Report No. EMB-3

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# THE APPLICATION OF THE THREE-DIMENSIONAL METHOD OF CHARACTERISTICS TO THE SOLUTION OF THE FLOW ABOUT AN AXIALLY SYMMETRIC BODY OF REVOLUTION.

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## Introduction

A previous report dealt with this method as applied to a nozzle solution (Ref. 1). The solution as discussed in Reference 1 did not take into account an occurrence of a shock wave such as appears at the nose of a missile moving at supersonic velocity. This report discusses the application of the three-dimensional method of characteristics to problems which have a shock wave present. The occurrence of a shock wave necessitates the use of a shock polar in the velocity plane which serves to satisfy the demands of the Rankine-Hugoniot equations relating flow velocities before and after the shock wave and shock inclination.

This report will cover the application of this method to finding the velocity field around an axially symmetric body of revolution moving at a supersonic velocity. Because known relations exist connecting the velocity at the surface of a missile with the pressure at the surface, this solution can be used to obtain the pressure distribution over the body.

The solution will determine a Mach net made up of the two families of characteristic lines in the physical plane and the velocities associated with each intersection of the characteristic lines. The angle of the shock wave will also be determined as it decreases from the shock angle associated with the tip of the missile investigated to the free stream Mach angle at large distances from the tip. This procedure is a step-by-step solution in a meridian plane.

## Symbols

$\alpha$ = local Mach angle = $\sin^{-1} 1/M$

c = local velocity of sound

c* = velocity of sound when M = 1

$\gamma = \frac{C_p}{C_v} \cong 1.4$ for air

M = Mach number = $\frac{w}{c}$

u = component of velocity along the local Mach line = $w \cos\alpha$

v = component of velocity across the local Mach line = $w \sin\alpha$

w = local free stream velocity

$W_{max}$ = maximum velocity attainable by exhausting adiabatically into a vacuum.

## Discussion

This graphical-numerical solution makes use of a physical plane and a velocity plane. These two planes are connected by difference equations which will be discussed later. The method requires a large amount of arithmetical calculation and careful measuring. It is recommended that a scale of at least 1 inch = 3 inches be used in the physical plane, and a scale of at least 10 inches for the critical velocity of $w^* = \frac{w}{c^*} = 1$ be used in the velocity plane. If a larger size surface than the standard drafting table is available, it should be used in order that the scale be as large as possible. The large scale helps to eliminate inaccuracies that can occur when measuring the small quantities that arise in using the difference equations. Tables are used throughout the method of keep the procedure as orderly as possible and to prevent redundant calculations.

The fundamental difference equations employed throughout this graphical solution arise from a mathematical manipulation of the hyperbolic supersonic differential equation for axially symmetrical flow. R. Sauer and others have treated the mathematical background thoroughly and methodically in several references. References 2 and 4 present such discussions in detail.

The working equations derived through manipulation of the axially symmetrical supersonic differential equation are given below. These equations connect the physical plane and the velocity plane.

$$dp = \frac{v}{r} \sin^2\alpha \; d\eta \qquad (1a)$$

$$dq = \frac{v}{r} \sin^2\alpha \; d\xi \qquad (1b)$$

where, as illustrated in Figure 1,

- dp - increment of the component of stream velocity in the direction of the local Mach line at the angle $(\delta - \alpha)$ with x axis.
- dq - increment of the component of stream velocity in the direction of the local Mach line at the angle $(\delta + \alpha)$ with x axis.
- v - radial component of the stream velocity or component perpendicular to x axis.
- $\delta$ - angle of stream velocity with the x axis.

$\alpha$ - local Mach angle = $\sin^{-1} \frac{1}{M}$

r - the radial distance of a point in the physical field from the x axis.

$d\eta$ - length of side of Mach quadrangle in the Mach line direction at the angle ( $\delta + \alpha$ ) with x axis.

$d\xi$ - length of side of Mach quadrangle in the Mach line direction at the angle ( $\delta - \alpha$ ) with x axis.

x axis - axis of symmetry.

All of these terms appear in Figure 1. Presume that the velocities of points 1 and 2 are known and that the velocity at point 3 is desired. By making use of equations (1) which connect the physical and velocity planes the velocity at unknown point 3 can be determined. That is, through measuring v, r, $d\eta$, $\alpha$, and $d\xi$, enough quantities in equation (1) are known to determine $dp_{13}$ and $dq_{23}$ which locate the velocity of point 3 in the velocity plane.

Before proceeding with a detailed discussion of the various techniques employed in this solution, a few recommendations will be made.

1. Use a graph paper with lines spaced 20 to the inch if a scale for $W^* = 1$ of 10 inches is used.

2. Use a drafting machine with the scales marked in tenths of an inch and finer divisions. This makes angular measurements and fractional measurements simpler.

3. Use tables throughout to keep the numerous measurements and calculations in order and prevent needless repetition of measurements. Figure 2 illustrates the scheme of such a table.

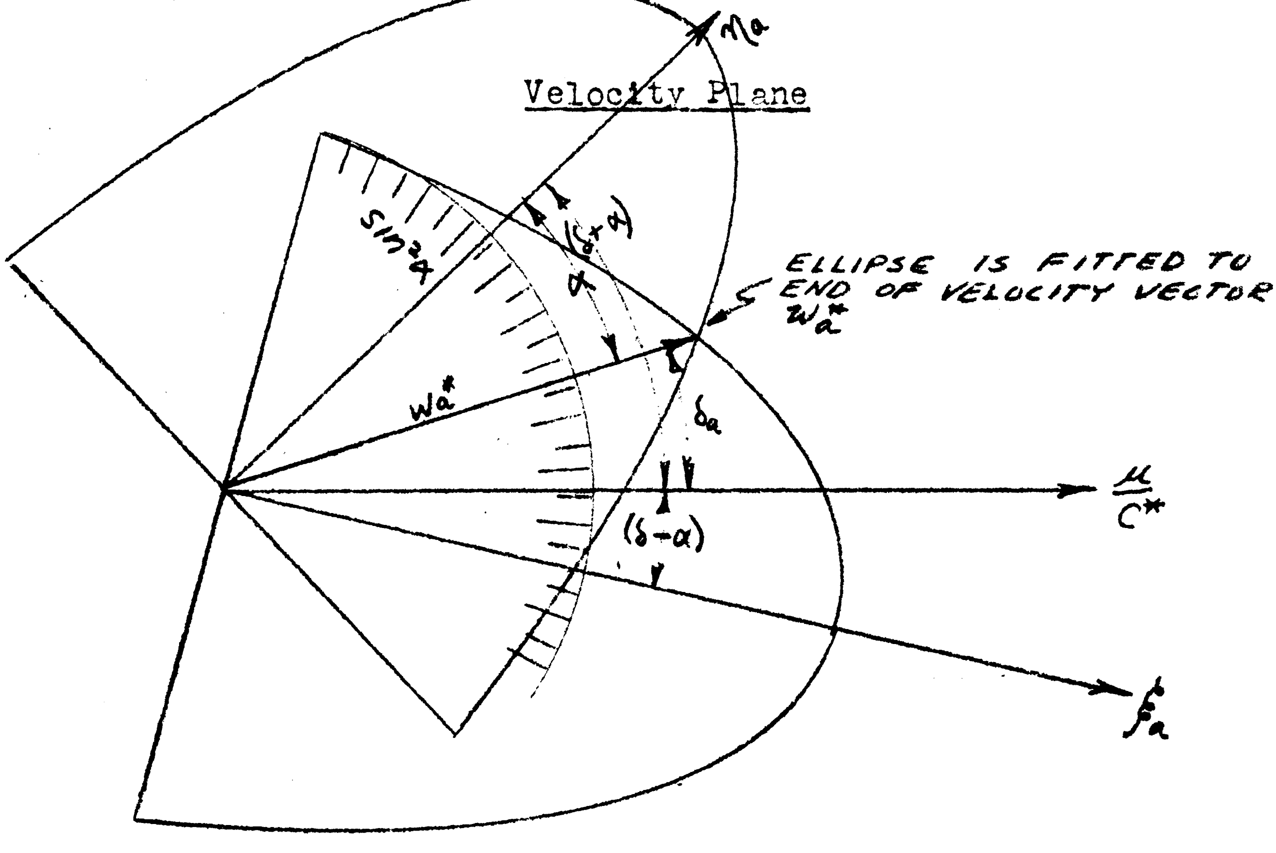
4. Use the "adiabatic ellipse" throughout to measure $\sin^2\alpha$ in the velocity plane. This eliminates a tremendous amount of calculation which would occur if these quantities were to be determined individually. The ellipse should be drawn on transparent paper. The adiabatic ellipse is discussed in simple detail in Reference 4, pages 55-56. The equation for the ellipse arises from the energy equation in the form:

$$\frac{u^2 + v^2}{2} + \frac{c^2}{\gamma - 1} = \frac{w^2_{max}}{2} \qquad (2)$$

This can be reduced to the equation of the ellipse:

$$\frac{u^2}{w^2_{max}} + \frac{v^2}{c^{*2}} = 1 \qquad (3)$$

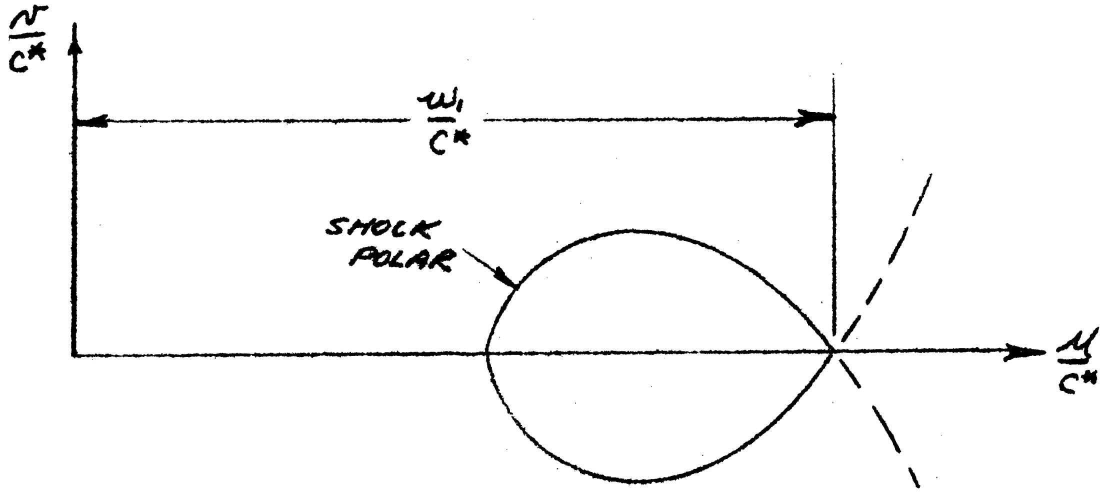
Because the velocities in the velocity plane are represented in terms of $c^*$, the velocity of sound when $M = 1$, plot the semi-major axis as $\frac{u}{c^*} = \sqrt{\frac{\gamma + 1}{\gamma - 1}} = 2.45$, and the semi-minor as $\frac{v}{c^*} = 1$ to be consistent. This ellipse can be utilized in the velocity plane to determine $\alpha$ and $\sin^2 \alpha$ for any particular velocity vector as shown in the sketch below:




When the ellipse is adjusted to a velocity vector as shown above, the local Mach angle $\alpha$ and the corresponding $\sin^2 \alpha$ can be read on the scale as the angle between the velocity vector and the major axes of the ellipse. $\sin^2 \alpha$ appears in the difference equations (1).

In the solution, use is made of the shock polar in the velocity plane. This polar depends upon the free stream Mach number $M_1$ and is constructed in the velocity plane using the geometrical construction of a strophoid described in Reference 4, pp. 107-110, expressing all velocities in terms of c*. This diagram will appear in the velocity plane as shown by the sketch below.

Velocity Plane




To make use of this method, the velocities of at least two points behind the shock wave must be known. Taylor and Maccoll in Reference 7 have presented the solution for an axially symmetric cone. Use is made of this valuable solution to give the velocities at certain points behind the shock by approximating the tip of the missile to be investigated with a cone. This can be done without sacrificing an undesirable amount of accuracy in the region of the tip. Reference 5 presents this data in excellent detail for a large range of Mach numbers.

After using the information from the above sources to determine two or more initial velocities behind the shock, the solution employing the fundamental working equations (1) proceeds. During the solution, three distinct cases occur which differ in detail in their handling. These are:

1. The case where the velocity to be determined lies in the free stream behind the shock wave.

2. The case where the velocity to be determined lies on the boundary or surface.

3. The case where the velocity to be determined lies on the shock wave.

Each of these different cases merits a discussion on the technique of handling them when they arise.

Consider Case 1. Assume the velocities are known at points 1 and 2 and the velocity at point 3 is to be determined as is shown in Figure 1. Here use is made of the table in Figure 2. Fill in columns (1) through (6) for points 1 and 2 by measuring the appropriate quantities in the physical and velocity plane as shown in Figures 1 and 2. Next, roughly estimate the position of the end point of velocity vector 3 in the velocity plane and thus establish the approximate values of columns (1), (2), (3), (4), and (5) for point 3. Now take the mean value of $(\delta + \alpha)$ and $(\delta - \alpha)$ for points 1 and 3 and points 2 and 3 and put in columns (9) and (10) for row $\overline{13}$ and $\overline{23}$, i.e.

$$(\delta + \alpha)_{\overline{23}} = \frac{(\delta + \alpha)_2 + (\delta + \alpha)_3}{2}$$

Construct the first approximation to the Mach quadrangle by drawing $\xi_{\overline{23}}$ line from 2 at angle $(\delta - \alpha)_{\overline{23}}$ and $\eta_{\overline{13}}$ line from 1 at angle $(\delta - \alpha)_{\overline{13}}$. The intersection of these two lines locates the first approximation to point 3 in the physical plane and enables the value of $r_3$ for point 3 in column (6) to be measured as well as the values $d\xi_{\overline{23}}$ and $d\eta_{\overline{13}}$. That is, $d\xi_{\overline{23}}$ is the distance between points 2 and 3 and $d\eta_{\overline{13}}$ is the distance between points 1 and 3. These values are put in proper columns for $\overline{13}$ and $\overline{23}$ as shown in Figure 2. Now all columns (1) through (6) are filled in for rows $\overline{13}$ and $\overline{23}$ by taking the mean of the values for points 1 and 3 and 2 and 3. Then all of the quantities are determined for equations (1). That is, for $\overline{13}$ and $\overline{23}$ columns (1) and (12):

$$dp_{\overline{13}} = \frac{V_{\overline{13}}}{r_{\overline{13}}} \sin^2\alpha_{\overline{13}} \, d\eta_{\overline{13}} \qquad (1a)$$

$$dq_{\overline{23}} = \frac{V_{\overline{23}}}{r_{\overline{23}}} \sin^2\alpha_{\overline{23}} \, d\xi_{\overline{23}} \qquad (1b)$$

Now, $dp_{\overline{13}}$ is layed off along a line from point 1 in the velocity plane in direction $(\delta - \alpha)_{\overline{13}}$. A perpendicular is from the endpoint as shown in Figure 1. $dq_{23}$ is layed from point 2 in the direction $(\delta + \alpha)_{23}$ and a perpendicular erected from the end point. The intersection of the perpendiculars locates velocity point 3. If the velocity point 3 corresponds closely with the first estimated location, no further calculation is needed. However, this occurs rarely and further calculations are necessary. Using the values of this velocity point 3 in columns (1) through (5), the process described above is repeated. If the results of this recalculation yields a point 3 closely approximating the location of the previously calculated point 3, then no further calculations are needed as point 3 is determined within the accuracy of this method. All points in the flow field behind the shock wave not on either the boundary or shock wave are determined in this manner.

Consider case 2 where the point to be determined lies on the boundary. Referring to Figure 3, assume that the velocity has been determined at point 5 and that the velocity at some point 6 on the surface is to be determined. Columns (1) through (6) and (9) and (10) in Figure 2 for known point 5 can be filled in. Draw a line from point 5 at angle $(\delta - \alpha)_5$ to determine a first approximation to the location of point 6 on the surface, $(6_1)$. Measure the angle from the axis of symmetry made by the tangent to the surface at point 6. Because the flow is tangent to the surface, this angle is a first approximation to the desired angle $\delta_6$ for the velocity plane. Next take the mean of this $\delta_6$ with $\delta_5$ and find $(\delta_{56} - \alpha_5)$ and lay off another line in the physical plane from 5 at this angle to give a closer approximation to point 6 on the surface (i.e. $6_2$ in Figure 3). Again measure the angle of the tangent and record it in the table as $\delta_6$ for point 6. Lay this off in the velocity plane and make an estimate of the location of velocity point 6 somewhere along it. Measure the values for columns (1), (3), (4), and (5) from the velocity plane and the value for column (6) in the physical plane and record for point 6. Take the means of each of these values and record in row $\overline{56}$ columns (1) though (6). Measure $d\xi_{56}$ in the physical plane and record in column (7). Use columns (2) and (3) row $\overline{56}$ to determine columns (9) and (10) row $\overline{56}$. All quantities are now known to determine $dq_{56}$ using equation (1b) i.e.

$$dq_{\overline{56}} = \frac{V_{\overline{56}}}{r_{\overline{56}}} \sin^2\alpha_{\overline{56}} \, d\xi_{\overline{56}} \qquad (1b)$$

This quantity is layed off in the velocity plane from known velocity point 5 in the direction of $(\delta + \alpha)_{56}$. A perpendicular is erected from the end point of $dq_{56}$ as shown in Figure 3 and its intersection with the $\delta_6$ line locates point 6. If this point 6 coincides with point 6 estimated, no further calculations are necessary. If not, use the new values for point 6 columns (1) through (5) to construct a new line $(\delta - \alpha)_{56}$ in the physical plane to locate a new point 6 (i.e. point $6_3$ in Figure 3). Measure a new $\delta_6$ here and repeat the process determining $dq_{56}$ exactly as done for the first approximation making use of the values for point 6 resulting for the first approximation. Generally this second approximation will yield a location of point 6 in the velocity plane very close to the previously located velocity point 6 indicating that no further calculations are necessary. All points on the missile surface are found using this method. It is of interest to note here that $\delta_5$ will not necessarily always be of larger magnitude than $\delta_6$ as is shown in Figure 3.

The last case to be considered in detail is the case where the velocity to be found is on the shock wave. This is the case that makes use of the shock polar in the velocity plane. Assume that the velocities for points 8 and 9 in Figure 4 are known. Record the values of columns (1) through (6) in Figure 2 for points 8 and 9. We know that any point that appears on the shock wave in the physical plane must be represented on the shock polar in the velocity plane. Since the shock wave bends in the clockwise sense, the angle $\theta$ as measured in the physical plane is decreasing. This determines that the point 10 must be on the shock polar below point 8 as $\theta_{10}$ will be less than $\theta_8$ as shown in Figure 4.

The first step in case three is to estimate the position of point 10 on the shock polar. Measure and record columns (1) through (5) in Figure 2. Measure $\theta_8$ and $\theta_{10}$ in the velocity plane as shown in Figure 4 and take the mean to get $\theta_{\overline{810}}$ as shown in Figure 4. This is the first approximation to the shock line segment $\overline{810}$. Fill in the columns (2) through (5) for row $\overline{910}$ by averaging the appropriate values for points 9 and 10, columns (2) through (5). Obtain columns (10) and (11) row $\overline{910}$ by making use of columns (2) and (3). Draw a line from known point 9 in the physical plane at the angle $(\delta + \alpha)_{\overline{910}}$. Where this line intersects the shock line $\overline{810}$ locates the first approxination to the position of point 10 in the physical plane. Measure $r_{10}$ and record it in column (6) row 10 and use it to determine $r_{\overline{910}}$. Measure $d\eta_{\overline{910}}$ and record it in row $\overline{910}$. Now calculate $dp_{\overline{910}}$ i.e.

$$dp_{\overline{910}} = \frac{V_{\overline{910}}}{r_{\overline{910}}} \sin^2 \alpha_{\overline{910}} \, d\eta_{\overline{910}} \qquad (1a)$$

Lay this value off in the velocity plane along a line from known point 9 in the direction of $(\delta - \alpha)_{\overline{910}}$. Erect a perpendicular at the end of $dp_{\overline{910}}$. Where the perpendicular intersects the shock polar as shown in Figure 4 locates the new position of velocity point 10. If this point closely approximates the original estimated position of point 10 no further calculation is needed. However, this rarely occurs and the procedure is repeated making use of the new velocity point 10 throughout instead of the velocity point 10 originally estimated. One more calculation is usually sufficient to converge the velocity point to the solution within the accuracy of the method.

This completes the discussion on the three variations of solution encountered in dealing with configurations of the axially symmetric type. This step by step solution yields results in the physical plane similar to Figure 5. The corresponding velocities associated with each Mach line or characteristic line intersection is recorded in the tables used in the method such as Figure 2 column (1). The velocities of points on the surface can be used to determine the pressure coefficient at each point. All points will provide data for a plot of the pressure coefficient in a meridian plane. Such a plot is shown in Figure 6 for an ogival nose at a Mach number slightly less than 3.

This solution is time-consuming and requires careful attention to details. The results obtained, however, are very accurate and closely approximate experimental results.

## Conclusion

This report presents the essential details of the mechanics of application of the three-dimensional method of characteristics in determining the velocity distribution over an axially symmetric body immersed in a supersonic stream. The report treats in detail the technique employed in determining an unknown velocity behind a shock wave for three distinctly different cases:

1. A point in the free stream behind the shock wave.

2. A point located on the surface of the body.

3. A point located on the shock wave.

This method is time-consuming and laborious in application, but may well merit use in certain specific cases. The purpose of this paper is to present an explanation of the procedure of the method including certain refinements which help to keep the labor involved to a minimum.

It is anticipated that this method will be employed to some extent to determine the pressure distribution over missile configurations prior to and in conjunction with wind tunnel tests.

References

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2. JPL-R-4-34 "Characteristic Methods for Quasi-linear Hyperbolic Differential Equations and an Application to Axially Symmetric Supersonic Flow Past an Ogive." H. K. Forster.

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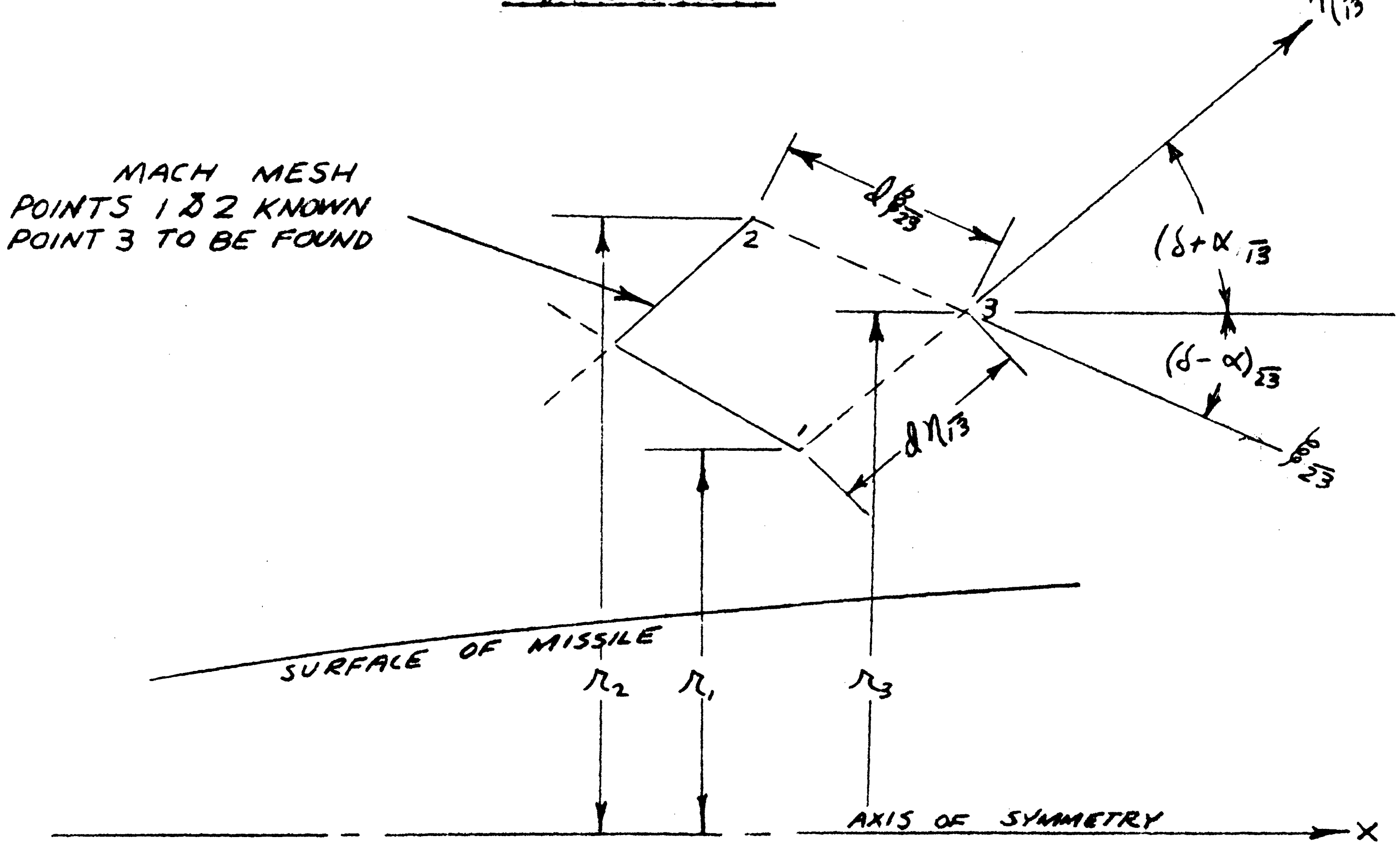
5. MIT-TR-1 "Supersonic Flow of Air Around Cones."

6. "Aerodynamics of a Compressible Fluid." Liepmann and Puckett. Wiley Bros.

7. "The Air Pressure on a Cone Moving at High Speeds." G. I. Taylor and J. W. Maccoll. The Royal Society, 1932.

physical plane




velocity plane

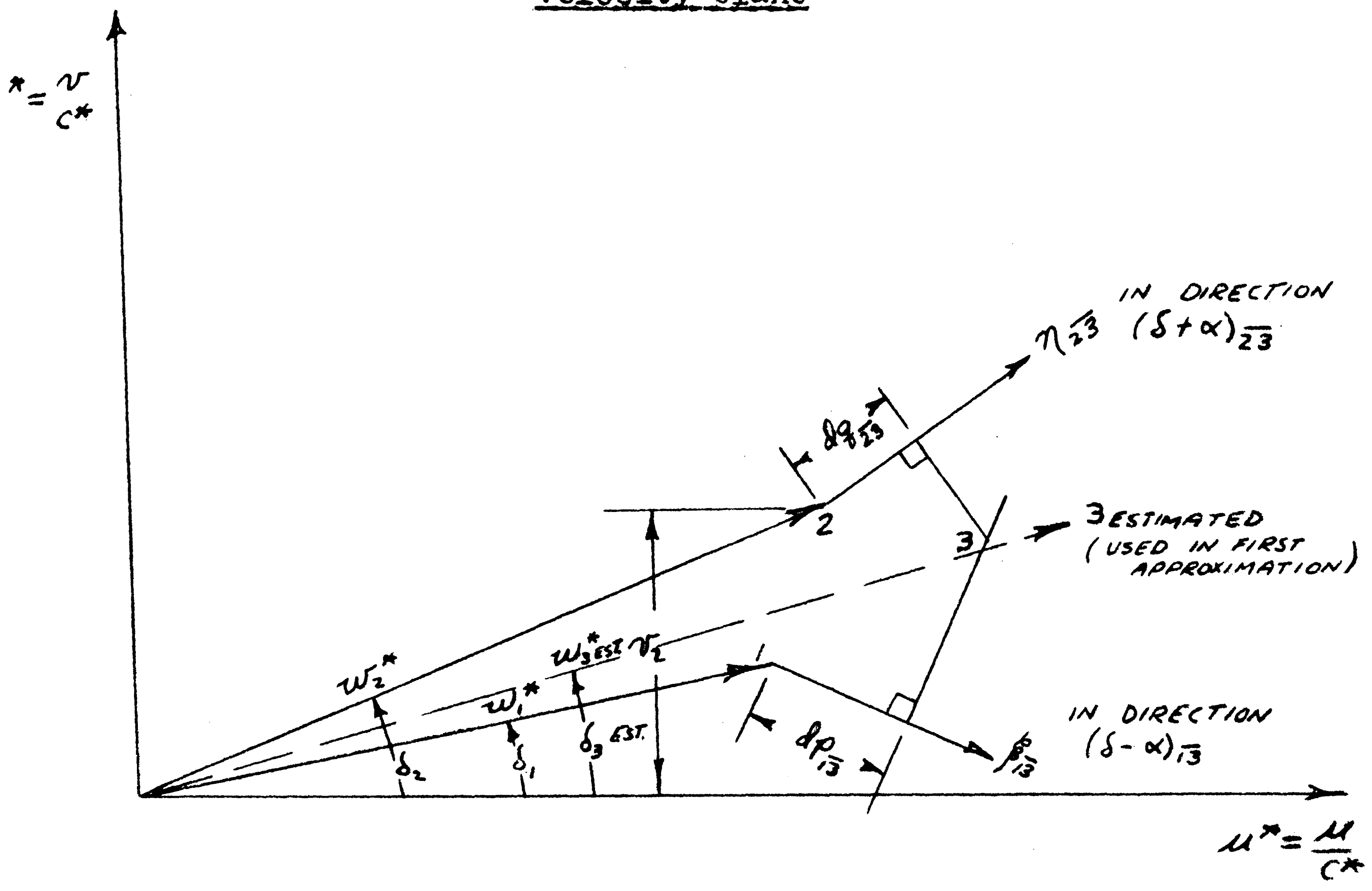



Figure 1

| | ① | ② | ③ | ④ | ⑤ | ⑥ | ⑦ | ⑧ | ⑨ | ⑩ | ⑪ | ⑫ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| POINT | $w^*$ | $\delta$ | $\alpha$ | $\sin^2\alpha$ | $v$ | $r$ | $d\xi$ | $d\eta$ | $\delta+\alpha$ | $\delta-\alpha$ | $dp$ | $dq$ |
| | | CASE 1 | | | | | | | | | | |
| 1 | $w_1^*$ | $\delta_1$ | $\alpha_1$ | $\sin^2\alpha_1$ | $v_1$ | $r_1$ | | | | | | |
| 2 | $w_2^*$ | $\delta_2$ | $\alpha_2$ | $\sin^2\alpha_2$ | $v_2$ | $r_2$ | | | | | | |
| 3 | $w_3^*$ | $\delta_3$ | $\alpha_3$ | $\sin^2\alpha_3$ | $v_3$ | $r_3$ | | | | | | |
| $\overline{13}$ | — | $\delta_{\overline{13}}$ | $\alpha_{\overline{13}}$ | $\sin^2\alpha_{\overline{13}}$ | $v_{\overline{13}}$ | $r_{\overline{13}}$ | — | $d\eta_{\overline{13}}$ | $(\delta+\alpha)_{\overline{13}}$ | $(\delta-\alpha)_{\overline{13}}$ | $dp_{\overline{13}}$ | — |
| $\overline{23}$ | — | $\delta_{\overline{23}}$ | $\alpha_{\overline{23}}$ | $\sin^2\alpha_{\overline{23}}$ | $v_{\overline{23}}$ | $r_{\overline{23}}$ | $d\xi_{\overline{23}}$ | — | $(\delta+\alpha)_{\overline{23}}$ | $(\delta-\alpha)_{\overline{23}}$ | — | $dq_{\overline{23}}$ |
| | | CASE 2 | | | | | | | | | | |
| 5 | $w_5^*$ | $\delta_5$ | $\alpha_5$ | $\sin^2\alpha_5$ | $v_5$ | $r_5$ | | | | | | |
| 6 | $w_6^*$ | $\delta_6$ | $\alpha_6$ | $\sin^2\alpha_6$ | $v_6$ | $r_6$ | | | | | | |
| $\overline{56}$ | — | $\delta_{\overline{56}}$ | $\alpha_{\overline{56}}$ | $\sin^2\alpha_{\overline{56}}$ | $v_{\overline{56}}$ | $r_{\overline{56}}$ | $d\xi_{\overline{56}}$ | — | $(\delta+\alpha)_{\overline{56}}$ | $(\delta-\alpha)_{\overline{56}}$ | — | $dq_{\overline{56}}$ |
| | | CASE 3 | | | | | | | | | | |
| 8 | $w_8^*$ | $\delta_8$ | $\alpha_8$ | $\sin^2\alpha_8$ | $v_8$ | $r_8$ | | | | | | |
| 9 | $w_9^*$ | $\delta_9$ | $\alpha_9$ | $\sin^2\alpha_9$ | $v_9$ | $r_9$ | | | | | | |
| 10 | $w_{10}^*$ | $\delta_{10}$ | $\alpha_{10}$ | $\sin^2\alpha_{10}$ | $v_{10}$ | $r_{10}$ | | | | | | |
| $\overline{910}$ | — | $\delta_{\overline{910}}$ | $\alpha_{\overline{910}}$ | $\sin^2\alpha_{\overline{910}}$ | $v_{\overline{910}}$ | $r_{\overline{910}}$ | — | $d\eta_{\overline{910}}$ | $(\delta+\alpha)_{\overline{910}}$ | $(\delta-\alpha)_{\overline{910}}$ | $dp_{\overline{910}}$ | — |
| | | | | | | | | | | | | |
| | | THESE REPRESENT THE FIRST APPROXIMATION ONLY | | | | | | | | | | |
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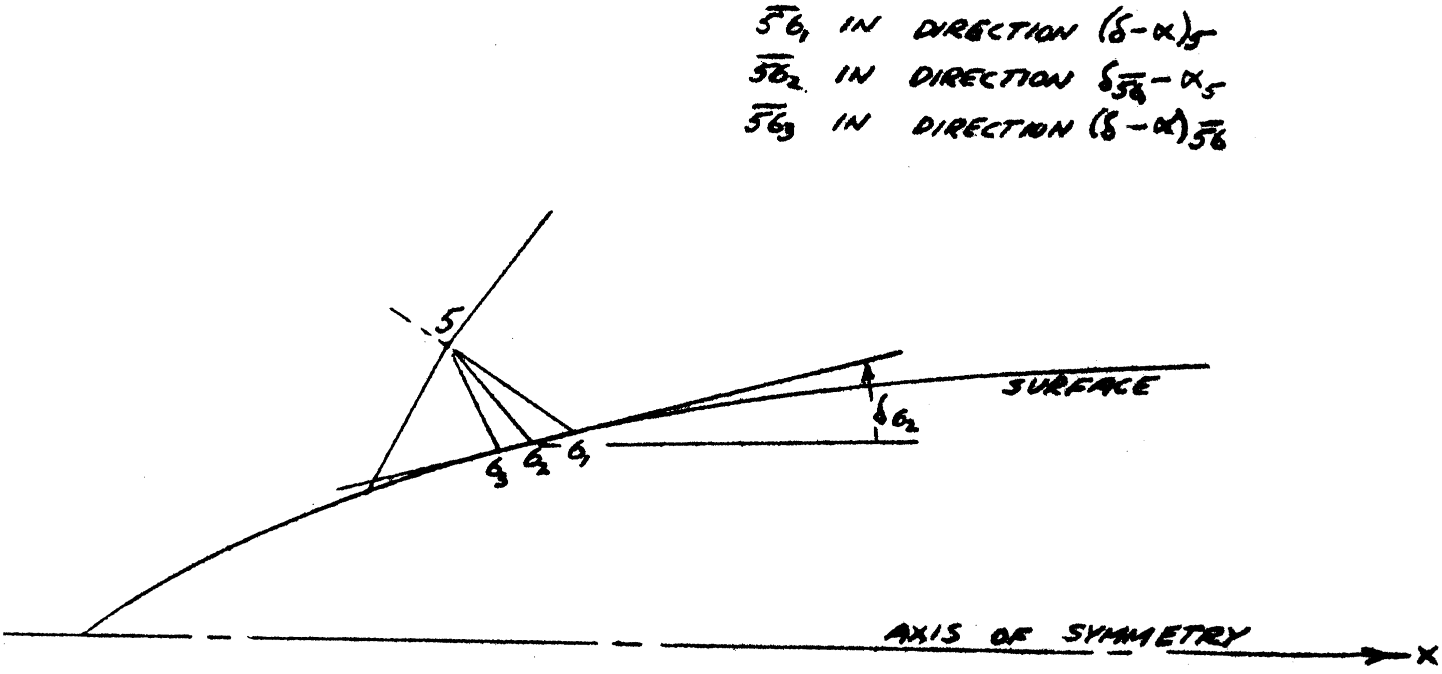
Figure 2

physical plane




velocity plane

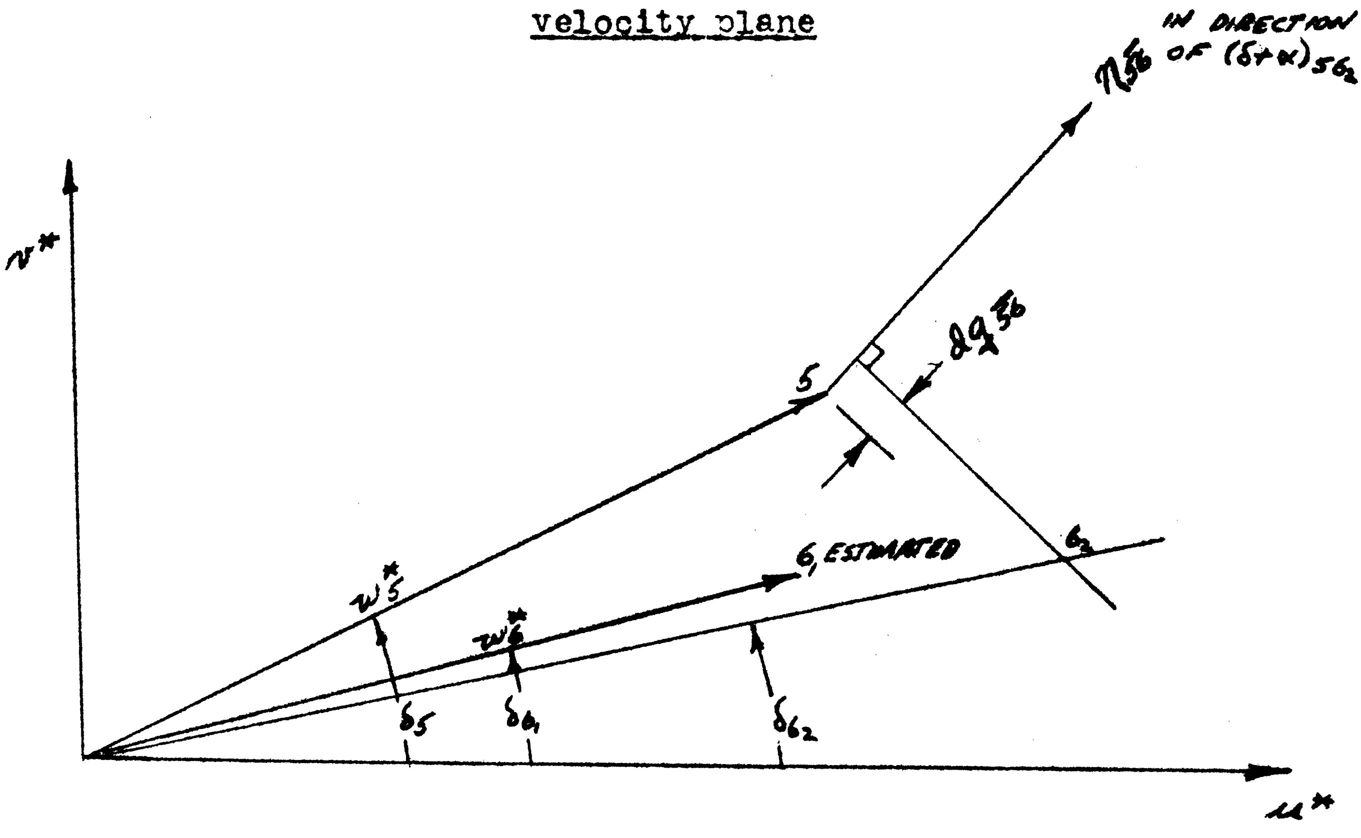
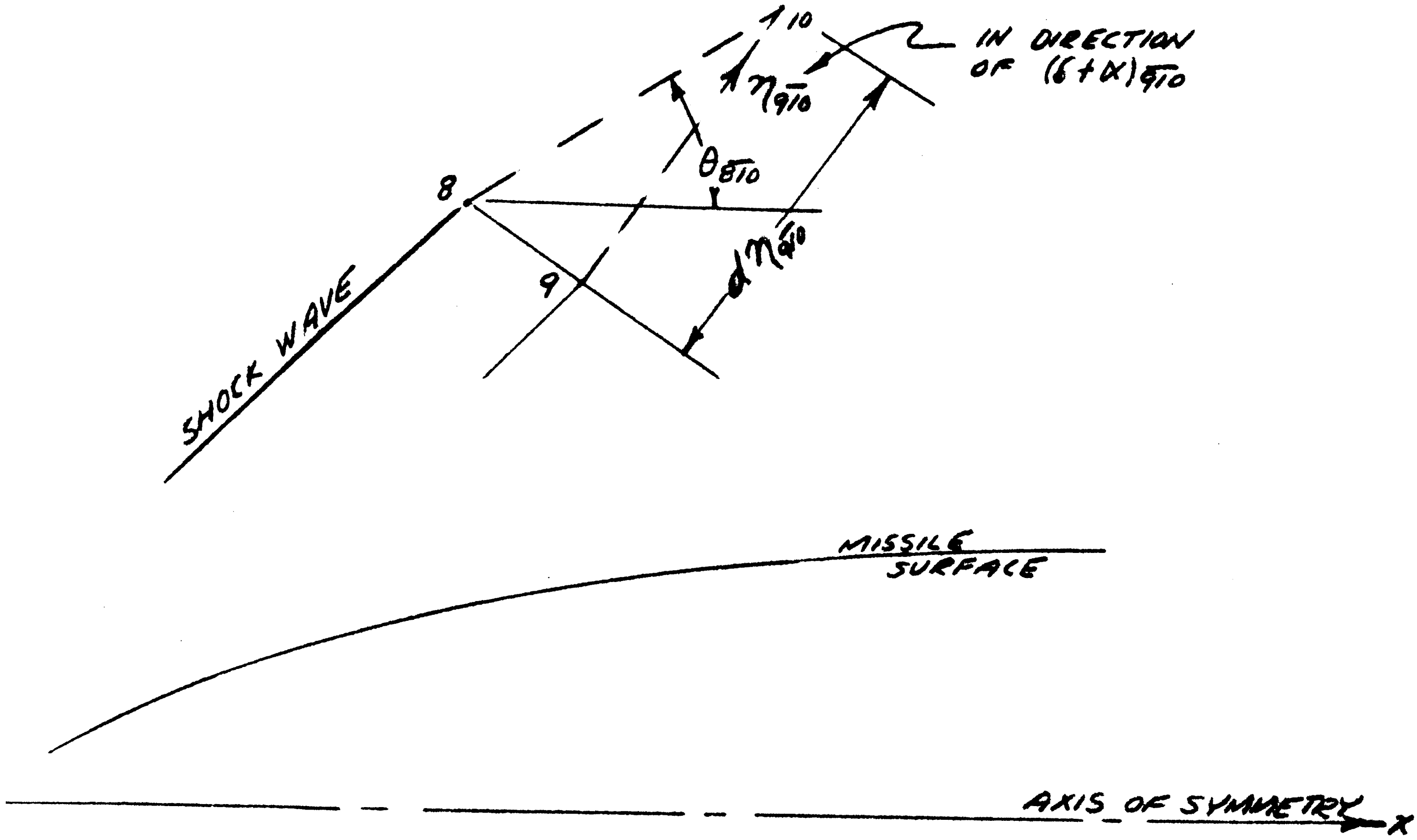



Figure 3

physical plane




velocity plane

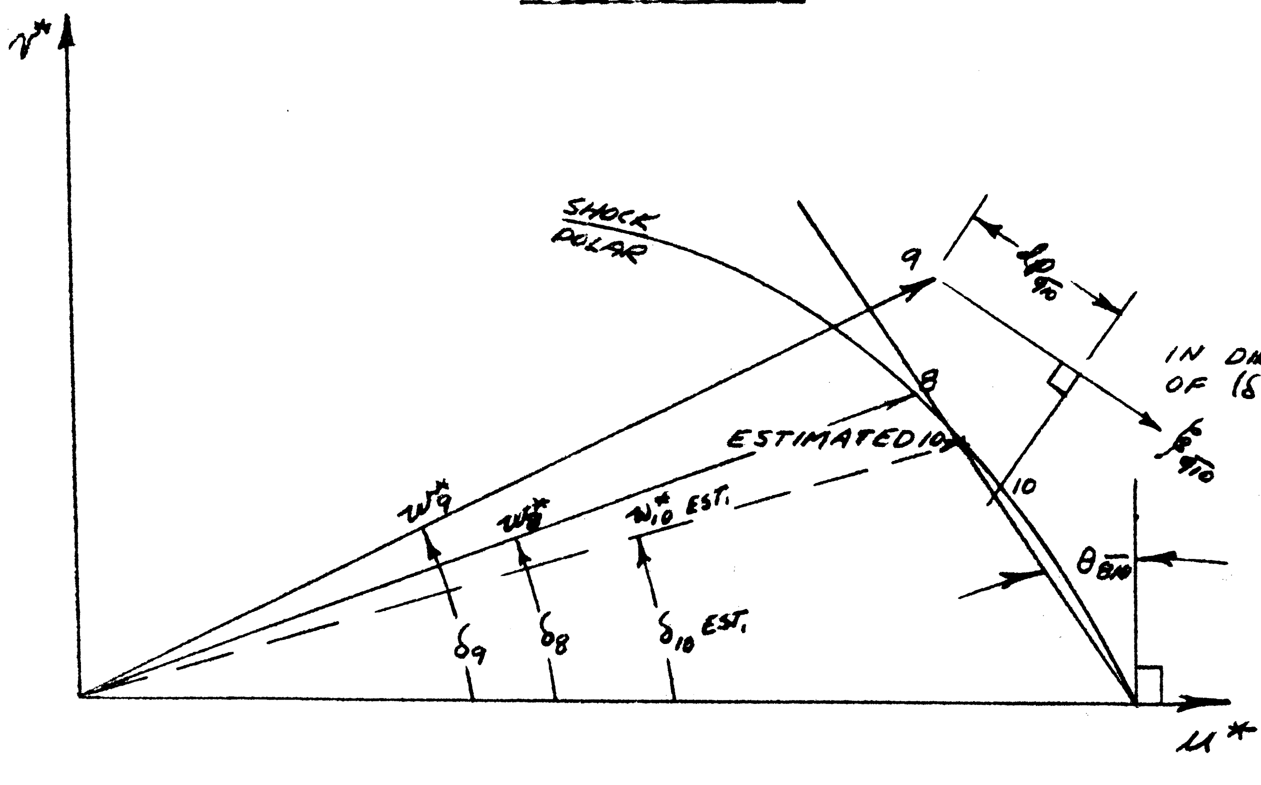
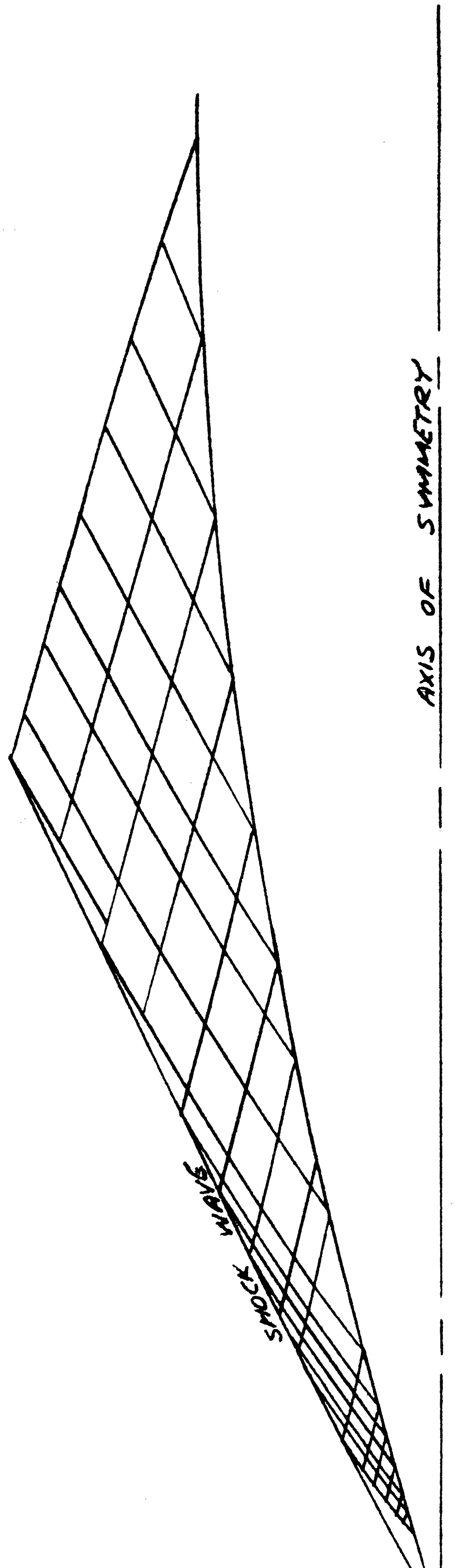
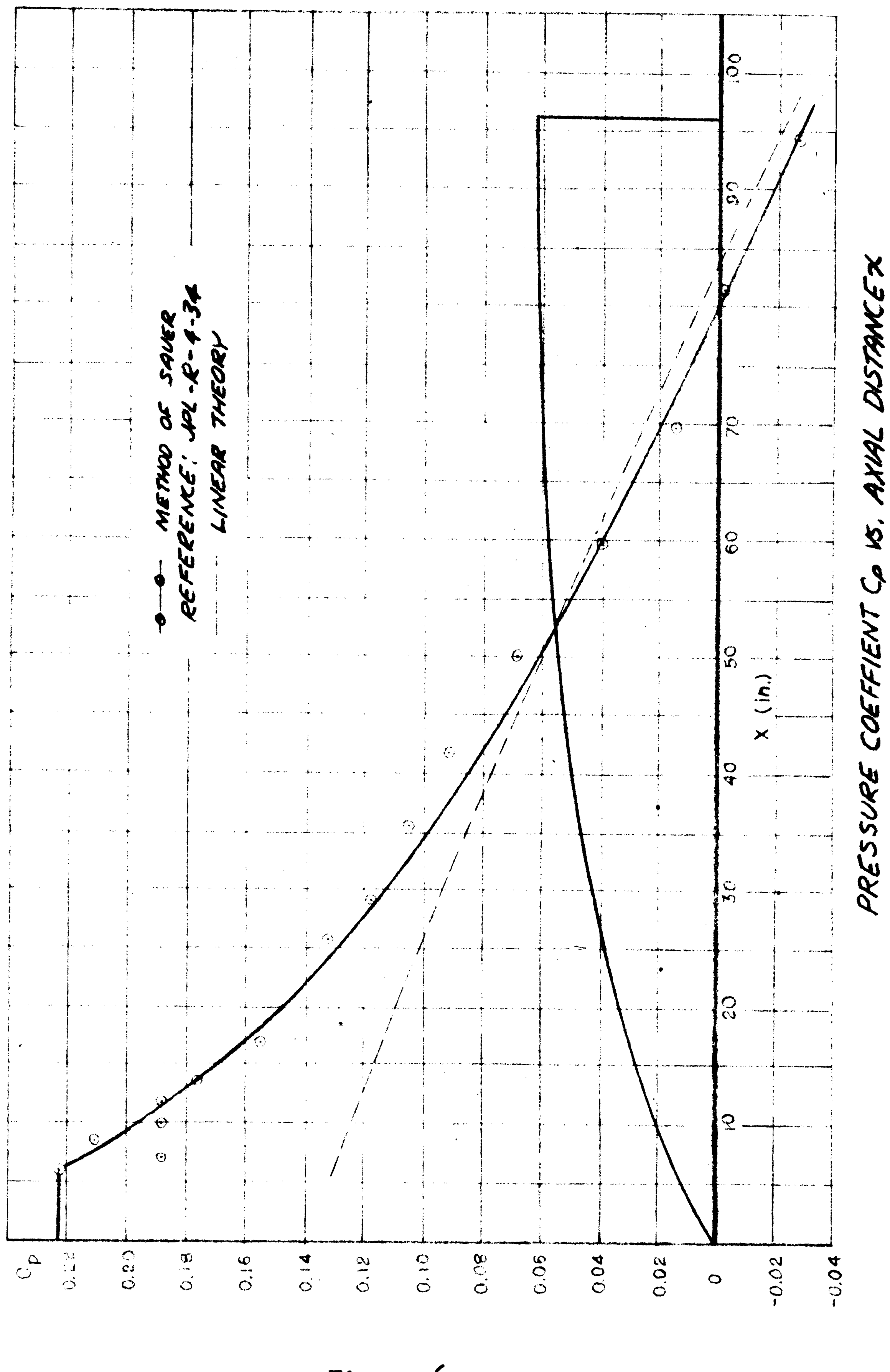



Figure 4




A TYPICAL MACH NET OVER AN OGIVE FOR $M \cong 3$
REFERENCE: JPL-R-4-34

Figure 5



METHOD OF SAUER
REFERENCE; APL-R-4-34
LINEAR THEORY
PRESSURE COEFFIENT Cp VS. AXIAL DISTANCE x
FOR OGIVE AT M=3
X (in.)
Cp
0.22
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
-0.02
-0.04
10
20
30
40
50
60
70
80
90
100


Figure 6