surfaces*

$$\partial M_\varepsilon = \{\rho = \varepsilon^2\}\,.$$

(2) *μ of ∂M_ε has an asymptotic expansion in ε^2, $0 < \varepsilon \ll 1$.*

$$\mu_\varepsilon \sim \frac{3}{16\pi\varepsilon^2}\int_X dA - \frac{1}{8\pi}\int_X k(x)dA + \sum_{l=1}^{\infty}\varepsilon^{2l}\int_X F_l(g_{ij}^{(k)})dA$$

where $k(x)$ is the scalar curvature, and
$\lambda^{2l+2}F_l(\lambda^2 g_{ij}^{(k)}) = F_l(g_{ij}^{(k)})$, for any nonzero real number λ.

(3) *There is no biholomorphic map from* M_{ε_1} *to* M_{ε_2} *if* $\varepsilon_1 \neq \varepsilon_2$, $0 < \varepsilon_1, \varepsilon_2 \ll 1$.

Proof. (3) Since μ is a global CR invariant, $\partial M_{\varepsilon_1}$ and $\partial M_{\varepsilon_2}$ are CR equivalent only if $\mu_{\varepsilon_1} = \mu_{\varepsilon_2}$ which, by (2), can't be true. A direct application of Fefferman's extension theorem [Fe] (any biholomorphic map between two compact, strictly pseudoconvex domains can be extended smoothly up to their boundaries) proves (3).

7 Grauert tubes with centers of constant sectional curvature

We have showed that there is no biholomorphic map between two Grauert tubes M_{ε_1} and M_{ε_2}, for $\varepsilon_1, \varepsilon_2$ small enough, although they clearly are homotopically equivalent. We would like to discuss more about the geometric properties of the Grauert tubes, and see to what extend the inequivalence holds. The result is not clear for general Riemannian manifolds, but, we do have a definite answer for those Grauert tubes constructed above centers of constant curvature.

The first case we will consider is when the center X is exactly the two-sphere. The discussion in Sect. 6 shows

$$M_r : \begin{cases} z_1^2 + z_2^2 + z_3^2 = 1 \\ |z_1|^2 + |z_2|^2 + |z_3|^2 < r\,, \quad r > 1 \end{cases} \tag{7.1}$$

are Monge–Ampère models which have the two-sphere $x_1^2 + x_2^2 + x_3^2 = 1$ as their common center. The invariant μ_r of level set ∂M_r is, by (6.6),

$$\mu_r = \frac{-1}{2} + \frac{6t^2}{(1-t^2)^2}\,, \quad r > 1\,. \tag{7.2}$$

(7.2), together with the fact that $t = \sqrt{r^2 - 1} - r$ and r are in one-one correspondence, proves that μ_r is a strictly decreasing function with

$$\lim_{r \to 1} \mu_r = \infty, \quad \lim_{r \to \infty} \mu_r = \frac{-1}{2}\,.$$

In other words, $\mu_{r_1} \neq \mu_{r_2}$ whenever $r_1 \neq r_2$. Thus any two Grauert tubes associated to the unit sphere with different radius can't be biholomorphically equivalent.

Among all CR structures, the spherical ones – those that are locally CR equivalent to the three-sphere in $\mathbb{C}^2$ – are especially interesting geometrically. We would like to see whether there is any spherical Grauert tube or not. In [B-E 1], the authors showed that the critical points of μ, viewed as functional on the space of CR structures, are exactly the spherical structures. Take derivative of (6.9),

$$\frac{d\mu_t}{dt} = \frac{12t(1+t^2)}{(1-t^2)^2}\,, \tag{7.3}$$

which is zero only at $t = 0$, i.e., μ_r can only be stationary if $r = \infty$. In other words, there can't be any spherical structure for $r < \infty$. As r goes to 0, we make a holomorphic change of coordinates, $Z_j = \frac{z_j}{r}$, $j = 1, 2, 3$. Then

$$\partial M_\infty : \begin{cases} Z_1^2 + Z_2^2 + Z_3^2 = 0 \\ |Z_1|^2 + |Z_2|^2 + |Z_3|^2 = 1 . \end{cases} \tag{7.4}$$

We claim ∂M_∞ is locally biholomorphic to the unit sphere

$$S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} ,$$

by defining the map

$$\begin{aligned} &\varphi : S^3 \to \partial M_\infty \\ &(z,w) \to \left(\frac{z^2 - w^2}{\sqrt{2}}, \frac{i(z^2 + w^2)}{\sqrt{2}}, \frac{2zw}{\sqrt{2}} \right) = (Z_1, Z_2, Z_3) . \end{aligned} \tag{7.5}$$

φ is clearly well-defined, holomorphic, onto. Furthermore, S^3/G is CR diffeomorphic to ∂M_∞ where $G = \{I, -I\}$, i.e., ∂M_∞ is locally biholomorphic to S^3, and $\mu(\partial M_\infty) = \frac{1}{2}\mu(S^3) = \frac{-1}{2}$.

More generally, this is also true if the center X is a compact Riemannian manifold of positive constant curvature k. Then X is isometric to $(S^2/\Gamma, \frac{1}{m}g)$, where Γ are discrete subgroups of the group $O(3)$ of isometries of S^2, which act freely and properly discontinuously on S^2, and g is the inherited metric of S^2 from $\mathbb{R}^3$. Actually, there are not many of them: S^2 and P^2R are the only two complete, two-dimensional manifolds of constant positive curvature.

Since θ and $\theta_{\bar{1}}$ are $O(3)$ invariant (see Sect. 6), the invariant μ_Γ is well-defined on the boundaries of these new Grauert tubes (it also follows from the Remark after Theorem 4.1). They are

$$\mu_{r,\Gamma} = \int_{\partial M_r/\Gamma} \tilde{T}C_2(\Pi) = \frac{1}{|\Gamma|}\mu_r . \tag{7.6}$$

Thus, $\mu_{r,\Gamma}$ preserves the same decreasing property of μ_r on the quotient space $\partial M_r/\Gamma$.

We next turn to the flat case. We will consider the spaces, for $r > 0$,

$$\partial M_r : \begin{cases} y_1^2 + y_2^2 = r^2 , \\ (x_1, x_2) \in \mathbb{R}^2 . \end{cases}$$

The pseudo-hermitian structure

$$\theta = \frac{-1}{2r}[y_1\,dz_1 + y_2\,dz_2], \quad \theta_{\bar{1}} = \frac{-i}{2r}[y_2\,dz_1 - y_1\,dz_2]$$

is preserved by the isometry group of $\mathbb{R}^2$, as are $\omega = -i\theta$, $\tau = i\theta_{\bar{1}}$ and $R = 1$. So, μ_r is well-defined on $\partial M_r/\Gamma$, for any discrete subgroup Γ of the isometry group of $\mathbb{R}^2$, which operates freely and properly discontinuously on $\mathbb{R}^2$.

$$\mu_r = \frac{3}{16\pi^2} \int_{\partial M_r/\Gamma} \theta \wedge d\theta = \tfrac{3}{16\pi^2 r^2} c(\Gamma)$$

is a strictly decreasing function of r,.

$$\lim_{r \to 0} \mu_r = \infty, \quad \lim_{r \to \infty} \mu_r = 0 .$$

where $c(\Gamma)$ is a positive constant depending on Γ only.

$$\frac{d\mu_r}{dr} = \frac{-9}{16\pi^2 r^2} c(\Gamma) < 0 .$$

Therefore, there is not any spherical structure. As r approaches ∞,

$$M_\infty = \lim_{r \to \infty} M_r$$

is a Reinhardt domain, whereas it is not so clear what ∂M_r looks like as r goes to ∞, comparing with (7.4), (7.5) above.

We sum these results up as follows:

Theorem 7.1. *Let X be a two-dimensional, compact Riemannian manifold of constant curvature $k \geqq 0$. Then X can be complexified to obtain an unbounded Monge–Ampère model (M,X,φ). The Grauert tubes $\{\varphi < v_1\}$ and $\{\varphi < v_2\}$ enclosed by different Monge–Ampère levels can't be biholomorphically equivalent. Furthermore, on the level surface $\{\varphi = v\}$, $v > 0$, one has*

(1) *For the case $k > 0$: The pseudo-hermitian curvature R_v is always positive, decreasing from ∞ to 1 as the radius gets larger. The invariant μ_v decreases from ∞ to $\frac{-1}{2}$. There is no spherical CR structure on $\{\varphi = v\}$, $v < \infty$ whereas the CR structures are becoming spherical as v goes to 0.*

(2) *For the case $k = 0$: The curvarture R_v is a constant 1 for every $v > 0$. The invariant μ_v is a positive, decreasing function, and there is no spherical CR structure on any $\{\varphi = v\}$.*

In the sequel, the Monge–Ampère models whose centers possess constant negative curvature will be our chief objects. Quite naturally, the hyperbolic space H^2 which is given by $x_1^2 + x_2^2 - x_3^2 = -1$, $x_3 > 0$ with $ds^2 = dx_1^2 + dx_2^2 - dx_3^2$ is the first one to be thought about. Complexify it, then take the intersection with $|z_1|^2 + |z_2|^2 - |z_3|^2 = r$, we obtain

$$\text{(7.7)} \qquad \partial M_r : \begin{cases} z_1^2 + z_2^2 - z_3^2 = 1 \\ |z_1|^2 + |z_2|^2 - |z_3|^2 = r, \quad r \in (-1,1) . \end{cases}$$

In terms of real coordinates, ∂M_r could be viewed as an embedded submanifold of $\mathbb{R}^6$ defined by equations

$$(7.8) \qquad \partial M_r : \begin{cases} x_1^2 + x_2^2 - x_3^2 = \frac{r-1}{2}, \\ y_1^2 + y_2^2 - y_3^2 = \frac{r+1}{2}, \\ x_1 y_1 + x_2 y_2 - x_3 y_3 = 0, \end{cases}$$

which shows ∂M_r are the tangent sphere bundles of the hyperbolic space H^2. ∂M_r degenerates to the two-dimensional real hyperbolic space as r approaches -1. Since the goal is to find the Monge–Ampère solution u on M_r so that u is constant on each level set ∂M_r, we might therefore assume that u is a function of $r = |z_1|^2 + |z_2|^2 - |z_3|^2$. We reduce the Monge–Ampère condition to

$$2u'' + \frac{2r}{r^2 - 1} u' = 0 .$$

Then

$$u(r) = \int (r^2 - 1)^{\frac{-1}{2}} dr = cos^{-1} r + d ,$$

d is a certain constant, determined by the initial condition $u(center) = 0$. If we fix the angle branch as $[\pi, 2\pi]$, the initial condition $u^{-1}(-1) = 0$ will imply $d = -\pi$. For our convience, we make a change of variable and then take Taylor series of u^2 with respect to the new variable s. Let $s = r + 1 \in (0, 2)$. Then

$$(7.9) \qquad u^2(r) = [c\, cos^{-1}(s-1) - c\pi]^2 = 2c^2 s + \tfrac{1}{3} c^2 s^2 + \tfrac{4}{45} c^2 s^3 + O(s^4) .$$

So,

$$(7.10) \qquad \left.\frac{\partial^2 u^2}{\partial z_i \partial \bar{z}_j}\right|_{H^2 = \{s=0\}} = \left.\left(2c^2 \frac{\partial^2 s}{\partial z_i \partial \bar{z}_j} + \frac{1}{3} c^2 \frac{\partial^2 s^2}{\partial z_i \partial \bar{z}_j} \right)\right|_{s=0} .$$

We consider local coordinates obtained by projection as described in Sect. 6, the metric (g_{ij}) – induced from the quadratic form $dx_1^2 + dx_2^2 - dx_3^2$ and $\mathbb{R}^3$ – of H^2 is then

$$(7.11) \qquad g_{11} = \frac{1 + x_2^2}{1 + x_1^2 + x_2^2}, \quad g_{12} = \frac{-x_1 x_2}{1 + x_1^2 + x_2^2}, \quad g_{22} = \frac{1 + x_1^2}{1 + x_1^2 + x_2^2} .$$

On the other hand, since

$$s = r + 1 = |z_1|^2 + |z_2|^2 - \sqrt{(1 + z_1^2 + z_2^2)(1 + \bar{z}_1^2 + \bar{z}_2^2)} + 1 ,$$

then

$$\left.\frac{\partial^2 s}{\partial z_1 \partial \bar{z}_1}\right|_{H^2} = g_{11}, \quad \left.\frac{\partial^2 s}{\partial z_1 \partial \bar{z}_2}\right|_{H^2} = g_{12}, \quad \left.\frac{\partial^2 s}{\partial z_2 \partial \bar{z}_2}\right|_{H^2} = g_{22}, \quad \left.\frac{\partial^2 s^2}{\partial z_i \partial \bar{z}_j}\right|_{H^2} = 0 .$$

So, the Kähler metric is

$$\left.\frac{\partial^2 u^2}{\partial z_i \partial \bar{z}_j}\right|_{H^2} = 2c^2 \left.\frac{\partial^2 s}{\partial z_1 \partial \bar{z}_2}\right|_{H^2} = 2c^2 g_{ij} .$$

Now the condition of compatibility implies $c = \frac{1}{\sqrt{2}}$, and the Monge–Ampère solution for this hyperbolic model is

$$u(z_1, z_2, z_3) = \frac{1}{\sqrt{2}} cos^{-1}(|z_1|^2 + |z_2|^2 - |z_3|^2) - \frac{1}{\sqrt{2}}\pi \,, \tag{7.12}$$

which is preserved under the group action of $O_+(2,1)$, and can't be extended beyond $r \in (-1,1)$.

A close examination checks that all of the above calculation can be done in higher-dimensional spaces. Thus, they provide bounded Monge–Ampère models to any complete manifolds of negative constant curvature.

We could also view $\partial M_r/\Gamma$ as an one-parameter family of locally homogeneous CR manifolds, and compute the invariant μ_r on $\partial M_r/\Gamma$. Again, Γ, a subgroup of $O_+(2,1)$ acts freely and properly discontinuously on H^2. The map

$$(x_1, y_1, x_2, y_2, x_3, y_3) \to g = \begin{pmatrix} cA_1 & by_1 & ax_1 \\ cA_2 & by_2 & ax_2 \\ cA_3 & -by_3 & -ax_3 \end{pmatrix} \in SO(2,1) \,,$$

gives a diffeomorphism from M_r to $SO(2,1)$, where

$$(A_1, A_2, A_3) = ((x_1, x_2, x_3) \times (y_1, y_2, y_3)) \,,$$

and

$$a \equiv \sqrt{\frac{2}{1-r}} \,, \quad b \equiv \sqrt{\frac{2}{1+r}} \,, \quad c \equiv \frac{2}{\sqrt{1-r^2}} \,.$$

The Cartan connection form on this group is

$$\Omega = g^{-1}dg = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ \beta & \gamma & 0 \end{pmatrix} \,,$$

where

$$\alpha = bc(A_1 dy_1 + A_2 dy_2 + A_3 dy_3) \,,$$
$$\beta = ac(A_1 dx_1 + A_2 dx_2 + A_3 dx_3) \,,$$
$$\gamma = ab(y_1 dx_1 + y_2 dx_2 - y_3 dx_3) \,.$$

are three independent left-invariant one-forms on $SO(2,1)$. The fundamental property of Cartan connection implies

$$d\alpha = -\beta \wedge \gamma, \quad d\beta = \gamma \wedge \alpha, \quad d\gamma = \alpha \wedge \beta \,.$$

Taking

$$\theta = -\gamma, \quad \theta_1 = \frac{1}{\sqrt{2}}(\alpha - i\beta) \,,$$

then

$$d\theta = i\theta_1 \wedge \theta_{\bar{1}}, \quad d\theta_1 = i\theta \wedge \theta_{\bar{1}} \,.$$

Therefore, $\{\theta, \theta_1, \theta_{\bar{1}}\}$ provides an $O_+(2,1)-$invariant CR structure on ∂M_r, and the embedded CR structures could be obtained through a deformation $\{\theta, \theta_1^t, \theta_{\bar{1}}^t\}$ of this CR structure, with

$$\theta_1^t = (1-t^2)^{\frac{-1}{2}}(\theta_1 + t\theta_{\bar{1}}), \quad t \in (-1,1).$$

$$t = \begin{cases} \frac{1-\sqrt{1-r^2}}{r}, & r \in (-1,0), \\ 0, & r = 0, \\ \frac{1-\sqrt{1-r^2}}{r}, & r \in (0,1), \end{cases} \tag{7.13}$$

gives the embedded CR structure at the corresponding level ∂M_r.

The connection ω, torsion τ and curvature R are

$$\omega = \frac{2it}{1-t^2}\theta, \quad \tau = i\frac{1+t^2}{1-t^2}\theta_{\bar{1}}^t, \quad R = \frac{-2t}{1-t^2}.$$

The invariant μ_r of $\partial M_r/\Gamma$ is then

$$\mu_r = \int_{\partial M_r/\Gamma} \tilde{T}C_2(\Pi) = \frac{1}{4\pi^2}\left[1 + \frac{3t^2}{(1-t^2)^2}\right]\int_{\partial M_r/\Gamma} \theta \wedge d\theta \tag{7.14}$$

Similar computation as in (6.7) and (6.8) shows

$$\int_{\partial M_r/\Gamma} \theta \wedge d\theta = c(\Gamma) > 0$$

is a r-independent constant. A calculation gives directly:

$$\frac{d\mu_r}{dt} = \frac{12t(1+t^2)}{(1-t^2)^2}c(\Gamma),$$

which obtains zero at $t = 0$. So, there is at most one spherical structure at $r = 0$. On ∂M_0, $x_1^2 + x_2^2 + x_3^2 = \frac{-1}{2}$, therefore $x_3 \neq 0$. There are actually two symmetric, connected pieces in ∂M_0, one has $x_3 > 0$, the other one has $x_3 < 0$. We will consider the $x_3 > 0$ piece in the sequel. Since

$$\partial M_0 : \begin{cases} z_1^2 + z_2^2 - z_3^2 = -1 \\ |z_1|^2 + |z_2|^2 - |z_3|^2 = 0, \end{cases} \tag{7.15}$$

$|z_3|^2 = |z_1|^2 + |z_2|^2 \neq 0$. The map

$$f : \partial M_0 \to S^3 - \{S^3 \cap \mathbb{R}^2\}$$

$$(z_1, z_2, z_3) \to \left(\frac{z_1}{z_3}, \frac{z_2}{z_3}\right) = (Z_1, Z_2)$$

is locally biholomorphic. This shows ∂M_0, as well as its quotient space $\partial M_0/\Gamma$, is spherical.

Combining (7.13) and (7.14), we show that μ_r decreases from ∞ to a constant μ_0 as r goes from -1 to 0, then climbs up to ∞ at the same speed, that is to say

$$(7.16) \qquad \mu_s = \mu_{-s}, \quad s \in (0,1).$$

Since μ is a global CR invariant, (6.15) gives the necessary condition for M_{-r} to be biholomorphic to M_r. As for the sufficient condition, we need the help of another CR invariant, the CR invariant $\lambda(N)$ associated with every compact, orientable, strictly pseudoconvex $2n+1$ dimensional CR manifold N, defined by David Jerison and John Lee in [J-L].

Let u be any smooth real function on N, and let R be the Webster scalar curvature for a fixed one-form θ. The invariant $\lambda(N)$ is defined by

$$\lambda(N) = inf\{A_\theta(u) : B_\theta(u) = 1, u \in C^\infty(N)\}$$

where, when $n = 1$

$$A_\theta = \int_N (4|du|_\theta^2 + Ru^2)\theta \wedge d\theta$$

$$B_\theta = \int_N |u|^4 \theta \wedge d\theta .$$

When the CR structure has constant positive curvature R, $A_\theta(u) \geqq 0$, for every $u \in C^\infty(N)$; this implies that $\lambda(N) \geqq 0$. In the case $R \equiv -c < 0$, we can take a constant function u so that $B_\theta(u) = 1$, then the associated $A_\theta(u) = \int_N -cu^2\theta \wedge d\theta < 0$, therefore, $\lambda(N) < 0$.

The Webster scalar curvature R_{-r} is a negative constant on ∂M_{-r},

$$R_{-r} = \frac{r(1-\sqrt{1-r^2})}{\sqrt{1-r^2}-(1-r^2)},$$

whereas it is a positive constant R_r on ∂M_r,

$$R_r = -R_{-r} .$$

It is thus clear the ∂M_r and ∂M_{-r} have different invariant λ, therefore they can't be biholomorphically equivalent.

Finally, we summarize these results as a theorem, the negative curvature case of Theorem 7.1.

Theorem 7.2. *Let X be a two-dimensional, compact Riemannian manifold of constant curvature $k < 0$. Then the Monge–Ampère model (M,X,φ) is of bounded type with sup $\varphi = \frac{\pi}{\sqrt{-2k}}$. We have exactly one spherical level set $\{\varphi = v_s\}$, $v_s = \frac{\pi}{2\sqrt{-2k}}$ is where μ_v attains its minimum. The pseudo-hermitian curvature R_v is positive when $v < v_s$ and is negative when $v > v_s$. Furthermore, the tube $\{\varphi < v_1\}$ is not biholomorphic to $\{\varphi < v_2\}$, for $v_1 \neq v_2$.*

The calculation in this section has actually extended the construction of hyperbolic tubes by Lempert in [Lem]. Let us remark that in that paper, Lempert didn't require the compatibility of the metrics and proved that the function $\varphi_0(z) = 2tan^{-1}(tanh\, d_n(z))$ is a non-negative plurisubharmonic function on B^n/Γ, satisfies the Monge–Ampère equation on $B^n/\Gamma - \Delta^n/\Gamma$, equals 0 exactly at Δ^n/Γ, goes to $\frac{\pi}{2}$ as $z \to \partial B^n/\Gamma$, and φ^2 is a strictly plurisubharmonic function on B^n/Γ, where $B^n \in \mathbb{C}^n$ is the unit ball, Δ^n is a hyperbolic space considered as the unit ball in $\mathbb{R}^n$ endowed with the Caley–Klein metric, and $d_n(z)$ measures the Kobayashi distance of z to Δ^n. We add the compatability condition to his model $(B^n/\Gamma, \Delta^n/\Gamma, \varphi)$, then the uniqueness of Monge–Ampère model, together with Theorem 7.2, imples:

$$\varphi(z) = \frac{2}{\sqrt{-2k}} tan^{-1}(tanh\, d_n(z)),$$

$d_n(z)$ measures the Kobayashi distance of z to Δ^n.

Furthermore, Theorem 2.5 of [L-S] asserts that there is a biholomorphic map f which sends our previous model $(M_0/\Gamma, H^n/\Gamma, u)$ to Lempert's model with

$$u(z) = \varphi(f(z)).$$

On the level surface ∂M_r, the uniqueness implies

$$2\, tan^{-1}(tanh\, d_n(f(z))) = cos^{-1} r - \pi .$$

So, the Kobayashi distance from the image $f(\partial M_r)$ of ∂M_r to Δ^n is

$$d_n f(z) = tanh^{-1} \frac{1-r}{\sqrt{1-r^2}} = \frac{1}{2} log \left(\frac{\sqrt{1-r^2} + 1 - r}{\sqrt{1-r^2} - 1 + r} \right) .$$

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