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Multichannel Nonlinear Scattering for Nonintegrable Equations* **

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Abstract. We consider a class of nonlinear Schrödinger equations (conservative and dispersive systems) with localized and dispersive solutions. We obtain a class of initial conditions, for which the asymptotic behavior ($t \to \pm\infty$) of solutions is given by a linear combination of nonlinear bound state (time periodic and spatially localized solution) of the equation and a purely dispersive part (decaying to zero with time at the free dispersion rate). We also obtain a result of *asymptotic stability* type: given data near a nonlinear bound state of the system, there is a nonlinear bound state of nearby energy and phase, such that the difference between the solution (adjusted by a phase) and the latter disperses to zero. It turns out that in general, the time-period (and energy) of the localized part is different for $t \to +\infty$ from that for $t \to -\infty$. Moreover the solution acquires an extra constant asymptotic phase $e^{i\gamma^{\pm}}$.

1. Introduction

This paper deals with the scattering theory of a class of conservative nonlinear dispersive equations admitting more than one channel. By this we mean that the asymptotic behavior is given by a linear combination of a localized (in space), periodic (in time) wave (solitary or standing wave) and a dispersive part. For nonlinear flows which are completely integrable (e.g. one-dimensional cubic nonlinear Schrödinger, Korteweg-de Vries equations), some analysis of the asymptotic system of, for example, localized part (solitons) plus dispersion can be carried out using the inverse scattering transform [G–G–K–M, Z–S, Lax, C–K]. The inverse scattering transform decouples the localized from the dispersive part.

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The cases we consider are not integrable. The main new feature here is that the localized and dispersive parts are interacting at all times. The spatially localized part that emerges as $t \to \pm\infty$ is identified with an exact solitary wave solution or nonlinear bound state of the full nonlinear equation. For the above-mentioned integrable systems the analogue of the solitary wave is the *one-soliton*. The model we focus on is a class of two and three dimensional nonlinear Schrödinger equations (NLS). The methods we present can however be adapted to other nonlinear dispersive systems.

Our main results are (see also Sect. 4):

(i) Asymptotic Stability (Theorem 4.2): Given initial conditions which lie in a neighborhood of a solitary wave of energy E_0 and phase γ_0, the asymptotic state of the system $(t \to \pm\infty)$ is given by a solitary wave of nearby energy $E^{\pm}$ and phase $\gamma^{\pm}$ plus a remainder which disperses to zero, i.e. the solution converges asymptotically to a solitary wave, say in some $\mathbf{L}^p$ norm with $p > 2$.
(ii) Scattering (Theorem 4.1): There is a ball in a Banach space of initial conditions for which the asymptotic behavior $(t \to \pm\infty)$ of the solution is given by a linear combination of a solitary wave of energy $E^{\pm}$ and phase $\gamma^{\pm}$, plus a remainder which is dispersive. The remainder is *purely dispersive* in the sense that it satisfies local decay and $\mathbf{L}^p$ decay estimates of linear theory.

Previous results on the stability of solitary waves involves the use of energy norms, e.g. $\mathbf{H}^1$ (see for example Ben, Ca–Li, Sh–Str, We2, We3, Ro–We, G–S–S). A typical result of this type states that if the solution begins in some neighborhood of the solitary wave orbit, then it remains in a neighborhood. Since energy norms are insensitive to dispersive behavior, one cannot conclude, as above, that solutions converge asymptotically to a solitary wave.

Earlier work on nonlinear scattering has focused on the situation where there are no bound states. In the above terminology, these are problems with a single (dispersive) channel (see for example Str1, Str3, G–V).

Cast into precise mathematical form, we prove that for a class of initial conditions for the nonlinear Schrödinger equation (NLS), the solution $\Phi(t)$ is given by

$$\Phi(t) = e^{-i\Theta(t)}\psi_{E(t)} + \phi_d(t), \tag{1.1}$$

$$\Theta = \int_0^t E(s) - \gamma(t), \tag{1.2}$$

where ψ_E is a spatially localized solution of the nonlinear bound state equation (with energy E) and $\phi_d(t)$ is a purely dispersive wave. As $t \to \pm\infty$, we have that $E(t) \to E^{\pm}$ and $\gamma(t) \to \gamma^{\pm}$. In completely integrable problems, one has $E(t) \equiv E^+ = E^-$ and $\gamma(t) \equiv \gamma^+ = \gamma^-$. Their values are determined by the "scattering data." The decomposition of the phase Θ in (1.2) is reminiscent of Berry's dynamic and geometric phase components [Ber]. The part $\gamma(t)$ cannot be fully accounted for by dynamical considerations.

While there has been considerable progress in understanding *linear* multichannel scattering theory (see [En, Sig–Sof] and those cited therein) in the past ten years, little is known about the corresponding nonlinear situations. Questions like when a bound state (temporally periodic, spatially localized solution) breaks down due to nonlinear (e.g. repulsive) interaction, and the scattering theory of localized waves in the presence of impurities and inhomogeneous media are not understood beyond heuristic considerations or finite time approximations.

Our approach to the problem begins with the simple physical observation that if one starts with the linear Schrödinger equation which describes a bound state and a dispersive wave (corresponding to the continuous spectral part of the Hamiltonian), then the qualitative behavior should not change that much in response to a small nonlinear and Hamiltonian perturbation in the dynamics, i.e. we should still see a localized part which decouples after a long time from the dispersive part. We make an Ansatz which incorporates this observation, from which we derive equations governing the interaction of the two channels.

One set of equations describes the motion of the localized part of the solution through a two-parameter family of bound states of our system. Visualized in terms of the energy (E) and phase (γ), this is a slow evolution of the bound state parameters on a cylinder. The second is a nonlinear equation which describes a purely dispersive wave moving under the effect of the nonlinearity, as well as the effective potential coming from the presence of the localized part. We observe that, $\frac{d}{dt}E(t), \frac{d}{dt}\gamma(t) \in \mathbf{L}^1(\mathbf{R}^1)$ if the remainder wave is dispersive (with a sufficient decay rate) and that the remainder is dispersive if $\frac{d}{dt}E(t), \frac{d}{dt}\gamma(t) \in \mathbf{L}^1(\mathbf{R}^1)$. Therefore, solving the coupled equations gives the required results. The modulating energy and phase of the nonlinear bound state, $E(t)$ and $\gamma(t)$ (or $\Theta(t)$), which govern the localized part of the nonlinear evolution are sometimes referred to by physicists as *collective coordinates*. Equations for collective coordinates have been derived using various formalisms (e.g. averaging of conservation laws, direct perturbation theory [K–A, K–M, Ne]). These equations are sometimes referred to as *modulation equations*. In [We2] their validity was studied in the linear approximation for certain systems which are conservative or small perturbations of conservative systems (e.g. weakly dissipative). We believe that our present results are the first rigorous justification of the collective coordinate description on an infinite time interval for nonintegrable systems.

The system of equations describing the evolution of E and Θ has the form of a perturbation of an integrable Hamiltonian system with a single degree of freedom. Here E and Θ play the role of action and angle variables. In the large $|t|$ limit the coupling to the infinite dimensional radiation field tends to zero and the (E, Θ) system reduces to $\dot{E}(t) = 0$, $\dot{\Theta}(t) = E$.

A final remark is that the problem we consider can be viewed as a kind of restricted three body scattering, where the localized part corresponds to a bound pair and the dispersive part is the "third particle" moving away as $|t| \to \infty$. It is hoped that such an analogy can be developed further and may allow the application of some powerful methods of phase space analysis developed for the linear N-body case.

Notation. All integrals are assumed to be taken over $\mathbf{R}^n$ unless otherwise specified.

$\Re(z)$, $\Im(z)$-respectively, real and imaginary parts of the complex number z,
Δ = Laplacian on $\mathbf{L}^2(\mathbf{R}^n)$,
$\langle x \rangle = (1 + x \cdot x)^{1/2}$, where $x \in \mathbf{R}^n$,
$\langle f, g \rangle = \int f^* g$, where f^* denotes the complex conjugate of f,

$\mathbf{L}^p = \mathbf{L}^p(\mathbf{R}^n)$,
$\mathbf{H}^s = \{f : (I - \Delta)^{s/2} f \in \mathbf{L}^2\}$,
$\mathbf{B} = \{f : f \in \mathbf{H}^1, \langle x \rangle^{1+a} f \in \mathbf{L}^2\}$,
$\|f\|_{\mathbf{B}} = \|f\|_{\mathbf{H}^1} + \|\langle x \rangle^{1+a} f\|_2$,
$\mathbf{C}(\mathbf{I}; \mathbf{X})$ = the space of functions, $u(t, x)$, which are continuous in t, with values in $\mathbf{X}$.

2. The Initial Value Problem, Solitary Waves and Linear Propagator Estimates

2.1. *A Quick Review of* NLS *in* $\mathbf{H}^1$

We shall consider the initial value problem for the nonlinear Schrödinger equation (NLS) with a potential term:

$$i\frac{\partial \Phi(t)}{\partial t} = [-\Delta\Phi(t) + f(x, |\Phi(t)|)]\Phi(t),$$

$$\Phi(0) = \Phi_0. \tag{2.1}$$

Here $\Phi(t)$ is considered as an element of $\mathbf{H}^1(\mathbf{R}^n)$, where n is the spatial dimension. (In this paper we focus on dimensions $n = 2$ and $n = 3$.) Consequently, (2.1) is understood in the sense of the equivalent integral equation:

$$\Phi(t) = e^{i\Delta t}\Phi_0 - i\int_0^t e^{i\Delta(t-s)} f(\cdot, |\Phi(s)|)\Phi(s)\,ds. \tag{2.1'}$$

The theory of well-posedness for the initial value problem in $\mathbf{H}^1$ and in spaces with specified spatial decay rates has been considered for general nonlinearities in [G–V, K, H–N–T, C–W].

In the following $f(x, u)$ will be chosen so that the global existence of solutions to (2.1), perhaps under some restrictions on Φ_0, is known. We specialize here to the case where

$$f(x, u) = V(x) + \lambda|u|^{m-1}, \quad 1 < m < \frac{n+2}{n-2}, \tag{2.2}$$

although the analysis holds for more general nonlinearities.

For the choice (2.2), the existence theory implies:

(i) $\lambda > 0$ (repulsive nonlinearity) global solutions for all $\Phi_0 \in H^1$, i.e. $\Phi \in \mathbf{C}(\mathbf{R}^1; \mathbf{H}^1)$.
(ii) $\lambda < 0$ (attractive nonlinearity)
  (a) $m < 1 + 4/n$, global solutions for all $\Phi_0 \in \mathbf{H}^1$.
  (b) $m \geqq 1 + 4/n$, global solutions for all Φ_0 such that $\|\Phi_0\|_{\mathbf{H}^1}$ is sufficiently small.

Furthermore, solutions of class $\mathbf{C}([0, \mathbf{T}); \mathbf{H}^1)$ leave the following functionals constant in time:

$$\mathscr{H}[\varphi] \equiv \int \tfrac{1}{2}|\nabla\varphi(x)|^2 + \tfrac{1}{2}V(x)|\varphi(x)|^2 + \frac{\lambda}{m+1}|\varphi(x)|^{m+1}dx,$$

$$\mathscr{N}[\varphi] \equiv \int |\varphi(x)|^2 dx.$$

We shall require the following of the linear potential $V(x)$.

Hypotheses V. *Let $V:\mathbf{R}^n \to \mathbf{R}^1$ be a smooth function satisfying:*

(V1) $\langle x\rangle^{3+k+\varepsilon}|\partial^\alpha V(x)| \leqq C_k$ *for all multi-indices* $\alpha \in \mathbf{Z}^+$ *with* $|\alpha| = k \leqq 3$.
(V2) $-\Delta + V$ *has exactly one bound state (isolated eigenvalue) on* $\mathbf{L}^2(\mathbf{R}^n)$ *with strictly negative eigenvalue,* E_*.
(V3) V is a function of $|x|$.

As we shall see later the restriction (V3) appears to be a technical convenience which is a consequence of the available linear local decay estimates. Also, it is clear from our proofs that we can work with considerably milder smoothness and decay assumptions than in (V1).

Our approach will be to reduce the study of (2.1) to essentially two independent problems. The first is the study of existence and certain decay properties of the nonlinear bound states (solitary waves) of (2.1). Then, one has to study the evolution equation for the dispersive part of the solution which one gets by linearizing around a certain time-independent nonlinear bound state. In the small data case, this involves linear spectral analysis of a time independent *reference* Schrödinger Hamiltonian.

2.2. The Solitary Wave and Its Properties

We seek a time periodic, and spatially localized solution of (2.1) of the form

$$\phi(x,t) = e^{-iEt}\psi_E(x).$$

ψ_E then satisfies the equation:

$$-\Delta\psi_E(x) + f(x,|\psi_E(x)|)\psi_E(x) = E\psi_E(x), \quad \psi_E \in \mathbf{H}^2(\mathbf{R}^n). \tag{2.3}$$

We call an $\mathbf{H}^2$ solution of (2.3) a nonlinear bound state or solitary wave profile. The solutions of (2.3) have been studied by many authors (see for example [Str2, Be–Li, Ro–We] and those cited therein). We will concentrate on the case (2.2), with a radial potential $V(x) = V(|x|)$. The result we now state follows from variational and bifurcation methods.

Theorem 2.1. *For $\lambda > 0$, let $E \in (E_*, 0)$, and for $\lambda < 0$, let $E < E_*$. Then there exists a solution $\psi_E > 0$ such that*

(a) $\psi_E \in \mathbf{H}^2$.
(b) *The function* $E \mapsto \|\psi_E\|_{\mathbf{H}^2}$ *is smooth for* $E \neq E_*$, *and*

$$\lim_{E \to E_*} \|\psi_E\|_{\mathbf{H}^2} = 0,$$

i.e. (E, ψ_E) *bifurcates from the zero solution at* $(E_*, 0)$ *in* $\mathbf{H}^2$ *(and therefore, for $n = 2, 3$ in* $\mathbf{L}^p$, *where* $2 \leqq p \leqq \infty$*).*
(c) *For all* $\varepsilon > 0$,

$$|\psi_E(x)| \leqq C_{E,\varepsilon} \exp(-[|E| - \varepsilon]|x|),$$

and
(d) *As* $E \to E_*$,

$$\psi_E = (E - E_*)^{1/(m-1)}\left(\lambda \int \psi_*^{m+1}\right)^{-1/(m-1)}[\psi_* + \mathcal{O}(E - E_*)],$$

the expansion being valid in $\mathbf{H}^2$. *Here,* ψ_* *is the normalized* ($\|\psi_*\|_2 = 1$) *ground state of* $-\Delta + V$ *with corresponding eigenvalue* E_*.

Proof. Parts of (a), (b), and (d) follow from standard theory of bifurcation from simple eigenvalues (see for example [Nir]). To prove part (c) we observe, by the weighted estimates proved below (Theorem 2.3), that $|\psi_E(x)| \leqq C\langle x\rangle^{-2}$. It follows that ψ_E satisfies an equation of the form $[-\Delta + Q(x) - E]\psi_E = 0$, where $Q(x) = o(|x|^{-1})$ as $|x| \to \infty$. Part (c) now follows from linear theory (see for example [Ag]).

A consequence of Theorem 2.1 which will be used is

Corollary 2.2.
(a) *Let* $\lambda > 0$. *Then, for all* $E \in \Omega$, *any compact subinterval of* $(E_*, 0)$, *we have* $\|\psi_E\|_{\mathbf{H}^2} \leqq C_\Omega \|\psi_E\|_2$.
(b) *Let* $\lambda < 0$. *Then there is a* E_c, $-\infty < E_c < E_*$, *such that for* $E \in \Omega$, *any compact subinterval of* (E_c, E_*), $\|\psi_E\|_{\mathbf{H}^2} \leqq C_\Omega \|\psi_E\|_2$.

In our analysis of the dynamics of bound states, we will require various weighted estimates of ψ_E and $\partial_E\psi_E$. We summarize this in

Theorem 2.3. *Let for* $\lambda > 0, E \in (E_*, 0)$, *and for* $\lambda < 0, E < E_*$. *Also, let* E *lie in a sufficiently small neighborhood of* E_*. *Then, for* $k \in \mathbf{Z}_+$ *and* $s \geqq 0$:

$$\|\langle x\rangle^k \psi_E\|_{H^s} \leqq C_{k,s,n} \|\psi_E\|_{\mathbf{H}^s}, \tag{2.4}$$

$$\|\langle x\rangle^k \partial_E \psi_E\|_{H^s} \leqq C'_{k,s,n} |E - E_*|^{-1} \|\psi_E\|_{\mathbf{H}^s}. \tag{2.5}$$

Theorems 2.1 and 2.3 summarize our requirements on solutions of the time independent nonlinear bound state problem. These conditions are not optimal; they are dictated by the known local decay estimates for the Schrödinger propagator associated with $-\Delta + V$ (restricted to its continuous spectral part) which at present, are far from optimal. These technical questions are currently under investigation. Their resolution would enable us to relax restrictions on $f(x, \xi)$ considerably (e.g. removal of the assumption of spherical symmetry and certain limitations on the growth rate of the nonlinearity).

The proof of Theorem 2.3 has the following key ingredients:

1. commuting powers of $\langle x\rangle$ through the Laplacian to derive equations for $w_j = \langle x\rangle^j \psi_E$,
2. the observation that

$$L_E \partial_E \psi_E = \psi_E, \tag{2.6}$$

where

$$L_E = -\Delta + V + \lambda m \psi_E^{m-1} - E \tag{2.7}$$

acting on $\mathbf{L}^2(\mathbf{R}^n)$. (Equation (2.6) follows from differentiation of (2.3) with respect to E.)
3. derivation of equations for $v_j = \langle x\rangle^j \partial_E \psi_E$, and
4. obtaining energy estimates for the control of the $\mathbf{H}^2$ norms of w_j and v_j. The proofs are carried out in Appendix B.

Here we wish to remark as well that since $L_{E_*} \geqq 0$, we have in the repulsive case ($\lambda > 0$) that $L_E > 0$ and that L_E^{-1} is a positivity improving operator [R–S]. Therefore, the positivity of ψ_E implies that $\partial_E \psi > 0$ and that

$$\frac{d}{dE} \int |\psi_E|^2 \bigg|_{E=E_0} > 0 \quad \text{for} \quad \lambda > 0,$$

i.e. the ground state bifurcation curve is monotonically increasing. This simplifies certain analysis in the case $\lambda > 0$ and leads to arguments which are more global in E. These details are presented as well in Appendix B.

2.3. Linear Propagator Estimates

Let $L = -\Delta + V$ on $\mathbf{L}^2(\mathbf{R}^n)$ and assume V satisfies hypotheses (V) of Sect. 2.1. We denote by $P_c(L)$ the projection on the continuous spectral part of L ($\chi_{(0,\infty)}(L)$). We assume that V satisfies a nonresonance condition [J–K, Mu]. To explain this condition we state the following expansion obtained in these references for the free resolvent.

Let $\varepsilon(n) = 0$ for n odd and $\varepsilon(n) = 1$ for n even. Also, let $\sigma > -1/2$ and $s > \max(\sigma + 1, 2\sigma + 2 - n/2)$. Then one has the following expansion as $z \to 0$ with $\Im(z), \Im(z^{1/2}), \Im(\log z) \geqq 0$:

$$(-\Delta - z)^{-1} = \sum_{j=0}^{[(\sigma+1-n)/2]} F_j z^{(n/2)-1-j} (\log z)^{\varepsilon(n)} + \sum_{j=0}^{[\sigma]} G_j z^j + o(z^\sigma), \tag{2.8}$$

where F_j, G_j map $\mathbf{H}^{0,s}$ to $\mathbf{H}^{2,-s}$, where for $s, \sigma \in \mathbf{R}^1$,

$$\mathbf{H}^{\sigma,s} \equiv \{f \in \mathscr{S}' : \langle x \rangle^s (I - \Delta)^{\sigma/2} f \in \mathbf{L}^2\}.$$

We next introduce the generalized null space

$$\mathbf{M} \equiv \{\varphi \in \mathbf{H}^{2,n/2-2-0} : (I + G_0 V)\varphi = 0\} \qquad \text{for} \quad n \geqq 3,$$
$$\mathbf{M} \equiv \{\varphi \in \mathbf{H}^{2,n/2-2-0} : (I + G_0 V)\varphi \in \operatorname{Range}(F_0), F_0 V = 0\} \quad \text{for} \quad n \leqq 2,$$

where $G_0 = (-\Delta)^{-1}$. The nonresonance condition is then

(NR) $$\mathbf{M} = \{0\}.$$

Under these conditions we have the following *local decay estimate* [J–K, M]:

Theorem 2.4. *For $n > 2$,*

$$\| \langle x \rangle^{-\sigma} e^{-iLt} P_c(L) g \|_2 \leqq C(V) \langle t \rangle^{-1-\delta} \| \langle x \rangle^{1+a} P_c(L) g \|_2, \tag{2.9}$$

where $c(V)$ is a constant which depends continuously on $\| \langle x \rangle^{2+a} V \|$, $a > 0$ is arbitrary, $\sigma \geqq 1 + a$, and $\delta = \delta(n, a, \sigma) > 0$. For $n = 2$, $\langle t \rangle^{1+\delta}$ is replaced by $\langle t \ln^2 t \rangle$.

Furthermore, we can use Theorem 2.2 to establish the following $\mathbf{L}^p$ estimates:

Theorem 2.5. *Let $2 < p < \dfrac{2n}{n-2}$ for $n \geqq 3$ and $p > 2$ for $n = 2$. Then,*

$$\| e^{-iLt} P_c(L) g \|_p \leqq C(V) |t|^{(n/p - n/2)} (\| P_c(L) g \|_q + \| \langle x \rangle^{1+a} P_c(L) g \|_2), \tag{2.10}$$

$$\|e^{-iLt}P_c(L)g\|_p \leqq C(V)\langle t\rangle^{(n/p-n/2)}(\|P_c(L)g\|_q + \|g\|_{\mathbf{H}^1} + \|\langle x\rangle^{1+a}P_c(L)g\|_2) \tag{2.11}$$

for some a with $1 \gg a > 0$. *Here,* $p^{-1}+q^{-1}=1$.

To prove (2.10) we write the propagator e^{-iLt} as a perturbation of $e^{i\Delta t}$:

$$e^{-iLt}P_c(L)g = e^{i\Delta t}P_c(L)g - i\int_0^t e^{i\Delta(t-s)}Ve^{-iLs}P_c(L)g\,ds. \tag{2.12}$$

By the free propagator estimate [R–S],

$$\|e^{-i\Delta t}h\|_p \leqq C|t|^{(n/p-n/2)}\|h\|_q, \quad p^{-1}+q^{-1}=1,$$

we have

$$\|e^{-iLt}P_c(L)g\|_p \leqq C|t|^{(n/p-n/2)}\|P_c(L)g\|_q + C\int_0^t |t-s|^{(n/p-n/2)}\|Ve^{-iLs}P_c(L)g\|_q ds. \tag{2.13}$$

Now applying the local decay estimate (2.9) we have

$$\begin{aligned}\|e^{-iLt}P_c(L)g\|_p &\leqq C|t|^{(n/p-n/2)}\|P_c(L)g\|_q \\ &\quad + C'(V)\int_0^t |t-s|^{(n/p-n/2)}\|\langle x\rangle^{-\sigma}e^{-iLs}P_c(L)g\|_2 ds \\ &\leqq C|t|^{(n/p-n/2)}\|P_c(L)g\|_q \\ &\quad + C'(V)\int_0^t |t-s|^{(n/p-n/2)}\langle s\rangle^{-1-\delta}\|\langle x\rangle^{1+a}P_c(L)g\|_2 ds,\end{aligned}$$

from which (2.10) follows. It is straightforward to show that if g is more regular, then $|t|$ can be replace by $\langle t\rangle$ to obtain estimate (2.11).

3. The Equations for the Localized and Dispersive Parts

Equation (2.1) together with our special choice of nonlinearity $f(\cdot)$ can be written as

$$i\frac{\partial\Phi(t)}{\partial t} = [-\Delta + V(x) + \lambda|\Phi(t)|^{m-1}]\Phi(t),$$

$$\Phi(0) = \Phi_0 \in \mathbf{H}^1, \quad n \geqq 2. \tag{3.1}$$

To distinguish between localized and dispersive parts of Φ, we use the following Ansatz:

(α) Decomposition:

$$\begin{aligned}\Phi(t) &\equiv e^{-i\Theta}(\psi_{E(t)} + \phi(t)), \\ \Phi(0) &= e^{i\gamma_0}(\psi_{E_0} + \phi(0)), \\ \Theta &\equiv \int_0^t E(s)ds - \gamma(t), \\ E(0) &= E_0, \quad \gamma(0) = \gamma_0.\end{aligned} \tag{3.2}$$

Here, ψ_E denotes the ground state of (2.3):

$$H(E)\psi_E \equiv (-\Delta + V + \lambda|\psi_E|^{m-1})\psi_E = E\psi_E,$$
$$\psi_E \in \mathbf{H}^2, \quad \psi > 0 \tag{3.3}$$

for $E \in (E_*, 0)$ if $\lambda > 0$ and $E \in (-\infty, E_*)$ if $\lambda < 0$, where

$$E_* \equiv \inf \sigma(-\Delta + V) < 0.$$

(β) Orthogonality condition:

$$\langle \psi_{E_0}, \phi_0 \rangle = 0 \quad \text{and} \quad \frac{d}{dt}\langle \psi_{E_0}, \phi(t) \rangle = 0. \tag{3.4}$$

The orthogonality condition ensures that $\phi(t)$ lies in Range $P_c(H(E_0))$, where $H(E)$ is defined in (3.3). Furthermore, the above use of a *reference Hamiltonian*, $H(E_0)$, is not really a restriction on the initial data ϕ_0, as we shall see in Sect. 5.

Using the above Ansatz, we derive the following equation for ϕ:

$$\begin{aligned} i\frac{\partial \phi}{\partial t} &= [-\Delta + V(x) - E(t) + \dot{\gamma}(t)]\phi \\ &\quad + \lambda|\psi_{E(t)} + \phi|^{m-1}(\psi_{E(t)} + \phi) - \lambda\psi_{E(t)}^m \\ &\quad + \dot{\gamma}(t)\psi_{E(t)} - i\partial_E\psi_{E(t)}\dot{E}(t). \end{aligned} \tag{3.5}$$

We now rewrite (3.5) making $H(E_0)$, the reference Hamiltonian, explicit.

$$i\frac{\partial \phi}{\partial t} = (H(E_0) - E_0)\phi + (E_0 - E(t) + \dot{\gamma}(t))\phi + \mathbf{F}(t). \tag{3.6}$$

Here,

$$\mathbf{F} \equiv \mathbf{F}_1 + \mathbf{F}_2,$$
$$\mathbf{F}_1 \equiv \dot{\gamma}\psi_E - i\dot{E}\partial_E\psi_E,$$

and

$$\mathbf{F}_2 \equiv \mathbf{F}_{2,\mathrm{lin}} + \mathbf{F}_{2,nl}.$$

$\mathbf{F}_{2,\mathrm{lin}}$ is a term which is linear in ϕ:

$$\mathbf{F}_{2,\mathrm{lin}}(\phi, \psi) = \lambda\left(\frac{m+1}{2}\psi_E^{m-1} - \psi_{E_0}^{m-1}\right)\phi + \frac{m-1}{2}\psi_E^{m-1}\phi^*, \tag{3.7}$$

and $\mathbf{F}_{2,nl}$ is a term that is nonlinear in ϕ such that:

$$|\mathbf{F}_{2,nl}(\phi, \psi)| \leqq |\lambda| c[A(\psi)|\phi|^2 + |\phi|^m], \tag{3.8}$$

where $|A(s)|$ is bounded for s bounded, $|A(s)| \to 0$ as $s \to 0$, and c is independent of ψ and ϕ.

To impose (β) we multiply (3.5) by ψ_{E_0} and integrate over all space, equate the real and imaginary parts to zero (condition (β)) to get a coupled system for E and γ:

$$\dot{E}(t) = -\langle \partial_E\psi_E, \psi_{E_0}\rangle^{-1}\Im\langle \mathbf{F}_2, \psi_{E_0}\rangle, \tag{3.9a}$$
$$\dot{\gamma}(t) = \langle \psi_E, \psi_{E_0}\rangle^{-1}\Re\langle \mathbf{F}_2, \psi_{E_0}\rangle. \tag{3.9b}$$

Equations (3.5) and (3.9) comprise a coupled system for the dispersive channel, described by $\phi(t)$, and the bound state channel, described by $E(t), \gamma(t)$. The function $\phi(t)$ and the *collective coordinates* $E(t), \gamma(t)$ are used via (3.2) to construct the solution of the full system (3.1). In the next section we state our main results concerning this decomposition.

In studying the localized and dispersive parts of $\Phi(t)$, we shall work with the equivalent integral formulation of (3.6). To derive an integral equation for the dispersive part, $\phi(t)$, we introduce $U(t,s)$, the propagator associated with the homogeneous linear problem:

$$i\frac{\partial u(t)}{\partial t} = (H(E_0) - E_0)u(t) + (E_0 - E(t) - \dot{\gamma}(t))u(t), \tag{3.10}$$

$$u(s) = f,$$

that is

$$u(t) = U(t,s)f, \quad U(s,s) = Id.$$

Let

$$u(t) = \exp\left(-i\int_s^t [E_0 - E(s)]\,ds - i(\gamma(t) - \gamma(s)) \right) v(t).$$

Then,

$$v(t) = \exp(-i(H(E_0) - E_0)(t-s))f,$$

and therefore

$$U(t,s) = \exp\left(-i\int_s^t (E_0 - E(s))\,ds - i(\gamma(t) - \gamma(s)) \right) \exp(-i(H(E_0) - E_0)(t-s)). \tag{3.11}$$

Equation (3.6) can now be rewritten as

$$\phi(t) = U(t,0)\phi_0 - i\int_0^t U(t,s)\mathbf{F}(s)\,ds. \tag{3.12}$$

For purposes of estimation in $\mathbf{L}^p$ or in a weighted $\mathbf{L}^2$ space (see Sect. 5), we observe that

$$\| U(t,s)g \|_{\mathbf{X}} = \| \exp(-i(H(E_0) - E_0)(t-s))g \|_{\mathbf{X}}, \tag{3.13}$$

where $\mathbf{X}$ denotes any of these spaces.

4. Scattering and Asymptotic Stability Theorems

We assume, as before, that $n = 2$ or $n = 3$. $V(x)$ satisfies hypotheses (**V**) and we let $f(|x|, \Phi) = V(|x|) + \lambda|\Phi|^{m-1}$. We define the **B**-norm of a function g by

$$\| g \|_{\mathbf{B}} = \| g \|_{\mathbf{H}^1} + \| \langle x \rangle^{1+a} g \|_2,$$

where a can be chosen arbitrarily small.

Theorem 4.1 (Scattering). *For $n = 2$ and $n = 3$ let*

$$m > 1 + \frac{2}{n} + \frac{2}{n-1}$$

and for $n = 3$ we require, in addition that $m < 3$. There exists a number δ_0 such that if

(i) $\Phi(0) = \Phi_0(|x|)$,
(ii) $\| \Phi_0 \|_{\mathbf{B}} \leqq \delta_0$,
(iii) *There exists $E_0 \neq E_*$, and Θ_0 such that*

$$\langle e^{i\Theta_0}\psi_{E_0}, \Phi_0 - e^{i\Theta_0}\psi_{E_0} \rangle = 0,$$

(iv) *V satisfies the* (NR) *condition of Sect. 2.3,*

then

$$\Phi(t) = \exp\left(-i \int_0^t E(s)ds + i\gamma(t) \right)(\psi_{E(t)} + \phi(t)) \tag{4.1}$$

with

$$\frac{dE(t)}{dt} \in \mathbf{L}^1(\mathbf{R}^1) \quad \left(\textit{so that} \lim_{t \to \pm\infty} E(t) = E^{\pm} \textit{ exist} \right)$$

$$\frac{d\gamma(t)}{dt} \in \mathbf{L}^1(\mathbf{R}^1) \quad \left(\textit{so that} \lim_{t \to \pm\infty} \gamma(t) = \gamma^{\pm} \textit{ exist} \right),$$

and $\phi(t)$ is purely dispersive in the sense that

$$\| \langle x \rangle^{-\sigma} \phi(t) \|_2 = \mathcal{O}(\langle t \rangle^{-1-\delta}) \tag{4.2a}$$

for $\sigma > 2$ and some $\delta > 0$ if $n = 3$ and

$$\| \langle x \rangle^{-\sigma} \phi(t) \|_2 = \mathcal{O}(\langle t \ln^2 t \rangle^{-1}) \quad \textit{for} \quad n = 2. \tag{4.2b}$$

Moreover,

$$\| \phi(t) \|_{2m} = \mathcal{O}(\langle t \rangle^{(n/2m - n/2)}). \tag{4.3}$$

Remarks

1. In Sect. 5.4 it is shown that hypotheses (ii)–(iii) holds at least for all Φ_0 in an open cone-like region with vertex at the origin.
2. Hypothesis (NR) is satisfied by $gV(x)$ for all but a discrete set of g-values [Ra]. This hypothesis is a way of ensuring that the optimal local decay rates of Sect 2.3 apply to the dispersive part of the solution.
3. The use of the $\mathbf{L}^{2m}$ norm is dictated by the dependence of the linear local decay estimates on the weighted norm $\| \langle x \rangle^{1+a} f \|_2$ (see Sect. 2). This is the source of the restriction to the spherically symmetric case. Namely, we use the uniform spatial decay rate of $\mathbf{H}^1$ radial functions (see Appendix A) to estimate the weighted $\mathbf{L}^2$ norm of the nonlinear term. The restriction $m < 3$ for $n = 3$ is required in order to preclude local (in time) singularities in the estimate for $\| \phi(t) \|_{2m}$. (See also the discussion following the proof of Lemma 5.6.) It is believed that a variant of these estimates for the linear propagator holds with $\mathbf{L}^p$ norms instead of weighted norms. Such estimates would lead to extensions of our results to the non-spherically

symmetric case and the more natural upper bound on the nonlinearity $m<\frac{n+2}{n-2}$, for $n \geqq 3$.

The following is a related stability result which says that if the initial data for (2.1) lies near a particular nonlinear bound state of energy E_0, and phase γ_0, then the solution $\Phi(t)$ converges, as $t \to \pm\infty$, to a nearby nonlinear bound state of energy $E^\pm$ and phase $\gamma^\pm$.

Theorem 4.2 (Asymptotic Stability). *Let m and n be as in Theorem 4.1. Let $\Omega_\eta = (E_*, E_* + \eta\,\mathrm{sgn}(\lambda))$, where η is positive and sufficiently small. Then for all $E_0 \in \Omega_\eta$ and $\gamma_0 \in [0, 2\pi)$, there is a positive number $\varepsilon(\eta, E_0)$ such that if*

$$\Phi(0) = (\psi_{E_0} + \phi(0))e^{i\gamma_0},$$

where

$$\|\phi(0)\|_{\mathbf{B}} \leqq \varepsilon,$$

then $\Phi(t)$ decomposes into localized and dispersive parts as in (4.1), where $\frac{dE(t)}{dt}, \frac{d\gamma(t)}{dt}$ are in $\mathbf{L}^1(\mathbf{R}^1)$ and $\phi(t)$ obeys the linear dispersive and local decay estimates (4.2), (4.3).

5. The Coupled Channel Equations

5.1. Local Existence

It is straightforward to prove, by a contraction mapping argument, that (3.5)–(3.6), (3.9) together with initial conditions $\phi(0) = \phi_0 \in \mathbf{H}^1$, $\gamma(0) = \gamma_0$, and $E(0) = E_0$ has, for some $T > 0$ a unique local solution $\phi \in \mathbf{C}([0, \mathbf{T}); \mathbf{H}^1)$, $E(t), \gamma(t) \in \mathbf{C}^1[0, \mathbf{T})$, with $E(t) \in (E_*, 0)$ for $\lambda > 0$ and $E(t) \in (E_c, E_*)$ for $\lambda < 0$. Thus, $\Phi(t)$ given by (3.2) solves (3.1) and agrees with the unique $\mathbf{H}^1$ solution discussed in our summary of the existence theory in Sect. 2.1. In particular, the functionals $\mathscr{H}[\Phi]$ and $\mathscr{N}[\Phi]$ (Sect. 2.1) are invariant on $[0, T)$. It follows by Sobelev–Nirenberg–Gagliardo type estimates that

$$\|\Phi(t)\|_{\mathbf{H}^1} \leqq C(\|\Phi_0\|_{\mathbf{H}^1}) \tag{5.1}$$

for $0 \leqq t \leqq T$, where the upper bound in (5.1) is independent of T. If $\lambda < 0$ (attractive nonlinearity) and $m \geqq 1 + \frac{4}{n}$, we require, in addition that $\|\Phi_0\|_{\mathbf{H}^1}$ be small for (5.1) to hold with C, independent of T. For otherwise, solutions can become unbounded in $\mathbf{H}^1$ in finite time (*blow up*). See, for example [Gl, We1].

It follows from our Ansatz (3.2) that

$$\|\phi(t)\|_{\mathbf{H}^1} \leqq C'(\|\Phi_0\|_{\mathbf{H}^1}, \|\psi_{E(t)}\|_{\mathbf{H}^1}) \tag{5.2}$$

for $t \in [0, T)$.

5.2. A Priori Estimates

In this section we obtain á priori estimates on $\phi(t), E(t)$, and $\gamma(t)$, which arise in the decomposition of $\Phi(t)$, (3.2), and show that the decomposition persists for all time, t, with the desired properties.

Since $\phi(t)$, the solution of (3.5) is an $\mathbf{H}^1$ function, we interpret (3.5) in the sense of the equivalent integral equation:

$$\phi(t) = U(t,0)\phi_0 - i\int_0^t U(t,s)P_c(H(E_0))\mathbf{F}(s)ds. \tag{5.3}$$

Here, $U(t,s)$ denotes the propagator displayed in (3.11) and $\mathbf{F}(s) = \mathbf{F}(\phi(s), \psi_{E(s)})$ is displayed in (3.6)–(3.8).

The first step is to use the local decay and $\mathbf{L}^p$ decay estimates of the linear theory (in Sect. 2.3) to derive decay estimates for $\phi(t)$. Let

$$\begin{aligned} L &= -\Delta + V(x) + \lambda|\psi_{E_0}|^{m-1} - E_0 \\ &= H(E_0) - E_0. \end{aligned}$$

We will apply the propagator estimates of Sect. 2.3 to the associated unitary group e^{-iLt}. These estimates require that the potential of the operator L, $V(x) + \lambda|\psi_{E_0}|^{m-1}$, satisfy (NR). We claim this is not a restriction. This is seen as follows.

Suppose $V(x) + \lambda|\psi_{E_0}|^{m-1}$ does not satisfy (NR). Then, we solve the initial value problem (3.1) for some small time interval $[0, T_0]$, with the decomposition (3.2) augmented with the modified orthogonality condition

$$\langle\psi_{E_0}, \phi_0\rangle = 0 \quad \text{and} \quad \frac{d}{dt}\langle\psi_{E(t)}, \phi(t)\rangle = 0 \tag{3.4'}$$

in place of (3.4). Now consider the one-parameter family of potentials

$$Q(x; E(t)) = V(x) + \lambda|\psi_{E(t)}|^{m-1}, \quad t\in[0, T_0].$$

Proposition. *For generic ϕ_0, we have that $Q(x; E(t))$ satisfies* (NR) *for some $t\in[0, T_0]$.*

Proof. The implicit function theorem for analytic mappings can be used to show that ψ_E is equal to $(E - E_*)^{1/(m-1)}$ times an absolutely convergent power series in $E - E_*$, for E sufficiently near E_*. (See part (d) of Theorem 2.1.) Thus, the mapping $E \mapsto \psi_E^{m-1}$ has a holomorphic extension to a complex E – neighborhood of E_*. By an argument of J. Rauch [Ra, pp. 164–165], $V(x) + \lambda|\psi_{E_0}|^{m-1}$ satisfies (NR) at all but a discrete set of E – values. Thus, if $E(t)$ is not identically E_0, there will be some $t_0\in[0, T_0]$ for which $Q(x; E(t_0))$ is nonresonant. The case where $E(t) \equiv E_0$ is nongeneric, as this would require $\frac{d}{dt}E(t=0) = 0$, which by (3.9a) leads to a codimension condition. ■

Having found a t_0 at which $V + \lambda|\psi_{E(t_0)}|^{m-1}$ satisfies (NR), we continue the solution for $t \geqq t_0$ using the decomposition (3.2), (3.4). By uniqueness of solutions to the Cauchy problem, the solution obtained in this way corresponds to the solution $\Phi(t)$ of (2.1) with data at $t = 0$, $\Phi(0)$, as in (3.2).

Due to the presence of weighted $\mathbf{L}^2$ norms in our linear decay estimates, estimation of the integral term in (5.3) will lead to weighted $\mathbf{L}^2$ estimates of the nonlinear term $\mathcal{O}(|\phi|^m)$. It is therefore natural, to seek estimates for $\phi(t)$ in $\mathbf{L}^{2m}$.

Proposition 5.1.

$$\|\phi(t)\|_{2m} \leqq \|e^{-iLt}\phi_0\|_{2m} + \int_0^t |t-s|^{\varepsilon_{2m}-1}(c_1(\psi,\phi)\|\phi\|_{2m}^2 + c_2(\psi,\phi)\|\phi\|_{2m}^{m-1}$$
$$+ c_3(\psi,\phi)\|\phi\|_{2m}^{\beta r} + c_4(\psi,\phi)\|\langle x\rangle^{-\sigma}\phi(s)\|_2$$
$$+ c_{01}|\dot{E}(s)| + c_{02}|\dot{\gamma}(s)|)ds, \tag{5.4}$$

$$\|\langle x\rangle^{-\sigma}\phi(t)\|_2 \leqq \|\langle x\rangle^{-\sigma}e^{-iLt}\phi_0\|_2$$
$$+ \int_0^t \langle t-s\rangle^{-1-\delta}(d_1(\psi,\phi)\|\phi\|_{2m}^2 + d_2(\psi,\phi)\|\phi\|_{2m}^{m-1}$$
$$+ d_3(\psi,\phi)\|\phi\|_{2m}^{\beta r} + d_4(\psi,\phi)\|\langle x\rangle^{-\sigma}\phi(s)\|_2$$
$$+ d_{01}|\dot{E}(s)| + d_{02}|\dot{\gamma}(s)|)ds. \tag{5.5}$$

Here, $\beta r = m(1-\mu)$, where $\mu = \dfrac{2(1+a)}{(m-1)(n-1)}$. (See Proposition 5.4 below.)

Here, $c_i(\psi,\phi)$ and $d_i(\psi,\phi)$, $1 \leqq i \leqq 4$, are constants which depend on weighted norms of $\psi_{E(t)}$ and the $\mathbf{H}^1$ norm of $\phi(t)$ for $t \in [0,T)$. Such weighted norms are all controlled by the weighted estimates of Sect. 2.2. Also, $c_i(a,b)$ and $d_i(a,b)$ tend to zero as a tends to zero while b lies in a bounded set. The precise form of c_{0i} and d_{0i} is worth giving in detail for the purpose of understanding the behavior of the product with $\dot{E}$ and $\dot{\gamma}$ (see Proposition 5.5 below) as $E \to E_*$. We have

$$c_{01} = \mathcal{O}(\|\psi_E\|_{2m}), \qquad c_{02} = \mathcal{O}(\|\partial_E\psi_E\|_{2m}),$$
$$d_{01} = \mathcal{O}(\|\langle x\rangle^{-\sigma}\psi_E\|_2), \quad d_{02} = \mathcal{O}(\|\langle x\rangle^{-\sigma}\partial_E\psi_E\|_2).$$

Furthermore, $1-\varepsilon_p \equiv \dfrac{n}{2} - \dfrac{n}{p}$, and $\delta > 0$ is the number appearing in the linear estimate (2.9). For $n=2$, $\langle\xi\rangle^{1+\delta}$ is replaced by $\langle\xi \ln^2 \xi\rangle$

Proof of Proposition 5.1. We begin by estimating (5.3) in $\mathbf{L}^p$. Using (2.10)–(2.11), we have

$$\|\phi(t)\|_p \leqq \|e^{-iLt}\phi_0\|_p + \int_0^t \|e^{-iL(t-s)}P_c(L)\mathbf{F}(s)\|_p ds$$
$$\leqq C(V)\langle t\rangle^{\varepsilon_p-1}(\|\phi_0\|_q + \|\phi_0\|_{\mathbf{H}^1} + \|\langle x\rangle^{1+a}\phi_0\|)$$
$$+ C(V)\int_0^t |t-s|^{\varepsilon_p-1}(\|P_c(L)\mathbf{F}(s)\|_q + \|\langle x\rangle^{1+a}P_c(L)\mathbf{F}(s)\|_2)ds. \tag{5.6}$$

The projection operators, $P_c = P_c(L)$ can be removed at the expense of ψ dependent constants:

Lemma 5.2.

$$\|P_c g\|_q \leqq (1 + \|\psi\|_p\|\psi\|_q\|\psi\|_2^{-2})\|g\|_q$$
$$\|\langle x\rangle^{1+a}P_c g\|_2 \leqq (1 + \|\langle x\rangle^{1+a}\psi\|_2\|\psi\|_2^{-1})\|\langle x\rangle^{1+a}g\|_2.$$

Therefore, we have

$$\|\phi(t)\|_p \leqq C_1(V)\langle t\rangle^{\varepsilon_p-1}(\|\phi_0\|_q+\|\phi_0\|_{\mathbf{H}^1}+\|\langle x\rangle^{1+a}\phi_0\|_2)$$
$$+C_2(V,\psi)\int_0^t |t-s|^{\varepsilon_p-1}(\|\mathbf{F}(s)\|_q+\|\langle x\rangle^{1+a}\mathbf{F}(s)\|_2)ds. \tag{5.7a}$$

Similarly, we can estimate the weighted norm of $\phi(t)$:

$$\|\langle x\rangle^{-\sigma}\phi(t)\|_2 \leqq C_1'(V)\langle t\rangle^{-1-\delta}\|\langle x\rangle^{1+a}\phi_0\|_2$$
$$+C_2(V,\psi)\int_0^t \langle t-s\rangle^{-1-\delta}\|\langle x\rangle^{1+a}\mathbf{F}(s)\|_2 ds. \tag{5.7b}$$

To proceed, we require estimates on $\|\mathbf{F}\|_q$ and $\|\langle x\rangle^{1+a}\mathbf{F}\|_2$, where $\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2$ (see (3.6)–(3.8)). For the terms $\|\mathbf{F}_2\|_q$ and $\|\langle x\rangle^{1+a}\mathbf{F}_2\|_2$, with $A(\psi)$ as in (3.8) we have:

Proposition 5.3. *Let $p=2m, m>2$, and $p^{-1}+q^{-1}=1$. Then,*

$$\|\mathbf{F}_2\|_q \leqq \|\langle x\rangle^{\sigma}\psi^{m-1}\|_{r_1}\|\langle x\rangle^{-\sigma}\phi\|_2+\|A^{1/2}(\psi)\|_{r_2}^2\|\phi\|_{2m}^2+\|\phi\|_2\|\phi\|_{2m}^{m-1}, \tag{5.8}$$

where $r_1^{-1}=q^{-1}-2^{-1}$ and $r_2^{-1}=(2q)^{-1}-p^{-1}$.

Proposition 5.4. *Let $p=2m$, $m>2$, and $p^{-1}+q^{-1}=1$. Then,*

$$\|\langle x\rangle^{1+a}\mathbf{F}_2\|_2 \leqq \|\langle x\rangle^{1+a+\sigma}\psi^{m-1}\|_\infty\|\langle x\rangle^{-\sigma}\phi\|_2$$
$$+\|\langle x\rangle^{1+a}A(\psi)\|_{r_3}\|\phi\|_{2m}^2$$
$$+c(\|\phi\|_{2m}^m+\|\phi\|_2^{\alpha r}\|\phi\|_{\mathbf{H}^1}^{2(1+a)/(n-1)}\|\phi\|_{2m}^{\beta r}), \tag{5.9}$$

where $r_3^{-1}=2^{-1}-m^{-1}$, $\alpha=\dfrac{\mu}{r}$, $\beta=\dfrac{m(1-\mu)}{r}$, $r=m-\dfrac{2(1+a)}{n-1}$, and $\mu=\dfrac{2(1+a)}{(m-1)(n-1)}$.

The proofs of Propositions 5.3 and 5.4 are presented in Appendix A. Here, we only wish to remark that it is in handling the weighted norm of the nonlinear term $\mathcal{O}(|\phi|^m)$ that the restriction to radial solutions is used to derive (5.8) and (5.9).

The inhomogeneous term in the ϕ equation, $\mathbf{F}_1$, can be easily bounded as follows:

$$\|\mathbf{F}_1\|_q \leqq \|\psi\|_q|\dot{\gamma}|+\|\partial_E\psi_E\|_q|\dot{E}|, \tag{5.10}$$

$$\|\langle x\rangle^{1+a}\mathbf{F}_1\|_2 \leqq \|\langle x\rangle^{1+a}\psi\|_2|\dot{\gamma}|+\|\langle x\rangle^{1+a}\partial_E\psi_E\|_2|\dot{E}|. \tag{5.11}$$

Propositions 5.3 and 5.4 together with estimates (5.10)–(5.11) imply Proposition 5.1.

Our next step is to estimate $\dot{\gamma}$ and $\dot{E}$, which appear in (5.10) and (5.11), in terms of norms of ϕ and ψ.

Proposition 5.5. *Let $[0,T)$ denote the time interval of local existence for the system (3.5), (3.9). Then, for $0\leqq t\leqq T$,*

$$|\dot{E}(t)| \leqq C_E(\partial_E\psi_E,\psi_{E_0})|\lambda|[\|\langle x\rangle^{-\sigma}\phi(t)\|_2+\|\phi(t)\|_{2m}^2+\|\phi(t)\|_{2m}^m], \tag{5.12}$$

$$|\dot{\gamma}(t)| \leqq C_\gamma(\psi_E,\psi_{E_0})|\lambda|[\|\langle x\rangle^{-\sigma}\phi(t)\|_2+\|\phi(t)\|_{2m}^2+\|\phi(t)\|_{2m}^m], \tag{5.13}$$

where C_E and C_γ depend on $\|\psi_{E(t)}\|_{\mathbf{H}^2}$ and $\|\psi_{E_0}\|_{\mathbf{H}^2}$ and tend to zero as these norms approach zero.

Proof of Proposition 5.5. From (3.9) we have that

$$|\dot{E}(t)| \leqq \langle \psi_{E_0}, \partial_E \psi_E \rangle|^{-1} |\langle \mathbf{F}_2, \psi_{E_0} \rangle|, \tag{5.14}$$

$$|\dot{\gamma}(t)| \leqq \langle \psi_{E_0}, \psi_E \rangle|^{-1} |\langle \mathbf{F}_2, \psi_{E_0} \rangle|. \tag{5.15}$$

To estimate the term $|\langle \mathbf{F}_2, \psi_{E_0} \rangle|$, we use our estimates on $\mathbf{F}_2$ in Sect. 3. First, by (3.6),

$$\begin{aligned} |\langle \psi_{E_0}, \mathbf{F}_{2,\mathrm{lin}} \rangle| &\leqq |\lambda| \frac{m-1}{2} |\langle \psi_{E_0}, \psi_E^{m-1} \phi \rangle| \\ &\leqq |\lambda| \frac{m-1}{2} \| \langle x \rangle^{\sigma} \psi_{E_0} \psi_E^{m-1} \langle x \rangle^{-\sigma} \phi \|_1 \\ &\leqq C(\psi_E, \psi_{E_0}, m) \| \langle x \rangle^{-\sigma} \phi \|_2 . \end{aligned} \tag{5.16}$$

By (3.8),

$$|\langle \psi_{E_0}, \mathbf{F}_{2,nl} \rangle| \leqq C|\lambda| (\| \psi_{E_0} A(\psi_E) \|_{m'} \| \phi \|_{2m}^2 + \| \psi_{E_0} \|_2 \| \phi \|_{2m}^m), \tag{5.17}$$

where $m'^{-1} = 1 - m^{-1}$.

Use of (5.16)–(5.17) in (5.14)–(5.15), and noting the behavior of ψ_E for E near E_*, given in Sect. 2.2 yields the result.

Remarks

1. Our goal is to obtain a set of inequalities for norms that control the dispersion of $\phi(t)$. The above estimates suggest the use of the norms

$$M_1(T) = \sup_{|t| \leqq T} \langle t \rangle^{1-\varepsilon_p} \| \phi(t) \|_p, \tag{5.18}$$

$$M_2(T) = \sup_{|t| \leqq T} \langle t \rangle^{1+\delta} \| \langle x \rangle^{-\sigma} \phi(t) \|_2, \tag{5.19}$$

where $\langle t \rangle^{1+\delta}$ is replaced by $\langle t \ln^2 t \rangle$ when $n = 2$.

2. It turns out that with the linear local decay estimates we use, it is natural to choose $p = 2m$. Better local decay estimates would permit using $p = m + 1$ for *large* nonlinearities. This would improve the upper bound on range of nonlinearities for which the above results are valid.

3. To show that the limits $\lim_{t \to \pm\infty} E(t)$ and $\lim_{t \to \pm\infty} \gamma(t)$ exist, we prove that $\dot{E}$ and $\dot{\gamma}$ are in $\mathbf{L}^1(\mathbf{R}^1; \mathbf{dt})$. Linear theory suggests that the correct $\mathbf{L}^p$ decay rate is $\langle t \rangle^{\varepsilon_p - 1} = \langle t \rangle^{(n/p - n/2)}$. Therefore, the estimates (5.12)–(5.13) suggest that m be chosen so that $2(1 - \varepsilon_p) > 1$, and $m(1 - \varepsilon_p) > 1$, where $1 - \varepsilon_p \equiv \frac{n}{2} - \frac{n}{p} = \frac{n}{2} - \frac{n}{2m}$. These reduce to the constraint $m > \frac{n}{n-1}$. We shall see further constraints on m imposed in the following section.

5.3. Global Existence and Large Time Asymptotics

In this subsection we derive closed coupled inequalities for $M_1(T)$ and $M_2(T)$ (see (5.18)–(5.19)) which yield bounds on M_1 and M_2, independent of T. This implies a rate of dispersion of $\phi(t)$ which in turn implies that $E(t)$ and $\gamma(t)$ have asymptotic values as $t \to \pm\infty$.

We first apply the estimates (2.9)–(2.11) to the initial data terms in (5.4)–(5.5). We then multiply (5.4) by $\langle t\rangle^{1-\varepsilon_{2m}}$, (5.5) by $\langle t\rangle^{1+\delta}$ ($\langle t\ln^2 t\rangle$ for $n=2$) and take the supremum over all $|t|\leqq T$ to obtain:

$$\begin{aligned} M_1(T)\leqq{} & C(V)(\|\phi_0\|_q+\|\phi_0\|_{\mathbf{H}^1}+\|\langle x\rangle^{1+a}\phi_0\|_2) \\ & +C_4(\psi)M_2(T)+C_1(\psi)M_1^2(T) \\ & +C_2(\psi)M_1^{m-1}(T)+C_3(\|\phi(t)\|_{\mathbf{H}^1})M_1^{\beta r}(T) \\ & +C_5(\psi,\partial_E\psi)\sup_{|t|\leqq T}\langle t\rangle^{1+\delta}[|\dot\gamma(t)|+|\dot E(t)|]. \end{aligned} \tag{5.20}$$

In the above estimate $q^{-1}=1-(2m)^{-1}$. Similarly, we have

$$\begin{aligned} M_2(T)\leqq{} & C(V)\|\langle x\rangle^{1+a}\phi_0\|_2+D_4(\psi)M_2(T) \\ & +D_1(\psi)M_1^2(T)+D_3(\|\phi(t)\|_{\mathbf{H}^1})M_1^{\beta r}(T) \\ & +D_4(\psi,\partial_E\psi)\sup_{|t|\leqq T}\langle t\rangle^{1+\delta}[|\dot\gamma(t)|+|\dot E(t)|]. \end{aligned} \tag{5.21}$$

In the constants C_j and D_j are contained terms of the form

$$\langle t\rangle^{\eta}\int_0^t|t-s|^{-\alpha}\langle s\rangle^{-\beta}ds.$$

These terms are required to be bounded independently of t. The range of nonlinearities (powers of m) for which this occurs is determined with the aid of:

Lemma 5.6. *For $\alpha<1$,*

$$\int_0^t|t-s|^{-\alpha}\langle s\rangle^{-\beta}ds\leqq C(\alpha,\beta)\langle t\rangle^{-\min(\alpha,\alpha+\beta-1)}.$$

Proof.

$$\int_0^t|t-s|^{-\alpha}\langle s\rangle^{-\beta}ds=\int_0^{t/2}+\int_{t/2}^t=A+B.$$

Estimating A and B individually, we get

$$A\leqq(2/t)^{\alpha}\int_0^{t/2}\langle s\rangle^{-\beta}ds,\quad B\leqq(2/t)^{\beta}\int_{t/2}^t|t-s|^{-\alpha}ds,$$

from which (5.21) follows.

The most problematic term, regarding decay is the term $\|\phi\|_{2m}^{\beta r}$ in (5.9). This leads to the restriction

$$m>1+\frac{2}{n}+\frac{2(1+a)}{n-1},$$

where a is arbitrarily small and positive. Furthermore, the restriction $\alpha<1$ in Lemma 5.6 implies

$$\frac{n}{2}-\frac{n}{2m}<1.$$

The latter leads to the constraint $m<3$ in dimension $n=3$, as in the statements of Theorems 4.1 and 4.2.

To close the inequalities (5.20)–(5.21) we use (5.12)–(5.13) together with Lemma 5.6. This gives

$$\sup_{|t|\leqq T} |\dot{E}(t)| \leqq C_E(\psi_E, \psi_{E_0})|\lambda|[M_2(T) + M_1^2(T) + M_1^m(T)], \tag{5.22a}$$

$$\sup_{|t|\leqq T} |\dot{\gamma}(t)| \leqq C_\gamma(\psi_E, \psi_{E_0})|\lambda|[M_2(T) + M_1^2(T) + M_1^m(T)]. \tag{5.22b}$$

We then use (5.22) in (5.20)–(5.21). The results are summarized in

Proposition 5.7. *Let $(\phi(t), E(t), \gamma(t))$ be the unique solution of (3.5), (3.9) of class*

$$\mathbf{C}([0,\mathrm{T}); \mathbf{H}^1) \times \mathbf{C}^1[0,\mathrm{T}) \times \mathbf{C}^1[0,\mathrm{T}).$$

Then,

$$\begin{aligned} M_1(T) \leqq\ & C(V)(\|\phi_0\|_q + \|\phi_0\|_{\mathbf{H}^1} + \|\langle x\rangle^{1+a}\phi_0\|_2) + C_1'(\psi, \partial_E\psi)M_2(T) \\ & + C_2'(\psi, \partial_E\psi)[M_1^2(T) + C_3'(\|\phi(t)\|_{\mathbf{H}^1})M_1^m(T), \end{aligned} \tag{5.23}$$

$$\begin{aligned} M_2(T) \leqq\ & C(V)\|\langle x\rangle^{1+a}\phi_0\|_2 + D_1'(\psi, \partial_E\psi)M_2(T) \\ & + D_2'(\psi, \partial_E\psi)[M_1^2(T) + D_3'(\|\phi(t)\|_{\mathbf{H}^1})M_1^m(T)]. \end{aligned} \tag{5.24}$$

Here, C_i' and D_i' are controlled by the maximum over $|t| \leqq T$ of the $\mathbf{H}^2$ norm of ψ and $\partial_E\psi$.

To obtain closed inequalities for $M_1(T)$ and $M_2(T)$, we observe that as E approaches E_*, the coefficients $C_1'(\psi, \partial_E\psi)$ and $D_1'(\psi, \partial_E\psi)$ tend to zero and are uniformly bounded on any compact subinterval of $(E_*, 0)$ for $\lambda > 0$ and $(-\infty, E_*)$ for $\lambda < 0$. These properties of $C_i'(\psi, \partial_E\psi)$ and $D_i'(\psi, \partial_E\psi)$ follow from the bifurcation analysis of the continuum of solutions (E, ψ_E) in a neighborhood of E_* (Sect. 2.2).

To prove global existence for the system (3.5), (3.9) with the desired asymptotic behavior, we first choose initial conditions E_0, γ_0 and ϕ_0 so that on the interval of local existence, C_1' and D_1' are less than $\frac{1}{2}$ in magnitude. Then, by (5.24) we have

$$\begin{aligned} M_2(T) \leqq\ & 2C(V)\|\langle x\rangle^{1+a}\phi_0\|_2 \\ & + D_2'(\psi, \partial_E\psi)[M_1^2(T) + D_3'(\|\phi(t)\|_{\mathbf{H}^1})M_1^m(T)]. \end{aligned} \tag{5.25}$$

Substitution of (5.25) into (5.23) yields, after some manipulation,

$$M_1(T) \leqq C_0''(\|\phi_0\|_q + \|\phi_0\|_{\mathbf{H}^1} + \|\langle x\rangle^{1+a}\phi_0\|_2) + C_1''M_1^2(T) + C_2''M_1^m(T)), \tag{5.26}$$

where $C_0'' = C(V)(1 + 2C_1')$, $C_1'' = C_2' + C_1'D_2'$ and $C_2'' = C_2'C_3' + C_1'D_2'D_3'$.

We now rewrite (5.26) as

$$M_1(T)f(M_1(T)) \leqq D_0,$$

where

$$f(\alpha) = 1 - C_1''\alpha - C_2''\alpha^{m-1},$$

and the data term

$$D_0 = C_0''[\|\phi_0\|_q + \|\phi_0\|_{\mathbf{H}^1} + \|\langle x\rangle^{1+a}\phi_0\|_2].$$

Let $\alpha_* f(\alpha_*) = \max_{\alpha > 0} \alpha f(\alpha)$. Let $|E_0 - E_*| \equiv 2\eta$, where η will be chosen sufficiently small. We first require that η be such that ψ_E and $\partial_E\psi_E$ dependent constants in (5.22) are less than $\eta^{1/2}$. This is possible by the local analysis of ψ_E presented in Sect. 2.2.

Now choose ϕ_0, so that

$$D_0 \leqq \eta f(\eta) \leqq \alpha_* f(\alpha_*)/2,$$

and so that

$$M_1(0) = \|\phi_0\|_{2m} < \eta.$$

Then by the continuity of M_1, we have $M_1(T) \leqq \eta$, and therefore by (5.22) via (5.25) that

$$|\dot{E}(t)| \leqq C_E \eta^{3/2} \langle t \rangle^{-1-\delta}, \tag{5.27a}$$

$$|\dot{\gamma}(t)| \leqq C_\gamma \eta^{3/2} \langle t \rangle^{-1-\delta}. \tag{5.27b}$$

For $n=2$, $\langle t \rangle^{-1-\delta}$ is replaced by $\langle t \ln^2 t \rangle^{-1}$.

Integration of (5.27) yields

$$\int_{-T}^{T} |\dot{E}(t)| dt \leqq C''' \eta^{3/2}, \tag{5.28a}$$

$$\int_{-T}^{T} |\dot{\gamma}(t)| dt \leqq C''' \eta^{3/2}, \tag{5.28b}$$

where C''' is independent of T and η.

Thus if

$$T_m \equiv \sup \{t : |E(t) - E_0| < \eta\},$$

it follows that for η sufficiently small $T_m = \infty$. For the right-hand side of (5.28) is independent of T, and this ensures that

$$|E(t) - E_0| < \eta, \quad |t| \leqq T$$

provided η is sufficiently small. It follows that all constants $C(\psi, \partial_E \psi)$ and $D(\psi, \partial_E \psi)$ maintain their assumed bounds and we can take $T \to \infty$ to obtain

$$M_1(\infty) \leqq \eta \tag{5.29a}$$

and

$$M_2(\infty) \leqq C\eta \tag{5.29b}$$

for some $C > 0$.

5.4. *Decomposition of the Initial Data* Φ_0

Here we return to the Ansatz (3.2)–(3.4). Let $\tilde{E} \in (E_*, 0)$ for $\lambda > 0$ and $\tilde{E} \in (-\infty, E_*)$ for $\lambda < 0$. Consider the initial data which is nearby a nonlinear bound state:

$$\Phi_0 = e^{i\tilde{\gamma}} \psi_{\tilde{E}} + \delta\Phi. \tag{5.30}$$

In general $\langle \delta\Phi, \psi_{\tilde{E}} \rangle \neq 0$, so we write

$$\begin{aligned} \Phi_0 &= e^{i\gamma_0} \psi_{E_0} + [e^{i\tilde{\gamma}} \psi_{\tilde{E}} - e^{i\gamma_0} \psi_{E_0} + \delta\Phi] \\ &\equiv e^{i\gamma_0} \psi_{E_0} + \phi_0 \end{aligned}$$

with a view towards finding E_0 and γ_0 such that

$$\langle e^{i\gamma_0} \psi_{E_0}, \phi_0 \rangle = 0.$$

We shall then take ϕ_0 to be the initial data for the dispersive channel evolution.
Let

$$F[E_0,\gamma_0,\delta\Phi] \equiv \langle e^{i\gamma_0}\psi_{E_0},\phi_0\rangle$$
$$= \langle e^{i\gamma_0}\psi_{E_0}, e^{i\tilde{\gamma}}\psi_{\tilde{E}} - e^{i\gamma_0}\psi_{E_0} + \delta\Phi\rangle. \tag{5.31}$$

Then, $F[\tilde{E},\tilde{\gamma},0]=0$. We want to solve $F=0$ in a neighborhood of $(\tilde{E},\tilde{\gamma},0)$. Since F is complex-valued, the equation $F=0$ can be viewed as two real equations:

$$F_1[E_0,\gamma_0,\delta\Phi]=0, \quad F_2[E_0,\gamma_0,\delta\Phi]=0.$$

The Jacobian of this mapping at $(E_0,\gamma_0,0)$ is given by

$$\begin{pmatrix} 0 & -\frac{d}{dE}\int|\psi_E|^2\Big|_{E=E_0} \\ \int|\psi_E|^2|_{E=E_0} & 0 \end{pmatrix}. \tag{5.32}$$

By the results of Sect. 2.2 and Appendix B, we have that if $\tilde{E}\in(E_*,0)$, for $(\lambda>0)$, and $\tilde{E}\in(E_*-\varepsilon,E_*)$, for $(\lambda<0)$, the curve $E\mapsto\|\psi_E\|_2^2$ has no critical points. It follows from the implicit function theorem that in some $\mathbf{L}^2$ neighborhood of $\psi_{\tilde{E}}$, the decomposition

$$\Phi_0 = e^{i\gamma_0}\psi_{E_0}+\phi_0$$

with condition (5.30) holds. Furthermore, since on any compact subinterval of $(E_*,0)$, for $\lambda>0$, and $(E_*-\varepsilon,E_*)$, for $\lambda<0$, $\frac{d}{dE}\|\psi_E\|_2^2$ stays uniformly away from zero, the $\mathbf{B}$-neighborhood of $\psi_{\tilde{E}}$ can be chosen uniformly in $\tilde{E}$, where $\tilde{E}$ varies over such a compact subinterval. This resolves the question of initial decomposition for Stability Asymptotic Theorem 4.2.

The proof of Theorem 4.1 follows the above lines. The constraint

$$\langle e^{i\Theta_0}\psi_{E_0}, \Phi_0 - e^{i\Theta_0}\psi_{E_0}\rangle = 0 \tag{5.33}$$

(see the statement of Theorem 4.1) prescribes a choice of E_0 and γ_0, and therefore an initial decomposition, (3.2). By Theorem 4.2, for each E in a sufficiently small interval with E_* as endpoint, there is an open ball about ψ_E such that for all data in this ball, the solution decomposes as in (4.1). The radius of this open ball may shrink to zero, in general as E tends to E_*. Thus, the set of data Φ_0, on which the constraint (5.33) can be realized contains the union of these open balls over E near E_*, or a cone-like region.

Furthermore, if E and γ are such that the constraint (5.33) holds, then $\|\Phi_0\|_2 \geqq \|\psi_E\|_2$.

Finally, if $\|\cdot\|_X$ denotes any norm used to measure ψ_E (see Sect. 2.2), then we have

$$\|\psi_E\|_X \leqq C\|\psi_E\|_2 \leqq C\|\Phi_0\|_2,$$
$$\|\phi\|_X \leqq \|\Phi_0\|_X + \|\psi_E\|_X \leqq C\|\Phi_0\|_X.$$

Therefore, the smallness required of certain constants in the á priori estimates of Sect. 5.3, is ensured by a smallness condition on the initial data Φ_0.

6. Scattering Theory

The S matrix is constructed from the wave operators Ω_+ and Ω_- by the formula

$$\mathbf{S} = \Omega_+^\star \Omega_- .$$

In our case, for each value of phase γ and energy E near E_* we construct wave operators

$$\Omega_\pm^{E,\gamma}(\phi) = s - \lim_{t \to \pm\infty} V_{E,\gamma}(0,t)^\star e^{-iH(E_\pm)t} P_c(H(E_\pm))\phi,$$

where $V_{E,\gamma}(t,s)$ is the nonlinear evolution (the dispersive ϕ evolution) from time s to time t which is coupled to the bound state channel (see (6.3) below).

To conclude that the S matrix is unitary, we need to show that there is a δ, such that for all initial conditions, $(E^\pm, \gamma^\pm, \phi_\pm)$ satisfying

$$\begin{aligned} |E^\pm - E_*| &< \delta, \\ |\gamma^\pm| &\leqq \pi, \\ \|\phi_\pm\|_{\mathbf{H}^1} &< \delta, \\ P_c(H(E_\pm))\phi_\pm &= \phi_\pm . \end{aligned} \tag{6.1}$$

there exists states $\Phi_\pm \in \mathbf{B}$ with its asymptotic behavior given by

$$\Phi_\pm \approx e^{-iH(E_\pm)t}\phi_\pm + \exp\left(-i\int_0^t E(s)ds - i\gamma(t)\right)\psi(E^\pm), \tag{6.2a}$$

with

$$E(t) \to E^\pm \quad \text{and} \quad \gamma(t) \to \gamma^\pm \quad \text{as} \quad t \to \pm\infty. \tag{6.2b}$$

The existence of such Φ follows from the global existence of solutions of the following system of nonlinear integral equations:

$$\begin{aligned} E(t) &= E^- + \int_{-\infty}^t g_E(s)ds, \\ \gamma(t) &= \gamma^- + \int_{-\infty}^t g_\gamma(s)ds, \\ \phi_-(t) &= e^{-iH(E^-)t}\phi_- + \int_{-\infty}^t e^{-iH(E^-)(t-s)}\tilde{F}(\phi_-(s))ds. \end{aligned} \tag{6.3}$$

Here, g_E, g_γ, and $\tilde{F}$ are expressions like the source terms in (3.9a), (3.9b) and (3.5), respectively.

Remark. They are not exactly the source terms appearing in Sect. 3 for the following reason. Since $E(t) - E^\pm \notin \mathbf{L}^1(\mathbf{R}^1; \mathbf{dt})$ for the construction of $\phi_\pm$, it is convenient to work with the equations resulting, not from the Ansatz (3.2) but from the following Ansatz:

$$\Phi(t) = e^{-i\Theta}\psi_E + e^{-iE_0 t}\phi(t).$$

The proof of existence of global solutions runs along analogous lines to the *one-channel* case (cf. [R–S]). We view the system (6.3) as a mapping of a space $\mathbf{M}$

of vector valued functions to itself and seek a fixed point. We consider only the case $t \to -\infty$. The case $t \to \infty$ is similar.

Let

$$\tilde{\mathbf{M}}_\eta = \{v \equiv (E(\cdot), \gamma(\cdot), \phi_-) : v \in \mathbf{C}^1(\mathbf{R}) \times \mathbf{C}^1(\mathbf{R}) \times \mathbf{C}^0(\mathbf{R}; \mathbf{H}^1), |||v||| \leqq \eta\},$$

where

$$|||v||| = \sup_{\mathbf{R}^1} (\langle t \rangle^{1+\delta} [|\dot{E}(t)| + |\dot{\gamma}(t)| + \| \langle x \rangle^{-\sigma} \phi_-(t) \|_2] + \| \phi_- \|_{\mathbf{H}^1}). \tag{6.4}$$

We then define

$$\mathbf{M}_\eta = \left\{ f \in \tilde{\mathbf{M}}_\eta : \phi_-(\cdot, |x|) = P_c(H(E^-))\phi_-, \lim_{t \to -\infty} E(t) = E^-, \lim_{t \to -\infty} \gamma(t) = \gamma^- \right\}.$$

For each $(E(\cdot), \gamma(\cdot), \phi_-) \in \mathbf{M}_\eta$ we set

$$\Phi(t) = \exp\left(i\gamma(t) - i \int_0^t E(u)du \right) \psi(E(t)) + \phi_-(t)$$

and a mapping K on $\mathbf{M}_\eta$:

$$K(E(\cdot), \gamma(\cdot), \phi_-) = (\bar{E}, \bar{\gamma}, \bar{\phi}).$$

One can check that the estimates used to prove Theorem 4.1 apply in this context to establish that K maps $\mathbf{M}_\eta$ to itself and for η sufficiently small, it has a fixed point, the solution of (6.3). We then take this solution at $t = 0$ as Φ_-. Φ_+ is constructed similarly.

Appendix A: Some Estimates of Nonlinear Terms

In this section we prove Propositions 5.3 and 5.4. To prove these estimates we recall that $\mathbf{F}_2$ is given by (3.6)–(3.8). We have that

$$|\mathbf{F}_2| \leqq c|\lambda|(\psi^{m-1}|\phi| + A(\psi)|\phi|^2 + |\phi|^m). \tag{A.1}$$

Proof of Proposition 5.3. We start by noting that

$$\| \mathbf{F}_2 \|_q \leqq c|\lambda| [\| \psi^{m-1}\phi \|_q + \| A(\psi)\phi^2 \|_q + \| \phi \|_{mq}^m]. \tag{A.2}$$

We now estimate the three terms on the right-hand side of (A.2) individually. First,

$$\begin{aligned} \| \psi^{m-1}\phi \|_q &= \| \langle x \rangle^\sigma \psi^{m-1} \langle x \rangle^{-\sigma} \phi \|_q \\ &\leqq \| \langle x \rangle^\sigma \psi^{m-1} \|_{r_1} \| \langle x \rangle^{-\sigma} \phi \|_2, \end{aligned} \tag{A.3}$$

where $q^{-1} = 2^{-1} + r_1^{-1}$.

For the next term in (A.2) we have

$$\| A(\psi)\phi^2 \|_q = \| A^{1/2}(\psi)\phi \|_{2q}^2 \leqq \| A^{1/2}(\psi) \|_{r_2}^2 \| \phi \|_p^2, \tag{A.4}$$

if $(2q)^{-1} = p^{-1} + r_2^{-1}$.

Finally, for the last term in (A.2) we have

$$\| \phi \|_{mq}^m \leqq \| \phi \|_2 \| \phi \|_{2m}^{m-1}. \tag{A.5}$$

Recall $q^{-1} = 1 - (2m)^{-1} = 1 - p^{-1}$. We have also used the simple interpolation result:

Lemma A.1. *If* $0 < \theta < 1$, *and* $r = \theta a + (1 - \theta)b$, *then*

$$\| f \|_r \leqq \| f \|_a^{\theta a/r} \| f \|_b^{(1-\theta)b/r}.$$

Proposition 5.3 follows from (A.3)–(A.5).

Proof of Proposition 5.4. The quantity to be estimated is

$$\| \langle x \rangle^{1+a} \mathbf{F}_2 \|_2 \leqq C|\lambda| [\| \langle x \rangle^{1+a} \psi^{m-1} \phi \|_2 + \| \langle x \rangle^{1+a} A(\psi) \phi^2 \|_2 + \| \langle x \rangle^{1+a} \phi^m \|_2]. \tag{A.6}$$

As in the previous proof, we estimate these three terms individually. First,

$$\| \langle x \rangle^{1+a} \psi^{m-1} \phi \|_2 = \| \langle x \rangle^{1+a+\sigma} \psi^{m-1} \langle x \rangle^{-\sigma} \phi \|_2 \leqq \| \langle x \rangle^{1+a+\sigma} \psi^{m-1} \|_\infty \| \langle x \rangle^{-\sigma} \phi \|_2. \tag{A.7}$$

For the second term in (A.6) we have

$$\| \langle x \rangle^{1+a} A(\psi) \phi^2 \|_2 \leqq \| \langle x \rangle^{1+a} A(\psi) \|_{r_3} \| \phi \|_{2m}^2, \tag{A.8}$$

where $2^{-1} = r_3^{-1} + m^{-1}$.

The estimate of the last term in (A.6), $\| \langle x \rangle^{1+a} \phi^m \|_2$, is more involved due to the absence of a spatially localizing factor. It is here that the assumption that the potential $V = V(|x|)$, and the initial conditions be spherically symmetric (thus, giving rise to spherically symmetric solutions) is used. Namely, we have the following [Str2].

Lemma A.2. *Let* $f \in \mathbf{H}^1(R^n)$ *and* $f = f(|x|)$. *Then,*

$$|f(|x|)| \leqq C_n |x|^{(1-n)/2} \| f \|_{\mathbf{H}^1}. \tag{A.9}$$

Now for the last term in (A.6) we have

$$\| \langle x \rangle^{1+a} \phi^m \|_2 \leqq C(\| \phi \|_{2m}^m + \| |x|^{1+a} \phi^m \|_2), \tag{A.10}$$

so it remains to estimate $\| |x|^{1+a} \phi^m \|_2$. Writing

$$|x|^{2(1+a)} |\phi|^{2m} = (|x|^{(n-1)/2} |\phi|)^{4(1+a)/(n-1)} |\phi|^{2m - 4(1+a)/(n-1)},$$

and using Lemma A.2, we have

$$|x|^{2(1+a)} |\phi|^{2m} \leqq \| \phi \|_{\mathbf{H}^1}^{4(1+a)/(n-1)} |\phi|^{2m - 4(1+a)/(n-1)}. \tag{A.11}$$

It follows that

$$\| |x|^{1+a} \phi^m \|_2 \leqq \| \phi \|_{\mathbf{H}^1}^{2(1+a)/(n-1)} \| \phi \|_{2(m - 2(1+a)/(n-1))}^{m - 2(1+a)/(n-1)}. \tag{A.12}$$

Finally, we interpolate the last factor on the right-hand side of (A.12) between $\mathbf{L}^2$ and $\mathbf{L}^{2m}$:

$$\| \phi \|_{2r} \leqq \| \phi \|_2^\alpha \| \phi \|_{2m}^\beta,$$

$$r = m - \frac{2(1+a)}{n-1}, \quad \alpha = \mu/r \quad \beta = m(1-\mu)/r, \quad \text{and} \quad \mu = \frac{2(1+a)}{(m-1)(n-1)}.$$

Appendix B: Weighted Estimates of Nonlinear Bound States

In this section we prove the weighted estimates stated in Sect. 2.2. We shall derive equations for weighted nonlinear bound states and their derivatives with respect to the energy parameter, E:

$$w_j = \langle x\rangle^j \psi_E, \quad v_j = \langle x\rangle^j \partial_E \psi_E. \tag{B.1}$$

To obtain such equations, we must commute powers of $\langle x\rangle$ through the Laplacian. For this we use the following simple observation:

$$[\langle x\rangle, \Delta]f = -2\frac{x}{\langle x\rangle}\cdot\nabla f + \frac{n+(n-1)|x|^2}{\langle x\rangle^3}f, \tag{B.2}$$

where $[A,B] \equiv AB - BA$ denotes the commutator of the operators A and B. We shall restrict ourselves to spatial dimensions $n=2,3$, the weights $j=0,1,2$, and the spaces $\mathbf{H}^s$ with $s=0,1,2$. This is what is required in the present paper. Our proofs carry over in a straightforward manner to the general case of $n>3$, $j>2$, and $\mathbf{H}^s$ with $s>2$, though with a bit of calculation and induction.

We begin with the equation of a nonlinear bound state u, and $\mathbf{H}^2$ solution of

$$-\Delta u + Vu - Eu + \lambda|u|^{m-1}u = 0, \tag{B.3}$$

which bifurcates from an eigenvalue, E_*, in $\mathbf{H}^2$, i.e. $\|u_E\|_{\mathbf{H}^2} \to 0$ as $E \to E_*$. Multiplication of (B.3) by $\langle x\rangle$ and application of (B.2) yields

$$\begin{aligned} &-\Delta w_1 + Vw_1 - Ew_1 + \lambda|u|^{m-1}w_1 \\ &= [\langle x\rangle, \Delta]u \\ &= -2\frac{x}{\langle x\rangle}\cdot\nabla u + \frac{n+(n-1)|x|^2}{\langle x\rangle^3}u. \end{aligned}$$

Similarly, we can obtain an inhomogeneous equation for any w_j. For w_2 we obtain

$$\begin{aligned} &-\Delta w_2 + Vw_2 - Ew_2 + \lambda|u|^{m-1}w_2 \\ &= [\langle x\rangle, \Delta]w_1 - 2x\cdot\nabla u + \frac{n+(n-1)|x|^2}{\langle x\rangle^2}u \\ &= -2\frac{x}{\langle x\rangle}\cdot\nabla w_1 + \frac{n+(n-1)|x|^2}{\langle x\rangle^3}w_1 \\ &\quad - 2x\cdot\nabla u + \frac{n+(n-1)|x|^2}{\langle x\rangle^2}u. \end{aligned} \tag{B.4}$$

(a) $\mathbf{H}^1$ estimate of w_1:

The next step is to derive energy estimates which, for E near E_*, will give control of the $\mathbf{H}^1$ norm of w_j. Multiplication of (B.3) by w_1 and integration over all space gives:

$$\begin{aligned} &\int(|\nabla w_1|^2 + V|w_1|^2 + \lambda|u|^{m-1}|w_1|^2 - E|w_1|^2)dx \\ &= n\int|u|^2dx + \int\frac{n+(n-1)|x|^2}{\langle x\rangle^2}|u|^2dx. \end{aligned} \tag{B.5}$$

From (B.5) we obtain

$$\int(|\nabla w_1|^2 - E|w_1|^2)dx$$
$$= n\int|u|^2dx + \int\left(\frac{n+(n-1)|x|^2}{\langle x\rangle^2}|u|^2 - \langle x\rangle^2 V|u|^2 - \lambda|u|^{m-1}|w_1|^2\right)dx$$
$$\leqq C(\|u\|_2^2 + |\lambda|\,\|u\|_{2(m-1)}^{m-1})\|w_1\|_{\mathbf{H}^1}^2 + \|\langle x\rangle^2 V\|_\infty\|u\|_2^2. \tag{B.6}$$

From (B.6) it follows that by choosing E sufficiently near E_*, that

$$\|w_1\|_{\mathbf{H}^1} \leqq C\|u_E\|_2. \tag{B.7}$$

(b) $\mathbf{H}^2$ estimate of w_1:

To estimate w_1 in $\mathbf{H}^2$ we differentiate (B.3) with respect to x_k, $k=1,2,\dots,n$, to obtain an equation for $\partial_k w_1$:

$$-\Delta\partial_k w_1 + V\partial_k w_1 - E\partial_k w_1 + \lambda|u|^{m-1}\partial_k w_1$$
$$= -\partial_k V w_1 - \lambda\partial_k u^{m-1}w_1 - \partial_k\left[2\frac{x}{\langle x\rangle}\cdot\nabla u - \frac{n+(n-1)|x|^2}{\langle x\rangle^3}u\right]. \tag{B.8}$$

The $\mathbf{H}^2$ estimate for w_1 is now derived from an energy estimate of the kind used above, now for $\partial_k w_1$, the solution of (B.8). Thus multiplication of (B.8) by $\partial_k w_1$ and integration over all space yields:

$$\int(|\nabla\partial_k w_1|^2 + V|\partial_k w_1|^2 + \lambda|u|^{m-1}|\partial_k w_1|^2 - E|\partial_k w_1|^2)dx$$
$$= -\int\Bigg(\partial_k w_1\partial_k V w_1 - \lambda\partial_k w_1 w_1\partial_k u^{m-1}$$
$$-\partial_k w_1\partial_k\left[2\frac{x}{\langle x\rangle}\cdot\nabla u - \frac{n+(n-1)|x|^2}{\langle x\rangle^3}u\right]\Bigg)dx. \tag{B.9}$$

Estimates of the type used to establish (B.7) can now be applied to conclude, for E sufficiently near E_*, that

$$\|w_1\|_{\mathbf{H}^2} \leqq C(V)\|u_E\|_{\mathbf{H}^2}. \tag{B.10}$$

A similar analysis can be applied to w_2 and $\partial_k w_2$ to conclude, using the $\mathbf{H}^2$ estimates on w_1, that for $|E-E_*|$ sufficiently small,

$$\|w_2\|_{\mathbf{H}^2} \leqq C\|u_E\|_{\mathbf{H}^2}. \tag{B.11}$$

We shall next outline the derivation of estimates for $\langle x\rangle^j\partial_k\partial_E u_E$ ($j=0,1,2$ and $k=0,1,2$) in $\mathbf{H}^2$. First, we recall from Theorem 2.1, that there is a bifurcation curve $(u(\varepsilon), E(\varepsilon))$, where

$$u(\varepsilon) = \varepsilon^{1/(m-1)}\tilde{u}(\varepsilon) \tag{B.12}$$

where

$$E(\varepsilon) = E_* + a_1\varepsilon + \mathcal{O}(\varepsilon^2), \quad a_1 = (\lambda\textstyle\int\psi_*^{m+1})^{1/(m-1)}, \tag{B.13}$$

and $\tilde{u}$ satisfies the following equation:

$$(-\Delta + V + \lambda\varepsilon\tilde{u}^{m-1} - E)\tilde{u} = 0. \tag{B.14}$$

Now, since $\partial_\varepsilon \approx a_1 \partial_E$, differentiation of (B.12) with respect to ε yields

$$\partial_E u_E \approx a_1^{-1} \varepsilon^{1/(m-1)} \partial_\varepsilon \tilde{u} + (m-1)^{-1} a_1^{-1} \varepsilon^{1/(m-1)-1} \tilde{u}.$$

From the estimates for $\langle x \rangle^j u_E$ it follows that $\| \langle x \rangle^j \tilde{u} \|_{\mathbf{H}^2}$ is uniformly bounded for $|E - E_*|$ sufficiently small. We therefore focus on $\| \langle x \rangle^j \partial_\varepsilon \tilde{u} \|_{\mathbf{H}^2}$. Differentiation of (B.14) with respect to ε yields

$$(-\Delta + V + \lambda \varepsilon m \tilde{u}^{m-1} - E) \partial_\varepsilon \tilde{u} = -\lambda \tilde{u}^m + \partial_\varepsilon E \tilde{u}. \tag{B.15}$$

As with w_j, we can now study the equation for $v_j \equiv \langle x \rangle^j \partial_E \tilde{u}$ by commuting powers of $\langle x \rangle$ through the Laplacian in (B.15) and using the commutator relation (B.2). We then derive energy estimates implying uniform control of the $\mathbf{H}^1$ norm of v_j and $\partial_k v_j$, using that $|E - E_*|$ is sufficiently small and the Sobelev inequality:

$$\| f \|_p \leqq \| f \|_{\mathbf{H}^1}, \quad 2 \leqq p < \frac{2n}{n-2}.$$

In this way the proof of the weighted estimates of nonlinear bound states in Sect. 2.2 is completed.

Finally, in the repulsive case ($\lambda > 0$) we observe that certain arguments can be made more globally in E so we give the details.

Proposition B.1. *Let $\lambda > 0$. Then for all $k, l > 0$,*

$$\lim_{E \to E_*} \sup_x \langle x \rangle^k \psi_E^l(x) = 0. \tag{B.16}$$

Proof. Suppose not. That is, there are sequences $E_j \downarrow E_*$ and $x_j \to \infty$, such that

$$\langle x_j \rangle^k \psi_{E_j}(x_j)^l \geqq \kappa > 0$$

for all $j \geqq 1$. (If x_j forms a bounded sequence we have an immediate contradiction.) Since $\partial_E \psi > 0$, we have that

$$\langle x_j \rangle^k \psi_{E_1}(x_j)^l \geqq \kappa > 0.$$

This contradicts the exponential decay of $\psi_E(x)$.

Proposition B.2. *Let $n \leqq 3$ and $\lambda > 0$. Then, for any $k \geqq 0$ and $p \geqq 1$*

$$\lim_{E \downarrow E_*} \| \langle x \rangle^k \psi_E(\cdot) \|_p = 0. \tag{B.17}$$

Proof.

$$\begin{aligned} \| \langle x \rangle^k \psi_E(\cdot) \|_p^p &= \int \langle x \rangle^{kp} |\psi_E(\cdot)|^p dx \\ &\leqq \sup_x (\langle x \rangle^{kp+\eta} |\psi_E(\cdot)|^\varepsilon) \int \langle x \rangle^{-\eta} |\psi_E(\cdot)|^{p-\varepsilon} dx \\ &\leqq C_\eta \sup_x (\langle x \rangle^{kp+\eta} |\psi_E(\cdot)|^\varepsilon) \| \psi_E(\cdot) \|_{\mathbf{H}^2}^{p-\varepsilon} \to 0 \end{aligned}$$

as $E \to E_*$. Here, we take $\eta > n$.

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Note added in proof. The authors have proved Scattering Theorem 4.1 and Asymptotic Stability Theorem 4.2 for a large class of potentials $V(x)$ and data $\Phi_0(x)$, which are not necessarily isotropic. The results hold for spatial dimensions $n \geqq 3$ and in the case of power nonlinearity, $|\Phi|^{m-1}\Phi$, for $m_*(n) < m < \frac{n+2}{n-2}$. A paper with the details is in preparation. A key ingredient is an $L^p - L^q$ estimate for $\exp(-iHt)$ obtained in the recent paper of Journé, J-L., Soffer, A. and Sogge, C.: "Decay estimates for Schrödinger operators", to appear in Commun. Pure Appl. Math.