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Smooth Plurisubharmonic Functions Without Subextension

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§1. Introduction

In this paper we will establish the following result on plurisubharmonic (psh) functions.

Theorem. *Let $\Omega \Subset \mathbb{C}^n$ be a smoothly bounded domain. There exists a smooth, psh function ψ on Ω such that for any domain $\tilde{\Omega}$ with $\tilde{\Omega} \cap \partial\Omega \neq \phi$, there is no function $\tilde{\psi}$ psh on $\tilde{\Omega}$ such that $\tilde{\psi} \leqq \psi$ on $\tilde{\Omega} \cap \Omega$.*

It is evident, in particular, that ψ cannot be extended to be psh in any larger domain, and thus we recover the known fact that Ω is a domain of existence for psh functions (see [1, 2]). However, since psh functions arise in complex analysis through their use in inequalities, the problem of subextension seems more appropriate than the problem of extension. Fornaess and Sibony [3] showed that there is a psh function on the ring domain $\{z \in \mathbb{C}^n: 1 < |z| < R\}$ which cannot be subextended to the ball $\{|z| < R\}$. The function given in the Theorem improves this example by showing the function can be taken to be smooth and by showing that the failure of subextension is actually a local phenomenon.

The construction in the Theorem is based on Lemma 1 which shows that Lelong number can be both created and "propagated" by certain kinds of decrease. For example, if $\psi(z, w)$ is psh in a ball containing (0, 0) and if

$$\psi(z, w) \leqq \log\left(\|w\|^2 + e^{-\frac{1}{|z|^{1/4}}}\right)$$

holds for $\operatorname{Re} z \geqq 0$, then $\psi(z, w)$ must have positive Lelong number on the variety $\{w=0\}$. The impossibility of subextension then arises from the Theorem of Siu which shows that the set where the Lelong number is $\geqq \varepsilon$ is a (global) variety.

Let us remark also that the Theorem gives a result on super-extension of (1, 1) currents. As was noted in [3], if ψ is the function given in Theorem 1, then the (1, 1)-form $S = d\,d^c\psi$ has the property:

if $\tilde{\Omega}$ is any domain with $\tilde{\Omega} \cap \partial\Omega \neq \phi$, then there is no positive, closed (1, 1)-current $\tilde{S}$ on $\tilde{\Omega}$ with $\tilde{S} \geqq S$ on $\Omega \cap \tilde{\Omega}$.

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§ 2. Propagation of Lelong Number

We will let $z=z_1\in\mathbb{C}$ and $w=(z_2, \ldots, z_n)$ denote the coordinates of $(z, w)\in\mathbb{C}^n$. By $\|w\|$, we mean the Euclidean norm, $\|w\|^2=|z_2|^2+\ldots+|z_n|^2$. Our basic tool is the following.

Lemma 1. *Let $\varphi(z, w)$ be psh, $\varphi<0$, on a ball*

$$B=\{(z, w): |z|^2+\|w\|^2<1\}.$$

Suppose there is an open cone $C_\gamma=\{(z, w): \|w\|<\gamma \operatorname{Re} z\}$ about the positive Re *z-axis such that*

$$\varphi(z, w)\leqq \tfrac{1}{2}\log(\|w\|^2+r(z)^2), \qquad (z, w)\in C_\gamma, \tag{1}$$

where $r(z)=r(|z|)$ is a monotone function such that $\dfrac{r(z)}{|z|^m}\to 0$ as $|z|\to 0$ for each $m>0$. Then there exist constants $A, \eta>0$ such that

$$\varphi(z, w)\leqq \eta \log\|w\|+A \qquad \textit{for all } |z|^2+\|w\|^2<\eta^2.$$

Proof. We first show the hypotheses imply there is an estimate of the same form as (1),

$$\varphi(z, w)\leqq \frac{\eta}{2}\log(\|w\|^2+\tilde{r}(z)^2) \tag{2}$$

which holds for all (z, w) in a smaller ball $|z|^2+\|w\|^2<\eta^2$. We consider two cases. First, suppose $\|w\|>|z|$. The point $(\|w\|, 0)$ lies in the cone C_γ. Consider the circle with center (z, w) and boundary passing through $(\|w\|, 0)$,

$$(\zeta, \tau)=(z, w)+(\|w\|-z, -w)\, e^{i\theta}, \qquad -\pi\leqq\theta\leqq\pi.$$

Then

$$\|\tau\|=\|w\|\,|1-e^{i\theta}|=2\,\|w\|\left|\sin\frac{\theta}{2}\right|\leqq\|w\|\,|\theta|$$

and

$$\begin{aligned}\operatorname{Re}\zeta &= \operatorname{Re}(z(1-e^{i\theta}))+\|w\|\cos\theta\\ &\geqq \|w\|\cos\theta-|z|\,\theta\geqq\|w\|(\cos\theta-\theta).\end{aligned}$$

Hence, so long as

$$\gamma(\cos\theta-|\theta|)\geqq|\theta|$$

or

$$\cos\theta\geqq|\theta|+\frac{1}{\gamma}|\theta|$$

the points $(\zeta(\theta), \tau(\theta))$ lie in the cone C_γ. In particular, this happens on an interval $(-\theta_0, \theta_0)$, where θ_0 depends only on γ. Thus,

$$\varphi(z,w) \leqq \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(\zeta(\theta), \tau(\theta))\, d\theta$$

$$\leqq \frac{1}{2\pi} \int_{-\theta_0}^{\theta_0} \varphi(\zeta(\theta), \tau(\theta))\, d\theta$$

$$\leqq \frac{\theta_0}{2\pi} \max_{|\theta| \leqq \theta_0} \log((\tilde{r}(|\zeta(\theta)|)^2 + \|\tau(\theta)\|^2).$$

But, $|\zeta(\theta)| \leqq 2|z| + \|w\| \leqq 3\|w\|$, so $\tilde{r}(\zeta(\theta))^2 \leqq \|w\|^2$ for sufficiently small $\|w\|$. Also, $\|\tau(\theta)\| \leqq 2\|w\|$, so we conclude $\varphi(z,w) \leqq \frac{\theta_0}{2\pi} \log(5\|w\|^2)$, which implies (2) in this case.

In the other case, $\|w\| \leqq |z|$, we can select λ such that $\lambda \geqq 2 + 2/\gamma$, and note that on the circle $(z + \lambda |z| e^{i\theta}, w)$, we have

$$\gamma \operatorname{Re}(z + \lambda |z| e^{i\theta}) \geqq \gamma |z| (\lambda \cos\theta - 1) = \gamma |z| [(\lambda - 2)\cos\theta + (2\cos\theta - 1)] \geqq \|w\|$$

whenever $\cos\theta \geqq 1/2$, or $|\theta| \leqq \pi/3$. Thus, all the points on this circle lie inside C_γ when $|\theta| \leqq \pi/3$, so that

$$\varphi(z,w) \leqq \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(z + \lambda |z| e^{i\theta}, w)\, d\theta$$

$$\leqq \frac{1}{2\pi} \int_{-\pi/3}^{\pi/3} \frac{1}{2} \log(\|w\|^2 + r((1+\lambda)|z|)^2)\, d\theta$$

$$= \frac{1}{6} \log(\|w\|^2 + r((1+\lambda)|z|)^2)$$

which is an estimate of the form (2). Consequently, (2) holds for (z, w) near $(0, 0)$.

Let $m(\rho_1, \rho_2) = \sup\{\varphi(z,w) : |z| = \rho_1, \|w\| = \rho_2\}$, and $f(x,y) = \frac{1}{\eta} m(e^x, e^y)$. Then $f(x,y)$ is a convex function of (x,y) defined for large negative x and y, since m is psh. The inequality (2) for φ means that

$$f(x,y) \leqq \tfrac{1}{2} \log(e^{2h(x)} + e^{2y}) = F(x,y)$$

where $h(x)$ is a function such that $\frac{h(x)}{x} \to +\infty$ as $x \to -\infty$.

Suppose that an affine function $Ax + By + C$ satisfies

$$Ax + By + C \leqq F(x,y) \tag{3}$$

for all $x \leqq x_0$, $y \leqq y_0$. Then we claim $B \geqq 1$. To see this, note that

$$Ax + C \leqq \psi(x) := \inf_{y \leqq y_0} F(x,y) - By. \tag{4}$$

We must have $B \geqq 0$, since if $B<0$, then $\psi(x)=-\infty$ for some finite x. If $B=0$, then $\psi(x)=h(x)\geqq Ax+C$, contrary to the assumption that $\frac{h(x)}{x}\to+\infty$ as $x\to-\infty$. The function $y\to F(x,y)-By$ is convex. If $B>0$, then $y\to F(x,y)-By$ tends to $+\infty$ as $y\to-\infty$. Thus the minimum of the function either occurs at $y=y_0$, or at the point $y<y_0$ where $\frac{\partial F}{\partial y}(x,y)=B$. A short calculation shows that at this point, we must have

$$e^{2y}=\frac{B}{1-B}\,e^{2h(x)}. \tag{5}$$

Hence, if $0<B<1$ and x is very large and negative, the minimum will occur at the point y for which (5) holds, so the infimum in (4) is equal to

$$h(x)-\tfrac{1}{2}\log(1-B).$$

But, this is impossible, for then

$$Ax+C\leqq h(x)+O(1) \quad \text{as} \quad x\to-\infty$$

contrary to our hypothesis that $\frac{h(x)}{x}\to+\infty$. Thus, we have $B\geqq 1$.

We can rewrite the affine function in (3) in the form $A(x-x_0)+B(y-y_0)+C$. Noting that when $B\geqq 1$, the minimum in (4) occurs for $y=y_0$, we have $A(x-x_0)+C\leqq F(x,y_0)$. Hence, the upper envelope of all affine functions satisfying (3) is at most $(y-y_0)+F(x,y_0)$. Since the convex function $f(x,y)$ is an upper envelope of affine functions satisfying (3), the last estimate of the lemma follows.

§ 3. A Smooth Function without Subextension

We first construct a continuous example and then modify it to give a smooth example.

For $0<\alpha<1/2$, let

$$v(z)=v_\alpha(z)=\operatorname{Re}-\frac{1}{z^\alpha}=-r^{-\alpha}\cos\alpha\theta, \qquad z=re^{i\theta}, \quad -\pi\leqq\theta\leqq\pi,\ r>0$$

and

$$r(z)=e^{v(z)}.$$

The function $v(z)$ is harmonic in $\mathbb{C}\setminus\mathbb{R}^-$, the complex plane with the negative real axis and 0 removed. It is continuous on $\mathbb{C}\setminus\{0\}$ and negative there, since $\alpha<1/2$. However, $v(z)$ is not subharmonic on the negative real axis. In fact, it is superharmonic there as locally it is the minimum of the two harmonic functions corresponding to different branches of z^α. The function $r(z)$ is continuous, nonnegative, and vanishes to infinite order at $z=0$, since

$$0\leqq r(z)\leqq\exp\left(-\frac{\cos\alpha\pi}{|z|^\alpha}\right).$$

Next, let $\mathcal{U}$ be a neighborhood of 0 and $g\colon \mathcal{U} \to \mathbb{C}^{n-1}$ a function analytic for $z \in \mathcal{U}$ such that

$$g(0)=0, \qquad \|g'(z)\| \leqq 1/4, \qquad z \in \mathcal{U}.$$

Consider the function u defined by

$$u(z, w)=u(z, w, \alpha, g)=\max\{v_\alpha(z), \log\|w-g(z)\|\}. \tag{6}$$

The following proposition lists several properties of $u(z, w)$.

Proposition 2. *The function u of* (6) *satisfies*

(i) *u is psh for $(z, w) \in [(\mathbb{C}\setminus\mathbb{R}^-) \cap \mathcal{U}] \times \mathbb{C}^{n-1}$ and on the open set*

$$\{(z, w) \in \mathcal{U} \times \mathbb{C}^{n-1} \colon \|w-g(z)\| > r(z)\}.$$

(ii) $u(z, w) \leqq \frac{1}{2}\log(\|w-g(z)\|^2 + r(z)^2)$.

(iii) *u is continuous on $\mathbb{C}^n\setminus(0, 0)$, and $u(z, w) \to -\infty$ as $(z, w) \to (0, 0)$.*

Proof. For $z \in \mathcal{U}$, z not on the negative real axis or $z=0$, $u(z, w)$ is clearly psh as the maximum of psh functions. The definition of $u(z, w)$ shows

$$u(z, w)=\begin{cases} \log\|w-g(z)\| & \text{if } \|w-g(z)\| > r(z) \\ \log r(z) & \text{if } \|w-g(z)\| \leqq r(z) \end{cases} \tag{7}$$

Thus, on the open set $\|w-g(z)\| > r(z)$, u is also psh because it is equal to $\log\|w-g(z)\|$, a psh function. The inequality of (ii) is also clear, since

$$\begin{aligned} u(z, w) &= \max\{\tfrac{1}{2}\log r(z)^2, \tfrac{1}{2}\log\|w-g(z)\|^2\} \\ &\leqq \tfrac{1}{2}\log(r(z)^2 + \|w-g(z)\|^2). \end{aligned}$$

The third assertion is clear.

The function $u(z, w)$ is essentially psh on the complement of the ball of radius 1 with center at $(-1, 0)$. The only problem is that u is only defined for z near 0. To get a globally defined function on a ring domain

$$\Omega_R=\{(z, w)\colon 1 < |z+1|^2 + \|w\|^2 < (1+R)^2\},$$

we want to take the function $u(z, w)$ for (z, w) near 0, and then modify it by some smooth function away from $(0, 0)$. Precisely, we have the following.

Proposition 3. *For any $\delta > 0$, there exist constants $C_1, C_2 > 0$ such that the function $U(z, w)$ defined by*

$$\begin{array}{ll} u(z, w) & \operatorname{Re} z \geqq -\delta \\ \max\{u(z, w), C_1 - C_2 \operatorname{Re} z\} & -\delta \geqq \operatorname{Re} z \geqq -2\delta \\ C_1 - C_2 \operatorname{Re} z & -2\delta \geqq \operatorname{Re} z \end{array}$$

is psh on Ω_R.

Now we note that the function $U(z, w)$ constructed on Ω_R is not smooth. However one may check that the singularities are locally of the form

$$U(z, w)=\max\{S(z, w), h(z, w)\}$$
$$=h(z, w)+\max\{0, S(z, w)-h(z, w)\}$$

i.e., they are locally the maximum of a smooth psh function $S(z, w)$ and a pluriharmonic function $h(z, w)$. In other words, at a singular point

$$U(z, w)=h(z, w)+\chi(S(z, w)-h(z, w))$$

where $\chi(t)=\max(0, t)$. If we replace χ by a C^∞ convex and increasing function $\tilde{\chi}$ with $\tilde{\chi}(t)=0$ for $t\leqq -\varepsilon$ and $\tilde{\chi}(t)=t$ for $t\geqq\varepsilon$, then the resulting function $\tilde{U}$ will be C^∞ and psh, and $|\tilde{U}-U|\leqq 2\varepsilon$. Further, the function is changed only on the set $\{-\varepsilon<S-h<\varepsilon\}$ so that if ε is taken to be small, $\tilde{U}=U$ holds near any other singularity.

We will now select a particular curve $g(z)$ so that the function $\tilde{U}$ constructed above does not subextend. Choose real numbers $(\beta_2, \ldots, \beta_n)$ such that the curve

$$c(t)=(e^{it}, e^{2\pi i\beta_2 t}, \ldots, e^{2\pi i\beta_n t}), \qquad -\infty<t<+\infty,$$

is dense in the n-torus. Then, for ε a small positive number, let $g(z)$ be the analytic curve defined for $|z|<1$ by $g(z)=(g_2(z), \ldots, g_n(z))$, where

$$g_i(z)=\varepsilon(1+z)^{\beta_i}-\varepsilon(1+z)$$

(and the principle branch of $(1+z)^{\beta_i}$ is used). The conditions $g(0)=0$, $\|g'(z)\|\leqq\frac{1}{4}$ are satisfied on $\{\operatorname{Re} z\geqq -2\delta, |z|<R\}$ if $\delta>0$ is sufficiently small. Thus, we obtain the function $\tilde{U}$ as above.

Proposition 3. *The function $\tilde{U}$ constructed as above is smooth and psh on the domain Ω_R, and for every $\gamma>0$, there is no psh function ϕ on $\{|z|^2+\|w\|^2<(1+\gamma)^2\}$ such that $\phi(z, w)\leqq\tilde{U}(z, w)$ holds on the set*

$$\tilde{C}_\gamma=\{(z, w)\colon \operatorname{Re} z>0, \|w\|<\gamma, |z|^2+\|w\|^2<(1+\gamma)^2\}.$$

Proof. We have already seen that $\tilde{U}$ is smooth and psh on Ω_R. If ϕ exists, then we have the estimate

$$\phi(z, w)\leqq\tilde{U}(z, w)\leqq\tfrac{1}{2}\log[\|w-g(z)\|^2+r(z)^2]$$

on $\tilde{C}_\gamma$. If we make the local biholomorphic change of coordinates $z'=z$, $w'=w-g(z)$, we have

$$\phi(z', w')\leqq\tfrac{1}{2}\log[|w'|^2+r(z)^2]$$

for all the points (z', w') in a cone C_γ about the $\operatorname{Re} z'$ axis in some small ball about (0, 0). Then, by Lemma 1, there exists $\eta>0$ such that

$$\varphi(z, w)\leqq\eta\log\|w-g(z)\|+\mathcal{O}(1)$$

for all (z, w) in a neighbourhood of (0, 0). Thus, the Lelong number of φ is at least η at every point of the variety $w=g(z)$ near to (0, 0).

By Siu's theorem [4], the set of points in $\{|z|^2+\|w\|^2<(1+\gamma)^2\}$ where the Lelong number is $\geqq \eta$ is an analytic variety, V. Since it contains the part of the curve $w=g(z)$ near (0, 0), it also contains all the points (z, w) one can connect to (0, 0) by analytic continuation along curves in the set

$$S=\{w=g(z)\}\cap\{|z|^2+\|w\|^2<(1+\gamma)^2\}.$$

However, we claim that this set is contained in no (proper) analytic variety. For, we can clearly follow a path in S to a small neighborhood of the branch point $z=-1$, $w=0$. Then for small ρ, analytic continuation on the path with $z(t)=-1+\rho e^{it}$, $-\infty<t<+\infty$, shows that V contains all the points $(z(t), w(t))$ where

$$w(t)=-\varepsilon\rho e^{it}(1, \ldots, 1)+\varepsilon[\rho^{\beta_2} e^{2\pi i\beta_2 t}, \ldots, \rho^{\beta_n} e^{2\pi i\beta_n t}].$$

This set is not contained in any (proper) variety in a neighborhood of $(-1, 0)$, since under the biholomorphic map $z'=(1+z)$, $w'=w+\varepsilon(1+z)=w+\varepsilon z'$ its image is a dense subset of the distinguished boundary of the polydisk with center at (0, 0) and polyradius $(\rho, \varepsilon\rho^{\beta_2}, \ldots, \varepsilon\rho^{\beta_n})$. This is a contradiction so no such function ϕ can exist.

We remark that in the proof one could use instead of Siu's theorem a weaker version due to Skoda [5], which asserts that this set is contained in a variety.

Proof of the Theorem. We use the notation

$$A((z_0, w_0), r_1, r_2)=\{(z, w)\in\mathbb{C}^2: r_1^2<|z-z_0|^2+\|w-w_0\|^2<r_2^2\}.$$

If Ω is smoothly bounded, then there is a sequence of domains

$$A_j=A((z_0^j, w_0^j), r_1^j, r_2^j)$$

such that $\Omega\subset\bigcap A_j$ and there is a point $p_j\in\partial\Omega\cap\partial A_j$. Clearly we can choose the A_j such that the set $\{p_j\}$ is dense in $\partial\Omega$. By Proposition 3 there exists ψ_j which is psh and smooth and cannot be locally subextended over p_j. Without loss of generality, we may assume $\psi_j<0$. If we choose $\varepsilon_j>0$ such that $\psi=\sum\varepsilon_j\psi_j$ converges in $C^\infty(\Omega)$, then ψ cannot be subextended over any neighborhood of $\partial\Omega$. For if $\tilde{\Omega}$ and $\tilde{\psi}$ are given, then there exists $p_j\in\tilde{\Omega}\cap\partial\Omega$, but $\tilde{\psi}\leqq\varepsilon_j\psi$ holds on $\tilde{\Omega}\cap\partial\Omega$, which is a contradiction.

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