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Instanton Expansion of Noncommutative Gauge Theory in Two Dimensions

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Abstract: We show that noncommutative gauge theory in two dimensions is an exactly solvable model. A cohomological formulation of gauge theory defined on the noncommutative torus is used to show that its quantum partition function can be written as a sum over contributions from classical solutions. We derive an explicit formula for the partition function of Yang-Mills theory defined on a projective module for an arbitrary noncommutativity parameter θ which is manifestly invariant under gauge Morita equivalence. The energy observables are shown to be smooth functions of θ. The construction of noncommutative instanton contributions to the path integral is described in some detail. In general, there are infinitely many gauge inequivalent contributions of fixed topological charge, along with a finite number of quantum fluctuations about each instanton. The associated moduli spaces are combinations of symmetric products of an ordinary two-torus whose orbifold singularities are not resolved by noncommutativity. In particular, the weak coupling limit of the gauge theory is independent of θ and computes the symplectic volume of the moduli space of constant curvature connections on the noncommutative torus.

Contents

1. Introduction and Summary

Quantum field theories on noncommutative spacetimes provide field theoretical contexts in which to study the dynamics of D-branes, while at the same time retaining the nonlocality inherent in string theory (see [1–3] for reviews). Recent studies of these field theories have raised many questions regarding their existence and properties, and even after extensive study there remain numerous questions concerning the new phenomena they exhibit even in the simplest cases. Of particular interest is Yang-Mills theory defined on a noncommutative torus which serves as an effective description of open strings propagating in flat backgrounds. In particular, noncommutative gauge theory on a two-dimensional torus describes codimension two vortex bound states of D-branes inside D-branes. In this paper we will show that this quantum field theory is exactly solvable and explicitly evaluate its partition function. Various non-trivial aspects of noncommutative gauge theories in two dimensions may be found in [4–10]

The commutative version of this theory has a well-known history as an exactly solvable model, which gives the first example of a confining gauge theory whose infrared limit can be reformulated analytically as a string theory (see [11, 12] for reviews). The key feature of two dimensions is that there are no gluons and the theory must be investigated on spacetimes of non-trivial topology or with Wilson loops in order to see any degrees of freedom. This suppression of degrees of freedom owes to the fact that the group of local symmetries of two-dimensional Yang-Mills theory contains not only local gauge invariance, but also invariance under area-preserving diffeomorphisms. Of the several different methods for solving this quantum field theory, a particularly fruitful approach is provided by the lattice formulation [13]. Using the area-preserving diffeomorphism invariance, the heat kernel expansion of the disk amplitude may be interpreted as a wavefunction for a plaquette. The fusion rules for group characters allow one to glue together disconnected plaquettes. The basic plaquette Boltzmann weight in this way turns out to be renormalization group invariant [14], so that the lattice gauge theory reproduces exactly the continuum answer.

While a lattice formulation of noncommutative Yang-Mills theory does exist [15], it does not exhibit an obvious self-similarity property as its commutative counterpart

does. The non-locality of the star-product mixes the link variables in the lattice action and the theory no longer has the nice Gaussian form that its commutative limit does. While under certain circumstances Morita equivalence can be used to disentangle the lattice star-product by mapping the noncommutative lattice gauge theory onto a commutative one, the continuum limit always requires a complicated double scaling limit to be performed with small lattice spacing and large commutative gauge group rank N, in order that the scale of noncommutativity θ remain finite in the continuum limit. A similar approach to solving gauge theory on the noncommutative plane has been advocated recently in [10]. Nevertheless, the lattice theory at finite N can be solved explicitly by mapping it onto a unitary two-matrix model [16], whose path integral can be reduced to a well-defined sum over integers [17]. This proves that the lattice model is exactly solvable, and thereby gives a strong indication that noncommutative gauge theory in two dimensions is a topological field theory (with no propagating degrees of freedom).

However, such an approach, like canonical quantization in the commutative case, is based almost entirely on the representation theory of the gauge group. This group is a somewhat mysterious object in noncommutative gauge theory whose full properties have not yet been unveiled. This infinite-dimensional Lie group is analyzed in [18–27] and it involves a non-trivial mixing of colour degrees of freedom with spacetime diffeomorphisms. A related difficulty arises in the diagonalization approach which requires fixing a gauge symmetry locally [28]. The resulting Faddeev-Popov functional determinants are difficult to analyze in the noncommutative setting. Hamiltonian methods are likewise undesirable because of problems associated with non-localities in time. An approach which doesn't rely on the (unknown) features of the noncommutative gauge group is thereby desired. We will see, however, that the basic geometric structure underlying this gauge group implies that the noncommutative theory is still invariant under area-preserving diffeomorphisms of the spacetime (though in a much stronger manner) and is thereby an exactly solvable model.

As we shall demonstrate, one technique of solving commutative $U(N)$ Yang-Mills theory which continues to be useful in the noncommutative case is that of non-Abelian localization [29]. This method takes advantage of the fact that in two dimensions a gauge fixed Yang-Mills theory is essentially a cohomological quantum field theory. A judicious deformation of the action by cohomologically exact terms allows one to reduce the quantum path integral defining the partition function to a sum over a discrete set of points which are in one-to-one correspondence with the critical points of the Yang-Mills action. Of course, these critical points are given by gauge field configurations which solve the classical equations of motion. Even though these solutions may be unstable, we will refer to any such configuration as an instanton. As a consequence, the quantum partition function can be evaluated as a sum over all instanton configurations of the gauge theory. In other words, the semi-classical approximation to this field theory is *exact*, provided that one sums over all critical points of the action. The feature which makes this approach work is the interpretation of noncommutative Yang-Mills theory as ordinary Yang-Mills theory (on a noncommutative space) with its infinite dimensional gauge symmetry group that is formally some sort of large N limit of $U(N)$.

In what follows we will derive an exact, nonperturbative expression for the partition function of quantum Yang-Mills theory defined on a projective module over the noncommutative two-torus. Using a combination of localization techniques and Morita duality, we are able to give an explicit formula written as the sum of contributions from the vicinity of instantons. The instantons themselves are parameterized by a collection of lists of pairs of integers $(\vec{p}\,,\vec{q}\,) \equiv \Big\{(p_k, q_k)\Big\}_{k\geq 1}$ which arise from partitions of the topological

numbers (p, q) of the projective module on which the gauge theory is defined. The result for the partition function $Z_{p,q}$ is then given as a sum, over all partitions, of terms involving the Boltzmann weights of the noncommutative Yang-Mills action $S(\vec{p}\,,\vec{q}\,;\theta)$ evaluated at its extrema, along with prefactors $W(\vec{p}\,,\vec{q}\,;\theta)$ which describe the quantum fluctuations about each instanton configuration. Schematically, we have

$$Z_{p,q} = \sum_{\text{partitions}} W(\vec{p}\,,\vec{q}\,;\theta)\ \mathrm{e}^{-S(\vec{p}\,,\vec{q}\,;\theta)}\ . \tag{1.1}$$

We will show that the full expression (1.1) is explicitly invariant under gauge Morita equivalence and that it is a smooth function of the noncommutativity parameter θ.

The formalism which we develop in this paper gives the tools necessary to explore and answer all questions about two-dimensional noncommutative Yang-Mills theory, and it gives a model which should capture some features of the more physical higher-dimensional theories, but within a much simplified setting. For example, the techniques developed here can be used to learn more about the observables of Yang-Mills theory on the noncommutative torus. The evaluation of the partition function as a sum of contributions from instantons is of course familiar from commutative Yang-Mills theory [30–33]. In that case there exists an equivalent expression via Poisson resummation which is interpreted as a sum over irreducible representations of the gauge group. For Yang-Mills theory on a noncommutative torus we have not been able to find an analogous group theoretical expansion though we believe it would give great insight into the representation theory of the noncommutative gauge group on the two-dimensional torus. The Yang-Mills action can be thought of as defining invariants of the star-gauge group, and the discrete sums over instantons as labelling its representations. The discrete nature of the action is necessary for it to be a Morse function and hence a candidate for the localization formalism [34], and it suggests that the noncommutative gauge group is compact. We expect to report on progress in understanding the details of the noncommutative gauge group on the torus in the near future.

1.1. Outline and summary of results. In the next section we shall begin with a review of the construction of gauge connections and Yang-Mills theory on the two-dimensional noncommutative torus. We include a brief discussion on the area-preserving nature of the noncommutative gauge symmetry which suggests that there are no local degrees of freedom in the noncommutative gauge theory, only global ones as in the commutative case. In Sect. 3 we give an overview of non-Abelian localization and how it applies to the evaluation of the quantum partition function of two-dimensional Yang-Mills theory on the noncommutative torus. We pay particular attention to rewriting the formalism in a manner which does not rely on the details of the noncommutative gauge group. We show in detail how the Yang-Mills action defines a system of Hamiltonian flows which coincide with the Lie algebra action of the group of noncommutative gauge transformations. This compatibility allows us to formally reduce the path integral defining the quantum partition function to a discrete sum. The procedure is applicable to Yang-Mills theory defined on a noncommutative torus with any value of the noncommutativity parameter θ, including vanishing, rational or irrational θ.

The localization of the path integral is onto gauge field configurations which are solutions of the classical equations of motion and provide critical points of the Yang-Mills action. In order to characterize these solutions and the spaces in which they are defined, in Sect. 4 we begin by giving a brief description of finitely-generated projective

(Heisenberg) modules over the noncommutative torus. We characterize all such classical solutions of Yang-Mills theory defined on a projective module for any value of the noncommutativity parameter in terms of partitions of the topological numbers of the projective module. These results serve to bridge previous constructions of classical solutions for two-dimensional Yang-Mills theory in the commutative case [33, 35] and in the noncommutative case for irrational θ [36].

In order to obtain explicit results for the partition function, in Sect. 5 we revisit Yang-Mills theory on the commutative torus and re-interpret the well-known evaluation of the quantum partition function in this case in terms of projective modules. In doing so we will find it necessary to make a distinction between the commonly known "physical" definition of two-dimensional Yang-Mills theory and a "module" definition where we restrict gauge field configurations to have a particular Chern (twist) number. The physical theory can then be recovered by summing over all such cohomological sectors. Given the partition function of ordinary Yang-Mills theory on the torus written in terms of projective modules, in Sect. 6 we use Morita equivalence to construct a mapping from the commutative theory to one with rational values of the noncommutativity parameter θ. Discarding the scaffolding of Morita equivalence, the result is an explicit expression for the quantum partition function of noncommutative Yang-Mills theory defined on a projective module with rational θ purely by the topological numbers of the module. Our construction also provides a more transparent interpretation of the Morita equivalence of Yang-Mills theories on commutative tori and ones with rational values of θ.

By exploiting the fact that the localization arguments hold irrespective of the particular value of θ, in Sect. 7 we propose a formula for the partition function at irrational values of θ by natural extension of the rational case. We give strong arguments in favour of this conjecture. The two independent constructions of this formula come from Morita equivalence, whereby the Morita invariant commutative partition function determines exactly the rational noncommutative one, and localization theory, which proves that the partition function is given by a sum over classical solutions for any θ. Further support for this proposal is provided by rational approximations to the irrational noncommutative gauge theory. We will find that the schematic expression (1.1) may be written explicitly as

$$Z_{p,q} = \sum_{\text{partitions}} \prod_{a\geq 1} \frac{(-1)^{\nu_a}}{\nu_a!} \left(\frac{g^2 A}{2\pi^2} \left(p_a - q_a\theta \right)^3 \right)^{-\nu_a/2} \times \exp\left[-\frac{2\pi^2}{g^2 A} \sum_{k\geq 1} (p_k - q_k\theta) \left(\frac{q_k}{p_k - q_k\theta} - \frac{q}{p - q\theta} \right)^2 \right] , \qquad (1.2)$$

where g is the Yang-Mills coupling constant and A is the area of the torus. The integer ν_a is the number of partition components (p_k, q_k) which have the same distinct values of the quantity $p_a - q_a\theta$. The sign factor in (1.2) is determined by a Morse index which measures the overall contribution from unstable modes in a given instanton configuration $(\vec{p}\,, \vec{q}\,)$. The exponential prefactors are the Gaussian fluctuation determinants, weighted with the appropriate permutation symmetry factors $\nu_a!$ associated with a partition. From (1.2) we see that the area dependence of the noncommutative gauge theory is similar to that of the commutative case. If $A \to \infty$ for fixed g and θ, then the theory is exponentially dominated by trivial instanton configurations. Essentially the energy of electric flux in the noncommutative theory is still proportional to the length of the flux line, and so the overall details of the dynamics (or lack thereof) are the same as in commutative

Yang-Mills theory. Thus, in direct analogy to the commutative situation, the gauge theory on the noncommutative plane is essentially trivial.

In Sect. 8 we develop a graphical method of analyzing the instanton contributions to Yang-Mills theory which works for θ irrational, rational or vanishing. This graphical approach is applied to the universal expression (1.2) for the partition function to show that the vacuum energy, along with a certain class of topological observables, of Yang-Mills theory on the noncommutative torus are smooth functions of θ. Finally, in Sect. 9 we end with a description of the moduli spaces of classical solutions of Yang-Mills theory on the noncommutative torus. The partition function in the weak coupling limit agrees with that of the commutative gauge theory, except that now it formally computes the symplectic volume of the moduli space of *all* (not necessarily flat) constant curvature gauge connections on the torus. The rearrangement of the series (1.2) into distinct gauge inequivalent instanton configurations is described. They are determined by rearranging the critical partition components (p_k, q_k) into distinct relatively prime pairs (p'_a, q'_a) of topological numbers with $(p_a, q_a) = N_a\,(p'_a, q'_a)$. We will see that the moduli space of such gauge orbits is given by

$$\mathcal{M}_{p,q} = \prod_{a\geq 1} \mathrm{Sym}^{N_a}\,\tilde{\mathbf{T}}^2\ , \tag{1.3}$$

where $\mathrm{Sym}^{N_a}\,\tilde{\mathbf{T}}^2$ is the symmetric product of a certain dual, ordinary two-torus $\tilde{\mathbf{T}}^2$. This generalizes the moduli space $\mathrm{Sym}^N\,\tilde{\mathbf{T}}^2$ of flat gauge connections in commutative $U(N)$ gauge theory. The instanton moduli space (1.3) has a natural physical interpretation in terms of that for a collection of distinct configurations of N_a free indistinguishable D0-branes in codimension two. In particular, the point-like instanton singularities are not resolved by noncommutativity. We will show how the orbifold singularities of (1.3) can be used to systematically construct the gauge inequivalent contributions to Yang-Mills theory. Such an explicit classification is only possible within the noncommutative setting. We shall find that, like for the instanton contributions to ordinary Yang-Mills theory, there are a finite number of quantum fluctuations about each gauge inequivalent classical solution. In contrast to the commutative case, however, for irrational θ there are infinitely many distinct instanton contributions to the path integral for fixed quantum numbers (p, q).

2. Noncommutative Gauge Theory in Two Dimensions

To set notation and conventions, we will start by reviewing some well-known facts about Yang-Mills theory on a noncommutative two-torus [1, 37, 38]. Our presentation will exhibit the interplay between the physical, quantum field theoretical approach and the mathematical approach within the framework of noncommutative geometry, as both descriptions will be fruitful for our subsequent analysis in later sections. We will also give the first indication that this theory is exactly solvable. For simplicity, we consider a square torus of radii R.

2.1. The noncommutative torus. The noncommutative two-torus may be defined as the abstract, noncommutative, associative unital $*$-algebra generated by two unitary operators $\hat{Z}_1$ and $\hat{Z}_2$ with the commutation relation

$$\hat{Z}_1\hat{Z}_2 = \mathrm{e}^{\,2\pi\,\mathrm{i}\,\theta}\ \hat{Z}_2\hat{Z}_1\ , \tag{2.1}$$

where θ is the real-valued, dimensionless noncommutativity parameter. Unless otherwise specified, we will assume that $\theta \in (0, 1)$ is an irrational number. The "smooth" completion $\mathcal{A}_\theta$ of the algebra generated by $\hat{Z}_1$ and $\hat{Z}_2$ consists of the power series

$$\hat{f} = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} f_{(m_1,m_2)}\ \mathrm{e}^{\,\pi\,\mathrm{i}\,\theta\,m_1 m_2}\ \hat{Z}_1^{m_1} \hat{Z}_2^{m_2}\ , \tag{2.2}$$

where the coefficients $f_{(m_1,m_2)}$ are Schwartz functions of $(m_1, m_2) \in \mathbb{Z}^2$, i.e. $f_{(m_1,m_2)} \to 0$ faster than any power of $|m_1| + |m_2|$ as $|m_1| + |m_2| \to \infty$. The phase factor in (2.2) is inserted to symmetrically order the operator product.

There are natural, anti-Hermitian linear derivations $\hat{\partial}_1$ and $\hat{\partial}_2$ of the algebra $\mathcal{A}_\theta$ which are defined by the commutation relations

$$\left[\hat{\partial}_1\ ,\ \hat{\partial}_2\right] = \mathrm{i}\,\Phi \cdot \mathbb{I}\ , \tag{2.3}$$

$$\left[\hat{\partial}_i\ ,\ \hat{Z}_j\right] = \frac{\mathrm{i}}{R}\,\delta_{ij}\,\hat{Z}_j\ ,\quad i, j = 1, 2\ , \tag{2.4}$$

where $\Phi \in \mathbb{R}$ can be interpreted as a background magnetic flux and $\mathbb{I}$ is the unit of $\mathcal{A}_\theta$. From (2.3) it follows that the Heisenberg Lie algebra $\mathcal{L}_\Phi$ acts on $\mathcal{A}_\theta$ by infinitesimal automorphisms. This action defines a Lie algebra homomorphism $X \mapsto \hat{\partial}_X$, $X \in \mathcal{L}_\Phi$, i.e. $\left[\hat{\partial}_X\ ,\ \hat{\partial}_Y\right] = \hat{\partial}_{[X,Y]}$, yielding a linear map

$$\hat{\partial}\ :\ \mathcal{A}_\theta\ \longrightarrow\ \mathcal{A}_\theta \otimes \mathcal{L}_\Phi^*\ . \tag{2.5}$$

The unique normalized trace on $\mathcal{A}_\theta$ is given by projection onto zero modes as

$$\mathrm{Tr}\ \hat{f} = f_{(0,0)}\ , \tag{2.6}$$

which defines a positive linear functional $\mathcal{A}_\theta \to \mathbb{C}$, i.e. $\mathrm{Tr}\ \hat{f}^\dagger \hat{f} \geq 0$ for any $\hat{f} \in \mathcal{A}_\theta$. The trace (2.6) satisfies $\mathrm{Tr}\ \hat{f}^\dagger = \overline{\mathrm{Tr}\ \hat{f}}$, and it is invariant under the action of the Lie algebra $\mathcal{L}_\Phi$ of automorphisms of $\mathcal{A}_\theta$, i.e.

$$\mathrm{Tr}\left[\hat{\partial}_i\ ,\ \hat{f}\right] = 0\ . \tag{2.7}$$

The conventional field theoretic approach employs a "dual" description to this analytic one in terms of functions on an ordinary torus $\mathbf{T}^2$. Let $x^1, x^2 \in [0, 2\pi R]$ be the coordinates of $\mathbf{T}^2$. Then given any element $\hat{f} \in \mathcal{A}_\theta$ with series expansion of the form (2.2), we can use the Schwartz sequence $f_{(m_1,m_2)}$ to define a smooth function on the torus by the Fourier series

$$f(x) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} f_{(m_1,m_2)}\ \mathrm{e}^{\,\mathrm{i}\,m_i x^i/R}\ . \tag{2.8}$$

This establishes a one-to-one correspondence between elements of the abstract algebra $\mathcal{A}_\theta$ and elements of the algebra $C^\infty(\mathbf{T}^2)$ of smooth functions on the torus. Under this correspondence, the noncommutativity of $\mathcal{A}_\theta$ is encoded in the multiplication relation

$$\hat{f}\,\hat{g} = \widehat{f \star g}\ , \tag{2.9}$$

where the star-product is given by

$$(f\star g)(x)=\sum_{n=0}^{\infty}\Big(-\pi\,\mathrm{i}\,R^2\theta\Big)^n\,\sum_{r=0}^{n}\frac{(-1)^r}{(n-r)!\,r!}\,\Big(\partial_1^r\,\partial_2^{n-r}f(x)\Big)\Big(\partial_1^{n-r}\,\partial_2^r g(x)\Big) \quad (2.10)$$

with $\partial_i=\partial/\partial x^i$. In addition, the actions of the derivations (2.4) correspond to ordinary differentiation of functions,

$$\Big[\hat\partial_i\ ,\ \hat f\,\Big]=\widehat{\partial_i f}\ , \quad (2.11)$$

while the canonical normalized trace (2.6) can be represented in terms of the classical average of functions over the torus,

$$\mathrm{Tr}\ \hat f=\frac{1}{4\pi^2R^2}\int \mathrm{d}^2x\ f(x)\ . \quad (2.12)$$

Integration by parts also shows that

$$\int \mathrm{d}^2x\ (f\star g)(x)=\int \mathrm{d}^2x\ f(x)\,g(x)\ . \quad (2.13)$$

Here and in the following, unless specified otherwise, all coordinate integrations extend over $\mathbf{T}^2$.

2.2. Gauge theory on the noncommutative torus. In the noncommutative setting, the generalizations of vector bundles are provided by projective modules, which are vector spaces on which the algebra is represented. Let $\mathcal{E}$ be a finitely-generated projective module over the algebra $\mathcal{A}_\theta$. We consider only right modules in the following. The free module $\mathcal{A}_\theta^M=\mathcal{A}_\theta\oplus\cdots\oplus\mathcal{A}_\theta$ consists of M-tuples $\hat\xi=(\hat f_1,\dots,\hat f_M)$ of elements $\hat f_a\in\mathcal{A}_\theta$. It is the analog of a trivial vector bundle. Let $\mathsf{P}\in\mathbb{M}_M(\mathcal{A}_\theta)$ be a projector with

$$\mathcal{E}=\mathsf{P}\,\mathcal{A}_\theta^M\ ,\ \ \mathsf{P}^2=\mathsf{P}=\mathsf{P}^\dagger\ , \quad (2.14)$$

where $\mathbb{M}_M(\mathcal{A}_\theta)=\mathcal{A}_\theta\otimes\mathbb{M}_M$ is the algebra of $M\times M$ matrices with entries in the algebra $\mathcal{A}_\theta$, whose multiplication is the tensor product of the multiplication in $\mathcal{A}_\theta$ with ordinary matrix multiplication. Alternatively, we may consider $\mathcal{E}$ as the subspace of elements $\hat\xi\in\mathcal{A}_\theta^M$ with $\mathsf{P}\,\hat\xi=\hat\xi$.

The endomorphism algebra $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E})=\mathcal{E}^*\otimes_{\mathcal{A}_\theta}\mathcal{E}$ of the module $\mathcal{E}$ is the algebra of linear maps $\mathcal{E}\to\mathcal{E}$ that commute with the right action of $\mathcal{A}_\theta$ on $\mathcal{E}$. It is isomorphic to the subalgebra of $\mathcal{A}_\theta$-valued matrices $\hat A\in\mathbb{M}_M(\mathcal{A}_\theta)$ which obey $\mathsf{P}\,\hat A\,\mathsf{P}=\hat A$. This means that the identity operator on $\mathcal{E}$ can be identified with the projector, $\mathbb{I}_\mathcal{E}=\mathsf{P}$. To simplify some of the formulas which follow, we shall frequently refrain from writing $\mathbb{I}_\mathcal{E}$ explicitly. Let $N\le M$ be the largest integer such that the module $\mathcal{E}$ can be represented as a direct sum $\mathcal{E}=\mathcal{E}'\oplus\cdots\oplus\mathcal{E}'$ of N isomorphic $\mathcal{A}_\theta$-modules. Then $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}')\cong\mathcal{A}_{\theta'}$ is also a noncommutative torus [37], where θ' is the *dual* noncommutativity parameter which depends on θ and the projective module $\mathcal{E}$, so that

$$\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E})\cong\mathbb{M}_N(\mathcal{A}_{\theta'})\ . \quad (2.15)$$

The derivations $\hat{\partial}_i$ naturally extend to operators on $\mathcal{A}_\theta^M$ via the definition

$$\left[\hat{\partial}_i\ ,\ \hat{\xi}\ \right] = \left(\left[\hat{\partial}_i\ ,\ \hat{f}_1\right]\ ,\ \ldots\ ,\ \left[\hat{\partial}_i\ ,\ \hat{f}_M\right]\right) \tag{2.16}$$

for $\hat{\xi} = (\hat{f}_1, \ldots, \hat{f}_M) \in \mathcal{A}_\theta^M$. Then $\mathsf{P} \circ \hat{\partial}_i \circ \mathsf{P}$ is a linear derivation on $\mathcal{E}$. The trace Tr on $\mathcal{A}_\theta$ also naturally extends to a trace on $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E})$ defined by

$$\mathrm{Tr}_{\,\mathcal{E}} = \mathrm{Tr} \otimes \mathrm{tr}_{\,M}\ , \tag{2.17}$$

where $\mathrm{tr}_{\,M}$ is the usual $M \times M$ matrix trace. On $\mathcal{E}$ there is a natural $\mathcal{A}_\theta$-valued inner product which is compatible with the $\mathcal{A}_\theta$-module structure of $\mathcal{E}$ and is defined on M-tuples $\hat{\xi} = (\hat{f}_1, \ldots, \hat{f}_M)$ and $\hat{\eta} = (\hat{g}_1, \ldots, \hat{g}_M)$ by

$$\left\langle \hat{\xi}\ ,\ \hat{\eta} \right\rangle_{\mathcal{A}_\theta} = \sum_{a=1}^{M} \hat{f}_a^\dagger\ \hat{g}_a\ . \tag{2.18}$$

The object

$$\left\langle \hat{\xi}\ ,\ \hat{\eta} \right\rangle = \mathrm{Tr} \left\langle \hat{\xi}\ ,\ \hat{\eta} \right\rangle_{\mathcal{A}_\theta} \tag{2.19}$$

then defines an ordinary Hermitian scalar product $\mathcal{E} \times \mathcal{E} \to \mathbb{C}$. This turns $\mathcal{E}$ into a separable Hilbert space. We will present the explicit classification of the projective modules over the noncommutative torus in Sect. 4.1.

We now define a connection on a module $\mathcal{E}$ over the noncommutative torus to be a pair of linear operators $\hat{\nabla}_1, \hat{\nabla}_2 : \mathcal{E} \to \mathcal{E}$ satisfying

$$\left[\hat{\nabla}_i\ ,\ \hat{Z}_j\right] = \frac{\mathrm{i}}{R}\,\delta_{ij}\,\hat{Z}_j\ ,\quad i, j = 1, 2\ , \tag{2.20}$$

where in this equation the $\hat{Z}_j$ are regarded as operators $\mathcal{E} \to \mathcal{E}$ representing the right action on $\mathcal{E}$ of the corresponding generators of $\mathcal{A}_\theta$. When acting on elements of $\mathcal{E}$, the requirement (2.20) is just the usual Leibnitz rule with respect to the derivations $\hat{\partial}_1$ and $\hat{\partial}_2$. In an analogous way to these operators, there is a linear map $X \mapsto \hat{\nabla}_X$, $X \in \mathcal{L}_\Phi$, which defines a vector space homomorphism

$$\hat{\nabla}\ :\ \mathcal{E}\ \longrightarrow\ \mathcal{E} \otimes_{\mathbb{C}} \mathcal{L}_\Phi^*\ . \tag{2.21}$$

This definition makes use of the bimodule structure on $\mathcal{A}_\theta \otimes \mathcal{L}_\Phi^*$. From the definitions (2.4) and (2.20) it follows that an arbitrary connection $\hat{\nabla}_i$ can be expressed in the form

$$\hat{\nabla}_i = \hat{\partial}_i + \hat{A}_i\ , \tag{2.22}$$

where $\hat{A}_i \in \mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E})$ are $N \times N\mathcal{A}_{\theta'}$-valued matrices which we will refer to as gauge fields. We stress that here, and below, the quantity $\hat{\partial}_i$ is implicitly understood as the operator $\mathsf{P} \circ \hat{\partial}_i \circ \mathsf{P}$ on $\mathcal{A}_\theta^M \to \mathcal{E}$. The same is true of similarly defined objects.

In the following we shall work only with connections which are compatible with the inner product (2.18), i.e. those which satisfy

$$\left\langle \hat{\nabla}_i\,\hat{\xi}\ ,\ \hat{\eta} \right\rangle_{\mathcal{A}_\theta} + \left\langle \hat{\xi}\ ,\ \hat{\nabla}_i\,\hat{\eta} \right\rangle_{\mathcal{A}_\theta} = \left[\hat{\partial}_i\ ,\ \left\langle \hat{\xi}\ ,\ \hat{\eta} \right\rangle_{\mathcal{A}_\theta}\right] \tag{2.23}$$

for any $\hat{\xi}, \hat{\eta} \in \mathcal{E}$. The compatibility condition (2.23) implies that $\hat{\nabla}_i$ is an anti-Hermitian operator with respect to the scalar product (2.19). It also implies that its curvature $\left[\hat{\nabla}_1\,,\,\hat{\nabla}_2\right]$, which is a two-form on the Heisenberg algebra $\mathcal{L}_\Phi$ with values in the space of linear operators on $\mathcal{E}$, commutes with the action of $\mathcal{A}_\theta$ on $\mathcal{E}$, and hence takes values in the space $\mathrm{End}^{\mathrm{H}}_{\mathcal{A}_\theta}(\mathcal{E})$ of anti-Hermitian endomorphisms of $\mathcal{E}$.[1] The space of all compatible connections on a module $\mathcal{E}$ will be denoted by $\mathsf{C}(\mathcal{E})$. From (2.20) and (2.23) it follows that $\mathsf{C}(\mathcal{E})$ is an affine space over the vector space of linear maps $\mathcal{L}_\Phi \to \mathrm{End}^{\mathrm{H}}_{\mathcal{A}_\theta}(\mathcal{E})$.

In this paper we will be interested in evaluating the partition function of two dimensional quantum Yang-Mills theory on the noncommutative torus, which is defined formally by the infinite-dimensional integral

$$Z(g^2,\theta,\Phi,\mathcal{E}) = \frac{1}{\mathrm{vol}\,\mathsf{G}(\mathcal{E})}\int\limits_{\mathsf{C}(\mathcal{E})} \mathrm{D}\hat{A}\ \mathrm{e}^{-S\left[\hat{A}\right]}\,, \tag{2.24}$$

where the Yang-Mills action on $\mathsf{C}(\mathcal{E})$ is defined for an arbitrary connection (2.22) by

$$S\left[\hat{A}\right] = S\left[\hat{\nabla}\right] = \frac{2\pi^2R^2}{g^2}\ \mathrm{Tr}_{\,\mathcal{E}}\left[\hat{\nabla}_1\,,\,\hat{\nabla}_2\right]^2, \tag{2.25}$$

with g the Yang-Mills coupling constant of unit mass dimension. The area factor $4\pi^2R^2$ is inserted to make the action dimensionless. Here $\mathsf{G}(\mathcal{E})$ is the group of gauge transformations, which will be described in the next subsection, and $\mathrm{vol}\,\mathsf{G}(\mathcal{E})$ is its volume. The measure $\mathrm{D}\hat{A}$, and also the volume $\mathrm{vol}\,\mathsf{G}(\mathcal{E})$, will be defined more precisely in Sect. 3. By using the operator-field correspondence of the previous subsection, we can express (2.24) in a more standard quantum field theoretical form as the Euclidean Feynman path integral

$$Z(g^2,\theta,\Phi,\mathcal{E}) = \frac{1}{\mathrm{vol}\,\mathsf{G}(\mathcal{E})}\int\limits_{\mathsf{C}(\mathcal{E})} \mathrm{D}A\ \mathrm{e}^{-S[A]}\,, \tag{2.26}$$

where

$$S[A] = \frac{1}{2g^2}\int \mathrm{d}^2x\ \mathrm{tr}_{\,N}\Big(F_A(x) + \Phi\cdot\mathbb{I}_{\mathcal{E}}\Big)^2 \tag{2.27}$$

with

$$F_A = \partial_1A_2 - \partial_2A_1 + A_1\star' A_2 - A_2\star' A_1, \tag{2.28}$$

the noncommutative field strength of the anti-Hermitian $U(N)$ gauge field A_i. The multiplication in (2.28) is the tensor product of the associative star-product (2.10), defined with θ replaced by its dual θ', and ordinary matrix multiplication. This extended star-product is still associative.

[1] Usually one would define the curvature to be a measure of the deviation of the mapping $X \mapsto \hat{\nabla}_X$ from being a homomorphism of the Lie algebra (2.3) of automorphisms of $\mathcal{A}_\theta$. This means that the curvature should be defined as $\left[\hat{\nabla}_1\,,\,\hat{\nabla}_2\right] - \Phi\cdot\mathbb{I}_{\mathcal{E}}$. However, later on we will wish to work with an action which is explicitly invariant under Morita duality, which can only be accomplished with the definition of curvature given in the text. This change of convention is mathematically harmless since it corresponds to a shift of the curvature by the central element of the Heisenberg algebra. Physically, it will only add constants to the usual gauge theory action and so will not affect any local dynamics, only topological aspects.

2.3. Gauge symmetry and area preserving diffeomorphisms. Let us now describe the symmetries of the noncommutative Yang-Mills action (2.25). It is invariant under any covariant transformation of the gauge connection of the form

$$\hat{\nabla}_i \;\longmapsto\; \hat{U}\,\hat{\nabla}_i\,\hat{U}^\dagger\,, \tag{2.29}$$

where $\hat{U} \in \mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E})$ is a unitary endomorphism of the projective module $\mathcal{E}$,

$$\hat{U}^\dagger\,\hat{U} = \hat{U}\,\hat{U}^\dagger = \mathbb{I}_{\mathcal{E}}\,, \tag{2.30}$$

which determines an inner automorphism of the right action of $\mathcal{A}_\theta$ on $\mathcal{E}$. In other words, $\hat{U} \in \mathbb{U}_N(\mathcal{A}_{\theta'})$, where $\mathbb{U}_N(\mathcal{A}_{\theta'})$ is the group of unitary elements of the algebra $\mathbb{M}_N(\mathcal{A}_{\theta'})$. These gauge transformations comprise operators of the form

$$\hat{U} = \mathbb{I}_{\mathcal{E}} + \hat{K}\,, \tag{2.31}$$

where $\hat{K}$ lies in an appropriate completion of the algebra of finite rank endomorphisms of $\mathcal{E}$ [24, 25]. These latter endomorphisms are defined as follows. For any $\hat{\eta}$, $\hat{\eta}' \in \mathcal{E}$, let $\big|\hat{\eta}\big\rangle\big\langle\hat{\eta}'\big|$ be the operator defined by

$$\big|\hat{\eta}\big\rangle\big\langle\hat{\eta}'\big|\,\hat{\xi} = \hat{\eta}\,\Big\langle\hat{\eta}'\,,\,\hat{\xi}\Big\rangle_{\mathcal{A}_\theta} \tag{2.32}$$

for $\hat{\xi} \in \mathcal{E}$, with adjoint $\big|\hat{\eta}'\big\rangle\big\langle\hat{\eta}\big|$. The $\mathcal{A}_\theta$-linear span of endomorphisms of the form (2.32) forms a self-adjoint two-sided ideal in $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E})$. Since, as mentioned before, $\mathcal{E}$ is a separable Hilbert space, this ideal is isomorphic to the infinite-dimensional algebra $\mathbb{M}_\infty$ of finite rank matrices. Its operator norm closure is the algebra $\mathrm{End}^\infty_{\mathcal{A}_\theta}(\mathcal{E})$ of compact endomorphisms of the module $\mathcal{E}$.

The Schwartz restriction on the expansion (2.2) implies that elements $\hat{f} \in \mathcal{A}_\theta$ act as compact operators on $\mathcal{E}$ [26]. Therefore, in (2.29) we should restrict to those unitary endomorphisms (2.31) with $\hat{K} \in \mathrm{End}^\infty_{\mathcal{A}_\theta}(\mathcal{E})$. We denote this infinite dimensional Lie group by $U^\infty(\mathcal{E})$. It is the operator norm completion of the infinite unitary group $U(\infty)$ obtained by taking $\hat{K}$ to be a finite rank endomorphism. [2] By Palais' theorem [39], these two unitary groups have the same homotopy type, and their homotopy groups are determined by Bott periodicity as

$$\pi_k\Big(U^\infty(\mathcal{E})\Big) = \pi_k\Big(U(\infty)\Big) = \begin{cases} \mathbb{Z} & ,\quad k \text{ odd}\,, \\ 0 & ,\quad k \text{ even}\,. \end{cases} \tag{2.33}$$

In particular, the gauge symmetry group is connected. It should be pointed out here that this is only a *local* description of the full gauge group of noncommutative Yang-Mills theory. The group of connected components of $\mathsf{G}(\mathcal{E})$ acts on the gauge orbit space, obtained by quotienting $\mathsf{C}(\mathcal{E})$ by the action of the group $\mathsf{G}_0(\mathcal{E})$ of smooth maps $\mathbf{T}^2 \to U^\infty(\mathcal{E})$, as a global symmetry group [1, 25, 27].

By using (2.22), one finds that the infinitesimal form of the gauge transformation rule (2.29) is $\hat{A}_i \mapsto \hat{A}_i + \delta_{\hat{\lambda}}\hat{A}_i$, where

$$\delta_{\hat{\lambda}}\hat{A}_i = -\Big[\hat{\partial}_i\,,\,\hat{\lambda}\Big] + \Big[\hat{\lambda}\,,\,\hat{A}_i\Big] \tag{2.34}$$

[2] The gauge group can also be chosen to be smaller than $U^\infty(\mathcal{E})$ by completing $U(\infty)$ in other Schatten norms [25–27]. The various choices all have the same topology and group theory, and so we shall work for definiteness with only the compact unitaries defined above.

and $\hat{\lambda}$ is an anti-Hermitian compact operator on $\mathcal{E}$. In terms of gauge potentials on the ordinary torus $\mathbf{T}^2$ this reads

$$\delta_\lambda A_i = -\partial_i \lambda + \lambda \star' A_i - A_i \star' \lambda \,, \tag{2.35}$$

where $\lambda(x)$ is a smooth, anti-Hermitian $N \times N$ matrix-valued field on $\mathbf{T}^2$. The noncommutative gauge transformations (2.35) mix internal, $U(N)$ gauge degrees of freedom with general coordinate transformations of the torus. Their geometrical significance has been elucidated in [26] by exploiting the relationship between appropriate completions of $U(\infty)$ and canonical transformations. The Lie algebra of noncommutative gauge transformations (2.35) is equivalent to the Fairlie-Fletcher-Zachos trigonometric deformation [40] of the algebra $w_\infty(\mathbf{T}^2)$ of area-preserving diffeomorphisms of $\mathbf{T}^2$.

Therefore, the gauge symmetry group of noncommutative Yang-Mills theory in two-dimensions consists of area-preserving diffeomorphisms, which "almost" makes it a topological field theory. Its gauge symmetry "almost" coincides with general covariance, thereby killing most of its degrees of freedom. From this feature we would expect the theory to contain no local propagating degrees of freedom, and hence to be exactly solvable. This reasoning is further supported by the Seiberg-Witten map [41] and the exact solvability of ordinary, commutative Yang-Mills gauge theory in two dimensions. Note however that the topological nature here is quite different than that of the commutative case, because in the noncommutative setting it arises due to the *gauge symmetry* of the theory, i.e. an inner automorphism of the algebra of functions, while in the commutative case it corresponds to an outer automorphism which preserves the local area element $4\pi^2 R^2 \, \mathrm{d}^2x$. For this reason, the partition function will only depend on the dimensionless combination $4\pi^2 \, g^2 \, R^2$ of the Yang-Mills coupling constant and the area of the surface. This fact makes it difficult to make sense of the theory on a non-compact surface.

In contrast, this argument breaks down for noncommutative tori of dimension larger than two. In any even dimension the transformations (2.35) generate symplectic diffeomorphisms [26], i.e. coordinate transformations which leave the symplectic two-form of the torus invariant. These are the diffeomorphisms which preserve the Poisson bivector defining the star-product in (2.10). In general, these transformations generate a group that is much smaller than the group of volume-preserving diffeomorphisms. In the D-brane interpretation, this latter group would be the natural worldvolume symmetry group of a static brane. However, particular to the two-dimensional case is the fact that canonical transformations and area-preserving diffeomorphisms are the same.

3. Localization of the Partition Function

The path integral (2.24) for quantum Yang-Mills theory on the noncommutative torus has several features in common with non-Abelian gauge theory defined on an ordinary, commutative torus. Formally, it can be regarded as a certain "large N limit" of ordinary $U(N)$ Yang-Mills where we have generalized the gauge fields to measurable operators. In this section we will exploit these similarities to show how one may compute exactly the partition function for noncommutative gauge theory on the torus via the technique of non-Abelian localization [29]. The first key observation we shall make is that the integration measure $\mathrm{D}\hat{A}$ in (2.24) may be naturally identified with the gauge invariant Liouville measure induced on the infinite dimensional operator space of compatible connections $\mathsf{C}(\mathcal{E})$ by a symplectic two-form $\omega[\cdot,\cdot]$. Moreover, the volume of the gauge group $\mathrm{vol}\,\mathsf{G}(\mathcal{E})$ is determined formally from the volume form on $\mathsf{G}(\mathcal{E})$ associated with

the metric $\left(\hat{\lambda}\,,\,\hat{\lambda}\right) = \mathrm{Tr}_{\mathcal{E}}\,\hat{\lambda}^2$ on $\mathrm{End}^{\mathrm{H},\infty}_{\mathcal{A}_\theta}(\mathcal{E})$. This metric also induces an invariant quadratic form $(\cdot,\cdot)$ on the dual Lie algebra $\left(\mathrm{End}^{\mathrm{H},\infty}_{\mathcal{A}_\theta}(\mathcal{E})\right)^*$, such that the noncommutative Yang-Mills action (2.25) is proportional to the square of the moment map μ corresponding to the symplectic action of $\mathsf{G}(\mathcal{E})$ on $\mathsf{C}(\mathcal{E})$. Equivalently, the Lie algebra action of the group of gauge transformations $\mathsf{G}(\mathcal{E})$ coincides with a system of Hamiltonian flows defined by the Yang-Mills action. In particular, this implies that the action (2.25) is a gauge-equivariant Morse function on $\mathsf{C}(\mathcal{E})$ [35, 36].

Consequently, the partition function of Yang-Mills theory defined on the noncommutative torus can be expressed formally as an infinite-dimensional statistical mechanics model

$$Z(g^2,\theta,\Phi,\mathcal{E}) = \frac{1}{\mathrm{vol}\,\mathsf{G}(\mathcal{E})}\int\limits_{\mathsf{C}(\mathcal{E})}\exp\left[\omega - \frac{\beta}{2}\left(\mu\,,\,\mu\right)\right]\,, \tag{3.1}$$

where

$$\beta = \frac{4\pi^2R^2}{g^2}\,. \tag{3.2}$$

As shown in [29], path integrals of the form (3.1) are formally calculable through a generalized non-Abelian localization technique. Here "localization" refers to the fact that the path integral (3.1) is given exactly by the sum over contributions from neighbourhoods of stationary points of the Yang-Mills action (2.25). If we denote the discrete set of all such critical points by $\mathcal{P}(\theta,\mathcal{E})$, then

$$Z(g^2,\theta,\Phi,\mathcal{E}) = \sum_{\hat{A}^{\mathrm{cl}}\in\mathcal{P}(\theta,\mathcal{E})} W\left[\hat{A}^{\mathrm{cl}}\right]\,\mathrm{e}^{-\frac{\beta}{2}\left(\mu[\hat{A}^{\mathrm{cl}}]\,,\,\mu[\hat{A}^{\mathrm{cl}}]\right)}\,, \tag{3.3}$$

where the function W gives the contributions due to the quantum fluctuations about the stationary points. In the remainder of this section we will derive all of these properties in some detail.

3.1. Symplectic structure. Let $\mathcal{E}$ be a finitely-generated projective module over the noncommutative torus, and consider the space $\mathsf{C}(\mathcal{E})$ of compatible connections on $\mathcal{E}$ introduced in Sect. 2.2. The group of gauge transformations $\mathsf{G}(\mathcal{E})$ acts on $\mathsf{C}(\mathcal{E})$ and it has Lie algebra $\mathrm{End}^{\mathrm{H},\infty}_{\mathcal{A}_\theta}(\mathcal{E})$ consisting of anti-Hermitian compact operators on $\mathcal{E}$. On this Lie algebra we introduce a natural invariant, non-degenerate quadratic form by

$$\left(\hat{\lambda}\,,\,\hat{\lambda}'\right) = \mathrm{Tr}_{\mathcal{E}}\,\hat{\lambda}\,\hat{\lambda}'\,,\quad \hat{\lambda},\hat{\lambda}'\in\mathrm{End}^{\mathrm{H},\infty}_{\mathcal{A}_\theta}(\mathcal{E})\,. \tag{3.4}$$

The infinitesimal gauge transformations (2.34) define a group action on $\mathsf{C}(\mathcal{E})$ because

$$\left[\delta_{\hat{\lambda}}\,,\,\delta_{\hat{\lambda}'}\right]\hat{A}_i = \delta_{\left[\hat{\lambda}\,,\,\hat{\lambda}'\right]}\hat{A}_i\,. \tag{3.5}$$

Consider the representation (2.3) of the Heisenberg algebra $\mathcal{L}_\Phi$ in the Lie algebra of derivations of $\mathcal{A}_\theta$, and let $\Lambda(\mathcal{L}_\Phi^*) = \bigoplus_{n\geq 0}\Lambda^n(\mathcal{L}_\Phi^*)$ be the $\mathbb{Z}_+$-graded exterior algebra of $\mathcal{L}_\Phi$. To this representation there corresponds the graded differential algebra

$$\Omega(\mathcal{E}) = \bigoplus_{n\geq 0}\Omega^n(\mathcal{E})\,,\quad \Omega^n(\mathcal{E}) = \mathrm{End}^{\mathrm{H},\infty}_{\mathcal{A}_\theta}(\mathcal{E})\otimes_{\mathbb{C}}\Lambda^n\left(\mathcal{L}_\Phi^*\right) \tag{3.6}$$

of left-invariant differential forms on $\exp(\mathcal{L}_\Phi)$ with coefficients in $\mathrm{End}^{\mathrm{H},\infty}_{\mathcal{A}_\theta}(\mathcal{E})$. For instance, the curvature $\left[\hat{\nabla}_1\,,\,\hat{\nabla}_2\right] = \hat{F}_{\hat{A}} + \Phi\cdot\mathbb{I}_{\mathcal{E}} \in \Omega^2(\mathcal{E})$ with $\mathsf{G}(\mathcal{E})$ acting infinitesimally through

$$\delta_{\hat{\lambda}}\hat{F}_{\hat{A}} = \left[\hat{\lambda}\,,\,\hat{F}_{\hat{A}}\right] \tag{3.7}$$

with the usual product on the differential algebra (3.6) implicitly understood. Functional differentiation at a point $\hat{A} \in \mathsf{C}(\mathcal{E})$ is then defined through

$$\frac{\delta}{\delta\hat{A}}f\left[\hat{a}\right] \equiv \frac{\mathrm{d}}{\mathrm{d}t}f\left[\hat{A}+t\,\hat{a}\right]\Big|_{t=0}\quad,\quad \hat{a}\in\Omega^1(\mathcal{E})\ . \tag{3.8}$$

As mentioned in Sect. 2.2, $\mathsf{C}(\mathcal{E})$ is an affine space over the vector space $\mathrm{End}^{\mathrm{H},\infty}_{\mathcal{A}_\theta}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{L}_\Phi^*$ of linear maps $\mathcal{L}_\Phi \to \mathrm{End}^{\mathrm{H},\infty}_{\mathcal{A}_\theta}(\mathcal{E})$, whose tangent space can be identified with the cotangent space $\mathrm{End}^{\mathrm{H},\infty}_{\mathcal{A}_\theta}(\mathcal{E})\otimes_{\mathbb{C}}\Lambda^1(\mathcal{L}_\Phi^*) = \Omega^1(\mathcal{E})$. A natural symplectic structure may then be defined on $\mathsf{C}(\mathcal{E})$ by the two-form

$$\omega\left[\hat{a},\hat{a}'\right] = \mathrm{Tr}_{\,\mathcal{E}}\,\hat{a}\wedge\hat{a}'\ ,\quad \hat{a},\hat{a}'\in\Omega^1(\mathcal{E})\ , \tag{3.9}$$

where[3]

$$\hat{a}\wedge\hat{a}' = \hat{a}_1\,\hat{a}_2' - \hat{a}_2\,\hat{a}_1'\ . \tag{3.10}$$

Since (3.9) is independent of the point $\hat{A}\in\mathsf{C}(\mathcal{E})$ at which it is evaluated, it is closed, i.e. $\delta\omega/\delta\hat{A} = 0$, and it is also clearly non-degenerate. In fact, because of the identities (2.12), (2.13) and (2.17), the symplectic two-form (3.9) coincides with the canonical, commutative one that is usually introduced in ordinary two-dimensional $U(N)$ Yang-Mills theory [35]. Its main characteristic is that it is invariant under the infinitesimal action

$$\delta_{\hat{\lambda}}\hat{a}_i = \left[\hat{\lambda}\,,\,\hat{a}_i\right] \tag{3.11}$$

of the gauge group $\mathsf{G}(\mathcal{E})$ on $\mathsf{C}(\mathcal{E})$,

$$\omega\left[\hat{a},\delta_{\hat{\lambda}}\hat{a}'\right] + \omega\left[\delta_{\hat{\lambda}}\hat{a},\hat{a}'\right] = 0\ . \tag{3.12}$$

3.2. Hamiltonian structure. Since $\mathsf{C}(\mathcal{E})$ is contractible and $\mathsf{G}(\mathcal{E})$ acts symplectically on $\mathsf{C}(\mathcal{E})$ with respect to the symplectic structure (3.9), there exists a moment map

$$\mu\ :\ \mathsf{C}(\mathcal{E})\ \longrightarrow\ \left(\mathrm{End}^{\mathrm{H},\infty}_{\mathcal{A}_\theta}(\mathcal{E})\right)^* \tag{3.13}$$

which naturally generates a system of Hamiltonians $H_{\hat{\lambda}}:\mathsf{C}(\mathcal{E})\to\mathbb{R}$ by

$$\left(\mu\left[\hat{A}\right]\,,\,\hat{\lambda}\right) = H_{\hat{\lambda}}\left[\hat{A}\right]\ . \tag{3.14}$$

[3] Note that components associated with the central elements of the Heisenberg algebra $\mathcal{L}_\Phi$ trivially drop out of all formulas such as (3.10), and hence will not be explicitly written in what follows.

To determine the moment map explicitly in the present case, we use the Hamiltonian flow condition

$$\frac{\delta}{\delta\hat{A}}H_{\hat{\lambda}}\left[\hat{a}\right] = -\omega\left[\delta_{\hat{\lambda}}\hat{A}\,,\,\hat{a}\right]\,,\quad \hat{a}\in\Omega^1(\mathcal{E})\,, \tag{3.15}$$

which is equivalent to the $\mathsf{G}(\mathcal{E})$-invariance (3.12). Using (2.34) and (3.9), the condition (3.15) reads

$$\frac{\delta}{\delta\hat{A}}H_{\hat{\lambda}}\left[\hat{a}\right] = -\,\mathrm{Tr}_{\,\mathcal{E}}\left[\hat{\nabla}\,,\,\hat{\lambda}\right]\wedge\hat{a}\,. \tag{3.16}$$

Since the trace is invariant under the natural action of the connection on $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E})$, i.e.

$$\mathrm{Tr}_{\,\mathcal{E}}\left[\hat{\nabla}_i\,,\,\hat{\lambda}\right] = 0\,, \tag{3.17}$$

using the Leibnitz rule we can write (3.16) equivalently as

$$\frac{\delta}{\delta\hat{A}}H_{\hat{\lambda}}\left[\hat{a}\right] = \mathrm{Tr}_{\,\mathcal{E}}\left[\hat{\nabla}\,\hat{,}\,\hat{a}\right]\hat{\lambda}\,. \tag{3.18}$$

Let us now compare (3.18) with the first order perturbation of the shifted field strength in a neighbourhood of a point $\hat{A}\in\mathsf{C}(\mathcal{E})$, which is easily computed to be

$$\hat{F}_{\hat{A}+t\,\hat{a}} + \Phi\cdot\mathbb{I}_{\mathcal{E}} = \hat{F}_{\hat{A}} + \Phi\cdot\mathbb{I}_{\mathcal{E}} + t\left[\hat{\nabla}\,\hat{,}\,\hat{a}\right] + O\left(t^2\right)\,. \tag{3.19}$$

By using (3.8) we may thereby write (3.18) as

$$\frac{\delta}{\delta\hat{A}}H_{\hat{\lambda}}\left[\hat{a}\right] = \frac{\delta}{\delta\hat{A}}\,\mathrm{Tr}_{\,\mathcal{E}}\left(\hat{F}_{\hat{A}} + \Phi\cdot\mathbb{I}_{\mathcal{E}}\right)\hat{\lambda}\,, \tag{3.20}$$

which is equivalent to

$$H_{\hat{\lambda}}\left[\hat{A}\right] = \left(\hat{F}_{\hat{A}} + \Phi\cdot\mathbb{I}_{\mathcal{E}}\,,\,\hat{\lambda}\right) \tag{3.21}$$

in the quadratic form (3.4). Comparing with (3.14) we see that the moment map for the action of the noncommutative gauge group on the space $\mathsf{C}(\mathcal{E})$ is the shifted noncommutative field strength,

$$\mu\left[\hat{A}\right] = \hat{F}_{\hat{A}} + \Phi\cdot\mathbb{I}_{\mathcal{E}}\,. \tag{3.22}$$

Since $\pi_2\Big(U^\infty(\mathcal{E})\Big) = 0$, the map $\hat{\lambda}\mapsto H_{\hat{\lambda}}$ determines a homomorphism from the Lie algebra $\mathrm{End}^{\mathrm{H},\infty}_{\mathcal{A}_\theta}(\mathcal{E})$ to the infinite-dimensional Poisson algebra induced on the space of functions $\mathsf{C}(\mathcal{E})\to\mathbb{R}$ by the symplectic two-form (3.9).

3.3. Cohomological formulation of noncommutative Yang-Mills theory. The fact that noncommutative gauge theory is so naturally a Hamiltonian system leads immediately to the localization of the path integral (2.24) onto the critical points of the action (2.25). We will now sketch the argument. First of all, the integration measure appearing in (2.24) is defined to be the Liouville measure corresponding to the symplectic two-form (3.9),[4]

$$\mathrm{D}\hat{A} = \mathrm{d}\hat{A} \int\limits_{\Pi\,\Omega^1(\mathcal{E})} \mathrm{d}\hat{\psi}\ \mathrm{e}^{-\mathrm{i}\,\omega\left[\hat{\psi}\,,\,\hat{\psi}\right]}\,, \tag{3.23}$$

where $\mathrm{d}\hat{A}$ is the "ordinary" Feynman measure which may be defined by using the identification (2.15) and the operator-field correspondence as

$$\mathrm{d}\hat{A} = \prod_{a=1}^{N}\prod_{b=1}^{N}\prod_{x\in\mathbf{T}^2}\mathrm{d}A_1^{ab}(x)\,\mathrm{d}A_2^{ab}(x)\,. \tag{3.24}$$

In (3.23), Π denotes the parity reversion operator, $\hat{\psi}$ are the odd generators of functions on the infinite dimensional superspace

$$\Pi(\mathcal{E}) = \mathsf{C}(\mathcal{E}) \oplus \Pi\,\Omega^1(\mathcal{E})\,, \tag{3.25}$$

and $\mathrm{d}\hat{A}\,\mathrm{d}\hat{\psi}$ is the corresponding functional Berezin measure.

The result of the previous subsection shows that the noncommutative Yang-Mills action (2.25) is proportional to the square of the moment map, in the quadratic form (3.4), for the symplectic action of the gauge group $\mathsf{G}(\mathcal{E})$ on $\mathsf{C}(\mathcal{E})$,

$$S\left[\hat{A}\right] = \frac{2\pi^2R^2}{g^2}\left(\mu\left[\hat{A}\right]\,,\,\mu\left[\hat{A}\right]\right)\,. \tag{3.26}$$

We can linearize the action (3.26) in μ via a functional Gaussian integration over an auxiliary field $\hat{\phi}\in\Omega^0(\mathcal{E})$, and by using (3.14) we can write the partition function (2.24) as

$$\begin{aligned} Z(g^2,\theta,\Phi,\mathcal{E}) = \frac{1}{\mathrm{vol}\,\mathsf{G}(\mathcal{E})} &\int\limits_{\Omega^0(\mathcal{E})} \mathrm{d}\hat{\phi}\ \mathrm{e}^{-\frac{1}{2\beta}\left(\hat{\phi}\,,\,\hat{\phi}\right)} \\ &\times \int\limits_{\Pi(\mathcal{E})} \mathrm{d}\hat{A}\,\mathrm{d}\hat{\psi}\ \mathrm{e}^{-\mathrm{i}\left(\omega[\hat{\psi},\hat{\psi}]-H_{\hat{\phi}}[\hat{A}]\right)}\,, \end{aligned} \tag{3.27}$$

with the measure $\mathrm{d}\hat{\phi}$ defined analogously to (3.24). Note that the operator $\hat{\phi}$ appears only quadratically in (3.27) and thereby essentially corresponds to a commutative field. Because the functional integration measures in (3.27) are the same as those which occur in the corresponding commutative case, the only place that noncommutativity is present

[4] To prove this formula, one needs to carefully study the gauge-fixed path integral measure. Since the gauge-fixed quantum action in two dimensions is Gaussian in the Faddeev-Popov ghost fields, and (3.9) coincides with the symplectic structure of the commutative case, the same arguments as in the commutative case [14] apply here and (3.23) is indeed the appropriate gauge-invariant measure to use on $\mathsf{C}(\mathcal{E})$.

is in the field strength which appears in the moment map (3.22). Indeed, it is essentially this feature that leads to the exact solvability of the model, in parallel with its commutative limit.

The representation (3.27) of noncommutative gauge theory is the crux of the matter. Notice first that in the weak coupling limit $g^2 = 0$ ($\beta = \infty$), the noncommutative field $\hat{\phi}$ appears linearly in (3.27), and its integration yields the constraint $\mu\left[\hat{A}\right] = 0$ which localizes the path integral onto gauge connections $\hat{A}$ of constant curvature $\hat{F}_{\hat{A}} = -\Phi\cdot\mathbb{I}_{\mathcal{E}}$. The partition function in this limit then formally computes the symplectic volume of the moduli space of constant curvature connections modulo noncommutative gauge transformations. The key property which enables this localization is that $\hat{A} \mapsto \mu\left[\hat{A}\right]$ is a complete Nicolai map which trivializes the integration over $\mathsf{C}(\mathcal{E})$. In this respect, the $g^2 = 0$ limit of (3.27) is a topological gauge theory, and indeed it coincides with a noncommutative version of BF theory in two dimensions [42]. What is remarkable though is that the same Nicolai map appears to trivialize to the full theory (3.27) at $g^2 \neq 0$ to a Gaussian integral over $\mathsf{C}(\mathcal{E})$. This works up to the points in $\mathsf{C}(\mathcal{E})$ where this map has singularities, which coincide with the solutions of the classical equations of motion of noncommutative gauge theory. Thus in the generic case the partition function receives only contributions from the classical noncommutative gauge field configurations.

To make these arguments precise, we first observe that the integral over the superspace $\Pi(\mathcal{E})$ in (3.27) is formally the partition function of an infinite dimensional statistical mechanics system, and, in the present situation whereby there is a symplectic group action generated by the Hamiltonian $H_{\hat{\phi}}\left[\hat{A}\right]$, it is known that such integrals can be typically reduced to finite dimensional integrals, or sums, determined by the critical points of $H_{\hat{\phi}}\left[\hat{A}\right]$ [34]. The main difference here is that there is no temperature parameter in front of the Hamiltonian through which to expand, but rather the noncommutative field $\hat{\phi}$. The argument for localization can nonetheless be carried through by adapting the non-Abelian localization principle [29] to the present noncommutative setting. This is achieved through a study of the cohomology of the infinite dimensional operator

$$\mathrm{Q}_{\hat{\phi}} = \mathrm{Tr}_{\,\mathcal{E}}\left(\hat{\psi}_i\,\frac{\delta}{\delta\hat{A}_i} + \left[\hat{\nabla}_i\,,\,\hat{\phi}\right]\frac{\delta}{\delta\hat{\psi}_i}\right) \tag{3.28}$$

which is defined on the space

$$\Omega_{\mathsf{G}(\mathcal{E})} = \mathrm{Sym}\,\Omega^0(\mathcal{E}) \otimes \Big(\mathsf{C}(\mathcal{E}) \oplus \Pi\,\Omega(\mathcal{E})\Big)\,, \tag{3.29}$$

where $\mathrm{Sym}\,\Omega^0(\mathcal{E})$ is the algebra of gauge-covariant polynomial functions on $\mathrm{End}^{\mathrm{H},\infty}_{\mathcal{A}_\theta}(\mathcal{E})$. The linear derivation (3.28) acts on the basic multiplet $\left(\hat{A}_i\,,\,\hat{\psi}_i\,,\,\hat{\phi}\right)$ of the noncommutative quantum field theory (3.27) through the transformation laws

$$\begin{aligned}\left[\mathrm{Q}_{\hat{\phi}}\,,\,\hat{A}_i\right] &= \hat{\psi}_i\,,\\ \left\{\mathrm{Q}_{\hat{\phi}}\,,\,\hat{\psi}_i\right\} &= \left[\hat{\nabla}_i\,,\,\hat{\phi}\right]\,,\\ \left[\mathrm{Q}_{\hat{\phi}}\,,\,\hat{\phi}\right] &= 0\,,\end{aligned} \tag{3.30}$$

and its square coincides with the generator of an infinitesimal gauge transformation with gauge parameter $\hat{\phi}$,

$$\left(\mathrm{Q}_{\hat{\phi}}\right)^2 = \delta_{\hat{\phi}} \,. \tag{3.31}$$

The key property of the operator (3.28) is that the Boltzmann weight over $\Pi(\mathcal{E})$ in (3.27) is annihilated by it,

$$\mathrm{Q}_{\hat{\phi}} \left(\mathrm{Tr}_{\,\mathcal{E}}\, \hat{\psi} \wedge \hat{\psi} - H_{\hat{\phi}}\left[\hat{A}\right]\right) = 0 \,, \tag{3.32}$$

where we have used the Hamiltonian flow equation (3.16). Via integration by parts over the superspace $\Pi(\mathcal{E})$, this implies that the partition function (3.27) is unchanged under multiplication of the Boltzmann factor by $\mathrm{Q}_{\hat{\phi}}\alpha$ for any gauge-invariant $\alpha \in \Omega_{\mathsf{G}(\mathcal{E})}$, i.e.

$$\left(\mathrm{Q}_{\hat{\phi}}\right)^2 \alpha = 0 \,. \tag{3.33}$$

In particular, we may write (3.27) in the form

$$\begin{aligned} Z(g^2,\theta,\Phi,\mathcal{E}) = \frac{1}{\mathrm{vol}\,\mathsf{G}(\mathcal{E})} \int\limits_{\Omega^0(\mathcal{E})} \mathrm{d}\hat{\phi}\ \mathrm{e}^{-\frac{1}{2\beta}\left(\hat{\phi},\hat{\phi}\right)} \\ \times \int\limits_{\Pi(\mathcal{E})} \mathrm{d}\hat{A}\ \mathrm{d}\hat{\psi}\ \mathrm{e}^{-\mathrm{i}\left(\omega[\hat{\psi},\hat{\psi}]-H_{\hat{\phi}}[\hat{A}]-t\,\mathrm{Q}_{\hat{\phi}}\alpha[\hat{A},\hat{\psi}]\right)} \,. \end{aligned} \tag{3.34}$$

That the right-hand side of (3.34) is independent of the parameter $t \in \mathbb{R}$ for gauge invariant α follows by noting that its derivative with respect to t vanishes upon integrating by parts over $\Pi(\mathcal{E})$, and using (3.32) and (3.33) along with the Leibnitz rule for the functional derivative operator (3.28). This will be true so long as the perturbation by $\mathrm{Q}_{\hat{\phi}}\alpha$ yields an effective action which has a nondegenerate kinetic energy term, and that it does not allow any new $\mathrm{Q}_{\hat{\phi}}$ fixed points to flow in from infinity in field space. The $t = 0$ limit of (3.34) coincides with the original partition function of noncommutative gauge theory, while its $t \to \infty$ limit yields the desired reduction for appropriately chosen α.

At this stage we will choose

$$\alpha\left[\hat{A}\,,\ \hat{\psi}\right] = 4\pi^2 R^2\ \mathrm{Tr}_{\,\mathcal{E}}\ \hat{\psi}^i\ \left[\hat{\nabla}_i\,,\ \mu\left[\hat{A}\right]\right] \,. \tag{3.35}$$

Substituting (3.35) into (3.34) using (3.28), performing the Gaussian integral over $\hat{\phi} \in \Omega^0(\mathcal{E})$, and taking the large t limit, we arrive at

$$\begin{aligned} &Z(g^2,\theta,\Phi,\mathcal{E}) \\ &\quad = \frac{1}{\mathrm{vol}\,\mathsf{G}(\mathcal{E})} \int\limits_{\Pi(\mathcal{E})} \mathrm{d}\hat{A}\ \mathrm{d}\hat{\psi}\ \mathrm{e}^{-\mathrm{Tr}_{\,\mathcal{E}}\left(\mathrm{i}\,\hat{\psi}\wedge\hat{\psi}+\frac{\beta}{2}\,\mu[\hat{A}]^2\right)} \\ &\qquad \times \lim_{t\to\infty} \exp\left(-\frac{(4\pi^2R^2)^3}{2g^2}\,t^2\ \mathrm{Tr}_{\,\mathcal{E}}\left[\hat{\nabla}^i\,,\left[\hat{\nabla}_i\,,\ \mu\left[\hat{A}\right]\right]\right]^2\right) \\ &\qquad \times \exp\left\{4\pi^2\,\mathrm{i}\,R^2\,t\ \mathrm{Tr}_{\,\mathcal{E}}\left(\mu\left[\hat{A}\right]\left[\hat{\psi}_i\,,\ \hat{\psi}^i\right] - \left[\hat{\nabla}_i\,,\ \hat{\psi}^i\right]\left[\hat{\nabla}\ \hat{,}\ \hat{\psi}\right]\right)\right\} , \end{aligned} \tag{3.36}$$

where we have further applied the Leibnitz rule along with (3.17), and also dropped overall constants for ease of notation. The $\hat{\psi}$ integrations in (3.36) produce polynomial functions of the parameter t, and the $\hat{A}$ integration is therefore suppressed by the Gaussian term in t as $t \to \infty$. Nondegeneracy of the quadratic form (3.4) implies that the functional integral thereby becomes localized near the solutions of the equation

$$\left[\hat{\nabla}^i \, , \, \left[\hat{\nabla}_i \, , \, \mu\left[\hat{A}\right]\right]\right] = 0 \, , \tag{3.37}$$

and it can be written as a sum over contributions which depend only on local data near the solutions of (3.37). Along with (3.17) and the Leibnitz rule, Eq. (3.37) implies

$$\begin{aligned} 0 = \; & \mathrm{Tr}_{\mathcal{E}} \, \mu\left[\hat{A}\right]\left[\hat{\nabla}^i \, , \, \left[\hat{\nabla}_i \, , \, \mu\left[\hat{A}\right]\right]\right]^2 \\ = \; & - \mathrm{Tr}_{\mathcal{E}} \left[\hat{\nabla}_i \, , \, \mu\left[\hat{A}\right]\right]^2 \, , \end{aligned} \tag{3.38}$$

which again by the non-degeneracy of (3.4) is equivalent to

$$\left[\hat{\nabla}_i \, , \, \mu\left[\hat{A}\right]\right] = 0 \, . \tag{3.39}$$

Since $\mu\left[\hat{A}\right] = \left[\hat{\nabla}_1 \, , \, \hat{\nabla}_2\right]$, Eqs. (3.39) coincide with the classical equations of motion of the action (2.25), i.e. $\delta S\left[\hat{A}\right]/\delta\hat{A} = 0$. This establishes the localization of the partition function (2.24) of noncommutative gauge theory in two dimensions onto the space of solutions of the noncommutative Yang-Mills equations. This space will be studied in detail in the next section.

Although the above technique leads to a formal proof of the localization of the partition function onto classical gauge field configurations, it does not yield any immediate useful information as to the precise form of the function W in (3.3) encoding the quantum fluctuations about the classical solutions. The infinite-dimensional determinants that arise from (3.36) have very large symmetries and are difficult to evaluate. The fluctuation determinants W will be determined later on by another technique. From a mathematical perspective, the action in (3.27) over $\Pi(\mathcal{E})$ is the $\mathsf{G}(\mathcal{E})$-equivariant extension of the moment map on $\mathsf{C}(\mathcal{E})$, the integration over $\Pi(\mathcal{E})$ defines an equivariant differential form, and the integral over $\hat{\phi} \in \Omega^0(\mathcal{E})$ defines equivariant integration of such forms. The operator (3.28) is the Cartan differential for the $\mathsf{G}(\mathcal{E})$-equivariant cohomology of $\mathsf{C}(\mathcal{E})$ [34]. The localization may then also be understood via a mapping onto a purely cohomological noncommutative gauge theory in the limit $t \to \infty$. These aspects will not be developed any further here.

4. Classification of Instanton Contributions

In the previous section we proved that the partition function is given by a sum over contributions localized at the classical solutions of the noncommutative gauge theory. In this section we will classify the instantons of two-dimensional gauge theory on the noncommutative torus, and later on explicitly evaluate their contribution to the partition function. By an "instanton" here we mean a solution $A_i = A_i^{\mathrm{cl}}$ of the classical noncommutative field equations

$$\partial_i F_A + A_i \star' F_A - F_A \star' A_i = 0 \tag{4.1}$$

which is not a gauge transformation of the trivial solution $A_i = 0$. Here F_A is the noncommutative field strength (2.28). Note that this definition also includes the unstable modes. In the commutative case, instanton contributions have a well-known geometrical classification based on the fundamental group of the spacetime [35]. In the noncommutative setting, however, the role of homotopy groups is played by the K-theory of the algebra and one must resort to an algebraic characterization of the contributing projective modules. For irrational values of the noncommutativity parameter θ, an elegant classification of the stationary points of noncommutative Yang-Mills theory has been given in [36]. In what follows we shall modify this construction somewhat to more properly suit our purposes.

4.1. Heisenberg modules. In order to classify the instanton solutions of gauge theory on the noncommutative torus, we need to specify the topological structures involved. This requirement leads us into the explicit classification of the projective modules over the algebra $\mathcal{A}_\theta$ [1, 38]. They are classified by the K-theory group [43]

$$\mathrm{K}_0(\mathcal{A}_\theta) = \pi_1\Big(\mathbb{U}_\infty(\mathcal{A}_\theta)\Big) = \mathbb{Z} \oplus \mathbb{Z} \,. \tag{4.2}$$

The cohomologically invariant trace $\mathrm{Tr} : \mathcal{A}_\theta \to \mathbb{C}$ induces an isomorphism $\mathrm{K}_0(\mathcal{A}_\theta) \to \mathbb{Z} + \mathbb{Z}\theta \subset \mathbb{R}$ of ordered groups. To each pair of integers $(p, q) \in \mathrm{K}_0(\mathcal{A}_\theta)$ there corresponds a virtual projector $\mathsf{P}_{p,q}$ with $\mathrm{Tr} \otimes \mathrm{tr}_M \mathsf{P}_{p,q} = p - q\theta$. However, given a projective module $\mathcal{E}$ determined by a Hermitian projector P, positivity of the trace implies

$$\dim \mathcal{E} = \mathrm{Tr}_{\mathcal{E}}\, \mathbb{I}_{\mathcal{E}} = \mathrm{Tr} \otimes \mathrm{tr}_M \mathsf{P} = \mathrm{Tr} \otimes \mathrm{tr}_M \mathsf{P}\mathsf{P}^\dagger \geq 0 \,, \tag{4.3}$$

and so the stable (rather than virtual) projective modules are classified by the positive cone of $\mathrm{K}_0(\mathcal{A}_\theta)$. Thus to each pair of integers (p, q) we can associate a *Heisenberg module* $\mathcal{E}_{p,q}$ [44] of positive Murray-von Neumann dimension

$$\dim \mathcal{E}_{p,q} = p - q\theta > 0 \,. \tag{4.4}$$

Such pairs of integers parameterize the connected components of the infinite dimensional manifold Gr_θ of Hermitian projectors of the algebra $\mathcal{A}_\theta$. In what follows we will be interested in studying the critical points of the noncommutative Yang-Mills action within a given homotopy class of Gr_θ. The integer

$$q = \frac{1}{2\pi\,\mathrm{i}}\ \mathrm{Tr} \otimes \mathrm{tr}_M \mathsf{P}_{p,q} \Big[\hat\partial\,,\, \mathsf{P}_{p,q}\Big] \wedge \Big[\hat\partial\,,\, \mathsf{P}_{p,q}\Big] \tag{4.5}$$

is the Chern number (or magnetic flux) of the corresponding gauge bundle [45]. In the case of irrational θ, any finitely generated projective module over the noncommutative torus is either a free module or it is isomorphic to a Heisenberg module [46]. We will view free modules as special instances of Heisenberg modules obtained by setting $q = 0$. Any two projective modules representing the same element of K-theory are isomorphic.

The main property of Heisenberg modules that we will exploit in the following is that they always admit a constant curvature connection $\hat\nabla^{\mathrm{c}} \in \mathsf{C}_{p,q} = \mathsf{C}(\mathcal{E}_{p,q})$,

$$\Big[\hat\nabla^{\mathrm{c}}_1\,,\, \hat\nabla^{\mathrm{c}}_2\Big] = \mathrm{i}\, f \cdot \mathbb{I}_{\mathcal{E}_{p,q}} \,, \tag{4.6}$$

where $f \in \mathbb{R}$ is a constant. In this subsection we shall set $\Phi = 0$, as the background magnetic flux can be reinstated afterwards by the shift $f \mapsto f + \Phi$. In the presence of supersymmetry, such a field configuration gives rise to a BPS state [47, 48]. It also leads to an explicit representation of the Heisenberg module $\mathcal{E}_{p,q}$ as the separable Hilbert space [38]

$$\mathcal{E}_{p,q} = L^2(\mathbb{R}) \otimes \mathbb{C}^q \ , \ \ q \neq 0 \ . \tag{4.7}$$

The Hilbert space $L^2(\mathbb{R})$ is the Schrödinger representation of the Heisenberg commutation relations (4.6). By the Stone-von Neumann theorem, it is the unique irreducible representation. The factor $\mathbb{C}^q$ defines the $q \times q$ representation of the Weyl-'t Hooft algebra in two dimensions,

$$\Gamma_1 \, \Gamma_2 = \mathrm{e}^{\,2\pi \, \mathrm{i} \, p/q} \, \Gamma_2 \, \Gamma_1 \ , \tag{4.8}$$

which may be solved explicitly by $SU(q)$ shift and clock matrices. The generators of the noncommutative torus are then represented on (4.7) as

$$\hat{Z}_i = \mathrm{e}^{\,\mathrm{i} \, f^{-1} \, \hat{\nabla}_i^{\mathrm{c}}/R} \otimes \Gamma_i \ , \tag{4.9}$$

and computing (2.1) using (4.6), (4.8) and the Baker-Campbell-Hausdorff formula thereby leads to a relation between the noncommutativity parameter θ and the constant flux f through

$$\theta = -\frac{1}{2\pi R^2 f} + \frac{p}{q} \ . \tag{4.10}$$

The $\mathcal{A}_\theta$-valued inner product on $\mathcal{E}_{p,q}$ is given by

$$\left\langle \hat{\xi}, \hat{\eta} \right\rangle_{\mathcal{A}_\theta} = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \mathrm{d}s \, \left[\Gamma_1^{m_1} \, \Gamma_2^{m_2} \, \xi \left(s - \frac{m_1}{2\pi R^2 f} \right) \right]^\dagger \, \eta(s) \ \mathrm{e}^{\,2\pi \, \mathrm{i} \, m_2} \right) \times \hat{Z}_1^{m_1} \hat{Z}_2^{m_2} \ . \tag{4.11}$$

For $q = 0$ we define $\mathcal{E}_{p,0}$ to be the free module of rank p, i.e.

$$\mathcal{E}_{p,0} = L^2(\mathbf{T}^2) \otimes \mathbb{C}^p \ . \tag{4.12}$$

The Heisenberg module $\mathcal{E}_{p,q}$ so constructed coincides, in the D-brane picture, with the Hilbert space of ground states of open strings stretching between a single Dr-brane and p Dr-branes carrying q units of D$(r-2)$-brane charge [41]. It is irreducible if and only if the integers p and q are relatively prime. The Weyl-'t Hooft algebra (4.8) has a unique irreducible representation (up to $SU(q)$ equivalence) of dimension $q/\gcd(p,q)$ [49, 50], and so the rank N of the resulting gauge theory as defined in Sect. 2.2 is given by

$$N = \gcd(p,q) \ . \tag{4.13}$$

Furthermore, the commutant $\mathbb{M}_N(\mathcal{A}_{\theta'})$ of $\mathcal{A}_\theta$ in $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}_{p,q})$ is Morita equivalent to the noncommutative torus with dual noncommutativity parameter θ' determined by the $SL(2,\mathbb{Z})$ transformation [38]

$$\theta' = \frac{n - s\theta}{p - q\theta} \, N \ , \tag{4.14}$$

where n and s are integers which solve the Diophantine equation

$$ps - qn = N \ . \tag{4.15}$$

4.2. Stationary points of noncommutative gauge theory. We will now describe the critical points of the noncommutative Yang-Mills action (2.25). Let us fix a Heisenberg module $\mathcal{E}_{p,q}$ over the noncommutative torus, which is labelled by a pair of integers (p,q) obeying the constraint (4.4). From (4.6) and (4.10) it follows that this projective module is characterized by a connection $\hat{\nabla}^{\rm c} \in \mathsf{C}_{p,q}$ of constant curvature

$$\hat{F}_{\hat{A}^{\rm c}} = \frac{1}{2\pi R^2}\,\frac{q}{p-q\theta}\cdot \mathbb{I}_{\mathcal{E}_{p,q}}\ . \tag{4.16}$$

Such constant curvature connections are of fundamental importance in finding solutions of the noncommutative Yang-Mills equations because they not only solve (4.1), but they moreover yield the absolute minimum value of the Yang-Mills action on the module $\mathcal{E}_{p,q}$ [15, 37]. This follows by using (3.19) to compute the infinitesimal variation $\hat{F}_{\hat{A}^{\rm c}+t\,\hat{a}}$ about a constant curvature connection to get

$$\begin{aligned} S\left[\hat{\nabla}^{\rm c}+t\,\hat{a}\right] &= \frac{2\pi^2R^2}{g^2}\ {\rm Tr}_{\mathcal{E}_{p,q}}\left(\hat{F}_{\hat{A}^{\rm c}+t\,\hat{a}}+\Phi\cdot\mathbb{I}_{\mathcal{E}_{p,q}}\right)^2 \\ &= S\left[\hat{\nabla}^{\rm c}\right]+\frac{2\pi^2R^2\,t^2}{g^2}\ {\rm Tr}_{\mathcal{E}_{p,q}}\left[\hat{\nabla}^{\rm c}\ \hat{,}\ \hat{a}\right]^2+O\left(t^4\right)\ . \end{aligned} \tag{4.17}$$

The cross terms of order t in (4.17) vanish due to the property (3.17) and the fact that the field strength (4.16) is proportional to the identity operator on $\mathcal{E}_{p,q}$. Since the quadratic term in t is positive definite, we have $S\left[\hat{\nabla}^{\rm c}+\hat{a}\right] \geq S\left[\hat{\nabla}^{\rm c}\right]\ \ \forall \hat{a}\in\Omega^1(\mathcal{E}_{p,q})$. To establish that $\hat{\nabla}^{\rm c}$ is a *global* minimum, we can exploit the freedom of choice of the background flux Φ (see Sect. 6.1) to identify it with the constant curvature (4.16). Then $S\left[\hat{\nabla}^{\rm c}\right]=0$, and since (2.25) is a positive functional, the claimed property follows. This shifting of the curvature will be used explicitly below.

In addition to yielding the minimum of the Yang-Mills action, constant curvature connections can also be used to construct *all* solutions of the classical equations of motion [36]. The main observation is that insofar as solutions of the Yang-Mills equations are concerned, the module $\mathcal{E}_{p,q}$ may be considered to be a direct sum of submodules [35]. To see this, we note that the equations of motion (3.39) imply that, at the critical points $\hat{\nabla}=\hat{\nabla}^{\rm cl}$, the moment map $\mu\left[\hat{A}\right]$ is invariant under the induced action of the Heisenberg algebra $\mathcal{L}_\Phi$ of automorphisms on the algebra ${\rm End}_{\mathcal{A}_\theta}(\mathcal{E}_{p,q})$. In particular, it corresponds to the central element of the Heisenberg Lie algebra generated by $\hat{\nabla}_1^{\rm cl}$, $\hat{\nabla}_2^{\rm cl}$ and $\mu\left[\hat{A}^{\rm cl}\right]$. This feature provides a natural direct sum decomposition of the module $\mathcal{E}_{p,q}$ through the adjoint action of the moment map on $\Omega_{p,q}=\Omega(\mathcal{E}_{p,q})$. For this, we consider the self-adjoint linear operators $\Xi_{\hat{\nabla}}:\Omega_{p,q}\to\Omega_{p,q}$ defined for each connection $\hat{\nabla}\in\mathsf{C}_{p,q}$ by

$$\Xi_{\hat{\nabla}}(\hat{\alpha}) = \left[\mu\left[\hat{A}\right]\ ,\ \hat{\alpha}\right]\ ,\ \ \hat{\alpha}\in\Omega_{p,q}\ . \tag{4.18}$$

From the equations of motion (3.39) it follows that the $\mathcal{A}_\theta$-valued eigenvalues $\hat{c}_k$ of $\Xi_{\hat{\nabla}}$ are constant in the vicinity of a critical point $\hat{\nabla}=\hat{\nabla}^{\rm cl}$, and so there is a natural direct sum decomposition of the module $\mathcal{E}_{p,q}$ into projective submodules $\mathcal{E}_{p_k,q_k}$,

$$\mathcal{E}_{p,q}=\bigoplus_{k\geq1}\mathcal{E}_{p_k,q_k}\ , \tag{4.19}$$

corresponding to the eigenspace decomposition $\Omega_{p,q} = \bigoplus_{k\geq 1} \Omega_{p_k,q_k}$ with respect to $\Xi_{\hat{\nabla}}$.[5] On each Ω_{p_k,q_k} the operator $\Xi_{\hat{\nabla}}$ acts as multiplication by a fixed scalar c_k. Since $\mu\left[\hat{A}^{\rm cl}\right]$ commutes with $\hat{\nabla}^{\rm cl}$, the connection $\hat{\nabla}^{\rm cl}$ is also a linear operator on each $\mathcal{E}_{p_k,q_k} \to \mathcal{E}_{p_k,q_k}$, and its restriction $\hat{\nabla}^{\rm c}_{(k)} = \hat{\nabla}^{\rm cl}\Big|_{\mathcal{E}_{p_k,q_k}}$ has constant curvature $\mu\left[\hat{A}^{\rm cl}\right]\Big|_{\mathcal{E}_{p_k,q_k}}$.

Given such a direct sum decomposition[6] of the module $\mathcal{E}_{p,q}$, we can define a connection $\hat{\nabla}$ on $\mathcal{E}_{p,q}$ by taking the sum of connections on each of the submodules, $\hat{\nabla} = \bigoplus_{k\geq 1} \hat{\nabla}_{(k)}$. The noncommutative Yang-Mills action is additive with respect to this decomposition,

$$S\left[\bigoplus_{k\geq 1} \hat{\nabla}_{(k)}\right] = \sum_{k\geq 1} S\left[\hat{\nabla}_{(k)}\right] \ . \tag{4.20}$$

It follows that for the particular choice of constant curvature connections $\hat{\nabla}^{\rm c}_{(k)}$ on each of the submodules $\mathcal{E}_{p_k,q_k}$, the Yang-Mills action has a critical point

$$\hat{\nabla}^{\rm cl} = \bigoplus_{k\geq 1} \hat{\nabla}^{\rm c}_{(k)} \tag{4.21}$$

on $\mathcal{E}_{p,q}$. Moreover, from the above arguments it also follows that every Yang-Mills critical point on $\mathcal{E}_{p,q}$ is of this form.

This construction thereby exhausts all possible critical points, and is essentially the noncommutative version of the bundle splitting method of constructing classical solutions to ordinary, commutative gauge theory in two dimensions [35]. While there are many possibilities for the decomposition (4.19) of the given module $\mathcal{E}_{p,q}$ into submodules, there are two important constraints that must be taken into account. First of all, the (positive) Murray-von Neumann dimension of the module is additive with respect to the decomposition (4.19),

$$\dim \mathcal{E}_{p,q} = \sum_{k\geq 1} \dim \mathcal{E}_{p_k,q_k} = \sum_{k\geq 1} (p_k - q_k\theta) \ . \tag{4.22}$$

Secondly, since a module over the noncommutative torus is completely and uniquely determined (up to isomorphism) by two integers, we need an additional constraint. This is the requirement that the Chern number of the module be equal to the total magnetic flux of the direct sum decomposition. For the module $\mathcal{E}_{p,q}$ this gives the relation

$$q = \sum_{k\geq 1} q_k \ . \tag{4.23}$$

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[5] It should be stressed that (4.19) is *not* the statement that the given Heisenberg module is reducible. It simply reflects the behaviour of connections near a stationary point of the noncommutative Yang-Mills action, in which one may interpret the eigenspaces $\Omega_{p_k,q_k} = \Omega(\mathcal{E}_{p_k,q_k})$ as the differential algebras of submodules $\mathcal{E}_{p_k,q_k} \subset \mathcal{E}_{p,q}$. For more technical details of the decomposition (4.19) as an $\mathcal{A}_\theta$-module, we refer to [36]. Notice also that here we abuse notation by making no distinction between $\hat{\nabla}^{\rm c}$ acting on $\Omega_{p,q}$ or $\mathcal{E}_{p,q}$. Only the latter operator will be pertinent in what follows.

[6] In Sect. 7 we will give an elementary proof that such decompositions necessarily contain only a finite number of direct summands. See [36] for a functional analytic proof.

For irrational values of the noncommutativity parameter θ it is clear from (4.4) that the constraint (4.23) follows from (4.22). This is not the case for rational θ and the constraint on the Chern class makes necessary a distinction between physical noncommutative Yang-Mills theory and Yang-Mills theory defined on a particular projective module $\mathcal{E}_{p,q}$. The latter field theory imposes a K-theory charge conservation law for submodule decompositions (4.19), $(p,q) = \sum_{k\geq 1}(p_k, q_k)$. This distinction will be discussed further when the partition function for Yang-Mills theory on the noncommutative torus is calculated explicitly.

We can now summarize the classification of the critical points of the noncommutative Yang-Mills action as follows. For any value of the noncommutativity parameter θ, any solution of the classical equations of motion of Yang-Mills theory defined on the Heisenberg module $\mathcal{E}_{p,q}$ is completely characterized by a collection of pairs of integers $\Big\{(p_k, q_k)\Big\}_{k\geq 1}$ obeying the constraints

$$\begin{aligned} p_k - q_k\theta &> 0 \ , \\ \sum_{k\geq 1}(p_k - q_k\theta) &= p - q\theta \ , \\ \sum_{k\geq 1} q_k &= q \ . \end{aligned} \tag{4.24}$$

We will call such a collection of integers a "partition" [7] and will denote it by $(\vec{p}\,,\vec{q}\,) \equiv \Big\{(p_k, q_k)\Big\}_{k\geq 1}$. In order to avoid overcounting partitions which will contribute to the Yang-Mills partition function, we also need to introduce a partial ordering for submodules in a given partition based on the dimension of each submodule,

$$0 < p_1 - q_1\theta \leq p_2 - q_2\theta \leq p_3 - q_3\theta \leq \dots \ . \tag{4.25}$$

Any number of partitions which are identical after such an ordering will be regarded as equivalent presentations of the same partition. The set of all distinct partitions associated with the Heisenberg module $\mathcal{E}_{p,q}$ will be denoted $\mathcal{P}_{p,q}(\theta) = \mathcal{P}(\theta, \mathcal{E}_{p,q})$.

It remains to evaluate the Yang-Mills action at a solution of the classical equations of motion, which, by the arguments of the previous section, is one of the key ingredients in the computation of the partition function of noncommutative gauge theory. At a critical point, i.e. a partition $(\vec{p}\,,\vec{q}\,)$, according to (4.20) it is just the sum of contributions from constant curvature connections on each of the submodules of the partition,

$$\begin{aligned} S(\vec{p}\,,\vec{q}\,;\theta) &= S\left[\bigoplus_{k\geq 1}\hat{\nabla}^{\rm c}_{(k)}\right] \\ &= \frac{1}{2g^2R^2}\sum_{k\geq 1}(p_k - q_k\theta)\left(\frac{q_k}{p_k - q_k\theta} - \frac{q}{p - q\theta}\right)^2 \ , \end{aligned} \tag{4.26}$$

[7] This definition of partition is more general than that of [36]. It is the one that is the most useful for the computation of the noncommutative gauge theory partition function in the following. In particular, it contains contributions from reducible connections, as these will also turn out to contribute to the Yang-Mills partition function. These connections are in fact responsible for the orbifold singularities that appear in the instanton moduli spaces. These points, as well as how to avoid the overcounting of critical points through combinatoric factors in the partition function, will be described in detail in Sect. 9.

where we have used $\mathrm{Tr}_{\mathcal{E}_{pk,qk}} \mathbb{I}_{\mathcal{E}_{pk,qk}} = p_k - q_k\theta$ and fixed the value of the background field Φ to be [15]

$$\Phi = \Phi_{p,q} = -\frac{q}{2R^2(p - q\theta)} \,. \tag{4.27}$$

This value ensures that the constant curvature connection $\hat{\nabla}^{\mathrm{c}}$ on the module $\mathcal{E}_{p,q}$, which corresponds to a solution of the equations of motion parameterized by the trivial partition (p, q), gives a vanishing global minimum of the action, $S(p, q; \theta) = 0$. This is the natural boundary condition, and generally the inclusion of Φ ensures that the classical action is invariant under Morita duality [15, 41, 48, 51, 52].

5. Yang-Mills Theory on a Commutative Torus

As we mentioned in Sect. 3, while we can prove that noncommutative gauge theory on a two-dimensional torus is given exactly by a sum over classical solutions (instantons), evaluating directly the fluctuation factors, which multiply the Boltzmann weights of the corresponding critical action values computed in the previous section, is a difficult task. We will therefore proceed as follows. We start with the well-known exact solution for Yang-Mills theory on a *commutative* torus and identify quantities which are invariant under gauge Morita equivalence. This will yield the partition function of noncommutative Yang-Mills theory for any *rational* value of the noncommutativity parameter θ. From this expression we will then be able to deduce the corresponding expression for Yang-Mills theory defined on a noncommutative torus with arbitrary θ. In this section we will analyze the instanton contributions to commutative Yang-Mills theory in order to set up this construction.

The physical Hilbert space $\mathcal{H}_{\mathrm{phys}}$ of ordinary $U(p)$ quantum gauge theory defined on a (commutative) two-torus is the space of class functions

$$\mathcal{H}_{\mathrm{phys}} = L^2\Big(U(p)\Big)^{\mathrm{Ad}\left(U(p)\right)} \tag{5.1}$$

in the invariant Haar measure on the $U(p)$ gauge group. By the Peter-Weyl theorem, it has a natural basis $|\mathcal{R}\rangle$ determined by the unitary irreducible representations $\mathcal{R}$ of the unitary Lie group $U(p)$. The Hamiltonian is essentially the Laplacian on the group manifold of $U(p)$, and so the corresponding vacuum amplitude has the well-known heat kernel expansion [14, 53, 54]

$$Z(g^2, p) = \sum_{\mathcal{R}} \mathrm{e}^{\,-2\pi^2 R^2 g^2\, C_2(\mathcal{R})} \,, \tag{5.2}$$

where the Boltzmann weight contains the quadratic Casimir invariant $C_2(\mathcal{R})$ of the representation $\mathcal{R}$. This concise form does not have a direct interpretation in terms of a sum over contributions from critical points of the classical action that we expect from the arguments of Sect. 3. In order to find a more appropriate form, it is useful to make explicit the sum over irreducible representations as a sum over integers and perform a Poisson resummation of (5.2) [33].

The representations $\mathcal{R}$ of $U(p)$ can be labelled by sets of p integers

$$+\infty > n_1 > n_2 > \cdots > n_p > -\infty \tag{5.3}$$

which give the lengths of the rows of the corresponding Young tableaux. In terms of these integers the Casimir operator is given by

$$C_2(\mathcal{R}) = C_2(n_1, \dots, n_p) = \frac{p}{12}\left(p^2 - 1\right) + \sum_{a=1}^{p}\left(n_a - \frac{p-1}{2}\right)^2 , \tag{5.4}$$

and by using its symmetry under permutations of the integers n_a we can write (5.2) as

$$Z(g^2, p) = \frac{1}{p!} \sum_{n_1 \neq \cdots \neq n_p} \mathrm{e}^{-2\pi^2 R^2 g^2 C_2(n_1, \dots, n_p)} . \tag{5.5}$$

One can extend the sums in (5.5) over *all* integers $n_1, \dots, n_p$ by inserting the determinant

$$\det_{1 \le a,b \le p} \left(\delta_{n_a, n_b}\right) = \sum_{\sigma \in \Sigma_p} \mathrm{sgn}(\sigma) \prod_{a=1}^{p} \delta_{n_a, n_{\sigma(a)}} , \tag{5.6}$$

where Σ_p is the group of permutations on p objects. The permutation symmetry of (5.4) implies that all elements in the same conjugacy class of Σ_p yield the same contribution to the partition function. The sum over permutations (5.6) thereby truncates to a sum over conjugacy classes of Σ_p. They are labelled by the sets of p integers $0 \le \nu_a \le [p/a]$, each giving the number of elementary cycles of length a in the usual cycle decomposition of elements of Σ_p, and which define a partition of p, i.e.

$$\nu_1 + 2\nu_2 + \cdots + p\nu_p = p . \tag{5.7}$$

The parity of the elements of a conjugacy class $C[\vec{\nu}\,] = C[\nu_1, \dots, \nu_p]$ is

$$\mathrm{sgn}\, C[\vec{\nu}\,] = (-1)^{\sum_{a'} \nu_{2a'}} \tag{5.8}$$

and it contains

$$\left|C[\vec{\nu}\,]\right| = \frac{p!}{\prod_{a=1}^{p} a^{\nu_a}\, \nu_a!} \tag{5.9}$$

elements.

The sum over the n_a's in (5.5) then yields a theta-function, and the corresponding Jacobi inversion formula can be derived in the usual way by means of the Poisson resummation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{q=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}s\; f(s)\; \mathrm{e}^{2\pi\,\mathrm{i}\,qs} . \tag{5.10}$$

The Fourier transformations required in (5.10) are all Gaussian integrals in the present case, and after some algebra the partition function (5.5) can be expressed as a sum over dual integers q_k as [33] [8]

[8] We have corrected here a few typographical errors appearing in Eqs. (13)–(15) of [33].

$$Z(g^2,\,p)\,=\,\mathrm{e}^{\,-\frac{\pi^2 g^2 R^2}{6}\,p(p^2-1)}\sum_{\vec{\nu}\,:\,\sum_a a\nu_a=p}\;\sum_{q_1=-\infty}^{\infty}\;\cdots\;\sum_{q_{|\vec{\nu}\,|}=-\infty}^{\infty}\mathrm{e}^{\,\mathrm{i}\,\pi\left(\sum_{a'}\,\nu_{2a'}+(p-1)\sum_k q_k\right)}$$
$$\times\,\prod_{a=1}^{p}\frac{\left(2g^2R^2a^3\right)^{-\nu_a/2}}{\nu_a\,!}\;\mathrm{e}^{\,-S_{\vec{\nu}}\,(q_1,\ldots,q_{|\vec{\nu}\,|})}\;. \tag{5.11}$$

Here

$$|\vec{\nu}\,|\,=\,\nu_1+\nu_2+\cdots+\nu_p \tag{5.12}$$

is the total number of cycles contained in the elements of the conjugacy class $C[\vec{\nu}\,]$ of Σ_p, and the action is given by

$$S_{\vec{\nu}}\,(q_1,\,\ldots\,,\,q_{|\vec{\nu}\,|})\,=\,\frac{1}{2g^2R^2}\,\left(\sum_{k_1=1}^{\nu_1}\frac{q_{k_1}^2}{1}+\sum_{k_2=\nu_1+1}^{\nu_1+\nu_2}\frac{q_{k_2}^2}{2}+\sum_{k_3=\nu_1+\nu_2+1}^{\nu_1+\nu_2+\nu_3}\frac{q_{k_3}^2}{3}\right.$$
$$\left.+\,\cdots+\sum_{k_p=\nu_1+\cdots+\nu_{p-1}+1}^{|\vec{\nu}\,|}\frac{q_{k_p}^2}{p}\right)\,. \tag{5.13}$$

It is understood here that if some $\nu_a=0$, then $q_{\nu_1+\cdots+\nu_{a-1}+1}=\cdots=q_{\nu_1+\ldots\nu_{a-1}+\nu_a}=0$.

The remarkable feature of this rewriting is that the action (5.13) is precisely of the general form (4.26). Since $\mathrm{K}_0\Big(C^\infty(\mathbf{T}^2)\Big)\,=\,\mathbb{Z}\oplus\mathbb{Z}$, any finitely-generated projective module $\mathcal{E}=\mathcal{E}_{p,q}$ over the algebra $\mathcal{A}_0=C^\infty(\mathbf{T}^2)$ is determined (up to isomorphism) by a pair of relatively prime integers $(p,\,q)\in\mathbb{Z}_+\oplus\mathbb{Z}$ with dimension given by p and constant curvature q/p [44]. Geometrically, $\mathcal{E}_{p,q}$ is the space of sections of a vector bundle over the torus $\mathbf{T}^2$ of rank p, topological charge q, and with structure group $U(p)$. Consider a direct sum decomposition (4.19) of this module. We will enumerate submodules in a partition according to increasing dimension. Let ν_a be the number of submodules of dimension a, corresponding to the splitting of the bundle into sub-bundles of rank a, so that

$$\dim\,\mathcal{E}_{p,q}\,=\,\nu_1+2\nu_2+\cdots+p\nu_p\;. \tag{5.14}$$

This condition is simply the constraint (4.22) on the total dimension of the sum of submodules in this case, i.e. $p=\sum_{k\geq1}p_k$ with $1\leq p_k\leq p$. Therefore, the expression (5.11) is nothing but the localization of the partition function of commutative Yang-Mills theory onto its classical solutions. Note that here the magnetic charges q_k are dual to the lengths of the rows of the Young tableaux of the unitary group $U(p)$.

There are, however, two important differences here. First of all, the action (5.13) is evaluated for a topologically trivial bundle, i.e. $q=0$, which yields a vanishing background flux $\Phi_{p,q}$. Consequently, (5.11) is not the most general result. Secondly, and most importantly, the sum over Chern numbers $q_1,\ldots,q_{|\vec{\nu}\,|}$ in the partition function is not constrained to satisfy (4.23), which in view of our first point is the restriction $\sum_k q_k=0$. In fact, the partition function (5.11) for *physical* $U(p)$ Yang-Mills gauge theory on the commutative torus is a sum of contributions from topologically distinct bundles (of different Chern numbers) over the torus. In order to generalize the calculation of the partition function to the case of Yang-Mills theory defined on a projective module, we need to separate out of (5.11) the terms which are well-defined on a particular isomorphism class $\mathcal{E}_{p,q}$ of modules.

In order to facilitate the identification of such a module definition of Yang-Mills theory, we write the partition function (5.11) in terms of the topological numbers of the module $\mathcal{E}_{p,q}$. We will first enforce the constraint (4.23) on the magnetic charges. It is also useful for further calculations to re-interpret the parity factors $(-1)^{\sum_{a'} \nu_{2a'}}$ in terms of the rank p and the total number $|\vec{\nu}\,|$ of submodules in a given partition $(\vec{p}\,,\vec{q}\,) = \left\{(p_k, q_k)\right\}_{k=1}^{|\vec{\nu}\,|}$ labelling a critical point of the action. If p is odd (even) then there is an odd (even) number of submodules $\mathcal{E}_{p_k,q_k}$ with p_k odd. By considering all possible cases one can show that

$$\sum_{a'=1}^{[p/2]} \nu_{2a'} = p + |\vec{\nu}\,| \pmod 2 \,. \tag{5.15}$$

With these adjustments we are led to the module Yang-Mills theory with partition function $Z_{p,q}$ which is well-defined on $\mathcal{E}_{p,q}$,

$$Z(g^2,\, p) = \mathrm{e}^{\,-\frac{\pi^2 g^2 R^2}{6}\, p(p^2-1)} \sum_{q=-\infty}^{\infty} (-1)^{(p-1)q+p}\, Z_{p,q}(g^2, \theta = 0) \,, \tag{5.16}$$

where the module partition function is given by a sum over partitions associated with the module $\mathcal{E}_{p,q}$,

$$\begin{aligned} Z_{p,q}(g^2, \theta = 0) &= Z(g^2, \theta = 0, \Phi_{p,q}, \mathcal{E}_{p,q}) \\ &= \sum_{(\vec{p}\,,\vec{q}\,)\in\mathcal{P}_{p,q}(\theta=0)} (-1)^{|\vec{\nu}\,|} \prod_{a=1}^{p} \frac{\left(2g^2R^2a^3\right)^{-\nu_a/2}}{\nu_a!}\ \mathrm{e}^{\,-S(\vec{p}\,,\vec{q}\,;\theta=0)} . \end{aligned} \tag{5.17}$$

Note that the critical points of the action are defined by partitions obeying the constraints (4.24), including a restriction to submodules with total Chern number q. We have also generalized to the correct action (4.26) for Yang-Mills theory on a bundle with Chern number q which contains the non-vanishing value (4.27) for the background magnetic field Φ. Again, this latter change is equivalent to adding boundary terms to the action which do not contribute to the classical dynamics of the theory and hence are not relevant to our analysis based on instanton contributions. The only essential role of Φ is, as we will see, to set the zero-point of the Yang-Mills action in the instanton picture. Therefore, a shift in Φ will at most result in multiplying the fixed module partition function (5.17) by overall constants dependent only on the topological numbers (p, q).

6. Yang-Mills Theory on a Noncommutative Torus: Rational Case

Given the partition function (5.17) for Yang-Mills theory which is well-defined on a given module $\mathcal{E}_{p,q}$ of sections of some bundle, we can now use Morita equivalence to obtain an explicit formula for Yang-Mills theory on a torus with rational noncommutativity parameter θ from the commutative case. Morita equivalence in this case refers to the mapping between noncommutative tori which is generated by the infinite discrete group $SO(2,2,\mathbb{Z}) \cong SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$, where one of the $SL(2,\mathbb{Z})$ factors coincides with the discrete automorphism group of the ordinary torus $\mathbf{T}^2$. It provides a one-to-one

correspondence between modules associated with different topological numbers and noncommutativity parameters. Here we will be interested in the transformations from modules corresponding to rational values of θ to modules with vanishing θ. In fact, there is an extended version of the correspondence known as gauge Morita equivalence [41, 48] which augments the mapping of tori with transformations of connections between modules, and leads to a rescaling of the area and coupling constant to give a symmetry of Yang-Mills theory as we have defined it in (2.26). The entire noncommutative quantum field theory is invariant under this extended equivalence [15] which coincides with the standard open string T-duality transformations [41, 51]. We will use this invariance property to construct the noncommutative gauge theory for rational values of the deformation parameter θ.

6.1. Gauge Morita equivalence. We begin by summarizing the basic transformation rules of Morita equivalence of noncommutative gauge theories [1, 3, 44]. In two dimensions, Morita equivalences of noncommutative tori are generated by the group elements

$$\begin{pmatrix} m & n \\ r & s \end{pmatrix} \in SL(2,\mathbb{Z}) \ , \tag{6.1}$$

where we concentrate on the $SL(2,\mathbb{Z})$ subgroup which acts only on the Kähler modulus of $\mathbf{T}^2$. The full duality group acts on the K-theory ring $\mathrm{K}_0(\mathcal{A}_\theta) \oplus \mathrm{K}_1(\mathcal{A}_\theta)$ in a spinor representation of $SO(2,2,\mathbb{Z})$ and the topological numbers (p,q) of a module $\mathcal{E} = \mathcal{E}_{p,q}$ transform as

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} m & n \\ r & s \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \ . \tag{6.2}$$

The noncommutativity parameter θ transforms under a discrete linear fractional transformation

$$\theta' = \frac{m\theta + n}{r\theta + s} \ . \tag{6.3}$$

From these rules it follows that under the gauge Morita equivalence parameterized by (6.1) the dimensions of modules are changed according to

$$\dim \mathcal{E}' = \frac{\dim \mathcal{E}}{|r\theta + s|} \ . \tag{6.4}$$

The invariance of the noncommutative Yang-Mills action (2.27) then dictates the corresponding transformation rules for the area element of $\mathbf{T}^2$, the Yang-Mills coupling constant, and the magnetic background as

$$\begin{aligned} R' &= |r\theta + s|\, R \ , \\ g'^2 &= |r\theta + s|\, g^2 \ , \\ \Phi' &= (r\theta + s)^2\, \Phi - \frac{r(r\theta + s)}{2\pi R^2} \ . \end{aligned} \tag{6.5}$$

6.2. The partition function for rational θ. Let us now consider the effect of such transformations on the module partition function defined in (5.17). Under the gauge Morita equivalence parameterized by (6.1), the ordinary Yang-Mills gauge theory (5.17) is mapped onto a noncommutative gauge theory with rational-valued noncommutativity parameter $\theta = n/s$. The classical action of the theory is invariant, and constant curvature connections are mapped into one another [1, 48]. Thus Morita equivalence maps solutions of the equations of motion between the commutative and rational noncommutative cases. The localization of the partition function onto classical solutions is therefore not affected by the transformation. The topological numbers of the submodules comprising partitions which define solutions of the classical equations of motion also map into each other in the two cases. In particular, the total number $|\vec{\nu}\,|$ of submodules in a partition is invariant under the Morita duality.

The only component of the partition function (5.17) we have left to examine is the pre-exponential factor $\prod_{a\geq 1}(2g^2R^2a^3)^{-\nu_a/2}/\nu_a!$. The symmetry factors $\nu_a!$ associated with a partition are preserved, and so from the transformation rules (6.5) for $\theta = 0$ it follows that this component is invariant only if the integer a^3 appearing here transforms according to the scaling

$$a' = \frac{a}{|s|} \tag{6.6}$$

under the Morita equivalence. But (6.6) is exactly the rescaling (6.4) of the dimension of a projective module in this case. It follows that the indices a in the pre-exponential factors of (5.17) should be interpreted as the (integer) dimensions of submodules in the commutative gauge theory, and this fact provides a Morita covariant interpretation of these indices which leads immediately to the appropriate generalization of the formula (5.17) to rational-valued $\theta \neq 0$.

We are now in a position to write down an explicit expression for the partition function of quantum Yang-Mills theory on the module $\mathcal{E}_{p,q}$ corresponding to a rational, non-integer noncommutativity parameter θ. The only modifications required are the counting and dimensions of modules which, in contrast to the commutative case, are no longer integer-valued. We order the submodules in a given partition $(\vec{p}\,,\vec{q}\,)$ according to increasing dimension,

$$0 < \dim \mathcal{E}_{p_1,q_1} \leq \dim \mathcal{E}_{p_2,q_2} \leq \dim \mathcal{E}_{p_3,q_3} \leq \dots \ . \tag{6.7}$$

Let ν_a be the number of submodules in this sequence that have the a^{th} least dimension, which we denote by $\dim_a$. Then the integer

$$|\vec{\nu}\,| = \sum_{a\geq 1} \nu_a \tag{6.8}$$

still gives the total number of submodules in a partition, and we may write the partition function for rational θ as

$$Z_{p,q}(g^2,\theta) = \sum_{(\vec{p}\,,\vec{q}\,)\in\mathcal{P}_{p,q}(\theta)} (-1)^{|\vec{\nu}\,|} \prod_{a\geq 1} \frac{\left(2g^2R^2\,(\dim_a)^3\right)^{-\nu_a/2}}{\nu_a!}\ \mathrm{e}^{\,-S(\vec{p}\,,\vec{q}\,;\theta)}\ . \tag{6.9}$$

This expression provides a direct evaluation of Yang-Mills theory on a torus with rational noncommutativity parameter θ, without recourse to Morita equivalence with the commutative theory. Note that in the case when all submodules in the partitions have integer dimension, the formula (6.9) reduces to that for the partition function of Yang-Mills theory on a commutative torus in (5.17).

6.3. Relation between commutative and rational noncommutative gauge theories. The arguments which led to the expression (6.9) give an interesting way to see the well-known connections between Yang-Mills theory on a noncommutative torus with rational-valued θ and Yang-Mills theory defined on a commutative torus. Consider the gauge theory defined on the Heisenberg module $\mathcal{E}_{p,q}$ over the noncommutative torus with deformation parameter $\theta = n/s$, where n and s are relatively prime positive integers. It can be verified from the definition (4.4) that any projective module over such a torus has a Murray-von Neumann dimension of at least $1/s$. Since the total dimension of the module $\mathcal{E}_{p,q}$ is $p - nq/s$ in this case, a partition $(\vec{p}\,, \vec{q}\,)$ which obeys the constraints (4.24) and which consists of submodules of dimensions greater than or equal to $1/s$ has at most

$$\frac{p - nq/s}{1/s} = ps - qn \tag{6.10}$$

components. Since Morita equivalence preserves the number of submodules in a partition, any gauge theory which is dual to this rational noncommutative one must admit partitions with $ps - qn$ components. On the other hand, for $U(N)$ Yang-Mills theory defined on a commutative torus, we know that due to the constraints (4.24), the maximum number of submodules in a partition is N (corresponding to $\nu_1 = N$ and $\nu_a = 0\ \forall a > 1$). We conclude that Yang-Mills theory on a Heisenberg module $\mathcal{E}_{p,q}$ over the noncommutative torus with $\theta = n/s$ is Morita equivalent to a $U(N)$ commutative gauge theory of rank $N = ps - qn$. This result agrees with how the rank of the noncommutative gauge theory appeared at the end of Sect. 4.1.

Notice that Morita equivalence maps submodules of the $U(ps - qn)$ commutative gauge theory, as defined in the previous section, onto submodules of the noncommutative gauge theory on the Heisenberg module $\mathcal{E}_{p,q}$ as defined in Sect. 4.1. The effect of this mapping on dimensions of projective modules is to divide by s. This includes the irreducible finite-dimensional representation of the Weyl-'t Hooft algebra generated by the $\hat{Z}_i$ in (2.1) as follows. The infinite-dimensional center of the algebra $\mathcal{A}_{n/s}$ is generated by the elements $z_i = (\hat{Z}_i)^s$, $i = 1, 2$ which, in an irreducible unitary representation, can be taken to be complex numbers of unit modulus. The center can thereby be identified with the commutative algebra $C^\infty(\mathbf{T}^2)$ of smooth functions on the ordinary torus $\mathbf{T}^2$, i.e. $\mathcal{A}_{n/s}$ may be regarded as a twisted matrix bundle over $C^\infty(\mathbf{T}^2)$ of topological charge n whose fibers are $s \times s$ complex matrix algebras $\mathbb{M}_s$. In particular, there is a surjective algebra homomorphism $\pi : \mathcal{A}_{n/s} \to \mathbb{M}_s$, sending the $\hat{Z}_i$ to the corresponding $SU(s)$ shift and clock matrices, under which the entire center of $\mathcal{A}_{n/s}$ is mapped to $\mathbb{C}$. In the language of Heisenberg modules this representation corresponds to the finite-dimensional factor $\mathbb{C}^s$ of the separable Hilbert space $\mathcal{E}_s = L^2(\mathbf{T}^2) \otimes \mathbb{C}^s$, which allows for twisted boundary conditions on functions of the ordinary torus $\mathbf{T}^2$ leading to the appropriate Weyl-'t Hooft algebra in this case. The irreducible finite-dimensional representation of the algebra is thereby associated with a free module $\mathcal{E}_s = \mathcal{E}_{s,0}$ of vanishing Chern class. Therefore, the localization of the partition function of quantum Yang-Mills theory on a rational noncommutative torus is determined entirely by contributions from classical solutions associated with Heisenberg modules as we have described them above. By construction, this includes the Morita equivalent projective modules over the ordinary torus.

7. Yang-Mills Theory on a Noncommutative Torus: Irrational Case

Finally, we come to the case of irrational θ. We claim that the formula (6.9) gives the Yang-Mills partition function as a sum over partitions $(\vec{p}\,, \vec{q}\,)$ consisting of pairs of

integers satisfying the constraints (4.24). Before justifying this claim, let us describe the quantitative differences in the formula (6.9) between the rational and irrational cases. In fact, the analytical structure of the partition functions in the two cases is very different due to the drastic differences of the partitions in $\mathcal{P}_{p,q}(\theta)$ which contribute to the functional integral. Recall from the previous section that in the case of rational θ, all modules have dimension at least $1/s$, and this fact was the crux of the existence of the mapping between the rational and commutative gauge theories. In contrast, when θ is irrational, submodules with arbitrarily small dimension can contribute to a partition which characterizes a critical point of the Yang-Mills action. As such, there is no *a priori* upper bound on the number of submodules in a partition of $\mathcal{P}_{p,q}(\theta)$. While for deformation parameter $\theta = n/s$ all partitions contain at most $ps - qr$ submodules of $\mathcal{E}_{p,q}$ of dimension at least $1/s$, in the irrational case there are no such global bounds on the elements of $\mathcal{P}_{p,q}(\theta)$. It is this fact that prevents Yang-Mills theory on a noncommutative torus with irrational-valued θ from being Morita equivalent to some commutative gauge theory of finite rank, and indeed in this case the algebra $\mathcal{A}_\theta$ has a trivial center.

As a consequence, in contrast to the rational case, the Yang-Mills partition function on an irrational noncommutative torus receives contributions from partitions containing arbitrarily many submodules. However, it is possible to show that any partition corresponding to a fixed finite action solution of the noncommutative Yang-Mills equations of motion contains only finitely many components. By using a Morita duality transformation (6.5) we can transform the action so that $\Phi = 0$. Consider a partition $(\vec{p}\,,\vec{q}\,) \in \mathcal{P}_{p,q}(\theta)$ on which the Yang-Mills action has the value $S(\vec{p}\,,\vec{q}\,;\theta) = \xi \in \mathbb{R}_+$. Since (4.26) is a sum of positive terms, this implies that $q_k^2 \leq \xi\,(p_k - q_k\theta)$ for each $k \geq 1$. But the constraints (4.24) imply

$$0 < p_k - q_k\theta \leq p - q\theta \tag{7.1}$$

for each $k \geq 1$, and hence

$$q_k^2 \leq \xi\,(p - q\theta)\ . \tag{7.2}$$

From (7.2) it follows that q_k can range over only a finite number of integers, and hence from (7.1) the same is also true of p_k, which establishes the result. In particular, we can pick out the minimum dimension submodule $\mathcal{E}_{p_1,q_1}$ in a given partition $(\vec{p}\,,\vec{q}\,)$ and order the submodules according to increasing Murray-von Neumann dimension as in (6.7). The definition (6.8) still makes sense and hence so does the expression (6.9) for the partition function, provided that one now allows for partitions with arbitrarily large (but finite) numbers of submodules. Incidentally, this line of reasoning also shows that the set of values of the noncommutative Yang-Mills action on the critical point set $\mathcal{P}_{p,q}(\theta)$ is discrete, as is required of a Morse function [36].

Let us now indicate the reasons why (6.9) is the correct result for the partition function of Yang-Mills theory on a noncommutative torus with irrational θ. First of all, notice that the localization arguments of Sect. 3 which give the functional integral as a sum over critical points of the Yang-Mills action are independent of the particular value of θ. In a direct evaluation, the pre-exponential factors in (6.9) would be determined by performing the functional Grassmann integrations and taking the $t \to \infty$ limit in the localization formula (3.36). In this formula, θ is a continuous parameter and we do not expect the calculation of contributions from Gaussian fluctuations to depend on the rationality of θ. Thus the pre-exponential factors in (6.9) yield the value of the fluctuation determinant at each critical point in the semi-classical expansion of the partition function. Moreover, as emphasized in Sect. 6.3, the contributing submodules to this expansion are always

Heisenberg modules, which are the only projective modules in the irrational case. In this regard it is interesting to note the role of the alternating sign factors $(-1)^{|\vec{\nu}|}$ in (6.9). The global minimum of the action, which has $|\vec{\nu}| = 1$, is the only stable critical point of the theory. According to general stationary phase arguments [34], a classical solution with n unstable modes is always weighted with a phase $-\mathrm{e}^{\pi\,\mathrm{i}\,n/2}$ in our normalization. Thus each submodule in a partition which defines a critical point corresponds to a local extremum of the noncommutative Yang-Mills action which is unstable in two directions. Going back to the topological sum (5.16), we see that, as is the usual case in $U(p)$ gauge theory, each unit charge instanton configuration yields $2p-2$ negative modes. The instanton configurations will be studied in more detail in Sect. 9.

Secondly, consider an approximation to the partition function for irrational θ by rational theories. Formally, this requires a limit $\theta = \lim_m n_m/s_m$ with both $n_m \to \infty$ and $s_m \to \infty$ as $m \to \infty$. As we have seen in the previous section, the minimum dimension of a submodule which is permitted over the noncommutative torus $\mathcal{A}_{n_m/s_m}$ is $1/s_m$. Consequently, any rational approximation to the partition function would contain partitions of arbitrarily small dimension, as we expect to see for irrational values of the noncommutativity parameter θ. With these pieces of evidence at hand, we thereby propose that the partition function of noncommutative gauge theory on a Heisenberg module $\mathcal{E}_{p,q}$ over a two-dimensional torus is given for all values of the deformation parameter θ by the expression

$$Z_{p,q}(g^2,\theta) = \sum_{(\vec{p}\,,\vec{q}\,)\in\mathcal{P}_{p,q}(\theta)} (-1)^{|\vec{\nu}|} \prod_{a\geq 1} \frac{\left(2g^2R^2\,(p_a-q_a\theta)^3\right)^{-\nu_a/2}}{\nu_a!} \times \prod_{k=1}^{|\vec{\nu}|} \exp\left[-\frac{1}{2g^2R^2}\left(p_k-q_k\theta\right)\left(\frac{q_k}{p_k-q_k\theta}-\frac{q}{p-q\theta}\right)^2\right], \quad (7.3)$$

where the integer a labels the ν_a submodules of dimension $\dim_a = p_a - q_a\theta$. This formula exhibits the anticipated universality between the irrational and rational cases [37], a feature which we will see more of in the following. Note that the contributions from classical solutions containing submodules of very small Murray-von Neumann dimension are exponentially suppressed in (7.3). In what follows we will explore some applications of this formalism.

8. Smoothness in θ

An important issue surrounding noncommutative field theories in general is the behaviour of the partition function and observables as functions of the noncommutativity parameter θ. For example, the poles at $\theta = 0$ which arise from perturbative expansions are the earmarks of the UV/IR mixing phenomenon [55]. However, it is not yet clear in the continuum field theories whether this is an artifact of perturbation theory or if it persists at a nonperturbative level.[9] A clearer understanding of the behaviour of the nonperturbative theory as a function of θ is therefore needed to fully address such issues. Related to this problem is the question of approximation of irrational noncommutative field theories by rational ones. If the quantum field theory is at least continuous in θ

[9] In the lattice regularization of noncommutative field theories [15], UV/IR mixing persists at a fully nonperturbative level as a kinematical effect.

then it can be successively approximated by rational theories. In particular, this would lead to a hierarchy of Morita dual descriptions in terms of quasi-local degrees of freedom [56] and also finite-dimensional matrix model approximations to the continuum noncommutative field theory [16]. It has also been suggested that smooth behaviour of physical quantities in θ could, by Morita equivalence, imply very stringent constraints on ordinary large N gauge theories on tori [57]. These issues have been further addressed recently in [58].

While physically one would not expect to be able to measure a distinction between rational-valued and irrational-valued observables, it has been observed that in certain examples and at high energies, generic non-BPS physical quantities exhibit discontinuous effects as functions of the deformation parameter, due to the multifractal nature of the renormalization group flows in these cases [59]. For instance, when θ is an irrational number, the cascade of Morita equivalent descriptions is unbounded as the energy of the system increases and no quasi-local description of the theory is possible beyond a certain energy level. To provide some different insight into these problems, in this section we will analyze the behaviour of quantum Yang-Mills theory on the noncommutative two-torus as a function of θ, using its representation (7.3) as a sum over partitions associated with the Heisenberg module $\mathcal{E}_{p,q}$. As we have seen, each critical point of the Yang-Mills action is determined by a partition which is a list of pairs of integers $(\vec{p}\,,\vec{q}\,) = \Big\{(p_k,q_k)\Big\}_{k=1}^{|\vec{\nu}\,|}$ labelling submodules that obey the constraints (4.24) on their dimensions and Chern numbers. We will now develop a graphical technique for constructing solutions of these constraints which will serve as a useful method for obtaining solutions of the noncommutative Yang-Mills equations of motion. This method makes no distinction between rational or irrational θ and smoothly interpolates between the two cases. We will then use it to prove the *smoothness* of the Yang-Mills partition function (7.3) as a function of the noncommutativity parameter θ. This continuity result is in agreement with an analysis, based on continued fraction approximations, of the behaviour of classical averages on a fixed projective module [60].

8.1. Graphical determination of classical solutions. Consider the integral lattice $\mathrm{K}_0(\mathcal{A}_\theta)$ of K-theory charges, which we will view as a subset of the plane $\mathbb{R}^2$. Each point (p_k,q_k) on this lattice corresponds to an isomorphism class $\mathcal{E}_{p_k,q_k}$ of projective modules over the noncommutative torus. Through each such point we draw a line in $\mathbb{R}^2$ of constant (positive) dimension according to the equation

$$p-q\theta = p_k - q_k\theta\ ,\quad k=1,2,3,\dots\ . \tag{8.1}$$

These lines all have slope θ^{-1}. For irrational values of θ, there is a unique solution, $(p,q)=(p_k,q_k)$, to (8.1) for each k, and hence there is only one point of the integer lattice on each line. Consequently there is an infinite number of parallel lines of constant dimension in any region of the K-theory lattice. On the other hand, if $\theta = n/s$ is a rational number, then there are infinitely many solutions (p,q) of Eq. (8.1) for each k and hence a large degeneracy of lattice points lying on each line. In this case there are only a finite number of lines of constant dimension in any region of the K-theory lattice.

For a given Heisenberg module $\mathcal{E}_{n,m}$, there are two important lines of constant dimension which will enable the enforcing of the constraints (4.24). These are the lines $p-q\theta = 0$ and $p-q\theta = n-m\theta$. A partition which yields a critical point of the Yang-Mills

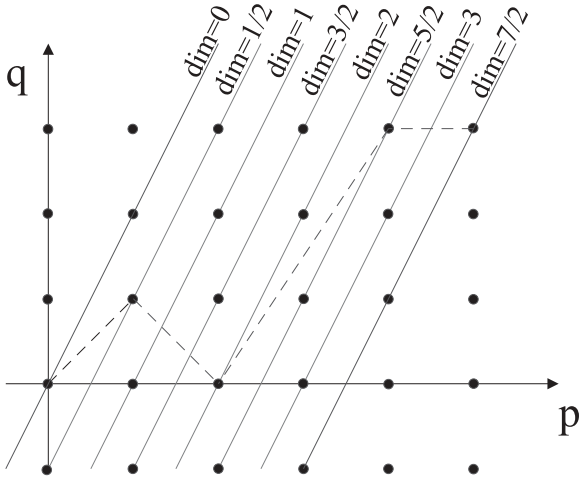

Fig. 1. Graphical representation of the partition $\left\{(1,1)\,,\ (1,-1)\,,\ (2,3)\,,\ (1,0)\right\} \in \mathcal{P}_{5,3}(\theta)$ which defines a solution of the noncommutative Yang-Mills equations of motion on the projective module $\mathcal{E}_{5,3}$. The sequence of lines in $\mathbb{R}^2$ of constant dimension for the case $\theta = 1/2$ is depicted. The dashed line goes through the sequence of points whose successive differences make up the elements of the partition

action on $\mathcal{E}_{n,m}$ is found by taking a sequence of points lying on lines of strictly increasing dimension, beginning at the origin $(0, 0)$ and terminating at the point (n, m). Taking the difference of the coordinates of successive points gives the topological numbers (n_k, m_k) of the submodules in the partition. The choice of a sequence of points which lie on lines of strictly increasing dimension guarantees that each submodule is of positive dimension. Fixing the initial and final points ensures that the constraints on the total dimension and Chern number are satisfied. An illustrative example of this procedure for the module $\mathcal{E}_{5,3}$ is depicted in Fig. 1. All finite sequences of points obeying these rules give all possible solutions of the constraints (4.24), and hence all critical points of the noncommutative Yang-Mills action corresponding to all solutions of the equations of motion. Note that the integer $|\vec{\nu}\,|$, counting the total number of submodules in a partition, may in this way be regarded as a topological invariant of the associated graphs.

8.2. Proof of θ-smoothness. Having determined all partitions graphically for fixed θ, we can now study how semi-classical quantities vary with a change of θ. From $n - m\theta = \dim \mathcal{E}_{n,m}$, we see that θ is the inverse slope of lines of constant dimension in the (p, q) plane. Thus a change in θ amounts to a change in slope of the lines of constant dimension. For partitions a small change in θ leads to a small change in the dimensions of submodules in a partition but leaves the number $|\vec{\nu}\,|$ of submodules and their topological numbers unchanged. The partition function (7.3) clearly varies smoothly under such variations of the noncommutativity parameter. This smooth behaviour terminates when a change in θ leads to a violation of the requirement that each submodule of a partition be of positive dimension. Such a condition can occur when a submodule of very small dimension to the right of the line $p - q\theta = 0$ is pushed through to negative dimension by an infinitesimal variation of θ. For example, the partition depicted in Fig. 1 represents a valid solution of Yang-Mills theory on the module $\mathcal{E}_{5,3}$ for all $\theta < 1$. As θ approaches unity, the dimension of the first submodule $\mathcal{E}_{1,1}$ vanishes. Thus at $\theta = 1$, the constraints

(4.24) defining partitions are violated and this partition is abruptly removed from the list $\mathcal{P}_{5,3}(\theta)$ of partitions which contribute to the partition function. Of course such an elimination occurs for any partition containing a submodule of vanishing dimension[10] and it would appear in general that this leads to a discontinuity in the partition function as a function of θ. There are also various "degenerate" cases that appear to lead to discontinuities, such as those partitions for which the dimension of a component which doesn't appear first in the list vanishes, or those where the dimensions of two submodules become equal when θ is varied. This leads to a reordering of the submodules and therefore a discontinuous change in the graphical representation of the previous subsection. However, these latter cases do not affect the partition sum in a discontinuous way, and thus only the former types of discontinuities appear to remain.

In fact this is not the case and the partition function is smooth in θ. The reason is that the contribution to the partition function (7.3) from partitions with submodules of vanishing dimension are exponentially suppressed, since the Boltzmann weight associated with such topological numbers (n, m) is of order $\mathrm{e}^{-1/\dim \mathcal{E}_{n,m}}$. Consequently, the partition function has already exponentially damped any contribution from a partition before it is discontinuously dropped due to the positive dimension constraint. It is easy to see that even though derivatives of the partition function with respect to θ will generate singular pre-exponential factors when submodule dimensions vanish, these singularities are all trumped by exponential suppression from the action. Thus all derivatives of the partition function with respect to θ are also finite and continuous. Note that, in the context of rational approximations to irrational values of the noncommutativity parameter, this analysis also shows that perturbations about any rational value of θ will miss exponentially small contributions to the partition function, which may be related to some of the peculiarities observed in the rational approximations of irrational noncommutative gauge theories [59].

The θ-smoothness proof can also be extended to physical (gauge invariant) observables which are at most polynomially singular in θ for modules of vanishing dimension. One such class of observables are the "topological" observables obtained by differentiating the partition function (3.27) with respect to the Yang-Mills coupling constant,

$$\begin{aligned}\left(\frac{\partial}{\partial g^2}\right)^n \ln Z_{p,q}(g^2,\theta) &= \left(\frac{1}{8\pi^2 R^2}\right)^n \left\langle \left(\hat{\phi}\,,\,\hat{\phi}\right)^n \right\rangle_{\mathrm{conn}} \\ &= \left(\frac{1}{8\pi^2 R^2}\right)^n \left\langle \prod_{r=1}^{n} \mathrm{tr}_N\, \phi(x_r)^2 \right\rangle_{\mathrm{conn}} , \end{aligned} \tag{8.2}$$

where the brackets $\langle\cdots\rangle_{\mathrm{conn}}$ denote connected correlation functions with respect to the functional integral (3.27), and $x_1, \ldots, x_n$ are arbitrary points on $\mathbf{T}^2$. For $n = 1$ this observable is proportional to the average energy of the system $\left\langle \mathrm{Tr}_{\mathcal{E}_{p,q}}\, \hat{F}_{\hat{A}}^2 \right\rangle$ on the Heisenberg module $\mathcal{E}_{p,q}$.

9. Instanton Moduli Spaces

The expansion (7.3) of the partition function of gauge theory on a noncommutative torus has a natural interpretation as a sum over noncommutative instantons in two dimensions,

[10] Recall that the components of a partition are partially ordered according to increasing submodule dimension.

in the sense that we have defined them at the beginning of Sect. 4. They are classified topologically by the homotopy classes (p, q) of the space of Hermitian projectors Gr_θ, and they are inherently nonperturbative since their action contribution to the path integral is of order e^{-1/g^2}. However, the semi-classical expansion does not organize the contributions from classical solutions into gauge orbits. Different partitions $(\vec{p}\,, \vec{q}\,)$ may give contributions which should be identified as coming from the same instanton. In this section we will discuss the rearrangement of the series (7.3) into a sum over gauge inequivalent critical points and describe the structure of the moduli spaces of instantons that arise, comparing them with those of ordinary Yang-Mills theory in two dimensions.

9.1. Weak coupling limit. We will begin with the weak coupling limit $g^2 \to 0$ of noncommutative gauge theory as it is the simplest case. Due to the invariance property (3.12), the moduli space $\mathcal{M}_{p,q}(\theta)$ of constant curvature connections on the Heisenberg module $\mathcal{E}_{p,q}$ modulo noncommutative gauge transformations has a natural symplectic structure inherited from the symplectic two-form (3.9) on $\mathsf{C}_{p,q}$. As shown in Sect. 3.3, the partition function $Z_{p,q}(g^2 = 0, \theta)$ formally computes the symplectic volume of $\mathcal{M}_{p,q}(\theta)$. Let us first describe this moduli space [37]. By using a Morita duality transformation (6.5) of the background magnetic flux Φ if necessary, we can assume that $f \neq 0$ in (4.6) without loss of generality. We therefore need to classify the irreducible representations determined by the Heisenberg module (4.7). As discussed in Sect. 4.1, the Weyl-'t Hooft algebra (4.8) has N irreducible components, where N is the rank of the noncommutative gauge theory given by (4.13). On the other hand, in Sect. 6.3 we saw that each such irreducible representation has a pair of complex moduli (z_1, z_2) generating the center of the Weyl-'t Hooft algebra. Thus the inequivalent irreducible representations of the matrix algebra (4.8) are labelled by a pair of complex numbers $\zeta = (z_1, z_2) \in \tilde{\mathbf{T}}^2$ which live on a commutative torus dual to the original noncommutative torus.

If $\mathcal{W}_\zeta \subset \mathbb{C}^q$, $\zeta \in \tilde{\mathbf{T}}^2$ are the corresponding irreducible representations, then the Heisenberg module (4.7) decomposes into irreducible $\mathcal{A}_\theta$-modules according to

$$\mathcal{E}_{p,q} = L^2(\mathbb{R}) \otimes \left(\mathcal{W}_{\zeta_1} \oplus \cdots \oplus \mathcal{W}_{\zeta_N}\right) . \tag{9.1}$$

Gauge transformations which preserve the constant curvature condition (4.6) are finite-dimensional unitary matrices in $U(q)$. Central elements of the Weyl-'t Hooft algebra are represented by diagonal matrices with respect to the decomposition (9.1). There is a residual gauge symmetry which acts by permutation of the N summands in (9.1) as the permutation group Σ_N, and therefore the moduli space of constant curvature connections associated with the module $\mathcal{E}_{p,q}$ over the noncommutative torus is the symmetric orbifold [37]

$$\mathcal{M}_{p,q}(\theta) = \mathrm{Sym}^N \tilde{\mathbf{T}}^2 \equiv \left(\tilde{\mathbf{T}}^2\right)^N / \Sigma_N \tag{9.2}$$

of dimension $2N$. This space is identical to the moduli space of flat bundles for commutative Yang-Mills theory on an elliptic Riemann surface with structure group $U(N)$ [35], i.e. $\mathcal{M}_N(\theta = 0) = \mathrm{Hom}\Big(\pi_1(\mathbf{T}^2)\,,\, U(N)\Big) / U(N)$, since the maximal torus of $U(N)$ is $U(1)^N$ consisting of diagonal matrices and its discrete Weyl subgroup is precisely the symmetric group Σ_N. The standard symplectic geometry on (9.2) possesses conical singularities on the coincidence locus, i.e. the "diagonal" subspace of $\left(\tilde{\mathbf{T}}^2\right)^N$.

Let us now consider this result in light of the instanton expansion. We take $\Phi = 0$ without loss of generality. The $g^2 \to 0$ limit of the Boltzmann factor in the partition function (7.3) is non-vanishing only in the zero instanton sector $q_k = 0 \ \forall k \geq 1$. But the constraints (4.24) with all $q_k = 0$ are just equivalent to those we encountered in Sect. 5 in the commutative limit $\theta = 0$, with rank $N = p$. The same is true of the fluctuation determinant factors in (7.3), and hence the partition function at weak coupling is given by

$$Z_{p,q}(g^2 = 0, \theta) \ = \ \lim_{g^2 \to 0} \sum_{\vec{\nu} \, : \, \sum_a a\nu_a = N} \prod_{a=1}^{N} \frac{(-1)^{\nu_a}}{\nu_a !} \left(2g^2 R^2 a^3 \right)^{-\nu_a/2} , \qquad (9.3)$$

where we have eliminated the (constant) exponential factor by a suitable renormalization of the quantum field theory [29]. The independence of (9.3) in the noncommutativity was observed in Sect. 3.3, where we saw that the auxiliary field ϕ in (3.27) was essentially a commutative field. In this way, the theory at $g^2 = 0$ eliminates all dependence on the parameters θ and R, and it is identical to topological Yang-Mills theory on the commutative two-torus [29]. This feature of the weakly coupled gauge theory is in agreement with the coincidence of the moduli space of the zero instanton sector in the commutative and noncommutative cases. It should be stressed though that, in contrast to the commutative case, by Morita equivalence the expressions for the moduli space (9.2) and partition function (9.3) over the noncommutative torus hold generically for *all* (not necessarily flat) constant curvature connections. In other words, in the noncommutative case the gauge quotiented level sets of the moment map μ on $\mathsf{C}_{p,q}$ are all equivalent.

The partition function has non-analytic behaviour in the Yang-Mills coupling constant as $g^2 \to 0$, with a pole of order $|\vec{\nu}\,|$ in g for each partition. We can relate the singularities arising in (9.3) in a very precise way to the orbifold singularities of the moduli space (9.2) which appear whenever the Heisenberg module $\mathcal{E}_{p,q}$ is reducible. The latter singular points come from the fixed point set of the action of the permutation group Σ_N on $\left(\tilde{\mathbf{T}}^2\right)^N$, which is straightforward to describe. As in Sect. 5, let $C[\vec{\nu}\,] = C[\nu_1, \ldots, \nu_N]$ be the conjugacy class of a given element $\sigma \in \Sigma_N$. An elementary cycle of length a leaves an N-tuple $(\zeta_1, \ldots, \zeta_N) \in \left(\tilde{\mathbf{T}}^2\right)^N$ invariant only if the a points on which it acts coincide. It follows that the fixed point locus of any permutation σ in the conjugacy class $C[\vec{\nu}\,]$ is given by

$$\left[\left(\tilde{\mathbf{T}}^2\right)^N \right]^{C[\vec{\nu}\,]} \ = \ \prod_{a=1}^{N} \left(\tilde{\mathbf{T}}^2\right)^{\nu_a} . \qquad (9.4)$$

On each such fixed point set there is still the action of the stabilizer subgroup $\mathcal{C}(\vec{\nu}\,)$ of Σ_N, which consists of all elements $\sigma' \in \Sigma_N$ that commute with σ and is given explicitly in terms of semi-direct products as

$$\mathcal{C}(\vec{\nu}\,) = \prod_{a=1}^{N} \Sigma_{\nu_a} \ltimes \mathbb{Z}_a^{\nu_a} . \qquad (9.5)$$

Here the symmetric group Σ_{ν_a} permutes the ν_a cycles of length a, while each cyclic group $\mathbb{Z}_a$ acts within one particular cycle of length a. Distinct singular points of the symmetric orbifold (9.2) then arise at the $\mathcal{C}(\vec{\nu}\,)$ invariants of the fixed point loci (9.4).

Only the subgroups Σ_{ν_a} of the centralizer (9.5) act non-trivially on (9.4). The singular point locus of the moduli space of constant curvature connections is thereby obtained as the disjoint union over the conjugacy classes $C[\vec{\nu}\,]$ of Σ_N of the strata

$$\left[\left(\tilde{\mathbf{T}}^2\right)^N\right]^{C[\vec{\nu}\,]} / \mathcal{C}(\vec{\nu}\,) \;=\; \prod_{a=1}^{N} \mathrm{Sym}^{\nu_a}\,\tilde{\mathbf{T}}^2\,. \tag{9.6}$$

Given the result (9.6), the interpretation of the singularities in the formal orbifold volume (9.3) is now clear. It is a sum over the connected components, labelled by conjugacy classes in Σ_N, of the total singular locus of the orbifold symmetric product (9.2). Within each conjugacy class (partition) $C[\vec{\nu}\,]$, the contribution from a gauge equivalence class of connections associated with a toroidal factor $\left(\tilde{\mathbf{T}}^2\right)^{\nu_a}$ in (9.6) is weighted by a singular fluctuation determinant $(-1)^{\nu_a}\,(2g^2R^2a^3)^{-\nu_a/2}$. Recall from Sect. 6.2 that the module dimension factor a^3 here ensures invariance under Morita duality. Gauge invariance dictates that the total contribution from the ν_a cycles of length a (submodules of rank a) be divided by the appropriate residual symmetry factor $\nu_a!$ which is the order of the local orbifold group Σ_{ν_a} acting in (9.6). Thus the conical singularities of the zero instanton sector are not smoothed out by the noncommutativity, as one might have naively expected [61, 63], and the moduli spaces of flat connections are the same in both commutative and noncommutative cases. The corresponding partition functions (9.3) represent the contribution of the global minimum $\mu^{-1}(0)$ to the localization formula for the functional integral. We shall now analyze how these properties change as one moves away from the weak coupling limit of the noncommutative gauge theory. As we will see, the orbifold singularities for coincident instantons on the moduli space still persist. Geometrically, the noncommutative instantons of two dimensional gauge theory on a torus remain point-like and hence have no smoothing effect on the conical singularities that occur on $\mathrm{Sym}^N\,\tilde{\mathbf{T}}^2$ where two or more points come together.

9.2. Instanton partitions. To count instantons labelled by a generic partition $(\vec{p}\,,\vec{q}\,)$ consisting of non-zero Chern numbers q_k, we need to arrange the expansion (7.3) into a sum over gauge inequivalent classical solutions. The essential problem which arises is the isomorphism $\mathcal{E}_{mp,mq} \cong \oplus^m \mathcal{E}_{p,q}$ of projective modules. Partitions of either side of this isomorphism lead to gauge equivalent contributions to the partition function and, in particular, from (4.16) it follows that the minimizing connections on $\mathcal{E}_{mp,mq}$ and $\mathcal{E}_{p,q}$ have the same constant curvature. Thus we need to refine the definition of partition given in Sect. 4.2 somewhat so as to combine submodules which yield the same constant curvature and hence prevent the over-counting of distinct noncommutative Yang-Mills stationary points [36]. This we do by writing any submodule dimension in the form

$$p_k - q_k\theta = N_k\left(p'_k - q'_k\theta\right), \tag{9.7}$$

where $N_k = \gcd(p_k, q_k)$, and the integers p'_k and q'_k are relatively prime. The corresponding curvature (4.16) is independent of the noncommutative rank N_k, and so we should also restrict to submodules for which each K-theory charge (p'_k, q'_k) is distinct. Therefore, we restrict the counting of critical points of the noncommutative Yang-Mills action to the sets of integers $(\vec{N}, \vec{p}\,', \vec{q}\,') \equiv \left\{(N_a, p'_a, q'_a)\right\}_{a\geq 1}$ which satisfy, in addition to the constraints (4.24), the requirements that $N_a > 0$, p'_a and q'_a are relatively prime,

and the pairs of integers (p'_a, q'_a) are all distinct. We shall refer to such a collection of integers as an "instanton partition". The additional constraints imposed on an instanton partition guarantee that we do not count as distinct those partitions which contain some submodules that can themselves be decomposed into irreducible components.

Let us look at the structure of the moduli space $\mathcal{M}_{p,q}(\vec{N}, \vec{p}\,', \vec{q}\,'; \theta)$ associated with an instanton partition $(\vec{N}, \vec{p}\,', \vec{q}\,')$ [36]. We want to determine the space of gauge orbits of the associated critical point connections $\hat{\nabla}^{\rm cl} = \bigoplus_{a\geq 1} \hat{\nabla}^{\rm c}_{(a)}$ on submodule decompositions

$$\mathcal{E}_{p,q} = \bigoplus_{a\geq 1} \mathcal{E}_{N_a p'_a, N_a q'_a} \ . \tag{9.8}$$

Since each constant curvature on $\mathcal{E}_{N_a p'_a, N_a q'_a}$ is distinct and any gauge transformation $\hat{U} \in \mathsf{G}(\mathcal{E}_{p,q})$ preserves the constant curvature conditions, every $\hat{U}$ is also a unitary operator on each instanton submodule $\mathcal{E}_{N_a p'_a, N_a q'_a} \to \mathcal{E}_{N_a p'_a, N_a q'_a}$. It follows that the instanton moduli space is given by

$$\mathcal{M}_{p,q}(\vec{N}, \vec{p}\,', \vec{q}\,'; \theta) = \prod_{a\geq 1} \mathcal{M}_{N_a p'_a, N_a q'_a}(\theta) \ , \tag{9.9}$$

where each $\mathcal{M}_{N_a p'_a, N_a q'_a}(\theta)$ is the moduli space of constant curvature connections on the Heisenberg module $\mathcal{E}_{N_a p'_a, N_a q'_a}$. From (9.2) we thus find that (9.9) can be written in terms of a product of symmetric orbifolds as [36]

$$\mathcal{M}_{p,q}(\vec{N}, \vec{p}\,', \vec{q}\,'; \theta) = \prod_{a\geq 1} {\rm Sym}^{N_a} \tilde{\mathbf{T}}^2 \ . \tag{9.10}$$

This result generalizes the instanton moduli space (9.2) which corresponds to the global minimum of the noncommutative Yang-Mills action on $\mathcal{E}_{p,q}$.

9.3. Examples. To get a feel for how the moduli spaces (9.10) classify the reorganization of the partition function (7.3) into a sum over distinct instanton contributions, let us consider two very simple examples [33]. For $\theta = 0$ and $p = 2$ the partition function is easily written in the form

$$Z_{2,q}(g^2, 0) = -\frac{{\rm e}^{\,-q^2/4g^2R^2}}{\sqrt{16g^2R^2}} + \frac{1}{4g^2R^2} \sum_{q_1=-\infty}^{\infty} {\rm e}^{\,-\frac{1}{2g^2R^2}\left(q_1^2+(q-q_1)^2\right)} \ . \tag{9.11}$$

When the Chern number q is odd, this is a sum over inequivalent instanton configurations, and the two terms in (9.11) are associated respectively with the smooth moduli spaces

$$\mathcal{M}_{2,q}(1, 2, q; 0) = \tilde{\mathbf{T}}^2 \ , \tag{9.12}$$

$$\mathcal{M}_{2,q}\Big((1, 1, q_1)\,,\ (1, 1, q_2)\,;\ 0\Big) = \tilde{\mathbf{T}}^2 \times \tilde{\mathbf{T}}^2 \ . \tag{9.13}$$

Heuristically, with the appropriate symmetry factors, each factor $\tilde{\mathbf{T}}^2$, representing the single instanton moduli space, contributes a mode with fluctuation determinant

$-1/\sqrt{16g^2R^2}$. On the other hand, when $q = 2q'$ is even there is a term in the infinite series in (9.11) which yields the same Boltzmann weight as the first term, and so these two terms should be combined to give

$$Z_{2,2q'}(g^2,0) = \mathrm{e}^{-q'^2/g^2R^2}\left(-\frac{1}{\sqrt{16g^2R^2}}+\frac{1}{4g^2R^2}\right) + \frac{1}{4g^2R^2}\sum_{q_1\neq q'}\mathrm{e}^{-\frac{1}{2g^2R^2}\left(q_1^2+(2q'-q_1)^2\right)} . \tag{9.14}$$

Again the last term in (9.14) may be attributed to contributions from instantons in the smooth moduli space (9.13) with $q = 2q'$ and $q_1 \neq q_2$. The two gauge equivalent instanton contributions to the first term are attributed with the singular moduli space in this case,

$$\mathcal{M}_{2,2q'}(2,1,q';0) = \mathrm{Sym}^2\,\tilde{\mathbf{T}}^2 . \tag{9.15}$$

The singular locus of the symmetric orbifold (9.15) is $\mathrm{Sym}^2\,\tilde{\mathbf{T}}^2 \amalg \tilde{\mathbf{T}}^2$ with the disjoint sets corresponding to the identity and order two elements of the cyclic group $\mathbb{Z}_2$, respectively. As in Sect. 9.1, the sum of contributions to the first term in (9.14) are readily seen to be those associated with the components of the total singular point locus of (9.15).

For $\theta = 0$ and $p = 3$ the partition function is given by

$$Z_{3,q}(g^2,0) = -\frac{\mathrm{e}^{-q^2/6g^2R^2}}{\sqrt{54g^2R^2}} + \frac{1}{32g^2R^2}\sum_{q_1=-\infty}^{\infty}\mathrm{e}^{-\frac{1}{4g^2R^2}\left(2q_1^2+(q-q_1)^2\right)} - \frac{1}{6(2g^2R^2)^{3/2}}\sum_{q_1=-\infty}^{\infty}\sum_{q_2=-\infty}^{\infty}\mathrm{e}^{-\frac{1}{2g^2R^2}\left(q_1^2+q_2^2+(q-q_1-q_2)^2\right)} . \tag{9.16}$$

For any $q \notin 3\,\mathbb{Z}$ the expression (9.16) can be written as a sum over distinct instanton contributions as

$$\begin{aligned} Z_{3,q}(g^2,0) &= -\frac{\mathrm{e}^{-q^2/6g^2R^2}}{\sqrt{54g^2R^2}} + \left(\frac{1}{32g^2R^2}-\frac{1}{6(2g^2R^2)^{3/2}}\right)\sum_{q_1=-\infty}^{\infty}\mathrm{e}^{-\frac{1}{4g^2R^2}\left(2q_1^2+(q-q_1)^2\right)} \\ &\quad - \frac{1}{6(2g^2R^2)^{3/2}}\sum_{q_1=-\infty}^{\infty}\sum_{q_2\neq q_1}\mathrm{e}^{-\frac{1}{2g^2R^2}\left((2q_1-q)^2+(2q_1-q_2)^2+q_2^2\right)} \\ &\quad - \frac{1}{6(2g^2R^2)^{3/2}}\sum_{q_1\neq q \bmod 2}\sum_{q_2=-\infty}^{\infty}\mathrm{e}^{-\frac{1}{2g^2R^2}\left(q_1^2+q_2^2+(q-q_1-q_2)^2\right)} \end{aligned} \tag{9.17}$$

corresponding respectively to the instanton moduli spaces

$$\mathcal{M}_{3,q}(1,3,q;0) = \tilde{\mathbf{T}}^2 , \tag{9.18}$$

$$\mathcal{M}_{3,q}\Big((1,1,q_1)\,,\,(2,1,q_2)\,;\,0\Big) = \tilde{\mathbf{T}}^2 \times \mathrm{Sym}^2\,\tilde{\mathbf{T}}^2 , \tag{9.19}$$

$$\mathcal{M}_{3,q}\Big((1,1,q_1)\,,\ (1,2,q_2)\,;\ 0\Big) = \tilde{\mathbf{T}}^2 \times \tilde{\mathbf{T}}^2\,, \tag{9.20}$$

$$\mathcal{M}_{3,q}\Big((1,1,q_1)\,,\ (1,1,q_2)\,,\ (1,1,q_3)\,;\ 0\Big) = \tilde{\mathbf{T}}^2 \times \tilde{\mathbf{T}}^2 \times \tilde{\mathbf{T}}^2\,. \tag{9.21}$$

Note again how the fluctuation determinants in (9.17) weight each factor of $\tilde{\mathbf{T}}^2$ in the corresponding moduli space, and how the second term incorporates the sum over singularities of the symmetric orbifold $\mathrm{Sym}^2\,\tilde{\mathbf{T}}^2$ in (9.19). For $q = 3q'$, the second term in (9.17) yields a contribution to the global minimum for $q_1 = q'$, and we have

$$\begin{aligned} Z_{3,3q'}(g^2,0) = {}& \mathrm{e}^{-3q'^2/2g^2R^2}\left(-\frac{1}{\sqrt{54g^2R^2}} + \frac{1}{32g^2R^2} - \frac{1}{6(2g^2R^2)^{3/2}}\right) \\ & + \left(\frac{1}{32g^2R^2} - \frac{1}{6(2g^2R^2)^{3/2}}\right) \sum_{q_1\neq q'} \mathrm{e}^{-\frac{1}{4g^2R^2}\left(2q_1^2+(3q'-q_1)^2\right)} \\ & - \frac{1}{6(2g^2R^2)^{3/2}} \sum_{q_1=-\infty}^{\infty} \sum_{q_2\neq q_1} \mathrm{e}^{-\frac{1}{2g^2R^2}\left((2q_1-3q')^2+(2q_1-q_2)^2+q_2^2\right)} \\ & - \frac{1}{6(2g^2R^2)^{3/2}} \sum_{q_1\neq 3q' \bmod 2} \sum_{q_2=-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2g^2R^2}\left(q_1^2+q_2^2+(3q'-q_1-q_2)^2\right)}\,. \end{aligned} \tag{9.22}$$

The last three terms in (9.22) may again be attributed to contributions associated with the instanton moduli spaces (9.19)–(9.21), respectively, with $q = 3q'$ and $q_1 \neq q_2 \neq q_3$. The first term represents the gauge equivalent instanton contributions coming from replacing the smooth moduli space (9.18) by the singular one

$$\mathcal{M}_{3,3q'}(3,1,q';0) = \mathrm{Sym}^3\,\tilde{\mathbf{T}}^2\,, \tag{9.23}$$

with each fluctuation determinant associated with the singular points of the orbifold (9.23) corresponding to the three conjugacy classes of the symmetric group Σ_3.

These two simple examples illustrate the general technique involved in reorganizing the sum (7.3) over critical points into distinct instanton contributions. They can be deduced, as above, from the singularity structures of the totality of instanton moduli spaces (9.10) corresponding to a Heisenberg module. The Boltzmann weight associated with an instanton partition $(\vec{N},\vec{p}\,',\vec{q}\,')$ is given by

$$\mathrm{e}^{-S(\vec{N},\vec{p}\,',\vec{q}\,';\theta)} = \prod_{a\geq 1} \mathrm{e}^{-\frac{1}{2g^2R^2}\,N_a q_a'^2/(p_a'-q_a'\theta)}\,, \tag{9.24}$$

and about it there is a finite number of quantum fluctuations representing a finite, but nontrivial, perturbative expansion in g^{-1}. These fluctuations are determined by the singular locus of the corresponding symmetric orbifolds in (9.10). The combinatorial problem of summing over all such instanton partitions is in general quite involved, especially for irrational values of θ when there are infinitely many partitions. However, repeating analogous arguments to those around (7.1,7.2) shows that an instanton partition contains only finitely many components. Thus the perturbative expansion around each instanton contribution contains only finitely many terms, although in the irrational case the exponential prefactor is no longer a polynomial of set order. It is amusing that, within

the class of noncommutative gauge theories, Morita equivalence allows such a moduli space classification of the instanton contributions even in the commutative case. Such a characterization is otherwise not possible because one only knows the structure of the moduli space of *flat* connections of commutative gauge theory on $\mathbf{T}^2$. Notice also that for $\theta \neq 0$ the instanton sums are no longer given by elementary theta-functions.

Finally, let us note that the instantons which contribute to the semi-classical expansion of noncommutative gauge theory that we have developed are reminiscent of the solitons on noncommutative tori which arise as solutions of open string field theory describing unstable D-branes wrapping a two-dimensional torus in the background of a constant B-field [64, 65]. An extremum of the tachyon potential is described by a projector of the algebra $\mathcal{A}_\theta$, and leads to an effective gauge theory on the corresponding projective module determined by the tachyon. The remaining string field equations of motion are then solved by direct sum decompositions of the given Heisenberg module as we have described them in this paper. A special instance of this are the fluxon solutions which describe the finite energy instantons, carrying quantized magnetic flux, of gauge theory on the noncommutative plane [4–7]. In the present setting these are the classical solutions associated with partitions consisting of only the full module, giving the global minimum of the Yang-Mills action. For the module $\mathcal{E}_{p,q}$, these solutions have gauge field strength (4.16) and partition function

$$Z^{\rm fluxon}_{N,q}(g^2,\tilde\theta) = \frac{{\rm e}^{\,-Nq'^2/2g^2R^2(p'-q'\theta)}}{\sqrt{2g^2R^2N^3(p'-q'\theta)^3}}\ . \tag{9.25}$$

In the large area limit $R \to \infty$ with the dimensionful noncommutativity parameter $\tilde\theta = 2\pi R^2\,\theta$ finite, this is the contribution to the functional integral, along with the appropriate Gaussian fluctuation factor, from a fluxon of magnetic charge $q = Nq'$ in gauge group rank N. The sum over all $q \in \mathbb{Z}$ in this limit determines the expansion of noncommutative gauge theory on $\mathbb{R}^2$ in terms of fluxons [10].

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