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THE DERIVATION OF FIRST ORDER EXPRESSIONS FOR LIFT COEFFICIENT, MOMENT COEFFICIENT AND CENTER OF PRESSURE LOCATION FOR AXIALLY SYMMETRIC BODIES IN A SUPERSONIC STREAM

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SYMBOLS

A_{base} = base area of body

A_{mean} = mean cross-sectional area of body

c.p. = center of pressure or point of zero aerodynamic moment

C_L = $\dfrac{\text{LIFT}}{\frac{\rho}{2}U_\infty^2 A_{base}}$ = lift coefficient

C_M = $\dfrac{\text{MOMENT}}{\frac{\rho}{2}U_\infty^2 A_{base}\, l}$ = coefficient of moment about body vertex

$f(\xi)$ = function representing distribution of source or sink strength along x axis at points ξ

$g(\xi)$ = function representing distribution of doublet strength along x axis at points ξ

l = length of body

$l_{c.p.}$ = distance from vertex to center of pressure of body

L = lift normal to x axis

M = moment of L about body vertex

p = static pressure

R = radius of body at station x

u,v,w = perturbation velocities in x, r, θ directions, respectively

U_∞ = free stream velocity

x,r,θ = cylindrical coordinate system used. x axis oriented along body axis of symmetry

GREEK SYMBOLS

α = angle of attack

$\beta = \sqrt{M_a^2 - 1}$ = compressibility factor

ξ = source, sink or doublet position along x axis

ρ_∞ = free stream density

φ = perturbation potential

φ_1 = perturbation potential for axially symmetrical flow

φ_2 = perturbation potential for spatial flow

Summary of Conclusions

Simple first order expressions for lift coefficient, pitching moment coefficient and center of pressure location, are derived, using solutions to the linearized potential equation for spatial supersonic flow. These expressions have been used as a basis for examining the trends and orders of magnitude of these aerodynamic parameters for various supersonic bodies. In all cases where these formulas are used, more rigorous solutions should be and have been used. It should be noted that these expressions simply point out the way to an optimum missile shape and are first order solutions only in the case of very thin bodies of revolution. These relations have their counterpart in subsonic incompressible flow theories such as the thin airfoil theory and the theory of airship bodies employing apparent mass concepts.

The expressions as derived are:

$$C_L = 2\alpha$$

$$C_M = 2\alpha\left[1 - \frac{Volume}{A_{base}\, l}\right]$$

$$l_{c.p.} = \left[1 - \frac{A_{mean}}{A_{base}}\right] L$$

In the aerodynamic design of supersonic missiles some theory has to be used to determine the conventional aerodynamic coefficients. Such theories employ solutions to the rigorous non-linear, hyperbolic, partial differential equation of flow or its linearized derivative. As yet, known solutions to the rigorous non-linear equation are few in number and restricted in nature. Such solutions include the Taylor-Maccoll solution for the flow around a cone, the Rankine-Hugoniot relations leading to two-dimensional deflection through a shock wave, and the Prandtl-Meyer solution for expansion flow around a convex corner. The method of characteristics for problems having a two-dimensional natural coordinate system is a solution to the rigorous equation only when the characteristic quadrangles become smaller approaching the limit. None of these rigorous solutions offers a ready tool for examining the spatial flow around supersonic bodies.

A standard approach to physical phenomena, obeying laws defined by non-linear differential equations, is to attempt to examine and justify the approximate solution to the linear differential equation derived from the more rigorous non-linear equation. In aerodynamics, numerous solutions have appeared for problems of bodies immersed in the flow field of the linear potential equation for supersonic flow. The solution dealt with in this paper is one of the earliest solutions of the linear equation for flow about axial symmetrical bodies of revolution. Because the method of linearizing the rigorous non-linear potential equation was the method of small perturbations, these solutions are restricted to slender pointed bodies at small angles of attack. The expressions derived herein are first order approximations to the more rigorous solutions of the linearized equation. As such their application is more restricted than are the more lengthy solutions. However, these expressions can be used to indicate design trends and orders of magnitude for preliminary analysis prior to obtaining the more lengthy solutions to the linearized equations appearing in Ref. 1. The following text contains the derivation of the first order expressions for lift, moment and center of pressure location for very slender axially symmetrical bodies in a supersonic stream.

The linear potential equation for spatial supersonic flow is:

$$(1-M^2)\frac{\partial^2\varphi}{\partial x^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = 0 \qquad (1)$$

Following in form the procedure outlined in Lamb (Ref. 2) the general solution to (1) when Mach number exceeds one is

$$\varphi = \sum_{s=1}^{n} \varphi_s r^s \cos s\theta + \psi_s r^s \sin s\theta \qquad (2)$$

where φ_s and ψ_s must satisfy the equation below obtained by substituting equation (2) into equation (1).

$$\frac{\partial^2}{\partial x^2}\left\{\varphi_s, \psi_s\right\} = \frac{1}{\beta^2}\left[\frac{\partial^2}{\partial r^2}\left\{\varphi_s, \psi_s\right\} + \frac{2s+1}{r}\frac{\partial}{\partial r}\left\{\varphi_s, \psi_s\right\}\right]. \tag{3}$$

A known solution to equation (3) is

$$\varphi_s = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^s \varphi_1 \tag{4}$$

where φ_1 is identically the solution to equation (5) given by Karman and Moore (Ref. 3).

$$(1-M^2)\frac{\partial^2\varphi}{\partial x^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial r^2} = 0 \tag{5}$$

Solution (2) to equation (1) was introduced by Lamb and further developed by Ferrari (Ref. 4) and Tsien (Ref. 5).

The solution to equation (5) is

$$\varphi_1 = \frac{1}{4\pi}\int_{\cosh^{-1}\frac{x}{\beta r}}^{0} f_1(x - \beta r \cosh u)\, du \tag{6}$$

where

$$\beta = \sqrt{M_a^2 - 1}$$

Solution (6) becomes more familiar when put in the form of a new variable ξ , where using the substitution below

$$\xi = x - \beta r \cosh u$$

equation (6) becomes

$$\varphi_1 = \frac{1}{4\pi}\int_0^{x-\beta r}\frac{f_1(\xi)\, d\xi}{\sqrt{(x-\xi)^2 - \beta^2 r^2}} \tag{7}$$

Equation (7) is the form of the source-sink solution often used in subsonic source-sink flow solutions. Here $f_1(\xi)$ is a function representing the distribution of strengths of the sources and sinks at points ξ of the x axis of the cylindrical coordinate system of equation (5). x and r are the coordinates of the particular field points for which the potential due to sources or sinks at points ξ is desired. According to (4), differentiation of (7) with respect to r will yield a solution for (1). However, (7) blows up at the upper limit and hence the differentiation must be performed on (7) in the form of (6). This is not the case in subsonic flow where the limits of (7) are constant.

Differentiate (6) with respect to r.

$$\varphi_s(x,r) = \frac{1}{r}\frac{\partial}{\partial r}\varphi_1(x,r) = \frac{-\beta}{4\pi r}\int_{\cosh^{-1}\frac{x}{\beta r}}^{0} f_1'(x-\beta r\cosh u)\cosh u \, du \tag{8}$$

By taking the index ς equal to one in equations (2) and (4) the corresponding solution to (1) is:

$$\varphi_2 = \frac{-\beta\cos\theta}{4\pi}\int_{\cosh^{-1}\frac{x}{\beta r}}^{0} g(x-\beta r\cosh u)\cosh u \, du \tag{9}$$

using $\xi = x - \beta r \cosh u$ (9) becomes

$$\varphi_2 = \frac{\cos\theta}{4\pi r}\int_{0}^{x-\beta r}\frac{g(\xi)(x-\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \tag{10}$$

(10) represents the potential for field point (x, r, θ) due to a system of doublets distributed at points ξ along the x axis according to the function

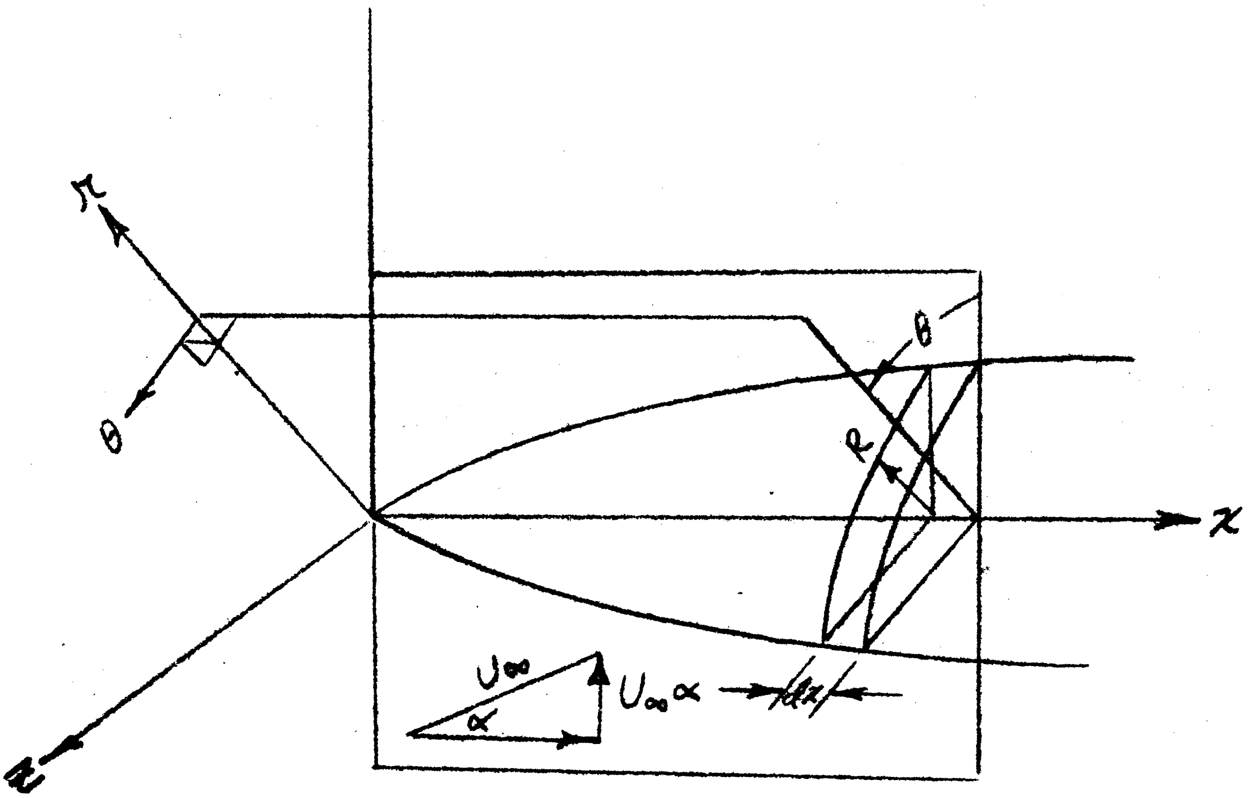
Now, visualize a slender body of revolution placed in the linearized compressible supersonic stream of equation (1) at an angle of attack. Assume a system of sources and sinks and doublets is distributed along the x axis at points ξ according to the rule of functions $f(\xi)$ and $g(\xi)$ respectively. These singularities are so arranged such that the effect of them in displacing the stream is to reproduce the effect of the body displacing the stream. This restriction upon the singularity distribution will be elaborated upon further in the statement of the boundary conditions of solutions to (1) and (5).

It is desired now to obtain a simple first order rule for the lift, moment about body vertex, and center of pressure location using solution (9).

Let L = lift of a slender axially symmetrical body in the direction normal to the x, z plane.

Let M = moment of the slender axially symmetrical body about the vertex of the body.

While determining a first order approximation to L and M, reference will be made to the sketch below.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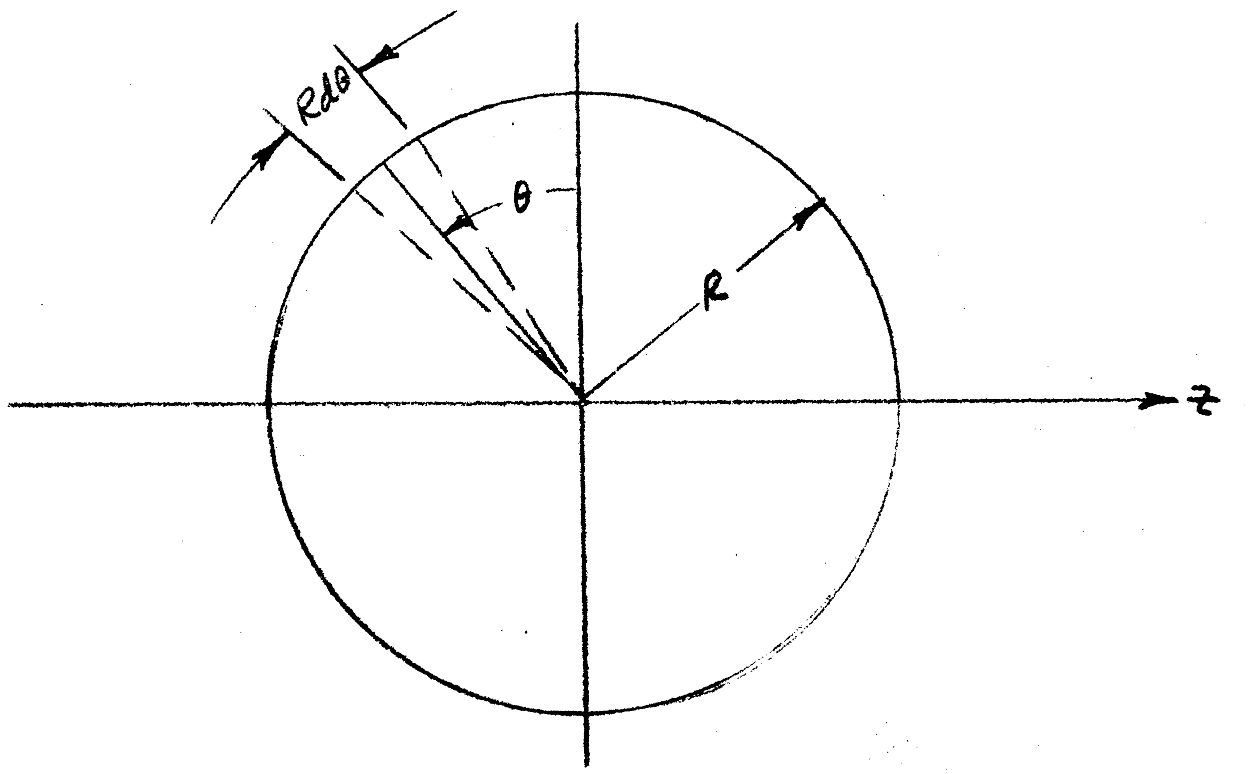
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The lift ΔL on the elemental surface is:

$$\Delta L = \Delta p\, R\, d\theta\, dx \cos\theta \tag{11}$$

where Δp = p surface - p free stream

Correspondingly, the moment of this elemental area is:

$$\Delta M = \Delta p\, R\, x \cos\theta\, d\theta\, dx \tag{12}$$

Integration over the surface now yields the general expressions for lift and moment about the vertex. That is,

$$L = \int_0^{l}\int_0^{2\pi} \Delta p\, R \cos\theta\, d\theta\, dx \tag{13}$$

and

$$M = \int_0^{l}\int_0^{2\pi} \Delta p\, R \cos\theta\, x\, d\theta\, dx \tag{14}$$

Within the rigor of the linearization process employed to obtain equation (1) from the rigorous non-linear flow equation, the expression for pressure coefficient is

$$C_p = \frac{2\Delta p}{\rho_\infty U_\infty^2} = -2\frac{u}{U_\infty} \tag{15}$$

where

$$u = \frac{\partial \varphi_2}{\partial x} \tag{16}$$

and hence,

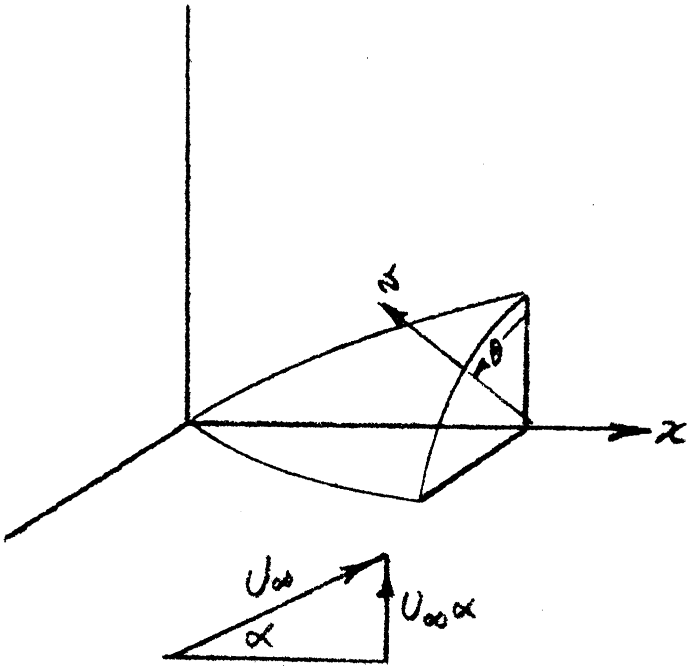
$$\Delta p = -\rho_\infty U_\infty \frac{\partial \varphi_2}{\partial x} \tag{17}$$

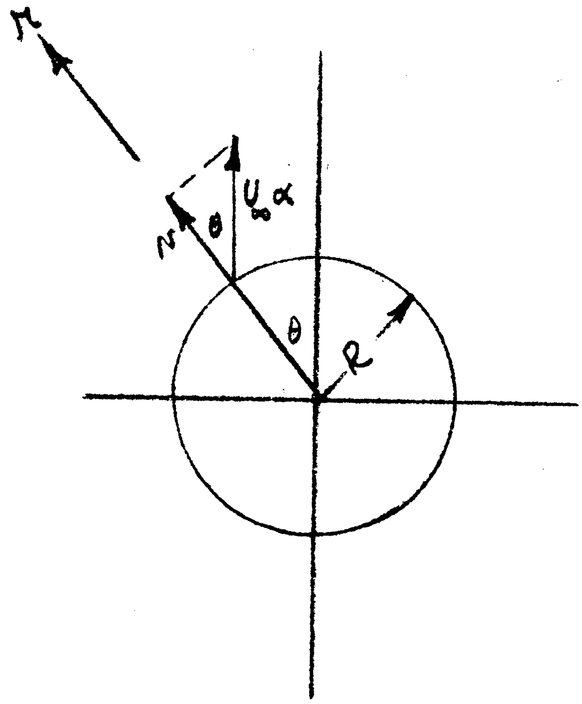
Substituting (17) into (13) and (14) yields

$$L = -\rho_\infty U_\infty \int_0^{l}\int_0^{2\pi} \frac{\partial \varphi_2}{\partial x} R \cos\theta\, d\theta\, dx \tag{18}$$

$$M = -\rho_\infty U_\infty \int_0^{l}\int_0^{2\pi} \frac{\partial \varphi_2}{\partial x}\, x\, R \cos\theta\, d\theta\, dx \tag{19}$$

We must now determine the value of $\frac{\partial \varphi_2}{\partial x}$ as $R = r \to 0$. (i.e. for a first order, very slender, body). Here the boundary conditions come into play. The boundary condition on solution (10) is that the derivative with respect to r (the r component of the perturbation velocity) must be equal and opposite to the component of the free stream velocity in the r direction in order that the flow be tangent to the surface of the body. To illustrate:





$$v = \frac{\partial \varphi_2}{\partial r} \tag{20}$$

and

$$\left(\frac{\partial \varphi_2}{\partial r}\right)_{r=R} = U_\infty \alpha \cos\theta \tag{21}$$

Where $r = R$, (10) becomes

$$\varphi_2 = \frac{\cos\theta}{2\pi R}\int_0^{x-\beta R} \frac{g(\xi)(x-\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2R^2}} \tag{22}$$

We must determine the value of $\left(\frac{\partial \varphi_2}{\partial r}\right)_{r=R}$ as $R \to 0$. Noticing again that in (22) the denominator vanishes at the upper limit, return to the more convenient form of (9) to perform the desired differentiation

$$\left(\frac{\partial \varphi_2}{\partial r}\right)_{r=R} = \frac{\beta^2 \cos\theta}{4\pi} \int_{\cosh^{-1}\frac{x}{\beta R}}^{0} g'(x - \beta R \cosh u) \cosh^2 u \, du \tag{23}$$

substituting $\xi = x - \beta R \cosh u$ yields

$$\left(\frac{\partial \varphi_2}{\partial r}\right)_{r=R} = \frac{\cos\theta}{4\pi R^2} \int_0^{x-\beta R} \frac{g'(\xi)(x-\xi)^2 \, d\xi}{\sqrt{(x-\xi)^2 - \beta^2 R^2}} \tag{24}$$

Let $R \to 0$ in (24) as for very slender bodies.

$$\left(\frac{\partial \varphi_2}{\partial r}\right)_{R\to 0} \longrightarrow \frac{\cos\theta}{4\pi R^2} \int_0^{x-\beta R} g'(\xi)(x-\xi) \, d\xi \tag{25}$$

The integral appearing in (25) can be integrated by parts. Performing this integration yields

$$\left(\frac{\partial \varphi_2}{\partial r}\right)_{R\to 0} \longrightarrow \frac{\cos\theta}{4\pi R^2} \left\{ \Big[(x-\xi) g(\xi) \Big]_0^{x} + \int_0^{x} g(\xi) \, d\xi \right\} \tag{26}$$

as $R \to 0$, $\xi = x - \beta R \cosh u \to x$. Noting that $g(0) = 0$ for pointed bodies, expression (26) becomes for slender bodies,

$$\left(\frac{\partial \varphi_2}{\partial r}\right)_{R\to 0} \longrightarrow \frac{\cos\theta}{4\pi R^2} \int_0^{x} g(x) \, dx \tag{27}$$

Substituting this value into (21), yields

$$\frac{\cos\theta}{4\pi R^2} \int_0^{x} g(x) \, dx = U_\infty \alpha \cos\theta \tag{28}$$

Now $\pi R^2 = A(x)$ the cross-sectional area of the body at x. Hence

$$\int_0^{x} g(x) \, dx = U_\infty \alpha \, 4 \, A(x) \tag{29}$$

Differentiate with respect to x, calling $g(0) = 0$ for a pointed body.

$$g(x) = 4 U_\infty \alpha \frac{d\,A(x)}{dx} \tag{30}$$

This establishes a rule connecting the doublet distribution g (x) with the shape of the body. i.e.,

$$A(x) = \pi R^2 \; ; \quad R = R(x) \tag{31}$$

Now, examine $\left(\frac{\partial \varphi_2}{\partial x}\right)_{r=R}$ as R → 0.

From (9)

$$\left(\frac{\partial \varphi_2}{\partial x}\right)_{r=R} \doteq -\frac{\beta \cos\theta}{4\pi R} \int_{\cosh^{-1}\frac{x}{\beta R}}^{0} g'(x - \beta R \cosh u) \cosh u \, du \tag{32}$$

if $\xi = x - \beta R \cosh u$

$$\left(\frac{\partial \varphi_2}{\partial x}\right)_{r=R} = + \frac{\cos\theta}{4\pi R} \int_0^{x-\beta R} \frac{g'(\xi)(x-\xi)\, d\xi}{\sqrt{(x-\xi)^2 - \beta^2 r^2}} \tag{33}$$

This becomes as R → 0.

$$\left(\frac{\partial \varphi_2}{\partial x}\right)_{R\to 0} \to + \frac{\cos\theta}{4\pi R} \int_0^{x} g'(\xi)\, d\xi \tag{34}$$

or

$$\left(\frac{\partial \varphi_2}{\partial x}\right)_{R\to 0} \to + \frac{\cos\theta}{4\pi R} \left\{ g(x) - g(0) \right\} = + \frac{\cos\theta \, g(x)}{4\pi R} \tag{35}$$

as g (0) is taken as zero for pointed bodies. Combining (35) and (30), yields

$$\left(\frac{\partial \varphi_2}{\partial x}\right)_{R\to 0} \to + \frac{U_\infty \alpha}{\pi R} \frac{d A(x)}{dx} \cos\theta \tag{36}$$

This is the desired value for $\frac{\partial \varphi_2}{\partial x}$ which must be substituted into the equations for lift and moment, (18) and (19).

Hence,

$$L = \frac{\rho U_\infty^2 \alpha}{\pi} \int_0^{l} \int_0^{2\pi} \frac{d A(x)}{dx} \cos^2\theta \, d\theta \, dx \tag{37}$$

and

$$M = \frac{\rho_\infty V_\infty^2 \alpha}{\pi} \int_0^l \int_0^{2\pi} \frac{dA(x)}{dx} x \cos^2\theta \, d\theta \, dx \tag{38}$$

Performing the indicated integrations yields

$$L = \rho_\infty V_\infty^2 \alpha \int_0^l \frac{dA(x)}{dx} dx \tag{39}$$

and

$$M = \rho_\infty V_\infty^2 \alpha \int_0^l \frac{dA(x)}{dx} x \, dx \tag{40}$$

Now

$$\int_0^l \frac{dA(x)}{dx} dx = \left[A(l) - A(0)\right] = A_{base} \tag{41}$$

since A (o) = 0 for a pointed body.

Then

$$L = \rho_\infty V_\infty^2 \alpha A_{base} \tag{42}$$

Define lift coefficient as

$$C_L = \frac{LIFT}{\frac{\rho_\infty}{2} V_\infty^2 A_{base}} \tag{43}$$

Then the first order approximation for very slender bodies is

$$C_L = 2\alpha \tag{44}$$

where α is in radians.

Define moment coefficient as

$$C_M = \frac{MOMENT}{\rho/2 \, V_\infty^2 A_{base} l} \tag{45}$$

Then the first order approximation for very slender bodies is

$$C_M = \frac{2\alpha}{A_{base} l} \int_0^l \frac{dA(x)}{dx} x \, dx \tag{46}$$

Now the integral

$$\int_0^l \frac{dA(x)}{dx}\, x\, dx$$

can be integrated by parts. That is, Pierce (Ref. 6) (19a):

$$\int u\, dv = uv - \int v\, du \tag{47}$$

Here let

$$u = x$$
$$dv = \frac{dA(x)}{dx}\, dx$$

then

$$\int_0^l \frac{dA(x)}{dx}\, x\, dx = \Big[x\Big]_0^l \int_0^l \frac{dA(x)}{dx}\, dx - \int_0^l A(x)\, dx \tag{48}$$

or

$$\int_0^l \frac{dA(x)}{dx}\, x\, dx = l A_{base} - Volume \tag{49}$$

Hence,

$$C_M = 2\alpha \left[1 - \frac{Volume}{A_{base}\, l}\right] \tag{50}$$

This simple formula for moment has been independently derived by Munk (Ref. 7), Tsien (Ref. 5), and Laitone (Ref. 8). Equations (50) and (44) can now be combined to yield one first order approximation for the center of pressure location.

This is:

$$\frac{C_M}{C_L} = \frac{l_{c.p.}}{l} = \left[1 - \frac{Volume}{A_{base}\, l}\right] \tag{51}$$

Now:

$$Volume = l \times A_{mean} \tag{52}$$

and hence

$$\frac{l_{c.p.}}{l} = \left[1 - \frac{A_{mean}}{A_{base}}\right] \tag{53}$$

This equation was used primarily by the Wizard Aerodynamics Group to predict the proper trend of shape of a body of revolution to yield favorable rearward c.p. location. In all cases, calculations based on these simple expressions were supported by the more rigorous solutions.

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