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Modulus and the Poincaré inequality on metric measure spaces

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Abstract The purpose of this paper is to develop the understanding of modulus and the Poincaré inequality, as defined on metric measure spaces. Various definitions for modulus and capacity are shown to coincide for general collections of metric measure spaces. Consequently, modulus is shown to be upper semi-continuous with respect to the limit of a sequence of curve families contained in a converging sequence of metric measure spaces. Moreover, several competing definitions for the Poincaré inequality are shown to coincide, if the underlying measure is doubling. One such characterization considers only continuous functions and their continuous upper gradients, and extends work of Heinonen and Koskela. Applications include showing that the p-Poincaré inequality (with a doubling measure), for $p \geq 1$, persists through to the limit of a sequence of converging pointed metric measure spaces — this extends results of Cheeger. A further application is the construction of new doubling measures in Euclidean space which admit a 1-Poincaré inequality.

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1 Introduction

1.1 Overview

The geometric flexibility of metric measure spaces under bi-Lipschitz and quasi-symmetric maps can be analyzed through, and sometimes characterized by, considerations of curves contained within the space. For example, Heinonen and Koskela [HK98] demonstrated that quasiconformal maps are quasisymmetric if (amongst other conditions) the metric space on which the maps are defined contains sufficiently many curves. Tyson has shown that one cannot use quasisymmetric maps to lower the Hausdorff dimension of any Ahlfors regular metric space which contains sufficiently many curves (see [Hei01, p.122]). Bonk and Kleiner [BK02] have shown that an Ahlfors regular space which is homeomorphic to the 2-sphere, and which contains sufficiently many curves, is actually quasisymmetric to the 2-sphere. This latter result is of particular interest for non-smooth uniformization procedures in connection with Thurston's Hyperbolization Conjecture (see [KB02, pp.24-26]). Related examples for controlling geometry through curves can be found in [Can94, CFP94, Sem96a, Tys00, Tys01b, BT01, Tys01a].

In all of the above cases, the authors use the concept of modulus to measure the amount of curves contained within a space. In particular, Heinonen and Koskela [HK98] employ modulus to give the following abstraction of a property of Euclidean space. They describe a metric measure space as being *Loewner* if the modulus of the collection of curves connecting each pair of compactum in the given space, is sufficiently large. Heinonen and Koskela demonstrated for Ahlfors regular metric measure spaces, that the Loewner condition coincides with the property of admitting their abstract formulation of a Poincaré inequality. This formulation expresses a scale invariant control for the oscillation of a real-valued function, defined over the metric space, in terms of the average integral of the infinitesimal behavior of the function.

The Poincaré inequality is of interest in its own right. There is an abundant collection of natural and exotic metric measure spaces which admit a Poincaré inequality. For a list of some such spaces see [HK98, Kei]. Moreover, a surprisingly rich structure can be deduced for a metric measure space, by merely knowing that it admits a Poincaré inequality with a doubling measure. This has been extensively studied, see [HKM93, HK95a, HK96, HK98, HK99, BMS01, HST01, Sha01, HKST01, KST01, Kei02]. In particular, Cheeger [Che99] demonstrated that spaces which admit a Poincaré inequality with a doubling measure, admit a sort of measurable differentiable structure similar to rectifiability. A consequence of Cheeger's work [Che99, Theorem 14.2] is that Ahlfors regular metric spaces which admit a Poincaré inequality and which admit a bi-Lipschitz embedding into Euclidean space, are actually rectifiable. This readily rules out the existence of bi-Lipschitz embeddings into Euclidean space for metric measure spaces which are not rectifiable (in the sense of Kirchheim [Kir94]), if they also admit a Poincaré inequality and are Ahlfors regular (see [Che99, p.504]). By the above mentioned work of Heinonen and Koskela, the same conclusion holds for metric measure spaces that are Loewner.

Again the property of a metric space containing sufficiently many curves imposes restrictive behavior on the (now not quasisymmetric, but rather this time bi-Lipschitz) maps defined on the space. The purpose of this paper is to further develop this understanding of the properties of, and interplay between, the Poincaré inequality, modulus, and the geometric flexibility of metric measure spaces under bi-Lipschitz and quasi-symmetric maps.

1.2 Statement of results

The main results of this paper will now be stated. We begin by showing for proper metric measure spaces with finite total mass, that modulus is equivalently defined regardless of whether an infimum is taken over all measurable functions, or whether an infimum is taken over all compactly supported Lipschitz functions. The rigorous version of this statement is technical and will be delayed until Proposition 6. This equivalence for competing definitions of modulus is used to prove the following upper semi-continuity property of modulus. See Section 4.1 and Section 2 for a description of the terminology.

Theorem 1. *Let $p \geq 1$, let $\{(X_n, d_n, \mu_n)\} \subset \mathcal{MM}$ be a sequence of compact metric measure spaces which converges to a compact metric measure space (X, d, μ) with $\mu(X) < \infty$, and let Γ_n be a family of rectifiable curves contained in X_n for each $n \in \mathbf{N}$. Then*

$$\operatorname{mod}_p\left(\limsup_{n\to\infty} \Gamma_n\right) \geq \limsup_{n\to\infty} \operatorname{mod}_p(\Gamma_n). \tag{1}$$

Remark 1. Notice that the statement of the above theorem is vacuous if $\Gamma = \limsup_{n\to\infty} \Gamma_n$ contains a degenerate curve. In this case there are no functions ρ admissible for Γ, and so $\operatorname{mod}_p(\Gamma) = \infty$.

The equivalence for competing definitions of modulus is also used to prove Proposition 7. Loosely speaking, Proposition 7 claims for metric measure spaces which are proper, geodesic and have finite total mass, that the capacity of a pair of compacta defined by taking an infimum over Lipschitz functions, is equal to the modulus of the curves connecting the given compacta. Similar conclusions have been obtained by other authors in a variety of differing circumstances. In Euclidean space equipped with Lebesgue measure, the connection between the conformal capacity of a ring and extremal length was given by Gehring [Geh61]. Ziemer [Zie69] generalized this equivalence to a statement, similar to Proposition 7, concerning capacity and modulus of pairs of compacta, again in Euclidean space equipped with Lebesgue measure. In the more abstract setting, for locally quasiconvex and compact metric measure spaces, Heinonen and Koskela [HK98, Proposition 2.17] demonstrated that modulus is equal to the capacity defined by taking an infimum over locally Lipschitz functions. Whereas Kallunki and Shanmugalingam [KS01] established the same conclusion for any domain of any complete metric measure space which admits a Poincaré inequality with a doubling measure. The difference between these results for metric measure spaces and Proposition 7 is that

Kallunki and Shanmugalingam [KS01] do not restrict their attention to metric measure spaces with finite total mass, but instead consider metric measure spaces which admit a Poincaré inequality. Also, Heinonen and Koskela [HK98, Proposition 2.17] consider more general metric measure spaces than those considered here, but then restrict their attention to capacity defined in a compact ball.

Proposition 7 is applied here to prove that various definitions of the Poincaré inequality coincide. Before stating this theorem, two competing definitions for the Poincaré inequality will now be given. Both definitions are based on the work of Heinonen and Koskela. See Section 2 and Section 7 for an explanation of the terminology used in the following definition and theorem.

Definition 1 (The Poincaré inequality). *Let* $p \geq 1$. *A metric measure space* (X, d, μ) *is said to admit a* p-Poincaré inequality for all measurable functions *(respectively, a* p-Poincaré inequality for all compactly supported Lipschitz functions and their compactly supported Lipschitz upper gradients*) with constants* $C, \lambda \geq 1$ *if the following holds: Every ball contained in X has measure in* $(0, \infty)$, *and we have*

$$\fint_{B} |u - u_B| \, d\mu \leq C \operatorname{diam} B \left(\fint_{\lambda B} \rho^p \, d\mu \right)^{1/p}, \tag{2}$$

whenever B is a ball in X, and for every pair of functions $u : X \longrightarrow \mathbf{R}$ *and* $\rho : X \longrightarrow [0, \infty]$ *where u is measurable, and* ρ *is an upper gradient for u (respectively, for every pair* $u, \rho \in \mathrm{LIP}_0(X)$ *with* ρ *an upper gradient for u).*

Theorem 2. *Let* $p \geq 1$, *and let* (X, d, μ) *be a complete metric measure space with* μ *doubling, and such that every ball in X has measure in* $(0, \infty)$. *Then the following are quantitatively equivalent:*

1. (X, d, μ) *admits the p-Poincaré inequality for all measurable functions,*
2. (X, d, μ) *admits the p-Poincaré inequality for all compactly supported Lipschitz functions and their compactly supported Lipschitz upper gradients,*
3. *there exists constants* $C, \lambda \geq 1$ *such that*

$$\fint_{B} |u - u_B| \, d\mu \leq C \operatorname{diam} B \left(\fint_{\lambda B} (\operatorname{Lip} u)^p \, d\mu \right)^{1/p}, \tag{3}$$

   for every $u \in \mathrm{LIP}_0(X)$ *and for every ball B in X,*
4. *there exists a constant* $C \geq 1$ *such that*

$$d(x, y)^{1-p} \leq C \operatorname{mod}_p(x, y; \mu_{xy}^C),$$

   for every pair of distinct points $x, y \in X$.

Further suppose (X, d) *admits a bi-Lipschitz embedding* ι *into some Euclidean space* $\mathbf{R}^N$, *with* $N \in \mathbf{N}$, *equipped with the standard Euclidean metric. Then each of the above conditions is equivalent to the following:*

5. There exists constants $C, \lambda \geq 1$ such that

$$\fint_{B} |u - u_B| \, d(\iota_* \mu) \leq C \operatorname{diam} B \left(\fint_{\lambda B} |\nabla u|^p \, d(\iota_* \mu) \right)^{1/p},$$

for every $u \in C^\infty(\mathbf{R}^N)$, and for every ball B in $\mathbf{R}^N$ that is centered in ιX.

Remark 2. If Condition (2) holds, Condition (1) and (3) then hold with exactly the same constants $C, \lambda \geq 1$. This is a consequence of Theorem 2, and the fact established by Cheeger (see [Che99, Section 5] and the proof of [Che99, Theorem 9.6]) that Condition (1) implies the set of compactly supported Lipschitz functions with continuous (or even compactly supported Lipschitz) upper gradients is suitably dense (in a Sobolev sense) within the class of functions relevant to Condition (1).

Remark 3. Hajłasz and Koskela have shown if a metric measure space is geodesic and the given measure is doubling, Condition (1) then implies Condition (1) with $\lambda = 1$ and a possibly inflated constant C (see [Hei01, Theorem 4.18]).

Remark 4. Each of Condition (1), (2), (3) and (4) is also equivalent to the requirement that Condition (4) hold almost everywhere, that is, that there exists a constant $C \geq 1$ such that

$$d(x, y)^{1-p} \leq C \operatorname{mod}_p(x, y; \mu^C_{xy}),$$

for μ almost every pair of distinct points $x, y \in X$. This is discussed further in Remark 10.

Consider a metric measure space which satisfies all the hypotheses of Theorem 2, excluding the assumption that the given metric measure space is complete. If Condition (3) holds, with the assumption $u \in \mathrm{LIP}_0(X)$ replaced by $u \in \mathrm{LIP}\,(X)$, or if instead we assume the stronger assumption that the metric measure space admits a Poincaré inequality for all Lipschitz functions and their Lipschitz upper gradients, each of Conditions (1) to (5) then hold in the completion of the metric measure space equipped with the push forward measure. Indeed, both the left-hand and right-hand side of (3) are not affected by passing to the completion. This simple consequence of Theorem 2 demonstrates the robust nature of the Poincaré inequality. For example, the completion of a totally disconnected metric measure space which observes Condition (3), and for which the given measure is doubling, admits a Poincaré inequality and is therefore quasiconvex [Che99, Appendix]. (A trivial example which demonstrates that this line of thinking is non-vacuous is given by Euclidean space less an appropriately chosen countable collection of hyperplanes.) Such statements are generally not true if the Poincaré inequality is formulated for a smaller class of functions. Koskela [Kos99] gave an example of a non-complete metric measure space for which the measure is doubling, which admits a Poincaré inequality for all Lipschitz functions but does not admit a Poincaré inequality for all measurable functions. By Theorem 2 and the above comments, such a space does not generally admit a Poincaré inequality for Lipschitz functions and their Lipschitz upper gradients.

Condition (5) of Theorem 2 gives a new criteria for establishing rectifiability, at least for Ahlfors regular subsets of Euclidean space. To see this, take an Ahlfors regular subset X of some Euclidean space, and suppose X satisfies the hypotheses of Condition (5), for some $p \geq 1$, with ι (the map in the hypotheses of Theorem 2) given by the identity map. Theorem 2 then implies X admits a p-Poincaré inequality for all measurable functions. This together with the fact that X is Ahlfors regular, implies X is rectifiable (this latter assertion is a consequence of [Che99, Theorem 14.2]).

The implication that Condition (2) implies Condition (1) is a significant part of Theorem 2. It extends a result of Heinonen and Koskela [HK99] which says that if a complete metric measure space with a doubling measure satisfies (2) for all Lipschitz functions and all their upper gradients, then the metric measure space admits a Poincaré inequality for all measurable functions and their upper gradients. The implication that Condition (2) implies Condition (1) is useful for showing a space admits a Poincaré inequality. It has been applied by Rajala to show that Alexandrov spaces with curvature bounded from below, admit a (local) Poincaré inequality [Raj]. The implication has also been used by Rajala and the author to show for complete metric measure spaces equipped with a doubling measure, that the definition of admitting a p-Poincaré inequality, for $p \geq 1$, due to Heinonen and Koskela (Definition 1 for all measurable functions) is equivalent to the definition of admitting p-Poincaré inequalities due to Semmes [Sem01, p.16]. The implication that Condition (2) implies Condition (1) is used in this paper to prove the following theorem concerning the persistence of the Poincaré inequality under converging metric measure spaces. See Section 2 for a description of the terminology.

Theorem 3. *Let $p \geq 1$, let $\{(X_n, d_n, q_n, \mu_n)\}$ be a sequence of complete pointed metric measure spaces, with μ_n a doubling measure for each $n \in \mathbf{N}$ with uniformly bounded doubling constant, which converges to the complete metric measure space (X, d, q, μ). Further suppose for each $n \in \mathbf{N}$, that (X_n, d_n, μ_n) admits a p-Poincaré, and does this with uniformly bounded constants. Then (X, d, μ) admits the p-Poincaré inequality with μ a doubling measure, and does this with constants which depend only on the previous uniform bounds.*

Remark 5. The proof of Theorem 3 also serves to establish the following alternate statement, useful for construction purposes. Let $p \geq 1$, and let $\{(X_n, d_n, q_n, \mu_n)\} \subset \mathcal{MM}$ be a sequence of complete pointed metric measure spaces which converges to the quasiconvex and complete pointed metric measure space (X, d, q, μ) with μ doubling. Further suppose for each $n \in \mathbf{N}$, that (X_n, d_n, μ_n) admits a p-Poincaré with uniform constants. Then (X, d, μ) admits the p-Poincaré inequality, with constants depending only on the uniform constants, the quasiconvexity constant of (X, d), and the doubling constant of μ. The point here is that the measures in (μ_n) are not necessarily doubling. Further generalizations of Theorem 3 are pursued in Remark 11.

Theorem 3 extends a result of Cheeger [Che99, Theorem 9.6] which states that under the assumptions of Theorem 3, and with the further assumption that $p > 1$, the space (X, d, μ) admits a q-Poincaré inequality for every $q > p$. Since proving Theorem 3 it has become apparent that both Koskela and Cheeger have also separately established Theorem 3. As far as I know, their proofs are unpublished, and are different to the proof presented here. Even the proof of [Che99, Theorem 9.6] is substantially different to the proof presented here, and in particular, unlike the proof here, does not rely on a result like Theorem 2 and Proposition 7.

Condition (4) of Theorem 2 is probably the condition with the most mysterious appearance. It says that the modulus of curves between every pair of distinct points of the space, is sufficiently large. Here modulus is taken with respect to the measure μ^C_{xy}, which is itself much like a restricted Riesz kernel of μ, see [Hei01, p.34]. In particular μ^C_{xy} blows up at the points x and y.

Condition (4) thus has some similarity to the Loewner property introduced by Heinonen and Koskela. Both properties are quantitative statements concerning the amount of curves connecting pairs of compactum in the space. However, the Loewner property specifies that there are many curves between every pair of compactum in the space, whereas Condition (4) only says that there are many curves between each pair of distinct points. Moreover, Heinonen and Koskela demonstrated for Ahlfors regular spaces that admitting a Poincaré inequality is equivalent to being Loewner [Hei01, Theorem 9.10]. Whereas here, not only does Theorem 2 make different claims, but it also assumes only the weaker assumption that the given measure is doubling.

Condition (4) of Theorem 1 is a key feature of this paper. It provides an essential stepping stone in the proof that Condition (2) implies Condition (1). Moreover, the equivalence of Condition (4) and Condition (1) in Theorem 2, together with Theorem 1, can be used to give an alternate proof (to the one presented here) of Theorem 3. This proof is left to the reader. Condition (4) can also be used to give new examples of doubling measures in Euclidean space which admit a 1-Poincaré inequality; one such collection of examples is presented in the following theorem. Such measures have been extensively studied (see [HKM93, Bjö 01]) and bear relation to strong A_∞ geometry, which consists of conformal deformations of Euclidean space [DS90, HK95b, Sem96b]. Observe that a 1-Poincaré inequality is the strongest inequality in that it implies the p-Poincaré inequality for every $p > 1$. For the following, let $n \in \mathbf{N}$, and given $\Omega \subset \mathbf{R}^n$ and $\alpha > 0$, define a measure $\mu_{\Omega,\alpha}$ on $\mathbf{R}^n$ by

$$\mu_{\Omega,\alpha}(A) = \int_{A\cap\Omega} d(x, \partial\Omega)^\alpha \, d\mathcal{L}(x),$$

for every Borel set $A \subset \Omega$. Here $\mathcal{L}$ denotes Lebesgue measure, we write d for the standard Euclidean metric in $\mathbf{R}^n$, and $\partial\Omega$ is the topological boundary of Ω. See Section 2 and Section 9 for a further description of the terminology employed in the following theorem.

Theorem 4. *Let $\alpha > 0$, and let $\Omega \subset \mathbf{R}^n$ be a uniform domain, for some $n \in \mathbf{N}$. Then $(\overline{\Omega}, d, \mu_{\Omega,\alpha})$ admits a 1-Poincaré inequality and $\mu_{\Omega,\alpha}$ is doubling as a measure on this space. Both the constants of the 1-Poincaré inequality and the doubling measure depend only on n, α, and the uniform domain constant of Ω.*

One striking feature of Theorem 4 is that it applies to measures given by weights against Lebesgue measure which can vanish on large well-behaved sets. For example, take Ω to be $\mathbf{R}^3$ less a circle (say, an isometrically embedded copy of the

1-sphere). In this case $\mu_{\Omega,\alpha}$ is given by a weight which vanishes along the given circle. In contrast, strong A_∞ weights do not vanish on rectifiable curves [Sem93, Proposition 3.12(a)]. As far as I understand, certain products and positive powers of strong A_∞ weights were previously the largest known collection of doubling measures on Euclidean space which admit a 1-Poincaré inequality (see [DS90, HK95b, Bjö 01]). Consequently, Theorem 4 provides new examples of doubling measures which admit a 1-Poincaré inequality.

1.3 Outline of Approach

Section 2 Section 2.1 recalls standard definitions, and Section 2.2 introduces definitions of convergence for curve families, (pointed) metric spaces, and (pointed) metric measure spaces. The latter definitions are equivalent up to a subsequence, to (pointed) Gromov-Hausdorff convergence and measured (pointed) Gromov-Hausdorff convergence, respectively. The reason for introducing these new definitions is that they are easy to work with, at least in the context of this paper.

Section 3 This section establishes a useful property of curves (Proposition 4), and useful extension results for continuous functions and their continuous upper gradients (Lemma 3, Lemma 4, and Proposition 5). Proposition 4 is applied in the proof of the equivalence for competing definitions of modulus (Proposition 6), and also in the proof of the upper semi-continuity of modulus (Theorem 1). Proposition 5 is applied in the proof of the equivalence for competing definitions of capacity (Proposition 7), and the proof of the persistence of the Poincaré inequality under converging metric measure spaces (Theorem 3). Lemma 3 is also applied in Proposition 7. Lemma 4 is applied to prove that metric measure spaces which admit any sort of Poincaré inequality with a doubling measure are then quasiconvex (Proposition 8).

Section 4 In this section the equivalence for competing definitions of capacity and modulus is proven, at least for certain collections of metric measure spaces. Specifically, Definition 10 and Definition 11 give several possible definitions for modulus and capacity, respectively. The alternate definitions for modulus are shown to coincide for metric measure spaces which are proper and have finite total mass (Proposition 6). The alternate definitions for capacity are shown to coincide for metric measure spaces which are geodesic, proper and which have finite total mass. Moreover, for such spaces, a relation is given between modulus and capacity (Proposition 7). The proof of Proposition 7 uses Proposition 6. In particular, the competing definitions for capacity are shown to coincide by trapping them between the (now known to be) equivalent definitions for modulus.

Section 5 This section establishes that modulus is upper semi-continuous with respect to the limit of a sequence of curve families contained in a converging sequence of metric measure spaces (Theorem 1). The precise statement utilizes the definitions for convergence of pointed metric measure spaces, and the convergence of families of curves, as presented in Section 2.2. The proof relies on the equivalence of alternate definitions of modulus (Proposition 6).

Section 6 In this section it is shown that metric measure spaces which admit any sort of Poincaré inequality with a doubling measure are then quasiconvex. To be precise, it is shown under the hypotheses of Theorem 2, that each of Conditions (1), (2), (3) and (5) guarantee that the given metric measure space is quasiconvex (Proposition 8). The argument that Condition (1) implies quasiconvexity is due to Semmes (see [Che99, Appendix]), and the corresponding argument for Condition (2) and (3) is similar and so left to the reader's discretion. The argument that Condition (5) implies quasiconvexity is new, but still draws upon ideas of the above mentioned proof of Semmes.

Section 7 This section gives a proof of the equivalence for several *a priori* different definitions of the Poincaré inequality (Theorem 2). The proof incorporates results of Heinonen and Koskela, and also uses the equivalence for competing definitions of capacity (Proposition 7). In order to apply Proposition 7, Proposition 8 is applied to guarantee that the given metric measure space is quasiconvex.

Section 8 In this section a proof for Theorem 3 is presented. That is, it is shown that the property of admitting a Poincaré inequality with a doubling measure persists under limits, if the associated constants are uniformly bounded. The statement of this theorem requires the definitions of convergence from Section 2.2. The proof relies on the assertion of Theorem 2 that in order to establish a metric measure space admits a Poincaré inequality, one need only consider continuous functions and their continuous upper gradients (that is, the proof relies on the fact that Condition (2) of Theorem 2 implies Condition (1)). The proof also applies the extension property for continuous functions and their continuous upper gradients (Proposition 5).

Section 9 In this section Theorem 4 is proven. To do this we verify both Condition (4) and the hypotheses of Theorem 2.

2 Preliminary definitions

2.1 Standard terminology

In this subsection we recall standard terminology. A ball in a metric space (X, d) centered at $x_0 \in X$ and with radius $r > 0$, is a set of the form

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}.$$

For $\lambda > 0$ we define

$$\lambda B(x_0, r) = B(x_0, \lambda r).$$

A metric space (X, d) is said to be *proper* if the closure of every ball in (X, d) is compact. Given $\epsilon > 0$, the *ϵ-neighborhood* of a subset A of a metric space (X, d) is the set

$$N_\epsilon(A) = \{x \in X : d(x, A) < \epsilon\}.$$

A *pointed metric space* (X, d, x) consists of a metric space (X, d) and a point $x \in X$.

A *metric measure space* (X, d, μ) consists of a set X, a metric d on X, and a Borel regular measure μ supported on X. For arbitrary $A \subset X$ with $0 < \mu(A) < \infty$ and a measurable function $f : X \longrightarrow [0, \infty]$, we write

$$\fint_A f = \frac{1}{\mu(A)} \int f \, d\mu \quad \text{and} \quad f_A = \fint_A f \, d\mu.$$

The measure μ is said to be *doubling* if μ is non-trivial and there exists $C > 0$ such that

$$\mu(B(x, 2r)) \leq C \, \mu(B(x, r)),$$

whenever $x \in X$ and $r > 0$. A *pointed metric measure space* (X, d, μ, x) consists of a metric measure space (X, d, μ) and a point $x \in X$.

A function $f : X \longrightarrow Y$ where (X, d) and (Y, ρ) are metric spaces is *Lipschitz* if there exists $C > 0$ such that

$$\rho(f(x), f(y)) \leq C \, d(x, y),$$

for every $x, y \in X$. In this case LIP f is defined to be the infimum of the values of $C > 0$ for which the above equation is true. A function is said to be *bi-Lipschitz* if it is Lipschitz and admits a Lipschitz inverse. An *isometry* is a 1-bi-Lipschitz map. The space of all real-valued Lipschitz functions on X is written LIP(X), whereas $\mathrm{LIP}_0(X)$ is the subspace of LIP(X) consisting of functions with compact support. For a real valued Lipschitz function f defined on a metric space (X, d), define

$$\mathrm{Lip}\, f(x) = \limsup_{\substack{y \to x \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x, y)},$$

for every $x \in X$. It is easy to see that Lip f is a Borel function (see [Kei]).

A *curve* in a metric space (X, d) is a continuous map γ of an interval $I \subset \mathbf{R}$ into X, and is said to be *rectifiable* if it has finite *length*, which we denote by $\ell(\gamma)$. A non-negative Borel function $\rho : X \longrightarrow [0, \infty]$ is said to be an *upper gradient* of a function $u : X \longrightarrow \mathbf{R}$ if

$$|u(x) - u(y)| \leq \int_\gamma \rho \, ds,$$

whenever γ is a rectifiable curve in X joining two points $x, y \in X$. A metric space (X, d) is said to be *λ-quasiconvex*, for $\lambda \geq 1$, if every pair of points in $x, y \in X$ can be joined by a curve γ such that $\ell(\gamma) \leq \lambda d(x, y)$. A 1-quasiconvex metric space is said to be *geodesic*. We refer the reader to [Hei01, Chapter 7] for a more full discussion of the above topics.

Given sets X, Y, Z, a function $g : X \longrightarrow Z$ and an injective function $f : X \longrightarrow Y$, the *pushforward* $f_*g : f(X) \longrightarrow Z$ of g by f is defined by $f_*g(y) = g \circ f^{-1}(y)$ for every $y \in f(X)$. Given a function $f : X \longrightarrow Y$ where (X, μ) is a measure space and Y is a set, the *pushforward* of μ by f is defined by $f_*\mu(A) = \mu(f^{-1}(A))$ for every $A \subset X$. If Γ is a family of curves in X, then $f\Gamma$ is the family consisting of all the curves $f \circ \gamma$, for every $\gamma \in \Gamma$. Given an injective function $f : X \longrightarrow Y$ where (X, d) is a metric space and Y is a set, the *pushforward* of d by f is a metric on $f(X)$ defined by $f_*d(x, y) = d(f^{-1}(x), f^{-1}(y))$ for every $x, y \in f(X)$.

2.2 Limit supremum of families of rectifiable curves, (pointed) metric spaces, and (pointed) metric measure spaces

In this section we introduce several notions of convergence. The reason for introducing new definitions for the latter two concepts is that they are easy to work with, at least in the context of this paper. The definitions for (pointed) Gromov-Hausdorff convergence and measured (pointed) Gromov-Hausdorff convergence are equivalent up to a subsequence, to the corresponding definitions presented here. Some of this is covered in [Gro81, Gro99, Pet93, BS94, Pet98, CY98], and the rest is left to the reader. Consequently, the new notions of convergence inherit the usual properties concerning uniqueness of limits and compactness. This is stated formally below.

Given $\epsilon, R > 0$ and a complete pointed metric space (X, d, x), define $N(\epsilon, R, X)$ as the maximal number of disjoint closed balls of radius ϵ which are contained in $B(x, R)$. Given a function $\eta : \mathbf{R}^2 \longrightarrow \mathbf{R}$ we let $\mathcal{M}_\eta$ be the collection of pointed metric spaces (X, d, x) such that $N(\epsilon, R, X) \leq \eta(\epsilon, R)$ for every $\epsilon, R > 0$. Further let $\mathcal{MM}_\eta$ be the collection of all complete pointed metric measure spaces whose underlying pointed metric space is contained in $\mathcal{M}_\eta$. In the following we omit mention of η, writing $\mathcal{M}$ and $\mathcal{MM}$ for $\mathcal{M}_\eta$ and $\mathcal{MM}_\eta$, respectively. In this case it is understood that some η is given and fixed. As an example, observe that a collection of complete metric measure spaces whose measures are doubling with the same doubling constant, all belong to the same family $\mathcal{MM}$.

When discussing compact pointed metric spaces $(X, d, p) \in \mathcal{M}$, we shall often omit mention of the point p, writing $(X, d) \in \mathcal{M}$. Similarly, we shall often refer to a compact pointed metric measure spaces as just compact metric measure spaces, writing $(X, d, \mu) \in \mathcal{MM}$. We do this to avoid introducing new notation. In these cases the point p will be superfluous.

Definition 2 (Convergence of subspaces of a metric space). *A sequence (F_n) of nonempty closed subsets of a metric space (Z, ρ) is said to* converge to *another nonempty closed subset F of Z if*

$$\lim_{n\to\infty} \sup_{z \in F_n \cap B(q,R)} \operatorname{dist}(z, F) = 0,$$

and

$$\lim_{n\to\infty} \sup_{z \in F \cap B(q,R)} \operatorname{dist}(z, F_n) = 0,$$

for all $q \in Z$ and $R > 0$. These suprema are interpreted to vanish when the relevant sets of competitors $F_n \cap B(z, R)$, and $F \cap B(z, R)$ are empty.

Definition 3 (Convergence of rectifiable curves of a metric space). *A sequence of rectifiable curves (γ_n) on a metric space (Z, ρ) is said to* converge to *another curve γ contained in Z, if there exists uniformly Lipschitz parameterizations of the given curves, all with the same domain, so that when viewed as functions the sequence (γ_n) converges uniformly to γ.*

We shall describe a family of rectifiable curves as being *closed* if the limit of every sequence of converging rectifiable curves in the family, is a member of the given family.

Definition 4 (Limit supremum of families of rectifiable curves of a metric space). *Given a sequence (Γ_n) of families of rectifiable curves in a metric space (X, ρ), we define the* limit supremum *of (Γ_n) (which we write as $\limsup_{n\to\infty} \Gamma_n$) to be the family of all curves γ such that there exists a sequence of rectifiable curves (γ_n), where $\gamma_n \in \Gamma_n$ for each $n \in \mathbf{N}$, with the property that a subsequence of (γ_n) converges to γ as curves.*

Notice that the limit supremum of a sequence of families of rectifiable curves is closed.

Definition 5 (Convergence of compact metric spaces). *A sequence of compact metric spaces $\{(X_n, d_n)\} \subset \mathcal{M}$ is said to* converge to *another compact metric space (X, d) if the following holds: There exists a compact metric space (Z, ρ) and isometric embeddings $\iota : X \longrightarrow Z$ and $\iota_n : X_n \longrightarrow Z$ for each $n \in \mathbf{N}$, such that $(\iota_n(X_n))$ converges to $\iota(X)$ as subspaces of Z.*

We say a sequence of measures (μ_n) defined on a metric space X *converges weakly* to some measure μ if for every continuous function $f : X \longrightarrow \mathbf{R}$ with bounded support, we have

$$\int_X f \, d\mu_n \to \int_X f \, d\mu,$$

as $n \to \infty$.

Definition 6 (Convergence of compact metric measure spaces). *A sequence of compact metric measure spaces $\{(X_n, d_n, \mu_n)\} \subset \mathcal{MM}$ is said to* converge to *another compact metric measure space (X, d, μ) if the following holds: There exists a compact metric space (Z, ρ) and isometric embeddings $\iota : X \longrightarrow Z$ and $\iota_n : X_n \longrightarrow Z$ for each $n \in \mathbf{N}$, such that $(\iota_n(X_n))$ converges to $\iota(X)$ as subspaces of Z, and such that $(\iota_n)_*\mu_n$ converges to $\iota_*\mu$ in the weak sense.*

Definition 7 (Convergence of pointed metric spaces). *A sequence of pointed metric spaces $\{(X_n, d_n, p_n)\} \subset \mathcal{M}$ is said to* converge to *another complete pointed metric space (X, d, p) if the following holds: There exists a proper pointed metric space (Z, ρ, q) and isometric embeddings $\iota : X \longrightarrow Z$ and $\iota_n : X_n \longrightarrow Z$ for each $n \in \mathbf{N}$, such that $\iota(p) = \iota_n(p_n) = q$, and such that $(\iota_n(X_n))$ converges to $\iota(X)$ as subspaces of Z.*

Definition 8 (Convergence of pointed metric measure spaces). *A sequence of pointed metric measure spaces* $\{(X_n, d_n, \mu_n, p_n)\} \subset \mathcal{MM}$ *is said to* converge to *another complete pointed metric measure space* (X, d, μ, p) *if the following holds: There exists a proper pointed metric space* (Z, ρ, q) *and isometric embeddings* $\iota : X \longrightarrow Z$ *and* $\iota_n : X_n \longrightarrow Z$ *for each* $n \in \mathbf{N}$ *such that* $\iota(p) = \iota_n(p_n) = q$*, such that* $(\iota_n(X_n))$ *converges to* $\iota(X)$ *as subspaces of* Z*, and such that* $(\iota_n)_*\mu_n$ *converges to* $\iota_*\mu$ *in the weak sense.*

Definition 9 (Generalized limit supremum of families of rectifiable curves). *Let* Γ_n *be a family of rectifiable curves contained in a pointed metric space* (X_n, d_n, p_n) *for each* $n \in \mathbf{N}$*, let* Γ *be a curve family contained in a pointed metric space* (X, d, p)*, and further suppose* $\{(X_n, d_n, p_n)\}$ *converges to* (X, d, p) *as pointed metric spaces. The family* Γ *is said to be the* limit supremum *of* (Γ_n) *(which we write as* $\Gamma = \limsup_{n\to\infty} \Gamma_n$*) if the following holds: There exists a proper metric space* (Z, ρ) *and isometric embeddings* $\iota : X \longrightarrow Z$ *and* $\iota_n : X_n \longrightarrow Z$ *for* $n \in \mathbf{N}$*, that satisfy the conditions of Definition 8, and are such that* $\iota\Gamma = \limsup_{n\to\infty}(\iota\Gamma_n)$.

As mentioned in the introduction of this section, the proofs of the following lemmas and propositions have been omitted.

Lemma 1 (Uniqueness of limits, pointed metric spaces). *Let* $\{(X_n, d_n, p_n)\} \subset \mathcal{M}$ *be a sequence of complete pointed metric spaces which converges to both the complete pointed metric spaces* (X, d, p) *and* (Y, ρ, q)*. Then there exists an isometry between* (X, d) *and* (Y, ρ) *which maps* p *to* q.

Lemma 2 (Uniqueness of limits, pointed metric measure spaces). *Let* $\{(X_n, d_n, p_n, \mu_n)\} \subset \mathcal{MM}$ *be a sequence of complete pointed metric measure spaces which converges to both the complete pointed metric measure spaces* (X, d, p, μ) *and* (Y, ρ, p, ν)*. Then there exists an isometry between* (X, d) *and* (Y, ρ) *which maps* p *to* q *and is such that* ν *is the push forward of* μ.

Proposition 1 (Existence of limits, pointed metric spaces). *Let* $\{(X_n, d_n, p_n)\} \subset \mathcal{M}$ *be a sequence of complete pointed metric spaces. Then there exists complete pointed metric space* $(X, d, p) \in \mathcal{M}$ *such that a subsequence of* $\{(X_n, d_n, p_n)\}$ *converges to* (X, d, p).

Proposition 2 (Existence of limits, pointed metric measure spaces). *Let* $\{(X_n, d_n, p_n, \mu)\} \subset \mathcal{MM}$ *be a sequence of complete pointed metric spaces such that*

$$\sup_n \mu_n(B(p_n, r)) < \infty,$$

for every $r > 0$*. Then there exists complete pointed metric measure space* $(X, d, p, \mu) \in \mathcal{MM}$ *such that a subsequence of* $\{(X_n, d_n, p_n, \mu_n)\}$ *converges to* (X, d, p, μ).

Remark 6. The following statement simplifies matters in the proof of Theorem 3. The statement is a consequence of the above propositions and lemmas, and is left to the reader. Let $\{(X_n, d_n, p_n, \mu_n)\}, \{(Y_n, \rho_n, q_n, \nu_n)\} \subset \mathcal{MM}$ be sequences of complete pointed metric measure spaces and let $\iota_n : X_n \longrightarrow Y_n$ be a

λ-bi-Lipschitz map with $\iota_n(p_n) = q_n$ and $\iota_* \mu_n = \nu_n$, for some fixed $\lambda \geq 1$ and for every $n \in \mathbf{N}$. Then after passing to a subsequence, there exists complete pointed metric measure spaces (X, d, p, μ) and (Y, d, q, ν) such that $\{(X_n, d_n, p_n, \mu_n)\}$ and $\{(Y_n, \rho_n, q_n, \nu_n)\}$ converge to (X, d, p, μ) and (Y, d, q, ν), respectively, and there exists a λ-bi-Lipschitz map $\iota : X \longrightarrow Y$ with $\iota(p) = q$ and $\iota_* \mu = \nu$.

The following statement is needed in the proof of Theorem 3.

Proposition 3 (Persistence of doubling measures under limits). *Let* $\{(X_n, d_n, \mu_n, p_n)\} \subset \mathcal{MM}$ *be a sequence of complete pointed metric measure spaces which converges to the complete pointed metric measure space* (X, d, μ, p). *Suppose that there exists* $C > 0$ *such that* μ_n *is doubling with doubling constant* C*, for every* $n \in \mathbf{N}$. *Then* μ *is doubling with doubling constant* C^4.

Proof. Adopt the notation of Definition 8, and let $x \in X$ and $r > 0$. Since (X_n) converges to X as subspaces in Z, there exists a sequence of points $(x_n) \subset Z$ converging to x, such that $x_n \in X_n$ for every $n \in \mathbf{N}$. Thus there exists $N > 0$ such that $n > N$ implies $B(x_n, r/4) \subset B(x, r/2)$ and $B(x, 3r) \subset B(x_n, 4r)$, and therefore that

$$\mu_n(B(x, 3r)) \leq \mu_n(B(x_n, 4r)) \leq C^4 \mu_n(B(x_n, r/4)) \leq C^4 \mu_n(B(x, r/2)).$$

Since (μ_n) weak converges to μ, we have that

$$\mu(B(x, 2r)) \leq \liminf_{n\to\infty} \mu_n(B(x, 3r)) \leq \liminf_{n\to\infty} C^4 \mu_n(B(x, r/2)) \leq C^4 \mu(B(x, r)).$$

This completes the proof. □

3 Properties of curves and upper gradients

This section establishes a useful property of curves (Proposition 4), and the existence of a useful extension for continuous functions and their continuous upper gradients when defined on quasiconvex spaces (Proposition 5).

3.1 A useful property for sequences of curves

The following proposition uses compactness arguments to ascertain the existence of a limiting curve which satisfies desirable limiting properties.

Proposition 4. *Let* (γ_n) *be a sequence of curves with uniformly bounded length in a compact metric space* (X, d)*, and let* (ρ_n) *be an increasing sequence of real valued continuous functions defined on* X *with pointwise limit* ρ. *Then there exists a (possibly degenerate) curve* γ *in* X*, a subsequence of* (γ_n) *that converges to* γ *as curves, and we have*

$$\int_\gamma \rho \, ds \leq \liminf_{n\to\infty} \int_{\gamma_n} \rho_n \, ds. \tag{4}$$

Remark 7. A consequence of the above proposition is that the limit of a sequence of geodesic, proper and complete metric spaces is itself geodesic. More generally, the limit of a sequence of λ-quasiconvex, proper and complete metric spaces, for $\lambda \geq 1$, is itself λ-quasiconvex. To see this, appeal to Definition 7 for pointed metric space convergence, and let ρ_n of Proposition 4 be defined by $\rho_n(z) = 1$ for every $z \in Z$ and $n \in \mathbf{N}$. Here Z is the metric space described in Definition 7.

Proof. First pass to a subsequence of (γ_n) so that the liminf in (4) is achieved as a finite limit; if this is not possible there is nothing to prove. By hypothesis there exists $l > 0$ which bounds the length of each curve of (γ_n). Represent each curve γ_n by the parameterization $\gamma_n : [0, l] \longrightarrow X$ so that $\gamma_n|_{[0,\ell(\gamma_n)]}$ is the length parameterization and $\gamma_n|_{[\ell(\gamma_n),l]}$ is a constant function. Since (X, d) is compact, the Ascoli-Arzelá theorem implies there exists a 1-Lipschitz function $\gamma : [0, l] \longrightarrow X$, such that after passing to a subsequence, the sequence of functions (γ_n) converges uniformly to γ. By passing to yet another subsequence of (γ_n) it can be arranged so that $\ell(\gamma_n) \to l_\infty$ for some $l_\infty \leq l$.

Since γ is 1-Lipschitz it follows that

$$\int_\gamma \rho\, ds \leq \int_0^{l_\infty} \rho \circ \gamma(t)\, dt.$$

A consequence of the parameterization of γ_n is that

$$\int_0^{\ell(\gamma_n)} \rho_n \circ \gamma_n(t)\, dt = \int_{\gamma_n} \rho_n\, ds,$$

for each $n \in \mathbf{N}$. Thus to prove (4), it remains to demonstrate that

$$\int_0^{l_\infty} \rho \circ \gamma(t)\, dt \leq \lim_{n\to\infty} \int_0^{\ell(\gamma_n)} \rho_n \circ \gamma_n(t)\, dt. \tag{5}$$

Fix $\epsilon > 0$. Recall (ρ_n) is an increasing sequence of functions. Therefore, the Monotone Convergence Theorem implies there exists $N \in \mathbf{N}$ such that

$$\int_0^{l_\infty} \rho \circ \gamma(t)\, dt \leq \int_0^{l_\infty} \rho_N \circ \gamma(t)\, dt + \epsilon.$$

Since ρ_N is continuous and because (γ_n) converges uniformly to γ, and $(\ell(\gamma_n))$ converge to l_∞, there exists $M \in \mathbf{N}$ such that $m > M$ implies

$$\int_0^{l_\infty} \rho_N \circ \gamma(t) \leq \int_0^{\ell(\gamma_m)} \rho_N \circ \gamma_m(t)\, dt + \epsilon.$$

Again using the fact that (ρ_n) is an increasing sequence of functions gives for $m > M, N$, that

$$\int_0^{\ell(\gamma_m)} \rho_N \circ \gamma_m(t)\, dt \leq \int_0^{\ell(\gamma_m)} \rho_m \circ \gamma_m(t)\, dt.$$

Bringing the three previous inequalities together, reveals for $m > N, M$, that

$$\int_0^{l_\infty} \rho \circ \gamma(t)\, dt \leq \int_0^{\ell(\gamma_m)} \rho_m \circ \gamma_m(t)\, dt + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, this proves (5) and so completes the proof. □

3.2 An extension for functions and their upper gradients

The next lemma demonstrates a well known extension method for continuous functions with continuous upper gradients. Further extension methods are also given in the following lemma and proposition.

Lemma 3. *Let u be a real-valued continuous function defined on a nonempty subset E of a geodesic metric space (Z, d), such that $\inf_E u > -\infty$. Let ρ be a non-negative continuous function defined on Z such that*

$$|u(x) - u(y)| \leq \int_\gamma \rho\, ds \tag{6}$$

whenever γ is a rectifiable curve in Z joining two points $x, y \in E$. Then there exists a continuous extension $\bar{u}$ of u to Z such that ρ is an upper gradient for $\bar{u}$.

Proof. Define a function $\bar{u}$ on Z by

$$\bar{u}(x) = \inf \int_\gamma \rho\, ds + u(e),$$

where the infimum is taken over all $e \in E$, and over all rectifiable curves γ connecting e to x. A consequence of this definition is that $\bar{u}$ is an extension of u. Moreover, since Z is geodesic, ρ is continuous, and $\inf_E u > -\infty$, the function $\bar{u}$ is well defined as a real-valued function.

We now show that ρ is an upper gradient of $\bar{u}$. Fix $x, y \in Z$. Without loss of generality suppose $\bar{u}(x) \leq \bar{u}(y)$. The definition of $\bar{u}$ then implies

$$\bar{u}(y) \leq \bar{u}(x) + \int_\gamma \rho\, ds,$$

where γ is any curve from x to y. Thus

$$|\bar{u}(y) - \bar{u}(x)| = \bar{u}(y) - \bar{u}(x) \leq \int_\gamma \rho\, ds,$$

and so ρ is an upper gradient for $\bar{u}$. Since ρ is continuous, and therefore locally bounded, and since Z is geodesic, we then conclude that u is continuous. This completes the proof. □

We denote the oscillation of a real-valued function f defined on a set E by

$$\operatorname*{osc}_E f = \sup_{x,y \in E} |f(x) - f(y)|.$$

Lemma 4. *Let E be a subset of a geodesic metric space (Z, d), let u be a continuous real-valued function defined on E such that $\operatorname{osc}_E u < \infty$, and let ρ be a bounded, continuous, and non-negative function defined on the neighborhood $N_{4\alpha}(E)$ for some $\alpha > 0$, such that ρ and u satisfy (6) whenever γ is a curve in $N_{4\alpha}(E)$ connecting two points $x, y \in E$. Then there exists continuous extensions $\bar{u}$ and $\bar{\rho}$ of $u|_E$ and $\rho|_{N_\alpha(E)}$ to Z, respectively, such that $\bar{\rho}$ is an upper gradient of $\bar{u}$, and such that $\bar{\rho}$ is bounded.*

Proof. Since ρ is a bounded, continuous and non-negative function defined on the closure of $N_{3\alpha(E)}$, we can extend $\rho|_{N_{3\alpha(E)}}$ to a bounded, continuous and non-negative function $\tilde{\rho}$ defined on Z. Define $\bar{\rho} = \tilde{\rho} + G$, where G is taken to be some fixed continuous non-negative function on Z, which vanishes on $N_\alpha(E)$, and which achieves its maximum $\operatorname{osc}_E u/\alpha$ on all of $Z \setminus N_{2\alpha}(E)$. Thus $\bar{\rho}$ is a bounded extension of $\rho|_{N_\alpha(E)}$.

Let γ be a curve in Z which connects two points $x, y \in N_\alpha(E)$. Due to Lemma 3, to complete the proof it suffices to show that (6) holds with ρ replaced by $\bar{\rho}$. If $\gamma \subset N_{3\alpha}(E)$, this inequality follows from the hypotheses. Otherwise if $\gamma \setminus N_{3\alpha}(E) \neq \emptyset$, we have that

$$\mathcal{H}^1(\gamma \cap (Z \setminus N_{2\alpha}(E)) > \alpha.$$

Here $\mathcal{H}^1$ denotes 1-Hausdorff measure. The construction of $\bar{\rho}$ guarantees that

$$\min_{Z \setminus N_{2\alpha}(E)} \bar{\rho} \geq \frac{1}{\alpha}|u(x) - u(y)|.$$

These last two equations together imply (6). This completes the proof. □

Proposition 5. *Let $\delta > 0$, let X be a λ-quasiconvex subset of a metric space (Z, d) for some $\lambda \geq 1$, and let $u, \rho \in$ LIP (X) be bounded functions such that* $\inf_X \rho > 0$, *such that* $\operatorname{osc}_X u < \infty$, *and such that ρ is an upper gradient for u in the sub-metric space (X, d). Then there exists continuous extensions $\bar{u}$ and $\bar{\rho}$ of u and ρ to Z, respectively, such that $(\lambda + \delta)\bar{\rho}$ is an upper gradient for $\bar{u}$, and such that $\bar{\rho}$ is bounded.*

Assume the hypotheses of Proposition 5. It is well known that every metric space can be isometrically embedded into a Banach space, see [DS97, p.21]. Therefore, in addition to the hypotheses of Proposition 5, we may as well assume that the metric space (Z, d) is actually a Banach space $(Z, |\cdot|)$.

Fix $0 < \sigma < 1$ such that

$$\lambda(1 + \sigma)^2 < (\lambda + \delta). \tag{7}$$

Let $\bar{\rho}$ be a Lipschitz extension of ρ to Z such that $\inf_X \rho \leq \bar{\rho} \leq \sup_X \rho$. This can be achieved by an appropriate truncation of the McShane extension of ρ (see [Hei01, pp.43–44]). Since $\inf_X \rho > 0$ and $\bar{\rho}$ is Lipschitz, there exists a positive number $\epsilon < \sigma$ such that for every $x, y \in Z$, we have

$$\bar{\rho}(y) < (1 + \sigma)\bar{\rho}(x) \quad \text{whenever} \quad |x - y| \leq 5\lambda\epsilon. \tag{8}$$

Set $\tau \ll \epsilon$ (for example, $\tau = \epsilon^2/2$).

Sublemma 5. *We have*

$$|u(x) - u(y)| \leq (\lambda + \delta) \int_\gamma \bar{\rho}\, ds, \tag{9}$$

whenever γ is a rectifiable curve in $N_\tau(X)$ connecting two points $x, y \in X$.

Proof. To prove the lemma, it suffices to show there is a curve β in X connecting x to y with the property that

$$\int_\beta \rho\, ds \le \lambda(1+\sigma)^2 \int_\gamma \bar{\rho}\, ds. \tag{10}$$

Sublemma 5 then follows from the fact that ρ is an upper gradient for u in X, and the fact that (7) holds.

Consider the case when γ is short, that is, when $\ell(\gamma) < 2\epsilon$. Since X is λ-quasiconvex, there exists a curve β in X that connects x and y, and is such that $\ell(\beta) \le \lambda|x-y|$. Therefore $\beta \subset B(x, 2\lambda\epsilon)$, and so

$$\int_\beta \bar{\rho}\, ds \le \lambda|x-y| \sup_{z\in B(x,2\lambda\epsilon)} \bar{\rho}(z).$$

We also have $\gamma \subset B(x, 2\epsilon)$, and so

$$|x-y| \inf_{z\in B(x,2\epsilon)} \bar{\rho}(z) \le \int_\gamma \bar{\rho}\, ds.$$

These last two inequalities together with (7) and (8) imply (10).

A similar method also works when γ is long, that is, when $\ell(\gamma) \ge 2\epsilon$. In this case, decompose γ into a union of consecutively connected subcurves $\{\gamma^i\}_{i=1}^N$ with $N \in \mathbf{N}$, so that $\epsilon \le \ell(\gamma^i) \le 2\epsilon$. For $1 \le i \le N$, let z_1^i and z_2^i be the endpoints of γ^i. A curve β which lies in X, and is very close to γ, will now be constructed. To do this use the fact that $\gamma^i \subset N_\tau(X)$, and so choose $x_1^i \in X$ with $|x_1^i - z_1^i| \le \tau$, for $2 \le i \le N$. Set $x_1^1 = x$ and $x_2^N = y$, and let $x_2^i = x_1^{i+1}$ for $1 \le i \le N-1$. As before, since X is λ-quasiconvex, there is a curve β^i in X connecting x_1^i to x_2^i such that $\ell(\beta^i) \le \lambda|x_1^i - x_2^i|$. Define the curve β to be the union of the consecutively intersecting curves β^i, for $1 \le i \le N$.

The closeness of the curve β to γ will now be used to establish (10). Fix $1 \le i \le N$. Observe that

$$\int_{\beta^i} \rho\, ds \le \lambda|x_1^i - x_2^i| \sup_{z\in B(x_1^i,2\lambda\epsilon)} \bar{\rho}(z).$$

The tiny choice for τ guarantees that

$$|x_1^i - x_2^i| \le (1+\epsilon)|z_1^i - z_2^i|,$$

We also have $\gamma^i \subset B(x_1^i, 3\lambda\epsilon)$, and so

$$|z_1^i - z_2^i| \inf_{z\in B(x_1^i,3\lambda\epsilon)} \bar{\rho}(z) \le \int_{\gamma^i} \bar{\rho}\, ds.$$

The last three inequalities together with (8) imply that

$$\int_{\beta^i} \rho\, ds \le \lambda(1+\sigma)(1+\epsilon) \int_{\gamma^i} \bar{\rho}\, ds.$$

Since the choice for $1 \le i \le N$ was arbitrary, we have (10) then follows from (7). This completes the proof. □

Proposition 5 follows by applying the above sublemma, and then applying Lemma 4 with $\alpha = \tau/4$ and $E = X$. This completes the proof of Proposition 5.

4 Equivalent notions of modulus and capacity

In this section several definitions for capacity and modulus are presented and then analyzed.

4.1 Equivalent definitions of modulus

We now present alternate definitions for modulus, and then show that several of these definitions coincide on proper metric measure spaces that have finite total mass.

Definition 10. *For a given curve family* Γ *in a metric measure space* (X, d, μ)*, a collection* $\mathcal{F}(X)$ *of Borel functions* $\rho : X \longrightarrow [0, \infty]$*, and* $p \geq 1$*, we define the* p-modulus of Γ in (X, d, μ) over $\mathcal{F}(X)$ *by*

$$\text{mod}_p(\Gamma; \mathcal{F}(X), \mu) = \inf \int_X \rho^p \, d\mu,$$

where the infimum is taken over all functions $\rho \in \mathcal{F}(X)$ *satisfying*

$$\int_\gamma \rho \, ds \geq 1, \tag{11}$$

for all locally rectifiable curves $\gamma \in \Gamma$*. Functions* ρ *which satisfy (11) are called* admissible functions *for* Γ*. If E and F are disjoint subsets of X, we define*

$$\text{mod}_p(E, F; \mathcal{F}(X), \mu) = \text{mod}_p(\Gamma; \mathcal{F}(X), \mu),$$

where Γ *is the collection of all curves contained in X which connects E to F. For the sake of neatness, we omit the mention of* μ *if this choice is contextually obvious. If* $\mathcal{F}(X)$ *is the set of all the Borel functions* $\rho : X \longrightarrow [0, \infty]$*, then* $\text{mod}_p(\Gamma; \mathcal{F}(X))$ *will be abbreviated as* $\text{mod}_p(\Gamma)$*, and* $\text{mod}_p(E, F; \mathcal{F}(X))$ *will be abbreviated as* $\text{mod}_p(E, F)$*. We shall often write* $\text{mod}_p(x, y)$ *when it is clear that we mean* $\text{mod}_p(\{x\}, \{y\})$*, for* $x, y \in X$*.*

Proposition 6. *Let* $p \geq 1$*, and let* Γ *be a closed family of curves contained in a proper metric measure space* (X, d, μ) *with* $\mu(X) < \infty$*, such that there exists a bounded subset of X that meets every curve in* γ*. Then*

$$\text{mod}_p(\Gamma) = \text{mod}_p(\Gamma; \text{LIP}_0(X)). \tag{12}$$

Further suppose (X, d) *admits an isometric embedding* ι *into some Euclidean space* $\mathbf{R}^N$*, for some* $N \in \mathbf{N}$*, equipped with the standard Euclidean metric. Then*

$$\text{mod}_p(\Gamma) = \text{mod}_p(\iota\Gamma; C^\infty(\mathbf{R}^N), \iota_*\mu). \tag{13}$$

To prove the above theorem we shall employ ideas of the classical proof by Ziemer [Zie69] that capacity equals modulus (see also [Ric93, p.54]). We need the following well known fact, provided without proof, concerning approximation of lower semi-continuous functions by Lipschitz functions.

Lemma 5. *Let ρ be a non-negative lower semi-continuous function defined on a metric space (X, d). Then there exists an increasing sequence (ρ_n) of non-negative valued Lipschitz functions defined on (X, d), which converges to ρ pointwise.*

Proof of Proposition 6. To prove (12) it suffice to show "$\geq$", since the converse inequality is trivial. Assume $\mathrm{mod}_p(\Gamma) < \infty$, and fix $\epsilon > 0$. The definition of p-modulus asserts the existence of a Borel function $\rho : X \longrightarrow [0, \infty]$ admissible for Γ, such that

$$\int_X \rho^p \, d\mu \leq \mathrm{mod}_p(\Gamma) + \epsilon. \tag{14}$$

Since $\mu(X) < \infty$, it can be assumed without loss of generality, that $\rho \geq \delta$ for some $\delta > 0$. In addition, the Vitali-Carathéodory theorem (see [Fol99, pp.209–213]) asserts that

$$\int_X f \, d\mu = \inf \left\{ \int_X g \, d\mu : g \text{ is lower semi-continuous and } g \geq f \right\},$$

and therefore it can be further assumed without loss of generality that ρ is lower semi-continuous.

Apply Lemma 5 to get an increasing sequence (ρ_n) of real valued Lipschitz functions which converges to ρ pointwise, and which satisfies $\rho_n \geq \delta$ for each $n \in N$. The Monotone Convergence Theorem implies

$$\lim_{n\to\infty} \int_X \rho_n^p \, d\mu = \int_X \rho^p \, d\mu. \tag{15}$$

By hypotheses there exists some ball $B_0 \subset X$ such that every curve in Γ intersects B_0. If X is bounded let $B = X$, otherwise define B to be the closure of a large ball such that $d(B_0, X \setminus B) > 1/\delta$. Define $\zeta : X \longrightarrow \mathbf{R}$ by

$$\zeta(x) = \max\{0, 1 - d(x, B)\}.$$

We claim that $(\zeta\rho_n)$ is admissible for Γ in a limiting sense. That is, we claim that

$$1 \leq \limsup_{n\to\infty} \inf_{\gamma\in\Gamma} \int_\gamma \zeta\rho_n \, ds. \tag{16}$$

It then follows from the claim and (15), that we have

$$\mathrm{mod}_p(\Gamma; \mathrm{LIP}_0(X)) \leq \int_X \rho^p \, d\mu.$$

This with (14) and the fact that the choice for ϵ was arbitrary, implies "$\geq$" in (12), and so will complete the proof.

It remains to verify the claim (16). Fix a curve $\gamma_n \in \Gamma$, for each $n \in \mathbf{N}$, such that

$$\lim_{n\to\infty} \int_{\gamma_n} \zeta\rho_n \, ds \leq \limsup_{n\to\infty} \inf_{\gamma\in\Gamma} \int_\gamma \zeta\rho_n \, ds. \tag{17}$$

After passing to a subsequence it can be assumed that $\int_{\gamma_n} \zeta\rho_n \, ds < 1$ for each $n \in \mathbf{N}$, otherwise (16) holds trivially. This with the fact that $\zeta\rho_n \geq \delta$ on B, implies $\ell(\gamma_n) \leq 1/\delta$, for $n \in \mathbf{N}$. Therefore $\gamma_n \subset B$ for all $n \in \mathbf{N}$. The hypothesis that (X, d) is proper implies that B is compact. It follows from Proposition 4 that there exists a curve γ contained in B, that a subsequence of (γ_n) converges to γ as curves, and that

$$\int_\gamma \rho \, ds = \int_\gamma \zeta\rho \, ds \leq \liminf_{n\to\infty} \int_{\gamma_n} \zeta\rho_n \, ds. \tag{18}$$

Since Γ is closed, we have $\gamma \in \Gamma$, and so ρ is admissible for γ. This together with (17) and (18) implies (16). This completes the proof of (12).

To prove (13) it suffices to replace the Lipschitz functions exhibited in Lemma 5 by compactly supported smooth functions, and then use these functions in the above proof. This completes the proof of Proposition 6. □

4.2 Equivalent definitions of capacity, and equivalence with modulus

Alternate definitions for capacity will now be presented. It will then be shown that several of these definitions coincide on metric measure spaces which are geodesic and proper. For such spaces, the capacity and modulus of a pair of compacta will be shown to coincide.

Definition 11. *For two disjoint subsets E and F of a metric measure space (X, d, μ), a collection $\mathcal{F}(X)$ of Borel functions $\rho : X \longrightarrow [0, \infty]$, and $p \geq 1$, we define the* p-capacity of the pair (E, F) in (X, d, μ) and over $\mathcal{F}(X)$ *by*

$$\operatorname{cap}_p(E, F; \mathcal{F}(X), \mu) = \inf \int_X \rho^p \, d\mu,$$

where the infimum is taken over all upper gradients in $\mathcal{F}(X)$, of each real valued function $u \in \mathcal{F}(X)$ that satisfies $u|_E \leq 0$ and $u|_F \geq 1$. For the sake of neatness, we omit mention of μ if this choice is contextually obvious. If $\mathcal{F}(X)$ is the set of all the Borel functions $\rho : X \longrightarrow [0, \infty]$, then $\operatorname{cap}_p(E, F; \mathcal{F}(X))$ *will be abbreviated by* $\operatorname{cap}_p(E, F)$. *We shall often write* $\operatorname{cap}_p(x, y)$ *when it is clear we mean* $\operatorname{cap}_p(\{x\}, \{y\})$, *for $x, y \in X$.*

The following proposition generalizes a similar result of Ziemer [Zie69] concerning the equality of the capacity and the modulus for condensers in Euclidean space (see [Ric93, p.54]).

Proposition 7. *Let E and F be disjoint compact subsets of a geodesic and proper metric measure space (X, d, μ) with $\mu(X) < \infty$. Then*

$$\operatorname{mod}_p(E, F) = \operatorname{cap}_p(E, F) = \operatorname{cap}_p(E, F; \mathrm{LIP}_0(X)). \tag{19}$$

Further suppose (X, d) admits a λ-bi-Lipschitz embedding ι into some Euclidean space $\mathbf{R}^N$ equipped with the standard Euclidean metric, for some $\lambda \geq 1$. Then each of the quantities in (19) is comparable to

$$\operatorname{cap}_p(\iota E, \iota F; C^\infty(\mathbf{R}^N)),$$

with comparability constant λ^{2p}.

Remark 8. The above assumption of geodesicity can be replaced by quasiconvexity to get a weaker conclusion. That is, let E and F be disjoint compact subsets of a λ-quasiconvex and proper metric measure space (X, d, μ) with $\mu(X) < \infty$, for some $\lambda \geq 1$. Then each of the quantities in (19) is comparable with comparability constant λ^2. This is seen by passing to the length space metric, which is necessarily geodesic (see [Hei01, pp.70–71]), and observing that this λ-bi-Lipschitz change of metric preserves each of the quantities from (19) up to a multiple of λ.

The equality

$$\text{mod}_p(E, F) = \text{cap}_p(E, F)$$

holds for any sets E and F contained in a metric space (see [Hei01, Theorem 7.31]). The inequality

$$\text{cap}_p(E, F) \leq \text{cap}_p(E, F; \text{LIP}_0(X)),$$

is trivially true. Thus to prove (19) holds under the hypothesis of Proposition 7, it remains to demonstrate that

$$\text{cap}_p(E, F; \text{LIP}_0(X)) \leq \text{mod}_p(E, F).$$

This inequality is a consequence of the following two lemmas.

Lemma 6. *Let E and F be disjoint subsets of a proper metric measure space (X, d, μ) with $\mu(X) < \infty$. Then*

$$\text{cap}_p(E, F; \text{LIP}_0(X)) = \text{cap}_p(E, F; \text{LIP}\,(X)). \tag{20}$$

Proof. To prove the lemma it suffices to establish "$\leq$" in (20), since the converse inequality is trivial. Let $u, \rho \in \text{LIP}\,(X)$ be such that ρ an upper gradient for u, and such that $u|_E = 0$ and $u|_F = 1$. Fix $\delta > 0$ and let B be a large open ball with radius greater than 1, which contains F, and is such that $\mu(X \setminus B) < \delta$. Define cut-off functions $\zeta_j : X \longrightarrow [0, \infty)$ by

$$\zeta_j(x) = \max\{0, 1 - d(x, jB)\},$$

for $x \in X$ and $j = 1, 2, 3$. Define $\bar{u} : X \longrightarrow [0, 1]$ by

$$\bar{u} = \zeta_2 \max\{0, \min\{1, u\}\}.$$

Observe that because X is proper, we have $\bar{u} \in \text{LIP}_0(X)$. Moreover, we have $\bar{u}|_E = 0$ and $\bar{u}|_F = 1$, and that $\bar{\rho} = \rho\zeta_2 + \zeta_3 - \zeta_1$ is an element of $\text{LIP}_0(X)$. Notice also that $\bar{\rho}$ is an upper gradient for $\bar{u}$. This can be seen by applying the product rule and observing that $\zeta_3 - \zeta_1$ is an upper gradient for ζ_2, and that $|u| \leq 1$. Also

$$\int_X \bar{\rho}^p \, d\mu = \int_X (\rho\zeta_2 + \zeta_3 - \zeta_1)^p \, d\mu \leq \int_X \rho^p \, d\mu + \mu(X \setminus B) \leq \int_X \rho^p \, d\mu + \delta.$$

Since the choice for $\delta > 0$ was arbitrary this completes the proof. □

Lemma 7. *Let E and F be disjoint compact subsets of a geodesic and proper metric measure space (X, d, μ) with $\mu(X) < \infty$. Then*

$$\text{cap}_p(E, F; \text{LIP}\,(X)) \leq \text{mod}_p(E, F).$$

Proof. Fix a function $\rho \in \mathrm{LIP}_0(X)$ which is admissible with respect to the family Γ of all curves which connect E to F. The sets E and F are compact, and therefore Γ is closed. Thus due to Proposition 6, to complete the proof, it suffices to show that

$$\mathrm{cap}_p(E, F; \mathrm{LIP}\,(X)) \leq \int_X \rho^p \, d\mu. \tag{21}$$

Define a function $u : E \cup F \longrightarrow \{0, 1\}$ by $u = 0$ on E, and $u = 1$ on F. Since ρ is admissible for Γ, we have

$$|u(x) - u(y)| \leq \int_\gamma \rho \, ds, \tag{22}$$

for every $x, y \in E \cup F$, and every curve γ contained in X which connects x and y. Lemma 3 asserts the existence of a continuous extension $\bar{u}$ of u such that ρ is an upper gradient of $\bar{u}$. Since ρ is bounded and (X, d) is geodesic, we have $\bar{u} \in \mathrm{LIP}\,(X)$. This verifies (21) and so completes the proof. □

It remains to verify the second statement of Proposition 7 concerning the bi-Lipschitz embedding of the given metric space into Euclidean space, in order to complete the proof of Proposition 7. We begin by recalling a standard mollification procedure, and then prove a useful property of this mollification. For $f \in L^1(\mathbf{R}^n)$, use Lebesgue measure to define the mollification $f_m = f * \eta_m$, for $m \in \mathbf{N}$, where $\eta_m(x) = \eta(xm)m^n$, and $\eta : \mathbf{R}^n \longrightarrow \mathbf{R}$ is a smooth compactly supported bump function with $\int_{\mathbf{R}^n} \eta \, d\mathcal{L} = 1$. Here the convolution is taken with respect to the Lebesgue measure $\mathcal{L}$ on $\mathbf{R}^n$.

Lemma 8. *Let f and g be continuous real-valued functions defined on $\mathbf{R}^n$ with g an upper gradient for f. Then there exist sequences of mollified functions (g_m), $(f_m) \subset C^\infty(\mathbf{R}^n)$ such that (f_m) and (g_m) converge to f and g locally uniformly, respectively, and such that g_m is an upper gradient for f_m for every $m \in \mathbf{N}$.*

Proof. We shall use the mollified functions defined above. It is a standard fact that (f_m) and (g_m) converge to f and g locally uniformly, respectively. Then use the fact that g is an upper gradient for f, that g is continuous, and the Dominated Convergence Theorem to conclude

$$\begin{aligned}
\mathrm{Lip}\, f_m(x) &= \lim_{r \to 0} \sup_{0<|x-y|<r} \frac{|f_m(x) - f_m(y)|}{|x-y|} \\
&\leq \lim_{r \to 0} \sup_{0<|x-y|<r} \int_{\mathbf{R}^n} \frac{|\eta_m(z)|\,|f(x-z) - f(y-z)|}{|x-y|} \, d\mathcal{L}(z) \\
&\leq \lim_{r \to 0} \sup_{0<|x-y|<r} \int_{\mathbf{R}^n} \frac{\eta_m(z)}{|x-y|} \int_{x-z}^{y-z} g \, ds \, d\mathcal{L}(z) \\
&\leq \int_{\mathbf{R}^n} \eta_m(z) g(x-z) \, d\mathcal{L}(z) \\
&= g_m(x),
\end{aligned}$$

for every $x \in \mathbf{R}^n$. Here $\int_x^y g\,ds$ is the line integral of g from x to y, for $x, y \in \mathbf{R}^n$. Since Lip f_m is an upper gradient for f_m, the above inequality verifies that g_m is an upper gradient for f_m. This completes the proof. □

Due to Lemma 7 and Lemma 6, to complete the proof of Proposition 7, it suffices to show that

$$\mathrm{cap}_p(\iota E, \iota F; C^\infty(\mathbf{R}^n)) \leq \lambda^{2p}\mathrm{cap}_p(E, F; \mathrm{LIP}_0(X)). \tag{23}$$

Let $u, \rho \in \mathrm{LIP}_0(X)$ be such that ρ an upper gradient for u, and such that $u|_E = 0$ and $u|_F = 1$. Fix $\delta > 0$. Proposition 5 asserts the existence of continuous extensions $\bar{\rho}$ and $\bar{u}$ of $\iota_*\rho + \delta$ and $\iota_* u$ to $\mathbf{R}^N$, respectively, such that $(\lambda + \delta)\bar{\rho}$ is an upper gradient for $\bar{u}$, and such that $\bar{\rho}$ is bounded. Lemma 8 provides sequences of smooth mollified functions $(\bar{u}_m)$ and $(\bar{\rho}_m)$ of $\bar{u}$ and $\bar{\rho}$, respectively, such that $(\bar{u}_m)$ and $(\bar{u}_m)$ converge locally uniformly to $\bar{u}$ and $\bar{\rho}$, respectively, and such that $(\lambda + \delta)\bar{\rho}_m$ is an upper gradient for $\bar{u}_m$ for every $m \in \mathbf{N}$. Moreover, since $\bar{\rho}$ is bounded, we have $(\bar{\rho}_m)$ is collection of uniformly bounded functions.

Since E and F are compact, we have

$$\limsup_{m\to\infty} \max_{x\in E} \bar{u}_m \leq 0,$$

$$\liminf_{m\to\infty} \min_{x\in F} \bar{u}_m \geq 1.$$

Since $\iota_*\mu(\mathbf{R}^n) < \infty$, we have

$$\lim_{m\to\infty} \int_{\mathbf{R}^n} \bar{\rho}_m^p \, d(\iota_*\mu) = \int_{\mathbf{R}^n} \bar{\rho}^p \, d(\iota_*\mu) = \int_X (\rho + \delta)^p \, d\mu \leq \int_X \rho^p \, d\mu + C\delta,$$

where $C > 0$ is a constant which is independent of δ. This implies (23) and so completes the proof of Proposition 7.

5 The upper semi-continuity of modulus

We now prove Theorem 1, which claims that modulus is upper semi-continuous with respect to the limit of a sequence of curve families contained in a converging sequence of metric measure spaces. See Section 2.2 for the associated definitions.

Proof of Theorem 1. Assume the hypothesis of Theorem 1, let Γ be the limit supremum of (Γ_n) (see Definition 9), and further assume that $\mathrm{mod}_p(\Gamma)$ is finite (and therefore that Γ contains no degenerate curves); otherwise there is nothing to prove. The definition of convergence for compact metric measure spaces implies there exists a compact metric space (Z, l) and isometric embeddings $\iota : X \longrightarrow Z$ and $\iota_n : X_n \longrightarrow Z$ for each $n \in \mathbf{N}$, such that $(\iota_n(X_n))$ converges to $\iota(X)$ as subspaces of Z, and such that $(\iota_n)_*\mu_n$ converges to $\iota_*\mu$ in the weak sense. Identify (X_n, μ_n) and (X, μ) with $(\iota X_n, \iota_*\mu_n)$ and $(\iota X, \iota_*\mu)$, respectively.

Fix $\delta > 0$. The limit supremum of a sequence of rectifiable curves is always closed. Therefore, Proposition 6 asserts the existence of a non-negative function $\rho \in \mathrm{LIP}(X)$ admissible for Γ, such that

$$\int_X \rho^p \, d\mu \leq \mathrm{mod}_p(\Gamma) + \delta. \tag{24}$$

Since $\mu(X) < \infty$, it can be assumed without loss of generality that ρ is strictly positive. Use the McShane extension (see [Hei01, p.43]) and a truncation argument to extend ρ to be a Lipschitz function defined on Z with the property that $m = \inf_{y \in Z} \rho(y) > 0$.

The weak convergence of the sequence (μ_n) to μ implies

$$\lim_{n\to\infty} \int_Z \rho^p \, d\mu_n = \int_Z \rho^p \, d\mu.$$

Thus to complete the proof it suffices to show that

$$\lim_{n\to\infty} \mathrm{mod}_p(\Gamma_n) \leq \lim_{n\to\infty} \int_Z \rho^p \, d\mu_n. \tag{25}$$

To prove (25) it suffices to show that

$$\limsup_{n\to\infty} \inf_{\gamma\in\Gamma_n} \int_\gamma \rho \, ds \geq 1. \tag{26}$$

Suppose that (26) does not hold, in order to get a contradiction. Then there exists $\epsilon > 0$ and $N \in \mathbf{N}$ such that for each $n > N$,

$$\int_{\gamma_n} \rho \, ds < 1 - \epsilon, \tag{27}$$

for some curve $\gamma_n \in \Gamma_n$. This with the fact that $\rho \geq m > 0$ implies that $\ell(\gamma_n) \leq 1/m$ for $n > N$. Since Z is compact, Proposition 4 asserts that there exists a (possibly degenerate) curve γ contained in Z, that a subsequence of (γ_n) converges to γ, and that

$$\int_\gamma \rho \, ds \leq \liminf_{n\to\infty} \int_{\gamma_n} \rho \, ds. \tag{28}$$

Since a subsequence of (γ_n) converges to γ, the definition of Γ implies $\gamma \in \Gamma$. Estimates (27) and (28) together contradict the admissability of ρ for Γ. This completes the proof. □

6 Quasiconvexity and the Poincaré inequality

In this section it is shown under the hypotheses of Theorem 2, that each of Conditions (1), (2), (3) and (5) guarantee that the given metric measure space is quasiconvex. This proof is based loosely on the argument due to Semmes (see [Che99, Appendix]) of the fact that under the hypothesis of Theorem 2, Condition (1) implies the given metric space is quasiconvex.

Proposition 8. *Under the assumptions of Theorem 2, each of the conditions (1), (2), (3) and (5) of Theorem 2 imply* (X, d) *admits a bi-Lipschitz map onto a geodesic metric space. Here the bi-Lipschitz constant of the map depends on the constants of the assumed condition and hypotheses of Theorem 2.*

Proposition 8 will be shown to be a consequence of the following lemma.

Lemma 9. *Under the assumptions of Theorem 2, each of the conditions (1), (2), (3) and (5) of Theorem 2 imply* (X, d) *is quasiconvex with the quasiconvexity constant depending only on the constants of the assumed condition and hypotheses of Theorem 2.*

To see Proposition 8 follows from Lemma 9, assume the hypotheses of Proposition 8. Due to Lemma 9, we have (X, d) is quasiconvex with quasiconvexity constant depending only on the implicit constants of Theorem 2. Also, since μ is doubling, we have (X, d) is totally bounded. This with the fact that (X, d) is complete implies (X, d) is proper. A proper and quasiconvex metric space is bi-Lipschitz to a geodesic metric space with the constant for the bi-Lipschitz map depending only on the quasiconvexity constant, see [Hei01, pp.70–71]. This completes the proof that Proposition 8 follows from Lemma 9.

The argument that conditions (1), (2) and (3) of Theorem 2 implies the conclusion of Lemma 9 is essentially contained in [Che99, Appendix] and will not be reproduced here. To complete the proof of Lemma 9 it remains to consider the case when Condition (5) of Theorem 2 holds. Let $p \geq 1$, and let (X, d, μ) be a complete metric measure space such that μ is doubling, such that every ball in X has measure in $(0, \infty)$, such that (X, d) admits a bi-Lipschitz embedding ι into some Euclidean space $\mathbf{R}^N$, with $N \in \mathbf{N}$, equipped with the standard Euclidean metric, and such that condition (5) of Theorem 2 holds. Since the assumptions on (X, d, μ) are invariant under bi-Lipschitz maps, it can be further assumed without loss of generality that ι is an isometry. Identify X and μ with their images ιX and $\iota_*\mu$, respectively, and let d denote the standard Euclidean metric on $\mathbf{R}^N$.

Let $x, y \in A \subset \mathbf{R}^N$. We say $(x_i)_{i=0}^n \subset A$, for $n \in \mathbf{N}$, is an *ϵ-chain in A connecting x to y* if $|x_i - x_{i+1}| < \epsilon$ for $i = 0, \ldots, n-1$, and if $x_0 = x$ and $x_n = y$. Two points $x, y \in X$ are said to lie in the same *ϵ-component of A* if there exists an ϵ-chain in A connecting x to y. Observe that lying in the same ϵ-component is an equivalence relation, and that each ϵ-component is open.

Sublemma 6. *For every $\epsilon > 0$, we have X consists of only one ϵ-component.*

Proof. Any two ϵ-components have at least a distance of ϵ from one another. Consequently, the characteristic function $u : X \longrightarrow \{0, 1\}$ of one such ϵ-component can be extended as a smooth function to $\mathbf{R}^N$ with the property that $|\nabla u| = 0$ on X. Since every ball B in $\mathbf{R}^N$ which is centered in X has measure in $(0, \infty)$, this contradicts condition (5) of Theorem 2. □

Due to Sublemma 6, we can define a function $u_{\epsilon,x} : N_\epsilon(X) \longrightarrow \mathbf{R}$ by

$$u_{\epsilon,x}(y) = \inf \sum_{i=0}^{n-1} d(w_i, w_{i+1}),$$

for $\epsilon > 0$ and $x \in X$. Here the infimum is taken over all ϵ-chains $(w_i)_{i=0}^n$ in $N_\epsilon(X)$ which connect x to y. For the rest of this section, we let C denote a positive variable whose value varies in each usage, but depends only on the constants of Condition (5) and the hypotheses of Theorem 2.

Sublemma 7. *Let $\epsilon, r > 0$, and $x \in X$. Then $(u_{\epsilon,x})_{B(x,r)} \leq Cr$.*

To prove the above sublemma we use the following proposition which is essentially due to Heinonen and Koskela [HK98], and can be found in the working of [Hei01, pp.68–73].

Proposition 9. *Let $p \geq 1$, let (X, d, μ) be a metric measure space with μ doubling such that every ball contained in X has measure in $(0, \infty)$, and let u be a real-valued continuous function and ρ be a real-valued Borel function, both defined on X, which satisfy (2) with $C, \lambda \geq 1$, for every ball B in X. Then there exists $L \geq 1$ depending only on C, λ and p, such that*

$$|u(x) - u_{B(x,r)}| \leq Lr \sup_{0<s<Lr} \left(⨍_{B(x,s)} \rho^p \, d\mu \right)^{1/p},$$

for every $x \in X$ and $r > 0$.

Proof of Sublemma 7. Observe that

$$|u_{\epsilon,x}(y) - u_{\epsilon,x}(z)| \leq |y - z|,$$

whenever $y, z \in N_\epsilon(X)$ with $|y - z| < \epsilon$. Therefore $u_{\epsilon,x}$ is continuous, and moreover the constant function $w \mapsto 1$ (which we write as $\mathbf{1}$) is an upper gradient for $u_{\epsilon,x}$ on $N_\epsilon(X)$.

Lemma 4 asserts the existence of continuous extensions $\bar{u}$ and $\bar{\rho}$ to $\mathbf{R}^N$ of $u|_K$ and $\mathbf{1}|_K$, respectively, such that $\bar{\rho}$ is an upper gradient for $\bar{u}$ on $\mathbf{R}^N$. Here we let $K = \overline{B(x, \lambda r) \cap X}$. (Observe that in order to make this extension we used the fact that $\mathbf{1}$ is an upper gradient for u over $N_\epsilon(X)$, and not just X.) Lemma 8 then provides sequences of mollified functions $(u_n), (\rho_n) \subset C^\infty(\mathbf{R}^N)$ which converge uniformly on K to $\bar{u}$ and $\bar{\rho}$, respectively, such that ρ_n is an upper gradient for u_n for every $n \in N$. In particular, the sequences (u_n) and (ρ_n) converge uniformly on K to u and $\mathbf{1}$, respectively.

By hypotheses, Condition (5) of Theorem 2 holds, and therefore (2) holds for the pair $(u_n, |\nabla u_n|)$. Since μ is a doubling measure on X and $x \in X$, Proposition 9 then asserts that

$$|u_n(x) - (u_n)_{B(x,r)}| \le Cr \sup_{0<s<Cr} \left(\fint_{B(x,s)} |\nabla u_n|^p \, d\mu \right)^{1/p},$$

for $n \in \mathbf{N}$. Since ρ_n is a continuous upper gradient of u_n, we have $|\nabla u_n| \le \rho_n$ for $n \in \mathbf{N}$. Thus the proof is completed by letting $n \to \infty$, and observing that $u_{\epsilon,x}(x) = 0$. □

Sublemma 8. *Let $x, y \in X$. Then there exists a curve γ with $\ell(\gamma) \le C|x-y|$ contained in X which connects x to some point $z \in B(y, |x-y|/2)$.*

Proof. Let $r = |x-y|/5$ and fix $\epsilon > 0$. It follows from Sublemma 7 and from the fact that μ is doubling that

$$\fint_{B(y,r)} u_{\epsilon,x} \, d\mu \le Cr.$$

Thus there exists $z_\epsilon \in B(y,r)$ such that $u_{\epsilon,x}(z_\epsilon) \le Cr$. By definition this implies there exists an ϵ-chain in $N_\epsilon(X)$ connecting x to z_ϵ. This is true for every $\epsilon > 0$. Use the fact that X is proper to attain a curve γ as a limit of a subsequence of this collection of ϵ-chains. By passing to another subsequence, attain the point z as the corresponding limit of a subsequence of $\{z_\epsilon\}_{\epsilon>0}$. The curve γ and point z clearly have the desired properties and so the proof is complete. □

Sublemma 9. *Let $x, y \in X$. Then there exists a curve γ contained in X which connects x to y and satisfies $\ell(\gamma) \le C|x-y|$.*

Proof. By applying the Sublemma 8 inductively, starting with x, we attain a sequence of points (x_i) converging to y, and curves (γ_i), such that γ_i connects x_i to x_{i+1} and has $\ell(\gamma_i) \le C2^{-i}|x-y|$. Thus the curve $\gamma = \cup_i \gamma_i$ lies in X, connects x to y, and has $\ell(\gamma) \le C|x-y|$. This completes the proof. □

The above sublemma directly verifies the claim of Lemma 9, and so completes the proof of Proposition 8.

7 Equivalent characterizations of the Poincaré inequality

This section verifies Theorem 2, which claims that if the given metric measure space is complete and the given measure is doubling, several competing definitions of the Poincaré inequality coincide. We begin by defining a variant of the Riesz kernel used in the statement of Theorem 2.

Definition 12 (Symmetric Riesz kernels). *Given $C > 0$ and two distinct points x and y contained in a metric measure space (X, d, μ), we define the* symmetric Riesz kernel *of μ at x and y to be the measure μ_{xy}^C given by*

$$\mu_{xy}^C(A) = \int_{A \cap B_{xy}} \frac{d(x,z)}{\mu(B(x,d(x,z)))} + \frac{d(y,z)}{\mu(B(y,d(y,z)))} \, d\mu(z), \qquad (29)$$

for every Borel $A \subset X$. *Here we have* $B_{xy} = B(x, 2Cd(x, y)) \cup B(y, 2Cd(x, y))$. *If* $C = 1$, *we shall abbreviate* μ_{xy}^C *by* μ_{xy}.

Remark 9. Observe that μ_{xy}^C is absolutely continuous with respect to μ, and that if μ is doubling, we have $\mu_{xy}^C(X) < \infty$. This can be seen by applying the standard method of decomposing the integral in (29) over annuli centered at x and y, see [Hei01, pp.71–72].

We need the following two propositions in order to prove Theorem 2. The first proposition is essentially due to Heinonen and Koskela [HK98] and can be found in the working of [Hei01, pp.68–73].

Proposition 10. *Let* $p \geq 1$, *let* (Y, l, ν) *be a geodesic metric measure space with* ν *doubling such that every ball contained in* Y *has measure in* $(0, \infty)$, *and let* u *be a real-valued continuous function and* ρ *be a real-valued Borel function, both defined on* Y, *which satisfy (2) with* $C, \lambda \geq 1$, *for every ball* B *in* Y. *Then there exists* $L \geq 1$ *depending only on* C, λ, p, *and the doubling constant of* ν, *such that*

$$|u(x) - u(y)|^p \leq L l(x, y)^{p-1} \int_Y \rho^p \, d\nu_{xy}^L, \tag{30}$$

for every pair of distinct points $x, y \in Y$.

Proposition 11. *Let* $p \geq 1$ *and let* (Y, l, ν) *be a metric measure space with* ν *doubling such that every ball contained in* Y *has measure in* $(0, \infty)$. *Further suppose that there exists* $L \geq 1$ *which satisfies the following property: For every pair of functions* $u : Y \longrightarrow [0, \infty)$ *and* $\rho : Y \longrightarrow [0, \infty]$, *where* u *is measurable, and where* ρ *is an upper gradient for* u, *we have Equation (30) holds for* ν *almost every pair of distinct points* $x, y \in Y$. *Then* (Y, l, ν) *admits a* p*-Poincaré inequality for all measurable functions, with constants depending only on* L, p, *and the doubling constant of* ν.

Proposition 11 follows from a combination of results due to Heinonen and Koskela, and Semmes. To see this, assume the hypotheses of Proposition 11. It follows from [Hei01, Theorem 9.5] that (2) holds with $C, \lambda \geq 1$ depending only on L and p, for every ball B contained in Y and all functions $u : Y \longrightarrow [0, \infty)$, $\rho : Y \longrightarrow [0, \infty]$, where u is continuous, and where ρ is an upper gradient for u. By an argument of Semmes [Che99, Appendix], we have (Y, l) is quasiconvex (see also Proposition 8). Therefore [Hei01, Theorem 9.22] implies that the conclusion of Proposition 11 holds. This completes the proof of Proposition 11. Another more direct way to verify Proposition 11 would be to apply the arguments in [HK99] for proving [Hei01, Theorem 9.22] directly to each given pair of functions u, ρ in the hypotheses of Proposition 11.

Theorem 2 is directly verified by the following three lemmas.

Lemma 10. *Under the hypotheses of Theorem 2, Condition (4) $\Longrightarrow$ Condition (1).*

Proof. Let u be a real-valued measurable function defined on X, and let ρ be an upper gradient for u. For distinct points $x, y \in X$ with $u(x) \neq u(y)$, define the function $\bar{u}$ by

$$\bar{u}(z) = \left| \frac{u(z) - u(x)}{u(x) - u(y)} \right|,$$

for every $z \in X$, and define $\bar{\rho}$ by

$$\bar{\rho}(z) = \frac{\rho(z)}{|u(x) - u(y)|},$$

for every $z \in X$. By applying the triangle inequality, we see that $\bar{\rho}$ is an upper gradient for $\bar{u}$. Observe that $\bar{u}(x) = 0$ and $\bar{u}(y) = 1$.

Condition (4) together with the discussion following Theorem 7 implies that

$$d(x, y)^{1-p} \leq C \operatorname{cap}_p(x, y; \mu_{xy}^C).$$

Here and after $C \geq 1$ is a constant depending only on the constants of Condition (4) and the hypotheses of Theorem 2. It therefore follows from the definition of capacity that

$$d(x, y)^{1-p} \leq C \int_X \bar{\rho}^p \, d\mu_{xy}^C. \tag{31}$$

Rewriting this equation in terms of u and ρ yields (30). This with Proposition 11 proves the result. □

Lemma 11. *Under the hypotheses of Theorem 2, Condition (1) $\Longrightarrow$ Condition (3) $\Longrightarrow$ Condition (2) $\Longrightarrow$ Condition (4).*

Proof. Proposition 8 asserts that if (X, d, μ) admits Conditions (1), (2) or (3) of Theorem 2 then (X, d) admits a bi-Lipschitz mapping onto a geodesic and proper metric space. Moreover, the bi-Lipschitz constant of the map depends on the constants of the assumed Condition and hypotheses of Theorem 2. Conditions (1), (2), (3) and (4) of Theorem 2 are quantitatively preserved under a bi-Lipschitz maps. Thus it can be assumed without loss of generality that (X, d) is geodesic and proper.

The implication that Condition (1) implies Condition (3) follows from the fact that if $u \in \mathrm{LIP}(X)$, we have $\mathrm{Lip}\, u$ is an upper gradient for u.

Suppose Condition (3) holds and let $u, \rho \in \mathrm{LIP}(X)$ with ρ an upper gradient for u. To prove Condition (2) it suffices to show $\mathrm{Lip}\, u \leq \rho$. Let γ be a geodesic in X connecting two distinct points $x, y \in X$. Then since ρ is an upper gradient of u,

$$\frac{|u(x) - u(y)|}{d(x, y)} \leq \fint_{\gamma} \rho \, ds.$$

Let $x \to y$ and use the fact that ρ is continuous to see that $\mathrm{Lip}\, u(x) \leq \rho(x)$.

Finally suppose Condition (2) holds. Condition (2) together with Proposition 10 implies

$$d(x, y)^{1-p} \leq C \int_X \rho^p \, d\mu_{xy}^C,$$

whenever $x, y \in X$ are distinct points, and whenever $u, \rho \in \mathrm{LIP}_0(X)$ are such that ρ is an upper gradient for u, and $u(x) = 0$ and $u(y) = 1$. Here $C \geq 1$ is a constant depending only on the constants of Condition (2) and the hypothesis of Theorem 2. Thus, we have

$$d(x, y)^{1-p} \leq C\mathrm{cap}_p(x, y; \mathrm{LIP}_0(X), \mu^C_{xy}).$$

Condition (4) then follows from Proposition 7 and the fact that $\mu^C_{xy}(X) < \infty$. □

Lemma 12. *Under the hypotheses of Theorem 2, suppose (X, d) admits a bi-Lipschitz embedding ι into some Euclidean space $\mathbf{R}^N$, where $N \in \mathbf{N}$, equipped with the standard Euclidean metric. Then Condition (1) $\Longrightarrow$ Condition (5) $\Longrightarrow$ Condition (4).*

Proof. Condition (1) of Theorem 2 implies Condition (5) because $\mathrm{Lip}\, u \circ \iota^{-1} \leq C|\nabla u|$ for every $u \in C^\infty(\mathbf{R}^N)$. Here $C \geq 1$ denotes the bi-Lipschitz constant for ι.

Suppose Condition (5) holds. With a similar argument to the proof of Lemma 11, and again due to Proposition 8, it can be assumed without loss of generality that (X, d) is geodesic and proper. Condition (5) together with Proposition 10 then implies

$$d(\iota x, \iota y)^{1-p} \leq C \int_X |\nabla u|^p \, d(\iota_*\mu)^C_{\iota x, \iota y},$$

whenever $x, y \in X$ are distinct points, and whenever $u \in C^\infty(\mathbf{R}^N)$ is such that $u(x) = 0$ and $u(y) = 1$. Here and below $C \geq 1$ is a varying constant whose value depends on the constants of Condition (5) and the doubling constant of μ. Therefore

$$d(\iota x, \iota y)^{1-p} \leq C\mathrm{cap}_p(x, y; C^\infty(\mathbf{R}^N), (\iota_*\mu)^C_{\iota x, \iota y}).$$

Since ι is bi-Lipschitz and μ is doubling, we have $(\iota_*\mu)^C_{\iota x, \iota y}$ is comparable to $\iota_*(\mu^C_{xy})$ with comparability constant C. Condition (5) of Theorem 2 then follows from Proposition 7 and the fact that $\mu^C_{xy}(X) < \infty$ (see Remark 9). □

Remark 10. To establish Remark 4 we need only make a small change to the proof of Lemma 10. The proof of Lemma 10 demonstrates that under the hypothesis of Theorem 2, Condition (4) implies that (31) holds for every pair of distinct points $x, y \in X$. The same argument shows again that under the hypothesis of Theorem 2, the condition presented in Remark 4 implies that (31) holds for almost every (and *a priori* not every) pair of distinct points $x, y \in X$. Fortunately, this is all that Proposition 11 requires. Therefore Condition (1) holds. This completes the verification of Remark 4.

8 The persistence of the Poincaré inequality under converging metric measure spaces

In this section we prove Theorem 3, which claims that if the associated measures are doubling with uniformly bounded constants, the Poincaré inequality persists under the convergence of pointed metric measure spaces. Remark 5 can also be readily deduced from the following proof.

Proof of Theorem 3. We begin the proof by recalling some facts. A complete metric measure space which admits a Poincaré inequality with a doubling measure is bi-Lipschitz to a geodesic space (see Proposition 8). Moreover the bi-Lipschitz constant of the map depends on the constants of the Poincaré inequality and the doubling measure. The property of admitting a Poincaré inequality and of a measure being doubling are both invariant under bi-Lipschitz maps. The limit of a sequence of geodesic metric spaces, is itself geodesic (see Remark 7). Bi-Lipschitz maps pass to the limit (see Remark 6). Therefore, in addition to assuming the hypotheses of Theorem 3, we can further assume without loss of generality that $\{(X_n, d_n)\}$ is a sequence of geodesic spaces, and consequently that (X, d) is geodesic.

The definition of convergence of pointed metric measure spaces implies there exists a proper pointed metric space (Z, ρ, l) and isometric embeddings $\iota : X \longrightarrow Z$ and $\iota_n : X_n \longrightarrow Z$ for each $n \in \mathbf{N}$, such that $\iota(q) = \iota_n(q_n) = l$, such that $(\iota_n(X_n))$ converges to $\iota(X)$ as subspaces of Z, and such that $(\iota_n)_*\mu_n$ converges to $\iota_*\mu$ in the weak sense. Identify (X_n, q_n, μ_n) and (X, q_n, μ) with $(\iota X_n, \iota q_n, \iota_*\mu_n)$ and $(\iota X, \iota q, \iota_*\mu)$, respectively.

We wish to show that (X, d, μ) admits a p-Poincaré inequality. Due to Theorem 2, it suffices to show that (2) holds whenever B is a ball in X, and whenever $u, \rho \in \mathrm{LIP}_0(X)$ with ρ an upper-gradient for u. Fix such a pair of functions, let B be a ball with center $x \in X$ and radius $r > 0$, and let $\delta > 0$. Proposition 5 asserts the existence of continuous extensions $\bar{u}$ and $\bar{\rho}$ of u and $\rho + \delta$ to Z, respectively, such that $(1+\delta)\bar{\rho}$ is an upper gradient for $\bar{u}$. It follows from the fact that (X_n) converges to X as subspaces of Z that there exists a sequence of points (x_n) converging to x in Z, with the property that $x_n \in X_n$ for each $n \in \mathbf{N}$. Write $B_n = B(x_n, r)$. Observe that there exists $N \in \mathbf{N}$ such that $2B \subset 4B_n \subset 6B$ and $2CB \subset 4CB_n \subset 6CB$, whenever $n > N$. Here and after $C > 10$ denotes a large positive constant whose value does not vary with each usage, and depends only on the constants of the p-Poincaré inequalities admitted by each element of $\{(X_n, d_n, \mu_n)\}$ and the doubling constant for each element of (μ_n). Thus our hypotheses that (X_n, d_n, μ_n) admits a p-Poincaré inequality implies that

$$
\begin{aligned}
\fint_{2B} |\bar{u} - \bar{u}_{4B_n}|\, d\mu_n &\le C \fint_{4B_n} |\bar{u} - \bar{u}_{4B_n}|\, d\mu_n \frac{\mu_n(4B_n)}{\mu_n(2B)} \\
&\le Cr \left(\fint_{4CB_n} (1+\delta)^p \bar{\rho}^p \, d\mu_n \right)^{1/p} \frac{\mu_n(6B)}{\mu_n(2B)} \\
&\le Cr \left(\fint_{6CB} (1+\delta)^p \bar{\rho}^p \, d\mu_n \right)^{1/p} \frac{\mu_n(6B)}{\mu_n(2B)} \left(\frac{\mu_n(6CB)}{\mu_n(4CB)} \right)^{1/p}
\end{aligned}
\tag{32}
$$

for $n > N$.

Recall (μ_n) is a sequence of doubling measures with uniformly bounded doubling constants, and that (μ_n) weak converges to μ. Therefore μ is doubling with doubling constant depending only on the uniform bound (see Proposition 3). With arguments similar to the proof of Proposition 3, the following statement can be easily deduced from the definition of weak convergence and the fact that μ is doubling: For every $\beta > 0$, there exists $M \in \mathbf{N}$ such that for $m > M$, the values

$\mu_n(\beta B)$ and $\mu(B)$ are comparable, with comparability constant depending only on β and the doubling constant of μ (and therefore depending only on the previous uniform bound for the doubling constants of (μ_n)). Also, after passing to a subsequence of (μ_n) we can arrange for $\{(\bar{u}_n)_{B_n}\}$ to converge to some $\alpha \in \mathbf{R}$. These properties of the sequence (μ_n) together with (32) imply that

$$\fint_B |\bar{u} - \alpha| \, d\mu \leq Cr \left(\fint_{8CB} (1+\delta)^p \bar{\rho}^p \, d\mu \right)^{1/p}.$$

Since μ is supported on X, then

$$\fint_B |u - u_B| \, d\mu \leq (1+\delta) Cr \left(\fint_{8CB} (\rho + \delta)^p \, d\mu \right)^{1/p}.$$

This completes the proof. □

Remark 11. Remark 5 stated that if we knew more information about the limit metric space, the doubling hypothesis for the measures in Theorem 3 could be relaxed without sacrificing the conclusion of the theorem. We can also relax the assumption that each of $\{(X_n, d_n, \mu_n)\}$ admits a Poincaré inequality. For example, instead of assuming the Poincaré inequality, we can ask that for each continuous function and its upper gradient, that the given sequence of metric measure spaces eventually observes equations like (2). This reversal of the quantifiers can be practical if we know, for example, that each of the given metric measure spaces admit a Poincaré inequality when viewed at a big enough resolution. And that this resolution converges to zero, as the sequence of spaces converges to the limit space. By resolution, we mean the precision with which we view the metric space, akin to the notion of approximating a metric space by epsilon nets (see [Gro99]).

This view of the Poincaré inequality is nicely expressed by the definition for admitting Poincaré inequalities due to Semmes [Sem01, pp.15–16]. Recall as stated in the introduction, Rajala and the author have shown for complete metric measure spaces equipped with a doubling measure, that this definition of Semmes is equivalent to Definition 1.

9 New doubling measures on Euclidean space which admit a 1-Poincaré inequality

In this section we prove Theorem 4, which claims that certain measures supported on uniform domains in Euclidean space, are doubling and admit a 1-Poincaré inequality. To do this we verify Condition (4) and the hypotheses of Theorem 2. This is achieved by using the fact that every two distinct points contained in a uniform domain is connected by a large collection of curves each of which satisfies the following: The curve has length comparable to the distance between the two points, and most of the curve does not pass too close to the boundary of the domain. We begin by recalling the definition of uniform domains.

Definition 13 (Uniform domains). *A domain $\Omega \subset \mathbf{R}^n$, for $n \geq 2$, is said to be* uniform *if there is a constant $C > 1$ such that every pair of distinct points $x, y \in \Omega$ can be joined by a curve γ in Ω whose length does not exceed $C|x - y|$, and is such that*

$$d(z, \partial\Omega) \geq C^{-1} \min\{|x - z|, |y - z|\},$$

for every $z \in \gamma$.

To prove Theorem 4 we use the following rephrased theorem of Martin. Recall that a function $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, for $n \in \mathbf{N}$, is said to be *L-conformally bi-Lipschitz* with scale factor λ, for $L \geq 1$ and $\lambda > 0$, if

$$L^{-1}\lambda|x - y| \leq |f(x) - f(y)| \leq L\lambda|x - y|,$$

for all $x, y \in \mathbf{R}^n$. We denote the standard orthonormal basis of $\mathbf{R}^n$ by $\{e_i\}_{i=1}^n$.

Theorem 10 ([Mar85, Theorem 5.1]). *Let x and y be distinct points in contained in the closure of a uniform domain $\Omega \subset \mathbf{R}^n$, with $n \in \mathbf{N}$. Then there exits $B_{xy} \subset \overline{\Omega}$, and there exists a surjective L-conformally bi-Lipschitz map $\iota : \overline{B(0, 1)} \longrightarrow B_{xy}$, such that $\iota(-e_1) = x$ and $\iota(e_1) = y$, and such that*

$$d(z, \mathbf{R}^n \setminus B_{xy}) \geq \min\{|x - z|, |y - z|\}/L,$$

for every $z \in B_{xy}$. Here $L \geq 1$ is a constant that depends only on the uniform domain constant of Ω.

Assume the hypotheses of Theorem 4. The following lemma verifies that the hypotheses of Theorem 2 is satisfied by (Ω, d, μ). Here we abbreviate $\mu_{\Omega,\alpha}$ by μ. For the remainder of this section $C > 0$ denotes a constant whose value varies in each usage, but depends only on n, α, and the uniform domain constant of Ω.

Lemma 13. *The measure μ is doubling with doubling constant depending only on C.*

Proof. Take $x \in \overline{\Omega}$ and $r > 0$. It needs to be shown that

$$\mu(B(x, 2r)) < C\mu(B(x, r)). \tag{33}$$

Consider the case when $B(x, 3r) \setminus \Omega \neq \emptyset$. Then $\mu(B(x, r)) \leq Cr^{\alpha+n}$. Fix $y \in B(x, 2r) \setminus B(x, r)$; if there are no such points then (33) is trivially true. Let B_{xy} be defined as in Theorem 10, and observe that there exists a ball $B \subset B_{xy} \cap B(x, r)$ with radius at least Cr, and such that $d(B, \partial\Omega) > Cr$. Therefore $\mu(B(x, r)) > Cr^{\alpha+n}$, and so (33) holds. Now consider the case when $B(x, 3r) \subset \Omega$. Then $d(B(x, r), \partial\Omega) > Cd(x, \partial\Omega)$, and therefore $\mu(B(x, r)) > Cr^n d(x, \partial\Omega)^\alpha$. However, we have $d(B(x, 2r), \partial\Omega) < Cd(x, \partial\Omega)$, and so $\mu(B(x, 2r)) < Cr^n d(x, \partial\Omega)^\alpha$, proving (33). This completes the proof. □

The following two lemmas will be used to show that $(\overline{\Omega}, d, \mu)$ admits a 1-Poincaré inequality. Recall that for Radon measures ν and τ on $\mathbf{R}^n$, the *derivative* of ν with respect to τ at $z \in \mathbf{R}^n$ is given by

$$\frac{D\nu}{D\tau}(z) = \lim_{r \to 0} \frac{\nu(B(z,r))}{\tau(B(z,r))},$$

and exists for τ almost every z. In the following we denote Lebesgue measure on $\mathbf{R}^n$ by $\mathcal{L}$.

Lemma 14. *We have*

$$\frac{D\mu_{xy}}{D\mathcal{L}_{xy}}(z) \geq C,$$

for every pair of distinct points $x, y \in \overline{\Omega}$, and for Lebesgue almost every $z \in B_{xy}$. (Here μ_{xy} and $\mathcal{L}_{xy}$ are the symmetric Riesz kernels given by Definition 12.)

Proof. Write $r = d(x, y)$, let z belong to the interior of $B_{xy} \setminus \{x, y\}$, and fix a ball B centered at z, which is sufficiently small so that $B \subset B_{xy}$ and

$$\operatorname{diam} B < \min\{|x - z|, |y - z|\}/(2L). \tag{34}$$

Recall L is defined in Theorem 10. To prove the lemma it suffices to show that

$$\frac{\mathcal{L}(B)}{\mathcal{L}(B(x, |x - z|))} \leq C \frac{\mu(B)}{\mu(B(x, |x - z|))}. \tag{35}$$

Consider the case when $B(x, 2r) \setminus \Omega \neq \emptyset$. Then

$$\mu(B(x, |x - z|)) < Cr^{\alpha} \mathcal{L}(B(x, |x - z|)).$$

A consequence of (34) and the definition of B_{xy} is that $\mu(B) > Cr^{\alpha}\mathcal{L}(B)$. These last two equations together establish (35). Now consider the case when $B(x, 2r) \subset \Omega$. Then $d(B, \partial\Omega) > Cd(x, \partial\Omega)$, and thus

$$\mu(B(x, r)) > C\mathcal{L}(B)d(x, \partial\Omega)^{\alpha}.$$

However $d(B(x, r), \partial\Omega) < Cd(x, \partial\Omega)$, and so

$$\mu(B(x, |x - z|)) < C\mathcal{L}\,(B(x, |x - z|))\, d(x, \partial\Omega)^{\alpha}.$$

These last two equations together prove (35). This completes the proof. □

In the following, the restriction of a measure μ to a set A will be denoted by $\mu \llcorner A$.

Lemma 15. *We have*

$$\operatorname{mod}_1(x, y; \mathcal{L}_{xy} \llcorner B_{xy}) > C,$$

for every pair of distinct points $x, y \in \overline{\Omega}$.

Proof. Let $\iota : \overline{B(0,1)} \longrightarrow B_{xy}$ be the surjective L-conformally bi-Lipschitz map provided by Theorem 10, and let $\lambda > 0$ be the scale factor of ι. Let $\rho : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a Borel function that is admissible for the collection of curves connecting x to y in $\mathbf{R}^n$. Then $\lambda L\rho \circ \iota$ is admissible for the collection of curves contained in $\overline{B(0,1)}$ which connect $-e_1$ to e_1. In particular $\lambda L\rho \circ \iota$ is admissible for each curve $\gamma_x : [0,2] \longrightarrow \overline{B(0,1)}$ given by

$$\gamma_x(t) = -e_1 + te_1 + \min\{t, 2-t\}y,$$

whenever $t \in [0,2]$ and $y \in Y$, and where

$$Y = \{y \in B(0,1) :< y, e_1 >= 0\}.$$

Since $\{\gamma_y\}_{y \in Y}$ is a collection of essentially disjoint curves, it follows by applying polar coordinates and Fubini's Theorem, that

$$\int_{B(0,1)} \lambda\rho \circ \iota(w) \left(\min\{|w-e_1|, |w+e_1|\}\right)^{1-n} \, d\mathcal{L}(w) > C.$$

The area formula together with the fact that ι is L-conformally bi-Lipschitz with scale factor λ, implies that

$$\int_{B_{xy}} \lambda\rho(z) \min\{|z-x|/\lambda, |z-y|/\lambda\}^{1-n} \, d\mathcal{L}(z)/\lambda^n > C.$$

Since λ cancels, this completes the proof. □

It follows from Lemma 14, Lemma 15, the definition of modulus, and the Radon-Nikodym theorem for Radon measures, that

$$\mathrm{mod}_1(x, y; \mu_{xy} \llcorner B_{xy}) \geq C,$$

for every pair of distinct points $x, y \in \overline{\Omega}$. Therefore

$$\mathrm{mod}_1(x, y; \mu_{xy}) \geq C,$$

for every pair of distinct points $x, y \in \overline{\Omega}$. The metric space $(\overline{\Omega}, d, \mu)$ thus satisfies Condition (4) of Theorem 2. This completes the proof of Theorem 4.

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