

# Gerbes over Orbifolds and Twisted K-Theory

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**Abstract:** In this paper we construct an explicit geometric model for the group of gerbes over an orbifold  $X$ . We show how from its curvature we can obtain its characteristic class in  $H^3(X)$  via Chern-Weil theory. For an arbitrary gerbe  $\mathcal{L}$ , a twisting  ${}^{\mathcal{L}}K_{\text{orb}}(X)$  of the orbifold  $K$ -theory of  $X$  is constructed, and shown to generalize previous twisting by Rosenberg [28], Witten [35], Atiyah-Segal [2] and Bowknegt et. al. [4] in the smooth case and by Adem-Ruan [1] for discrete torsion on an orbifold.

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## 1. Introduction

An orbifold is a very natural generalization of a manifold. Locally it looks like the quotient of an open set of a vector space divided by the action of a group, in such a way that the stabilizer of the action at every point is a finite group. Many moduli spaces, for example, appear with canonical orbifold structures.

Recently Chen and Ruan [8] motivated by their ideas in quantum cohomology and by orbifold string theory models discovered a remarkable cohomology theory of orbifolds that they have coined *orbifold cohomology*. Adem and Ruan [1] went on to define the corresponding *orbifold K-theory* and to study the resulting Chern isomorphism. One of the remarkable properties of the theory is that both theories can be twisted by what Ruan has called an *inner local system* coming from a third group-cohomology class called *discrete torsion*.

Independently of that, Witten [35], while studying  $K$ -theory as the natural recipient of the charge of a D-brane in type IIA superstring theories was motivated to define a twisting of  $K(M)$  for  $M$  smooth by a third cohomology class in  $H^3(M)$  coming from a codimension 3-cycle in  $M$  and Poincaré duality. This twisting appeared previously in the literature in different forms [11, 14, 28].

In this paper we show that if an orbifold is interpreted as a *stack* then we can define a twisting of the natural  $K$ -theory of the stack that generalizes both Witten's and Adem-Ruan's twistings. We also show how we can interpret the theory of *bundle gerbes* over a smooth manifold and their  $K$ -theory [25, 26, 4] in terms of the theory developed here.

Since the approach to the theory of stacks that we will follow is not yet published [3], we try very hard to work in very concrete terms and so our study includes a very simple definition of a gerbe over a stack motivated by that of Chatterjee and Hitchin [15] on a smooth manifold. This definition is easy to understand from the point of view of differential geometry, and of algebraic geometry.

Using results of Segal [31, 33] on the topology of classifying spaces of categories and of Crainic, Moerdijk and Pronk on sheaf cohomology over orbifolds [10, 23, 24, 20] we show that the usual theory for the characteristic class of a gerbe over a smooth manifold [5] extends to the orbifold case. Then we explain how Witten's arguments relating the charge of a  $D$ -brane generalize.

A lot of what we will show is valid for foliation groupoids and also for a category of *Artin stacks* - roughly speaking spaces that are like orbifolds except that we allow the stabilizers of the local actions to be Lie groups. In particular we will explain how the

twisting proposed here can be used to realize the Freed-Hopkins-Teleman twisting used in their topological interpretation of the Verlinde algebra [12].

## 2. A Review of Orbifolds

In this section we will review the classical construction of the category of orbifolds. This category of orbifolds is essentially that introduced by Satake [30] under the name of *V-manifolds*, but with a fundamental difference introduced by Chen and Ruan. They have restricted the morphisms of the category from orbifold maps to *good maps*, in fact Moerdijk and Pronk have found this category previously [23] where good maps go by the name of *strict maps*. This is the correct class of morphisms from the point of view of stack theory as we will see later.

*2.1. Orbifolds, good maps and orbibundles.* Following Ruan [29, 8] we will use the following definition for an *orbifold*.

**Definition 2.1.1.** *An  $n$ -dimensional uniformizing system for a connected topological space  $U$  is a triple  $(V, G, \pi)$  where*

- $V$  is a connected  $n$ -dimensional smooth manifold
- $G$  is a finite group acting on  $V$  smoothly ( $C^\infty$  automorphisms)
- $\pi : V \rightarrow U$  is a continuous map inducing a homeomorphism  $\tilde{\pi} : V/G \rightarrow U$ .

Two uniformizing systems,  $(V_1, G_1, \pi_1)$  and  $(V_2, G_2, \pi_2)$  are isomorphic if there exists a pair of functions  $(\phi, \lambda)$  such that:

- $\phi : V_1 \rightarrow V_2$  a diffeomorphism,
- $\lambda : G_1 \rightarrow G_2$  an isomorphism

with  $\phi$  being  $\lambda$ -equivariant and  $\pi_2 \circ \phi = \pi_1$ .

Let  $i : U' \hookrightarrow U$  be a connected open subset of  $U$  and  $(V', G', \pi')$  a uniformizing system of  $U'$ .

**Definition 2.1.2.**  $(V', G', \pi')$  is induced from  $(V, G, \pi)$  if there exist:

- a monomorphism  $\lambda : G' \rightarrow G$  inducing an isomorphism  $\lambda : \ker G' \xrightarrow{\cong} \ker G$ , where  $\ker G'$  and  $\ker G$  are the subgroups of  $G'$  and  $G$  respectively that act trivially on  $V'$  and  $V$ , and
- a  $\lambda$ -equivariant open embedding  $\phi : V' \rightarrow V$

with  $i \circ \pi' = \pi \circ \phi$ . We call  $(\phi, \lambda) : (V', G', \pi') \rightarrow (V, G, \pi)$  an injection.

Two injections  $(\phi_i, \lambda_i) : (V'_i, G'_i, \pi'_i) \rightarrow (V, G, \pi)$ ,  $i = 1, 2$ , are isomorphic if there exist:

- an isomorphism  $(\psi, \tau) : (V'_1, G'_1, \pi'_1) \rightarrow (V'_2, G'_2, \pi'_2)$  and
- an automorphism  $(\tilde{\psi}, \tilde{\tau}) : (V, G, \pi) \rightarrow (V, G, \pi)$

such that  $(\tilde{\psi}, \tilde{\tau}) \circ (\phi_1, \lambda_1) = (\phi_2, \lambda_2) \circ (\psi, \tau)$

*Remark 2.1.3.* Since for a given uniformizing system  $(V, G, \pi)$  of  $U$ , and any connected open set  $U'$  of  $U$ ,  $(V, G, \pi)$  induces a unique isomorphism class of uniformizing systems of  $U'$  we can define the *germ* of a uniformizing system localized at a point.

Let  $U$  be a connected and locally connected topological space,  $p \in U$  a point, and  $(V_1, G_1, \pi_1)$  and  $(V_2, G_2, \pi_2)$  uniformizing systems of the neighborhoods  $U_1$  and  $U_2$  of  $p$  respectively, then

**Definition 2.1.4.**  $(V_1, G_1, \pi_1)$  and  $(V_2, G_2, \pi_2)$  are equivalent at  $p$  if they induce uniformizing systems for a neighborhood  $U_3 \subset U_1 \cap U_2$  of  $p$

The germ of  $(V, G, \pi)$  at  $p$  is defined as the set of uniformizing systems of neighborhoods of  $p$  which are equivalent at  $p$  with  $(V, G, \pi)$ .

**Definition 2.1.5.** Let  $X$  be a Hausdorff, second countable topological space. An  $n$ -dimensional orbifold structure on  $X$  is a set  $\{(V_p, G_p, \pi_p) | p \in X\}$  such that

- $(V_p, G_p, \pi_p)$  is a uniformizing system of  $U_p$ , neighborhood of  $p$  in  $X$ ,
- for any point  $q \in U_p$ ,  $(V_p, G_p, \pi_p)$  and  $(V_q, G_q, \pi_q)$  are equivalent at  $q$ .

We say that two orbifold structures on  $X$ ,  $\{(V_p, G_p, \pi_p)\}_{p \in X}$  and  $\{(V'_p, G'_p, \pi'_p)\}_{p \in X}$ , are equivalent if for any  $q \in X$   $(V_q, G_q, \pi_q)$  and  $(V'_q, G'_q, \pi'_q)$  are equivalent at  $q$ .

**Definition 2.1.6.** With a given orbifold structure,  $X$  is called an orbifold.

Sometimes we will simply denote by  $X$  the pair  $(X, \{(V_p, G_p, \pi_p)\}_{p \in X})$ . When we want to make the distinction between the underlying topological space  $X$  and the orbifold  $(X, \{(V_p, G_p, \pi_p)\}_{p \in X})$  we will write  $\mathbf{X}$  for the latter.

For any  $p \in X$  let  $(V, G, \pi)$  be a uniformizing of a neighborhood around  $p$  and  $\bar{p} \in \pi^{-1}(x)$ . Let  $G_p$  be the stabilizer of  $G$  at  $p$ . Up to conjugation the group  $G_p$  is independent of the choice of  $\bar{p}$  and is called the isotropy group or local group at  $p$ .

**Definition 2.1.7.** An orbifold  $X$  is called reduced if the isotropy groups  $G_p$  act effectively for all  $p \in X$ .

In particular this implies that an orbifold is reduced if and only if the groups  $\ker G$  of Definition 2.1.2 are all trivial.

*Example 2.1.8.* Let  $X = Y/G$  be the orbifold which is the global quotient of the finite group  $G$  acting on a connected space  $Y$  via automorphisms. Then  $\{(X, G, \pi)\}$  is trivially an orbifold structure for  $X$ . We can also define another equivalent orbifold structure in the following way: for  $p \in X$ , let  $U_p \subset X$  be a sufficiently small neighborhood of  $p$  such that

$$\pi^{-1}(U_p) = \bigsqcup_{\alpha} V_p^{\alpha},$$

the disjoint union of neighborhoods  $V_p^{\alpha}$ , where  $G$  acts as permutations on the connected components of  $\pi^{-1}(U_p)$ .

Let  $V_p$  be one of these connected components, and let  $G_p$  be the subgroup of  $G$  which fixes this component  $V_p$  (we could have taken  $U_p$  so that  $G_p$  is the isotropy group of the point  $y \in \pi^{-1}(p) \cap V_p$ ) and take  $\pi_p = \pi|_{V_p}$ , then  $V_p/G_p \xrightarrow{\cong} U_p$  and  $(V_p, G_p, \pi_p)$  is a uniformizing system for  $U_p$ . This is a direct application of the previous remark.

Now we can define the notion of an orbifold vector bundle or orbibundle of rank  $k$ . Given a uniformized topological space  $U$  and a topological space  $E$  with a surjective continuous map  $pr : E \rightarrow U$ , a uniformizing system of a rank  $k$  vector bundle  $E$  over  $U$  is given by the following information:

- A uniformizing system  $(V, G, \pi)$  of  $U$ ,
- A uniformizing system  $(V \times \mathbf{R}^k, G, \tilde{\pi})$  for  $E$  such that the action of  $G$  on  $V \times \mathbf{R}^k$  is an extension of the action of  $G$  on  $V$  given by  $g(x, v) = (gx, \rho(x, g)v)$ , where  $\rho : V \times G \rightarrow \text{Aut}(\mathbf{R}^k)$  is a smooth map which satisfies

$$\rho(gx, v) \circ \rho(x, g) = \rho(x, h \circ g), \quad g, h \in G, x \in V,$$

- The natural projection map  $\tilde{pr} : V \times \mathbf{R}^k \rightarrow V$  satisfies  $\pi \circ \tilde{pr} = pr \circ \tilde{\pi}$ .

In the same way the orbifolds were defined, once we have the uniformizing systems of rank  $k$  we can define the germ of orbundle structures.

**Definition 2.1.9.** *The topological space  $E$  provided with a given germ of vector bundle structures over the orbifold structure of  $X$ , is an orbundle over  $X$ .*

Let's consider now orbifolds  $X$  and  $X'$  and a continuous map  $f : X \rightarrow X'$ . A *lifting* of  $f$  is the following: for any point  $p \in X$  there are charts  $(V_p, G_p, \pi_p)$  at  $p$  and  $(V_{f(p)}, G_{f(p)}, \pi_{f(p)})$  at  $f(p)$ , and a lifting  $\tilde{f}_p$  of  $f_{\pi_p(V_p)} : \pi_p(V_p) \rightarrow \pi_{f(p)}(V_{f(p)})$  such that for any  $q \in \pi_p(V_p)$ ,  $\tilde{f}_q$  and  $\tilde{f}_p$  define the same germ of liftings of  $f$  at  $q$ .

**Definition 2.1.10.** *A  $C^\infty$  map between orbifolds  $X$  and  $X'$  (orbifold-map) is a germ of  $C^\infty$  liftings of a continuous map between  $X$  and  $X'$ .*

We would like to be able to pull-back bundles using maps between orbifolds, but it turns out that with general orbifold-maps they cannot be defined. We need to restrict ourselves to a more specific kind of maps between orbifolds; they were named *good maps* by Chen and Ruan (see [8]). These good maps will precisely match the definition of a morphism in the category of groupoids (see Proposition 5.1.7).

Let  $\tilde{f} : X \rightarrow X'$  be a  $C^\infty$  orbifold-map whose underlying continuous function is  $f$ . Suppose there is a compatible cover  $\mathcal{U}$  of  $X$  and a collection of open subsets  $\mathcal{U}'$  of  $X'$  defining the same germs, such that there is a 1-1 correspondence between elements of  $\mathcal{U}$  and  $\mathcal{U}'$ , say  $U \leftrightarrow U'$ , with  $f(U) \subset U'$  and  $U_1 \subset U_2$  implies  $U'_1 \subset U'_2$ . Moreover, there is a collection of local  $C^\infty$  liftings of  $f$  where  $\tilde{f}_{UU'} : (V, G, \pi) \rightarrow (V', G', \pi')$  satisfies that for each injection  $(i, \phi) : (V_1, G_1, \pi_1) \rightarrow (V_2, G_2, \pi_2)$  there is another injection associated to it  $(v(i), v(\phi)) : (V'_1, G'_1, \pi'_1) \rightarrow (V'_2, G'_2, \pi'_2)$  with  $\tilde{f}_{U_1U'_1} \circ i = v(i) \circ \tilde{f}_{U_2U'_2}$ ; and for any composition of injections  $j \circ i, v(j \circ i) = v(j) \circ v(i)$  should hold.

The collection of maps  $\{\tilde{f}_{UU'}\}$  defines a  $C^\infty$  lifting of  $f$ . If it is in the same germ as  $\tilde{f}$  it is called a *compatible system* of  $\tilde{f}$ .

**Definition 2.1.11.** *A  $C^\infty$  map is called good if it admits a compatible system.*

**Lemma 2.1.12** [29, Lemma 2.3.2]. *Let  $pr : E \rightarrow X$  be an orbifold vector bundle over  $X'$ . For any compatible system  $\xi = \{\tilde{f}_{UU'}, v\}$  of a good  $C^\infty$  map  $f : X \rightarrow X'$ , there is a canonically constructed pull-back bundle of  $E$  via  $\tilde{f}$  (a bundle  $pr : E_\xi \rightarrow X$  together with a  $C^\infty$  map  $\tilde{f}_\xi : E_\xi \rightarrow E$  covering  $\tilde{f}$ .)*

**2.2. Orbifold cohomology.** Motivated by index theory and by string theory Chen and Ruan have defined a remarkable cohomology theory for orbifolds. One must point out that while as a group it had appeared before in the literature in several forms, its product is completely new and has very beautiful properties.

For  $X$  an orbifold, and  $p$  a point in  $U_p \subset X$  with  $(V_p, G_p, \pi_p)$ ,  $\pi(V_p) = U_p$  a local chart around it, the multi-sector  $\widetilde{\Sigma}_k X$  is defined as the set of pairs  $(p, (\mathbf{g}))$ , where  $(\mathbf{g})$  stands for the conjugacy class of  $\mathbf{g} = (g_1, \dots, g_k)$  in  $G_p$ .

For the point  $(p, (\mathbf{g})) \in \widetilde{\Sigma}_k X$  the multisector can be seen locally as

$$V_p^{\mathbf{g}}/C(\mathbf{g}),$$

where  $V_p^{\mathbf{g}} = V_p^{g_1} \cap V_p^{g_2} \cap \dots \cap V_p^{g_k}$  and  $C(\mathbf{g}) = C(g_1) \cap C(g_2) \cap \dots \cap C(g_k)$ .  $V_p^g$  stands for the fixed-point set of  $g \in G_p$  in  $V_p$ , and  $C(g)$  for the centralizer of  $g$  in  $G_p$ .

Its connected components are described in the following way. For  $q \in U_p$ , up to conjugation, there is an injective homomorphism  $G_q \rightarrow G_p$ , so for  $\mathbf{g}$  in  $G_q$  the conjugacy class  $(\mathbf{g})_{G_p}$  is well defined. In this way we can define an equivalence relation  $(\mathbf{g})_{G_q} \cong (\mathbf{g})_{G_p}$  and we call  $T_k$  the set of such equivalence classes. We will abuse notation and will write  $(\mathbf{g})$  to denote the equivalence class to which  $(\mathbf{g})_{G_p}$  belongs. Let  $T_k^0 \subset T_k$  be the set of equivalence classes  $(\mathbf{g})$  such that  $g_1 g_2 \dots g_k = 1$ .

$\widetilde{\Sigma}_k X$  is decomposed as  $\bigsqcup_{(\mathbf{g}) \in T_k} X_{(\mathbf{g})}$ , where

$$X_{(\mathbf{g})} = \{(p, (\mathbf{g}')_{G_p}) \mid \mathbf{g}' \in G_p \text{ \& } (\mathbf{g}')_{G_p} \in (\mathbf{g})\}.$$

$X_{(\mathbf{g})}$  for  $\mathbf{g} \neq 1$  is called a twisted sector and  $X_{(1)}$  the non-twisted one.

*Example 2.2.1.* Let's consider the global quotient  $X = Y/G$ ,  $G$  a finite group. Then  $X_{(\mathbf{g})} \cong Y^g/C(g)$  where  $Y^g$  is the fixed-point set of  $g \in G$  and  $C(g)$  is its centralizer, hence

$$\widetilde{\Sigma} X \cong \bigsqcup_{\{(g) \mid g \in G\}} Y^g/C(g)$$

An important concept in the theory is that of an *inner local system* as defined by Y. Ruan [29]. We will show below that the gerbes are models of these systems.

**Definition 2.2.2.** *Let  $X$  be an orbifold. An inner local system  $\mathcal{L} = \{L_{(\mathbf{g})}\}_{(\mathbf{g}) \in T_1}$  is an assignment of a flat complex line orbibundle  $L_{(\mathbf{g})} \rightarrow X_{(\mathbf{g})}$  to each twisted sector  $X_{(\mathbf{g})}$  satisfying the compatibility conditions:*

- (1)  $L_{(1)} = 1$  is trivial.
- (2)  $I^* L_{(g^{-1})} = L_{(g)}$
- (3) Over each  $X_{(\mathbf{g})}$  with  $(\mathbf{g}) \in T_3^0$  ( $g_1 g_2 g_3 = 1$ ),

$$e_1^* L_{(g_1)} \otimes e_2^* L_{(g_2)} \otimes e_3^* L_{(g_3)} = 1.$$

One way to introduce inner local systems is by discrete torsion. Let  $Y$  be the universal orbifold cover of the orbifold  $Z$ , and let  $\pi_1^{orb}(Z)$  be the group of deck transformations (see [34]).

For  $X = Z/G$ ,  $Y$  is an orbifold universal cover of  $X$  and we have the following short exact sequence:

$$1 \longrightarrow \pi_1(Z) \longrightarrow \pi_1^{orb}(X) \longrightarrow G \longrightarrow 1.$$

We call an element in  $H^2(\pi_1^{orb}(X), U(1))$  a discrete torsion of  $X$ . Using the previous short exact sequence  $H^2(G, U(1)) \rightarrow H^2(\pi_1^{orb}(X), U(1))$ , therefore elements  $\alpha \in H^2(G, U(1))$  induce discrete torsions.

We can see  $\alpha : G \times G \rightarrow U(1)$  as a two-cocycle satisfying  $\alpha_{1,g} = \alpha_{g,1} = 1$  and  $\alpha_{g,hk}\alpha_{h,k} = \alpha_{g,h}\alpha_{gh,k}$  for any  $g, h, k \in G$ . We can define its phase as  $\gamma(\alpha)_{g,h} := \alpha_{g,h}\alpha_{h,g}^{-1}$  which induces a representation of  $C(g)$ ,

$$L_g^\alpha := \gamma(\alpha)_{g,-} : C(g) \rightarrow U(1).$$

*Example 2.2.3.* In the case that  $Y \rightarrow X$  is the orbifold universal cover and  $G$  is the orbifold fundamental group such that  $X = Y/G$ , we can construct a complex line bundle  $L_g = Y^g \times_{L_g^\alpha} \mathbf{C}$  over  $X_{(g)}$ . We get that  $L_{tgt^{-1}}$  is naturally isomorphic to  $L_g$  so we can denote the latter one by  $L_{(g)}$ , and restricting to  $X_{(g_1, \dots, g_k)}$ ,  $L_{(g_1, \dots, g_k)} = L_{(g_1)} \cdots L_{(g_k)}$ ; then  $\mathcal{L} = \{L_{(g)}\}_{(g) \in T_1}$  is an inner local system for  $X$ .

To define the orbifold cohomology group we need to add a shifting to the cohomology of the twisted sectors, and for that we are going to assume that the orbifold  $X$  is almost complex with complex structure  $J$ ; recall that  $J$  will be a smooth section of  $End(TX)$  such that  $J^2 = -Id$ .

For  $p \in X$  the almost complex structure gives rise to an effective representation  $\rho_p : G_p \rightarrow GL_n(\mathbf{C})$  ( $n = dim_{\mathbf{C}} X$ ) that could be diagonalized as

$$diag \left( e^{2\pi \frac{m_{1,g}}{m_g}}, \dots, e^{2\pi \frac{m_{n,g}}{m_g}} \right),$$

where  $m_g$  is the order of  $g$  in  $G_p$  and  $0 \leq m_{j,g} < m_g$ . We define a function  $\iota : \widetilde{\Sigma X} \rightarrow \mathbf{Q}$  by

$$\iota(p, (g)_{G_p}) = \sum_{j=1}^n \frac{m_{j,g}}{m_g}.$$

It is easy to see that it is locally constant, hence we call it  $\iota_{(g)}$ ; it is an integer if and only if  $\rho_p(g) \in SL_n(\mathbf{C})$  and

$$\iota_{(g)} + \iota_{(g^{-1})} = rank(\rho_p(g) - I),$$

which is the complex codimension  $dim_{\mathbf{C}} X - dim_{\mathbf{C}} X_{(g)}$ .  $\iota_{(g)}$  is called the degree shifting number.

**Definition 2.2.4.** Let  $\mathcal{L}$  be an inner local system; the orbifold cohomology groups are defined as

$$H_{orb}^d(X; \mathcal{L}) = \bigoplus_{(g) \in T_1} H^{d-2\iota_{(g)}}(X_{(g)}; \mathcal{L}_{(g)}).$$

If  $\mathcal{L} = \mathcal{L}_\alpha$  for some discrete torsion  $\alpha$  we define

$$H_{orb,\alpha}^*(X, \mathbf{C}) = H_{orb}^*(X, \mathcal{L}_\alpha).$$

*Example 2.2.5.* For the global quotient  $X = Y/G$  and  $\alpha \in H^2(G, U(1))$ ,  $L_g^\alpha$  induces a twisted action of  $C(g)$  on the cohomology of the fixed point set  $H^*(Y^g, \mathbf{C})$  by  $\beta \mapsto L_g^\alpha(h)h^*\beta$  for  $h \in C(g)$ . Let  $H^*(Y^g, \mathbf{C})^{C^\alpha(g)}$  be the invariant subspace under this twisted action. Then

$$H_{orb,\alpha}^d(X; \mathbf{C}) = \bigoplus_{(g) \in T_1} H^{d-2\iota_{(g)}}(Y^g; \mathbf{C})^{C^\alpha(g)}.$$

**2.3. Orbifold  $K$ -theory.** In this section we will briefly describe a construction by Adem and Ruan of the so-called twisted orbifold  $K$ -theory. The following construction will generate a twisting of the orbifold  $K$ -theory by a certain class in a group cohomology group. We will recover this twisting later, as a particular case of a twisting of  $K$ -theory on a groupoid by an arbitrary gerbe.

The following constructions are based on projective representations. A function  $\rho : G \rightarrow GL(V)$ , for  $V$  a finite dimensional complex vector space, is a projective representation of  $G$  if there exists a function  $\alpha : G \times G \rightarrow \mathbf{C}^*$  such that  $\rho(x)\rho(y) = \alpha(x, y)\rho(xy)$ . Such  $\alpha$  defines a two-cocycle on  $G$ , and  $\rho$  is said to be  $\alpha$ -representation on the space  $V$ . We can take the sum of any two  $\alpha$ -representations, hence we can define the Grothendieck group,  $R_\alpha(G)$ , associated to the monoid of linear isomorphism classes of such  $\alpha$ -representations.

Let's assume that  $\Gamma$  is a semi-direct product of a compact Lie group  $H$  and a discrete group  $G$ . Let  $\alpha \in H^2(G, \mathbf{U}(1))$  so we have a group extension

$$1 \rightarrow \mathbf{U}(1) \rightarrow \tilde{G} \rightarrow G \rightarrow 1,$$

and  $\tilde{\Gamma}$  is the semi-direct product of  $H$  and  $\tilde{G}$ .

Suppose that  $\Gamma$  acts on a smooth manifold  $X$  such that  $X/\Gamma$  is compact and the action has only finite isotropy, then  $Y = X/\Gamma$  is an orbifold.

**Definition 2.3.1.** An  $\alpha$  twisted  $\Gamma$ -vector bundle on  $X$  is a complex vector bundle  $E \rightarrow X$  such that  $\mathbf{U}(1)$  acts on the fibers through complex multiplication extending the action of  $\Gamma$  in  $X$  by an action of  $\tilde{\Gamma}$  in  $E$ .

We define  ${}^\alpha K_\Gamma(X)$  the  $\alpha$ -twisted  $\Gamma$ -equivariant  $K$ -theory of  $X$  as the Grothendieck group of isomorphism classes of  $\alpha$  twisted  $\Gamma$ -bundles over  $X$ .

For an  $\alpha$ -twisted bundle  $E \rightarrow X$  and a  $\beta$ -twisted bundle  $F \rightarrow X$  consider the tensor product bundle  $E \otimes F \rightarrow X$ ; it becomes an  $\alpha + \beta$ -twisted bundle. So we have a product

$${}^\alpha K_\Gamma(X) \otimes {}^\beta K_\Gamma(X) \rightarrow {}^{\alpha+\beta} K_\Gamma(X).$$

And so we call the total twisted equivariant  $K$ -theory of a  $\Gamma$  space as:

$$TK_\Gamma(X) = \bigoplus_{\alpha \in H^2(G, \mathbf{U}(1))} {}^\alpha K_\Gamma(X)$$

When  $\Gamma$  is a finite group, there is the following decomposition theorem,

**Theorem 2.3.2** [29, Th. 4.2.6.]. Let  $\Gamma$  be a finite group that acts on  $X$ , then for any  $\alpha \in H^2(G, \mathbf{U}(1))$ ,

$${}^\alpha K_\Gamma^*(X) \otimes \mathbf{C} \cong H_{orb, \alpha}^*(X/\Gamma; \mathbf{C}).$$

The decomposition is as follows:

$${}^\alpha K_\Gamma^*(X) \otimes \mathbf{C} \cong \bigoplus_{(g)} (K(X^g) \otimes L_g^\alpha)^{C(g)} \cong \bigoplus_{(g)} H^*(X^g; \mathbf{C})^{C(g)} \cong H_{orb, \alpha}^*(X/\Gamma; \mathbf{C}).$$

**Definition 2.3.3.** In the case that  $Y \rightarrow X$  is the orbifold universal cover and  $\alpha \in H^2(\pi_1^{orb}(X), \mathbf{U}(1))$ , the  $\alpha$  twisted orbifold  $K$ -theory,  ${}^\alpha K_{orb}(X)$ , is the Grothendieck group of isomorphism classes of  $\alpha$ -twisted  $\pi_1^{orb}(X)$ -orbifold bundles over  $Y$  and the total orbifold  $K$ -theory is:

$$TK_{orb}(X) = \bigoplus_{\alpha \in H^2(\pi_1^{orb}(X); \mathbf{U}(1))} {}^\alpha K_{orb}(X).$$



2.4. *Twisted K-theory on smooth manifolds.* In [35] Witten shows that the  $D$ -brane charge for Type IIB superstring theories (in the case of 9-branes) should lie on a twisted  $K$ -theory group that he denotes as  $K_{[H]}(M)$ , where a 3-form  $H \in \Omega^3(M; \mathbf{R})$  models the Neveu-Schwarz  $B$ -field and  $[H] \in H^3(M; \mathbf{Z})$  is an integer cohomology class. The manifold  $M$  is supposed smooth and it is where the  $D$ -branes can be wrapped. The class  $[H]$  is not torsion, but in any case when  $[H]$  is a torsion class Witten gives a very elementary definition of  $K_{[H]}(M)$ . This will also be a particular class of the twisting of  $K$ -theory on a stack by a gerbe defined below.

The construction of  $K_{[H]}(M)$  is as follows. Consider the long exact sequence in simplicial cohomology with constant coefficients

$$\dots \rightarrow H^2(M; \mathbf{R}) \xrightarrow{i} H^2(M; \mathbf{U}(1)) \rightarrow H^3(M; \mathbf{Z}) \rightarrow H^3(M; \mathbf{R}) \rightarrow \dots \quad (2.4.1)$$

induced by the exponential sequence  $0 \rightarrow \mathbf{Z} \xrightarrow{i} \mathbf{R} \xrightarrow{\exp} \mathbf{U}(1) \rightarrow 1$ . Since  $[H]$  is torsion, it can be lifted to a class  $H^* \in H^2(M; \mathbf{U}(1))$ , and if  $n$  is its order, then for a fine covering  $\mathcal{U} = \{U_i\}_i$  of  $M$  the class  $H^*$  will be represented by a Čech cocycle  $h_{ijk} \in \check{C}^3(M)(\mathbf{Q}(\zeta_n))$  valued on  $n^{\text{th}}$  roots of unity.

Now we can consider a vector bundle as a collection of functions  $g_{ij} : U_{ij} \rightarrow \text{GL}_m(\mathbf{C})$  such that  $g_{ij}g_{jk}g_{ki} = id_{\text{GL}_m(\mathbf{C})}$ .

**Definition 2.4.1.** *We say that a collection of functions  $g_{ij} : U_{ij} \rightarrow \text{GL}_m(\mathbf{C})$  is an  $[H]$ -twisted vector bundle  $E$  if  $g_{ij}g_{jk}g_{ki} = h_{ijk} \cdot id_{\text{GL}_m(\mathbf{C})}$ . The Grothendieck group of such twisted bundles is  $K_{[H]}(M)$ .*

This definition does not depend on the choice of cover, for it can be written in terms of a Grothendieck group of modules over the algebra of sections  $\text{END}(E)$  of the endomorphism bundle  $E \otimes E^*$ , that in particular is an ordinary vector bundle [11].

In the case in which the class  $\alpha = [H]$  is not a torsion class one can still define a twisting and interpret it in terms of Fredholm operators on a Hilbert space. The following description is due to Atiyah and Segal [2]. Let  $\mathcal{H}$  be a fixed Hilbert space. We let  $\mathcal{B}(\mathcal{H})$  be the Banach algebra of bounded operators on  $\mathcal{H}$  and  $\mathcal{F}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$  be the space of Fredholm operators on  $\mathcal{H}$ , namely, those operators in  $\mathcal{B}(\mathcal{H})$  that are invertible in  $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ , where  $\mathcal{K}(\mathcal{H})$  is the ideal in  $\mathcal{B}(\mathcal{H})$  consisting of compact operators.

Then we have the following classical theorem of Atiyah, for  $X$  a topological space:

$$K(X) = [X, \mathcal{F}],$$

where the right-hand side means all the homotopy classes of maps  $X \rightarrow \mathcal{F}$ . In particular  $\mathcal{F} \simeq BU$ .

For a cohomology class  $\alpha \in H^3(X, \mathbf{Z})$  Atiyah and Segal construct a bundle  $\mathcal{F}_\alpha$  over  $X$  with fiber  $\mathcal{F}(\mathcal{H})$ , and then define the twisted  $K_\alpha$ -theory as

$$K_\alpha(X) = [\Gamma(\mathcal{F}_\alpha)], \quad (2.4.2)$$

namely the homotopy classes of sections of the bundle  $\mathcal{F}_\alpha$ .

To construct  $\mathcal{F}_\alpha$  we will use Kuiper's theorem that states that the group  $U(\mathcal{H})$  of unitary operators in  $\mathcal{H}$  is contractible and therefore one has

$$\mathbf{P}(\mathbf{C}^\infty) \simeq K(\mathbf{Z}, 2) \simeq BU(1) \simeq U(\mathcal{H})/U(1) = \mathbf{P}U(\mathcal{H}).$$

This fact immediately implies  $K(\mathbf{Z}, 3) \simeq \mathbf{BPU}(\mathcal{H})$ . Hence the class  $\alpha \in H^3(X, \mathbf{Z}) = [X, K(\mathbf{Z}, 3)] = [X, \mathbf{BPU}(\mathcal{H})]$  produces a Hilbert projective bundle  $\mathbf{P}_\alpha$ . The class  $\alpha$  is called the *Dixmier-Douady* class of the projective bundle. We define

$$\mathcal{F}_\alpha := \mathbf{P}_\alpha \times_{\mathbf{PU}(\mathcal{H})} \mathcal{F}(\mathcal{H}),$$

where  $\mathbf{PU}(\mathcal{H})$  acts on  $\mathcal{F}(\mathcal{H})$  by conjugation<sup>1</sup>. It is worthwhile to mention that J. Rosenberg has previously defined  $K_\alpha(X)$  in [28]. His definition is clearly equivalent to the one explained above.

### 3. Gerbes over Smooth Manifolds

*3.1. Gerbes.* As a way of motivation for what follows, later we will summarize the facts about gerbes over smooth manifolds; we recommend to see [5, 15] for a more detailed description of the subject. Just as a line bundle can be given by transition functions, a gerbe can be given by transition data, namely line bundles. But the “total space” of a gerbe is a stack, as explained in the appendix. The same gerbe can be given as transition data in several ways.

Let’s suppose  $M$  is a smooth manifold and  $\{U_\alpha\}_\alpha$  an open cover. Let’s consider the functions

$$g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \longrightarrow \mathbf{U}(1)$$

defined on the threefold intersections satisfying

$$g_{\alpha\beta\gamma} = g_{\alpha\gamma\beta}^{-1} = g_{\beta\alpha\gamma}^{-1} = g_{\gamma\beta\alpha}^{-1}$$

and the cocycle condition

$$(\delta g)_{\alpha\beta\gamma\eta} = g_{\beta\gamma\eta} g_{\alpha\gamma\eta}^{-1} g_{\alpha\beta\eta} g_{\alpha\beta\gamma}^{-1} = 1$$

on the four-fold intersection  $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\eta$ . All these data define a *gerbe*. We could think of  $g$  as a *Čech* cocycle of  $H^2(M, C^\infty(\mathbf{U}(1)))$  and therefore we can tensor them using the product of cocycles. It also defines a class in  $H^3(M; \mathbf{Z})$ ; taking the long exact sequence of cohomology

$$\dots \rightarrow H^i(M, C^\infty(\mathbf{R})) \rightarrow H^i(M, C^\infty(\mathbf{U}(1))) \rightarrow H^{i+1}(M, \mathbf{Z}) \rightarrow \dots$$

given from the exact sequence of sheaves

$$0 \rightarrow \mathbf{Z} \rightarrow C^\infty(\mathbf{R}) \rightarrow C^\infty(\mathbf{U}(1)) \rightarrow 1$$

and using that  $C^\infty(\mathbf{R})$  is a fine sheaf, we get  $H^2(M, C^\infty(\mathbf{U}(1))) \cong H^3(M, \mathbf{Z})$ . We might say that a gerbe is determined topologically by its characteristic class.

A trivialization of a gerbe is defined by functions

$$f_{\alpha\beta} = f_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow \mathbf{U}(1)$$

on the twofold intersections such that

$$g_{\alpha\beta\gamma} = f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha}.$$

In other words  $g$  is represented as a coboundary  $\delta f = g$ .

<sup>1</sup> The ordinary action of  $U(\mathcal{H})$  on  $\mathcal{F}(\mathcal{H})$  by conjugation clearly descends to  $\mathbf{PU}(\mathcal{H})$ .

The difference of two trivializations  $f_{\alpha\beta}$  and  $f'_{\alpha\beta}$  given by  $h_{\alpha\beta}$  becomes a line bundle ( $h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha} = 1$ ).

Over a particular open subset  $U_0$  we can define a trivialization, for  $\beta, \gamma \neq 0$  we take  $f_{\beta\gamma} := g_{0\beta\gamma}$  and because of the cocycle condition we have  $g_{\alpha\beta\gamma} = f_{\alpha\beta}f_{\beta\gamma}f_{\gamma\alpha}$ . Adding  $f_{0\beta} = 1$  we get a trivialization localized at  $U_0$  and we could do the same over each  $U_\alpha$ . Then on the intersections  $U_\alpha \cap U_\beta$  we get two trivializations that differ by a line bundle  $\mathcal{L}_{\alpha\beta}$ . Thus a gerbe can also be seen as the following data:

- A line bundle  $\mathcal{L}_{\alpha\beta}$  over each  $U_\alpha \cap U_\beta$
- $\mathcal{L}_{\alpha\beta} \cong \mathcal{L}_{\beta\alpha}^{-1}$
- A trivialization  $\theta_{\alpha\beta\gamma}$  of  $\mathcal{L}_{\alpha\beta}\mathcal{L}_{\beta\gamma}\mathcal{L}_{\gamma\alpha} \cong 1$ , where  $\theta_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \rightarrow U(1)$  is a 2-cocycle.

*Example 3.1.1* [15, Ex. 1.3]. Let  $M^{n-3} \subset X^n$  be an oriented codimension 3 submanifold of a compact oriented one  $X$ . Take coordinate neighborhoods  $U_\alpha$  of  $X$  along  $M$ ; we could think of them as  $U_\alpha \cong (U_\alpha \cap M) \times \mathbf{R}^3$ , and let  $U_0 = X \setminus N(M)$ , where  $N(M)$  is the closure of a small neighborhood of  $M$ , diffeomorphic to the disc bundle in the normal bundle. We have

$$U_0 \cap U_\alpha \cong U_\alpha \cap M \times \{x \in \mathbf{R}^3 : \|x\| > \epsilon\}$$

and let's define the bundle  $\mathcal{L}_{\alpha 0}$  as the pullback by  $x \mapsto x/\|x\|$  of the line bundle of degree 1 over  $S^2$ .

The line bundles  $\mathcal{L}_{\alpha\beta} = \mathcal{L}_{\alpha 0}\mathcal{L}_{0\beta}^{-1}$  are defined on  $(U_\alpha \cap U_\beta \cap M) \times \{x \in \mathbf{R}^3 : \|x\| > \epsilon\}$  and by construction  $c_1(\mathcal{L}_{\alpha\beta}) = 0$  over  $S^2$ , then they can be extended to trivial ones on the whole  $U_\alpha \cap U_\beta$ . This information provides us with a gerbe and the characteristic class of it in  $H^3(X, \mathbf{Z})$  is precisely the Poincaré dual to the homology class of the submanifold  $M$ . This is the gerbe that we will use to recover Witten's twisting of  $K$ -theory.

**3.2. Connections over gerbes.** We can also do differential geometry over gerbes [15] and we will describe what is a connection over a gerbe.

For  $\{U_\alpha\}$  a cover such that all finite non-empty intersection are contractible (a *Leray* cover), a connection will consist of 1-forms over the double intersections  $A_{\alpha\beta}$ , such that

$$iA_{\alpha\beta} + iA_{\beta\gamma} + iA_{\gamma\alpha} = g_{\alpha\beta\gamma}^{-1}dg_{\alpha\beta\gamma},$$

where  $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \rightarrow U(1)$  is the cocycle defined by the gerbe.

Because  $d(g_{\alpha\beta\gamma}^{-1}dg_{\alpha\beta\gamma}) = 0$  there are 2-forms  $F_\alpha$  defined over  $U_\alpha$  such that  $F_\alpha - F_\beta = dA_{\alpha\beta}$ ; as  $dF_\alpha = dF_\beta$  then we define a global 3-form  $G$  such that  $G|_{U_\alpha} = dF_\alpha$ . This 3-form  $G$  is called the curvature of the gerbe connection.

As the  $A_{\alpha\beta}$  are 1-forms over the double intersections, we could reinterpret them as connection forms over the line bundles. So, using the line bundle definition of gerbe, a connection in that formalism is:

- A connection  $\Delta_{\alpha\beta}$  on  $\mathcal{L}_{\alpha\beta}$  such that
- $\Delta_{\alpha\beta\gamma}\theta_{\alpha\beta\gamma} = 0$  where  $\Delta_{\alpha\beta\gamma}$  is the connection over  $\mathcal{L}_{\alpha\beta}\mathcal{L}_{\beta\gamma}\mathcal{L}_{\gamma\alpha}$  induced by the  $\Delta_{\alpha\beta}$
- A 2-form  $F_\alpha \in \Omega^2(U_\alpha)$  such that on  $U_\alpha \cap U_\beta$ ,  $F_\beta - F_\alpha$  equals the curvature of  $\Delta_{\alpha\beta}$ .

When the curvature  $G$  vanishes we say that the connection on the gerbe is *flat*.

### 4. Groupoids

The underlying idea of everything we do here is that an orbifold is best understood as a stack. A stack  $\mathcal{X}$  is a “space” in which we can’t talk of a point in  $\mathcal{X}$  but rather only of functions  $S \rightarrow \mathcal{X}$ , where  $S$  is any space, much in the same manner in which it makes no sense to talk of the value of the Dirac delta  $\delta(x)$  at a particular point, but it makes perfect sense to write  $\int_{\mathbf{R}} \delta(x) f(x) dx$ . To be fair there are points in a stack, but they carry automorphism groups in a completely analogous way to an orbifold. We refer the reader to the Appendix for more on this. In any case, just as a smooth manifold is completely determined by an open cover and the corresponding gluing maps, in the same manner a stack will be completely determined by a groupoid representing it. Of course there may be more than one such groupoid, so we use the notion of Morita equivalence to deal with this issue.

A groupoid can be thought of as a generalization of a group, a manifold and an equivalence relation. First an equivalence relation. A groupoid has a set of relations  $\mathcal{R}$  that we will think of as arrows. These arrows relate elements in a set  $\mathcal{U}$ . Given an arrow  $\overset{r}{\rightarrow} \in \mathcal{R}$  it has a source  $x = s(\overset{r}{\rightarrow}) \in \mathcal{U}$  and a target  $y = t(\overset{r}{\rightarrow}) \in \mathcal{U}$ . Then we say that  $x \overset{r}{\rightarrow} y$ , namely  $x$  is related to  $y$ . We want to have an equivalence relation, for example we want transitivity and then we will need a way to compose arrows  $x \overset{r}{\rightarrow} y \overset{s}{\rightarrow} z$ . We also require  $\mathcal{R}$  and  $\mathcal{U}$  to be more than mere sets. Sometimes we want them to be locally Hausdorff, paracompact, locally compact topological spaces, sometimes schemes.

Consider an example. Let  $X = S^2$  be the smooth 2-dimensional sphere. Let  $p, q$  be the north and the south poles of  $S^2$ , and define  $U_1 = S^2 - \{p\}$  and  $U_2 = S^2 - \{q\}$ . Let  $U_{12} = U_1 \cap U_2$  and  $U_{21} = U_2 \cap U_1$  be two disjoint annuli. Similarly take two disjoint disks  $U_{11} = U_1 \cap U_1$  and  $U_{22} = U_2 \cap U_2$ . Consider a category where the objects are  $\mathcal{U} = U_1 \uplus U_2$ , where  $\uplus$  means disjoint union. The set of arrows will be  $\mathcal{R} = U_{11} \uplus U_{12} \uplus U_{21} \uplus U_{22}$ . For example the point  $x \in U_{12} \subset \mathcal{R}$  is thought of as an arrow from  $x \in U_1 \subset \mathcal{U}$  to  $x \in U_2 \subset \mathcal{U}$ , namely  $x \overset{x}{\rightarrow} x$ . This is a groupoid associated to the sphere. In this example we can write the disjoint union of all possible triple intersections as  $\mathcal{R}_t \times_s \mathcal{R}$ .

4.1. *Definitions.* A groupoid is a pair of objects in a category  $\mathcal{R}, \mathcal{U}$  and morphisms

$$s, t : \mathcal{R} \rightrightarrows \mathcal{U}$$

called respectively source and target, provided with an identity

$$e : \mathcal{U} \longrightarrow \mathcal{R},$$

a multiplication

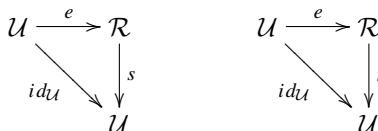
$$m : \mathcal{R}_t \times_s \mathcal{R} \longrightarrow \mathcal{R},$$

and an inverse

$$i : \mathcal{R} \longrightarrow \mathcal{R}$$

satisfying the following properties:

- (1) The identity inverts both  $s$  and  $t$ :



(2) Multiplication is compatible with both  $s$  and  $t$ :

$$\begin{array}{ccc}
 \mathcal{R}_t \times_s \mathcal{R} & \xrightarrow{m} & \mathcal{R} \\
 \pi_1 \downarrow & & \downarrow s \\
 \mathcal{R} & \xrightarrow{s} & \mathcal{U}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{R}_t \times_s \mathcal{R} & \xrightarrow{m} & \mathcal{R} \\
 \pi_2 \downarrow & & \downarrow t \\
 \mathcal{R} & \xrightarrow{t} & \mathcal{U}
 \end{array}$$

(3) Associativity:

$$\begin{array}{ccc}
 \mathcal{R}_t \times_s \mathcal{R}_t \times_s \mathcal{R} & \xrightarrow{id_{\mathcal{R}} \times m} & \mathcal{R}_t \times_s \mathcal{R} \\
 m \times id_{\mathcal{R}} \downarrow & & \downarrow m \\
 \mathcal{R}_t \times_s \mathcal{R} & \xrightarrow{m} & \mathcal{U}
 \end{array}$$

(4) Unit condition:

$$\begin{array}{ccc}
 \mathcal{R} & \xrightarrow{(es, id_{\mathcal{R}})} & \mathcal{R}_t \times_s \mathcal{R} \\
 id_{\mathcal{R}} \searrow & & \downarrow m \\
 & & \mathcal{R}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{R} & \xrightarrow{(id_{\mathcal{R}}, et)} & \mathcal{R}_t \times_s \mathcal{R} \\
 id_{\mathcal{R}} \searrow & & \downarrow m \\
 & & \mathcal{R}
 \end{array}$$

(5) Inverse:

$$\begin{aligned}
 i \circ i &= id_{\mathcal{R}}, \\
 s \circ i &= t, \\
 t \circ i &= s,
 \end{aligned}$$

with

$$\begin{array}{ccc}
 \mathcal{R} & \xrightarrow{(id_{\mathcal{R}}, i)} & \mathcal{R}_t \times_s \mathcal{R} \\
 s \downarrow & & \downarrow m \\
 \mathcal{U} & \xrightarrow{e} & \mathcal{R}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{R} & \xrightarrow{(i, id_{\mathcal{R}})} & \mathcal{R}_t \times_s \mathcal{R} \\
 t \downarrow & & \downarrow m \\
 \mathcal{U} & \xrightarrow{e} & \mathcal{R}
 \end{array}$$

We denote the groupoid by  $\mathcal{R} \rightrightarrows \mathcal{U} := (\mathcal{R}, \mathcal{U}, s, t, e, m, i)$ , and the groupoid is called *étale* if the base category is that of locally Hausdorff, paracompact, locally compact topological spaces and the maps  $s, t : \mathcal{R} \rightarrow \mathcal{U}$  are local homeomorphisms (diffeomorphisms). We will say that a groupoid is proper if  $s \times t : \mathcal{R} \rightarrow \mathcal{U} \times \mathcal{U}$  is a proper (separated) map. We can of course work in the category of schemes or of differentiable manifolds as well.

*Remark 4.1.1.* From now on we will assume that our groupoids are differentiable, étale and proper. The spaces  $\mathcal{R}$  and  $\mathcal{U}$  will be manifolds and the structure maps  $(s, t, e, m, i)$  will be smooth. And the maps  $s, t$  will be submersions in order for the space  $\mathcal{R}_t \times_s \mathcal{R}$  to be also a manifold.

*Example 4.1.2.* For  $M$  a manifold and  $\{U_\alpha\}$  an open cover, let

$$\mathcal{U} = \bigsqcup_{\alpha} U_{\alpha} \qquad \mathcal{R} = \bigsqcup_{(\alpha, \beta)} U_{\alpha} \cap U_{\beta} \quad (\alpha, \beta) \neq (\beta, \alpha)$$

$$s|_{U_{\alpha\beta}} : U_{\alpha\beta} \rightarrow U_\alpha, \quad t|_{U_{\alpha\beta}} : U_{\alpha\beta} \rightarrow U_\beta \quad e|_{U_\alpha} : U_\alpha \rightarrow U_\alpha$$

$$i|_{U_{\alpha\beta}} : U_{\alpha\beta} \rightarrow U_{\beta\alpha} \quad \& \quad m|_{U_{\alpha\beta\gamma}} : U_{\alpha\beta\gamma} \rightarrow U_{\alpha\gamma}$$

the natural maps. Note that in this example  $\mathcal{R}_t \times_s \mathcal{R}$  coincides with the subset of  $\mathcal{R}_t \times_s \mathcal{R}$  of pairs  $(u, v)$  so that  $t(u) = s(v)$ , namely the disjoint union of all possible triple intersections  $U_{\alpha\beta\gamma}$  of open sets in the open cover  $\{U_\alpha\}$ . We will denote this groupoid  $\mathcal{R} \rightrightarrows \mathcal{U}$  by  $\mathcal{M}(M, U_\alpha)$ .

*Example 4.1.3.* Let  $G$  be a group and  $U$  a set provided with a left  $G$  action

$$G \times U \longrightarrow U,$$

$$(g, u) \mapsto gu,$$

we put  $\mathcal{U} = U$  and  $\mathcal{R} = G \times U$  with  $s(g, u) = u$  and  $t(g, u) = gu$ . The domain of  $m$  is the same as  $G \times G \times U$ , where  $m(g, h, u) = (gh, u)$ ,  $i(g, u) = (g^{-1}, gu)$  and  $e(u) = (id_G, u)$ .

We will write  $G \times U \rightrightarrows U$  (or sometimes  $X = [U/G]$ ), to denote this groupoid.

**Definition 4.1.4.** A morphism of groupoids  $(\Psi, \psi) : (\mathcal{R}' \rightrightarrows \mathcal{U}') \longrightarrow (\mathcal{R} \rightrightarrows \mathcal{U})$  are the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{R}' & \xrightarrow{\Psi} & \mathcal{R} \\ s' \downarrow & & \downarrow s \\ \mathcal{U}' & \xrightarrow{\psi} & \mathcal{U} \\ t' \downarrow & & \downarrow t \end{array} \quad \begin{array}{ccc} \mathcal{R}' & \xrightarrow{\Psi} & \mathcal{R} \\ e' \uparrow & & \uparrow e \\ \mathcal{U}' & \xrightarrow{\psi} & \mathcal{U} \end{array}$$

$$\begin{array}{ccc} \mathcal{R}' \times_{t', s'} \mathcal{R}' & \xrightarrow{\Psi} & \mathcal{R}_t \times_s \mathcal{R} \\ m' \downarrow & & \downarrow m \\ \mathcal{R}' & \xrightarrow{\Psi} & \mathcal{R} \end{array} \quad \begin{array}{ccc} \mathcal{R}' & \xrightarrow{\Psi} & \mathcal{R} \\ i' \downarrow & & \downarrow i \\ \mathcal{R}' & \xrightarrow{\psi} & \mathcal{R} \end{array}$$

Now we need to say when two groupoids are “equivalent”.

**Definition 4.1.5.** A morphism of étale groupoids  $(\Psi, \psi)$  is called an étale Morita morphism whenever:

- The map  $s \circ \pi_2 : \mathcal{U}' \times_{\psi \times_t} \mathcal{R} \rightarrow \mathcal{U}$  is an étale surjection,
- The following square is a fibered product

$$\begin{array}{ccc} \mathcal{R}' & \xrightarrow{\Psi} & \mathcal{R} \\ (s', t') \downarrow & & \downarrow (s, t) \\ \mathcal{U}' \times \mathcal{U}' & \xrightarrow{\psi \times \psi} & \mathcal{U} \times \mathcal{U} \end{array}$$

where only the second condition is the required for a morphism of general groupoids to be Morita. When working on étale groupoids, the Morita morphisms are understood to be étale.

Two groupoids  $\mathcal{R}_1 \rightrightarrows \mathcal{U}_1, \mathcal{R}_2 \rightrightarrows \mathcal{U}_2$  are called *Morita equivalent* if there are Morita morphisms  $(\Psi_i, \psi_i) : \mathcal{R}' \rightrightarrows \mathcal{U}' \longrightarrow \mathcal{R}_i \rightrightarrows \mathcal{U}_i$  for  $i = 1, 2$ . This is an equivalence relation and in general we will consider the category of étale groupoids obtained by formally inverting the Morita equivalences (see [20] for details).

It is not hard to define principal bundles over groupoids where the fibers are groupoids (cf. [10]), but here we will restrict ourselves to the construction of principal  $G$  bundles over groupoids, where  $G$  is a Lie group (or an algebraic group). This will facilitate the construction of the desired twistings in  $K$ -theory.

We give ourselves a groupoid  $s, t : \mathcal{R} \rightrightarrows \mathcal{U}$ .

**Definition 4.1.6.** A principal  $G$ -bundle over the groupoid  $\mathcal{R} \rightrightarrows \mathcal{U}$  is the groupoid

$$\tilde{s}, \tilde{t} : \mathcal{R} \times G \rightrightarrows \mathcal{U} \times G$$

given by the following structure:

$$\tilde{s}(r, h) := (s(r), h), \quad \tilde{t}(r, h) := (t(r), \rho(r)h), \quad \tilde{i}(r, h) := (i(r), \rho(r)h),$$

$$\tilde{e}(u, h) := (e(u), h), \quad \text{and} \quad \tilde{m}((r, h), (r', \rho(r')h)) := (m(r, r'), \rho(m(r, r'))h),$$

where  $\rho : \mathcal{R} \rightarrow G$  is a map satisfying:

$$i^* \rho = \rho^{-1} \quad (\pi_1^* \rho) \cdot (\pi_2^* \rho) = m^* \rho.$$

**Definition 4.1.7.** For a group  $G$  we write  $\tilde{G}$  to denote the groupoid  $\star \times G \rightrightarrows \star$ .

**Proposition 4.1.8.** To have a principal  $G$ -bundle over  $\mathcal{G} = (\mathcal{R} \rightrightarrows \mathcal{U})$  is the same thing as to have a morphism of groupoids  $\mathcal{G} \rightarrow \tilde{G}$ .

This definition coincides with the one of orbifold given previously in Sect. 2.1 when we work with the groupoid associated to the orbifold, this will be discussed in detail in the next section.

## 5. Orbifolds and Groupoids

*5.1. The groupoid associated to an orbifold.* The underlying idea behind what follows is that an orbifold is best understood when it is interpreted as a stack. We will expand this idea in the Appendix. There we explain separately the procedures to go first from an orbifold to a stack, in such a way that the category of orbifolds constructed above turns out to be a full subcategory of the category of stacks; and then, from a stack to a groupoid, producing again an embedding of categories. But there is a more direct way to pass directly from the orbifold to the groupoid and we explain it now. We recommend to see [10, 22, 24] for a detailed exposition of this issue. Then we complete the dictionary between the orbifold approach of [8] and the groupoid approach.

Let  $X$  be an orbifold and  $\{(V_p, G_p, \pi_p)\}_{p \in X}$  its orbifold structure, the groupoid  $\mathcal{R} \rightrightarrows \mathcal{U}$  associated to  $X$  will be defined as follows:  $\mathcal{U} := \bigsqcup_{p \in X} V_p$  and an element  $g : (v_1, V_1) \rightarrow (v_2, V_2)$  (an arrow) in  $\mathcal{R}$  with  $v_i \in V_i, i = 1, 2$ , will be a equivalence class of triples  $g = [\lambda_1, w, \lambda_2]$ , where  $w \in W$  for another uniformizing system  $(W, H, \rho)$ , and the  $\lambda_i$ 's are injections  $(\lambda_i, \phi_i) : (W, H, \rho) \rightarrow (V_i, G_i, \pi_i)$  with  $\lambda_i(w) = v_i, i = 1, 2$  as in Definition 2.1.2.

For another injection  $(\gamma, \psi) : (W', H', \rho') \rightarrow (W, H, \rho)$  and  $w' \in W'$  with  $\gamma(w') = w$  then  $[\lambda_1, w, \lambda_2] = [\lambda_1 \circ \gamma, w', \lambda_2 \circ \gamma]$ .

Now the maps  $s, t, e, i, m$  are naturally described:

$$s([\lambda_1, w, \lambda_2]) = (\lambda_1(w), V_1), \quad t([\lambda_1, w, \lambda_2]) = (\lambda_2(w), V_2), \quad e(x, V) = [id_V, x, id_V],$$

$$i([\lambda_1, w, \lambda_2]) = [\lambda_2, w, \lambda_1], \quad m([\lambda_1, w, \lambda_2], [\mu_1, z, \mu_2]) = [\lambda_1 \circ \nu_1, y, \mu_2 \circ \nu_2],$$

where  $h = [\nu_1, y, \nu_2]$  is an arrow joining  $w$  and  $z$  (i.e.  $\nu_1(y) = w$  &  $\nu_2(y) = z$ ).

It can be given a topology to  $\mathcal{R}$  so that  $s, t$  will be étale maps, making it into a proper, étale, differentiable groupoid, and it is not hard to check that all the properties of groupoid are satisfied. A detailed proof of this fact can be found in [24, Thm 4.1.1].

*Remark 5.1.1.* Two equivalent orbifold structures (as in Def. 2.1.5) will induce Morita equivalent groupoids and vice versa. Thus, the choice of groupoid in the Morita equivalence class that we will use for a specific orbifold will depend on the setting, it may change once we take finer covers, but it will be clear that it represents the same orbifold. This fact is proven by Moerdijk and Pronk in [23].

This is a good place to note that an orbifold  $X$  given by a groupoid  $\mathcal{R} \rightrightarrows \mathcal{U}$  will be a smooth manifold if and only if the map  $(s, t) : \mathcal{R} \rightarrow \mathcal{U} \times \mathcal{U}$  is one-to-one.

Now we can construct principal  $\Gamma$ -bundles on the groupoid  $\mathcal{R} \rightrightarrows \mathcal{U}$  associated to the orbifold  $X$  getting,

**Proposition 5.1.2.** *Principal  $\Gamma$  bundles over the groupoid  $\mathcal{R} \rightrightarrows \mathcal{U}$  are in 1-1 correspondence with  $\Gamma$ -orbibundles over  $X$ .*

*Proof.* Let's suppose the bundles are complex, in other words  $\Gamma = GL_n(\mathbf{C})$ . The proof for general  $\Gamma$  is exactly the same.

For an  $n$ -dimensional complex bundle over  $\mathcal{R} \rightrightarrows \mathcal{U}$  we have a map  $\rho : \mathcal{R} \rightarrow GL_n(\mathbf{C})$  and a groupoid structure  $\mathcal{R} \times \mathbf{C}^n \rightrightarrows \mathcal{U} \times \mathbf{C}^n$  as in Definition 4.1.6. Let  $U$  be an open set of  $X$  uniformized by  $(V, G, \pi)$  which belongs to its orbifold structure; for  $g \in G$  and  $x \in V$ ,  $\xi = [id_G, x, g]$  is an element of  $\mathcal{R}$  ( via the identity on  $V$ , and the action of  $g$  in  $V$  and the conjugation by  $g$  on  $G$  thought of as an automorphism of  $V$ ) and we can define  $\rho_{V,G} : V \times G \rightarrow GL_n(\mathbf{C})$  by  $\rho_{V,G}(x, g) \mapsto \rho([id_G, x, g])$ . As  $m([id_G, x, g], [id_G, gx, hg]) = [id_G, x, hg]$ , we have  $\rho([id_G, gx, h]) \circ \rho([id_G, x, g]) = \rho([id_G, x, hg])$ , which implies  $\rho_{V,G}(gx, h) \circ \rho_{V,G}(x, g) = \rho_{V,G}(x, hg)$ . So  $(V \times \mathbf{C}^n, G, \tilde{\pi})$  with  $\rho_{V,G}$  extending the action of  $G$  in  $\mathbf{C}^n$  is a uniformizing system for the orbibundle we are constructing, we need to prove now that they define the same germs and then we would get an orbibundle  $E \rightarrow X$  using its bundle orbifold structure.

Let  $(\lambda_i, \phi_i) : (W, H, \mu) \rightarrow (V_i, G_i, \pi_i)$  be injections of uniformizing systems of  $X$ , with corresponding bundle uniformizing systems  $(W \times \mathbf{C}^n, H, \tilde{\mu})$  and  $(V_i \times \mathbf{C}^n, G_i, \tilde{\pi}_i)$ . For  $x \in W$ ,  $\xi \in \mathbf{C}^n$  and  $h \in H$ ,  $([id_H, x, h], \xi) \in \mathcal{R} \times \mathbf{C}^n$  and  $\tilde{t}([id_H, x, h], \xi) = (hx, \rho([id_H, x, h])\xi) = \rho_{W,H}(x, h)\xi$ . As  $[\lambda_i, x, \phi_i(h) \circ \lambda_i] = [id_H, x, h]$  for  $i \in \{1, 2\}$  then  $\rho_{V_1, G_1}(\lambda_1(x), \phi_1(h)) = \rho_{V_2, G_2}(\lambda_2(x), \phi_2(h))$ ; so the bundle uniformizing systems  $(V_i \times \mathbf{C}^n, G_i, \tilde{\pi}_i)$  define the same germs, thus they form a bundle orbifold structure over  $X$ .

Conversely, if we have the orbibundle structure for  $E \rightarrow X$  we need to define the function  $\rho : \mathcal{R} \rightarrow GL_n(\mathbf{C})$ . So, for injections  $(\tilde{\lambda}_i, \phi_i) : (W \times \mathbf{C}^n, H, \mu) \rightarrow (V_i \times \mathbf{C}^n, G_i, \pi_i)$  (where  $\tilde{\lambda}_i$  extends the  $\lambda_i$ 's previously defined),  $\rho([\lambda_1, x, \lambda_2])$  will be the element in  $GL_n(\mathbf{C})$  such that maps  $pr_2(\tilde{\lambda}_1(x, \xi)) \mapsto pr_2(\tilde{\lambda}_2(x, \xi))$ ; here  $pr_2$  stands for the projection on the second coordinate; in other words

$$\rho([\lambda_1, x, \lambda_2])pr_2(\tilde{\lambda}_1(x, \xi)) = pr_2(\tilde{\lambda}_2(x, \xi)).$$



Because this bundle uniformizing systems define the same germs,  $\rho$  satisfies the product formula; the inverse formula is clearly satisfied.  $\square$

**Proposition 5.1.3.** *Isomorphic  $\Gamma$ -bundles over  $\mathcal{R} \rightrightarrows \mathcal{U}$  correspond to isomorphic  $\Gamma$ -orbibundles over  $X$ , and vice versa.*

*Proof.* We will focus again on complex bundles. To understand what relevant information we have from isomorphic bundles, let's see the following lemmas:

**Lemma 5.1.4.** *An isomorphism of bundles over  $\mathcal{R} \rightrightarrows \mathcal{U}$  (with maps  $\rho_i : \mathcal{R} \rightarrow \mathrm{GL}_n(\mathbf{C})$  for  $i = 1, 2$ ) is determined by a map  $\delta : \mathcal{R} \rightarrow \mathrm{GL}_n(\mathbf{C})$  such that*

$$\begin{aligned} \mathcal{R} \times \mathbf{C}^n &\xrightarrow{\Psi} \mathcal{R} \times \mathbf{C}^n, & \mathcal{U} \times \mathbf{C}^n &\xrightarrow{\psi} \mathcal{U} \times \mathbf{C}^n, \\ (r, \xi) &\mapsto (r, \delta(r)\xi), & (u, \xi) &\mapsto (u, \delta(e(u))\xi), \end{aligned}$$

satisfying  $\delta(i(r))\rho_1(r) = \rho_2(r)\delta(r)$  and  $\delta(r) = \delta(es(r))$ .

*Proof.* It is easy to check that  $(\Psi, \psi)$  defined in this way is a morphism between the bundles; the equality  $\delta(r) = \delta(es(r))$  comes from the diagram of the *source* map and  $\delta(i(r))\rho_1(r) = \rho_2(r)\delta(r)$  from the one of the *target* map, the rest of the diagrams follow from those two.  $\square$

In the same way we could do this procedure for complex orbibundles:

**Lemma 5.1.5.** *An isomorphism of complex orbibundles over  $X$  (with maps  $\rho_{V,G}^i : V \times G \rightarrow \mathrm{GL}_n(\mathbf{C})$  for  $i = 1, 2$  and  $\{(V, G, \pi)\}$  orbifold structure of  $X$ ) is determined by the maps  $\tilde{\delta}_V : V \rightarrow \mathrm{GL}_n(\mathbf{C})$  such that*

$$\begin{aligned} V \times \mathbf{C}^n &\rightarrow V \times \mathbf{C}^n \\ (r, \xi) &\mapsto (r, \tilde{\delta}_V(r)\xi) \end{aligned}$$

satisfying  $\tilde{\delta}(gr)\rho_{V,G}^1(r, g) = \rho_{V,G}^2(r, g)\tilde{\delta}(r)$ . The  $\tilde{\delta}_V$ 's form a good map.

*Proof.* Because the underlying orbifold structure needs to be mapped to itself, we obtain the  $\delta_V$ 's. The equality  $\tilde{\delta}(gr)\rho_{V,G}^1(r, g) = \rho_{V,G}^2(r, g)\tilde{\delta}(r)$  holds because of the good map condition.  $\square$

The proof of the proposition is straightforward from these lemmas. The map  $\delta$  that comes from the isomorphism of the complex bundles determines uniquely the  $\tilde{\delta}_V$ 's, and vice versa.  $\square$

*Example 5.1.6.* The *tangent bundle*  $TX$  of an orbifold  $X$  is an orbibundle over  $X$ . If  $U = V/G$  is a local uniformizing system, then a corresponding local uniformizing system for  $TX$  will be  $TU/G$  with the action  $g \cdot (x, v) = (gx, dg_x(v))$ .

Similarly the *frame bundle*  $P(X)$  is a principal orbibundle over  $X$ . The local uniformizing system is  $U \times \mathrm{GL}_n(\mathbf{C})/G$  with local action  $g \cdot (x, A) = (gx, dg \circ A)$ . Notice that  $P(X)$  is always a *smooth manifold* for the local action, is free, and  $(s, t) : \mathcal{R} \rightarrow \mathcal{U} \times \mathcal{U}$  is one-to-one.

We want the morphism between orbifolds to be morphisms of groupoids, and this is precisely the case for the good maps given in Definition 2.1.11.

**Proposition 5.1.7.** *A morphism of groupoids induces a good map between the underlying orbifolds, and conversely, every good map arises in this way.*

*Proof.* For  $f : X \rightarrow X'$  a good map between orbifolds, we have a correspondence  $U \leftrightarrow U'$  between open subsets of a compatible cover of  $X$  and open subsets of  $X'$ , such that  $f(U) \subset U'$ , and  $U_1 \subset U_2$  implies  $U'_1 \subset U'_2$ . Moreover, we are provided with local liftings  $f_{UU'} : (V, G, \pi) \rightarrow (V', G', \pi')$  as in Definition 2.1.11. Let  $\mathcal{R} \rightrightarrows \mathcal{U}$  and  $\mathcal{R}' \rightrightarrows \mathcal{U}'$  be the groupoids constructed from the orbifold structures of  $X$  and  $X'$  respectively, determined by the compatible cover  $\{U_i\}_i$  of  $X$  and a cover of  $X'$  that uniformizes  $\{U'_j\}_j$ .

Define  $\psi : \mathcal{U} \rightarrow \mathcal{U}'$  such that  $\psi|_U = f_{UU'}$  and  $\Psi : \mathcal{R} \rightarrow \mathcal{R}'$  by  $\Psi([\lambda_1, w, \lambda_2]) = ([\nu(\lambda_1), \psi(w), \nu(\lambda_2)])$ , where the  $\lambda_i$ 's are injections between  $W$  and  $V_i$  and the  $\nu(\lambda_i)$ 's are the corresponding injections between  $W'$  and  $V'_i$  given by the definition of good map; because

$$\nu(\lambda_i) \circ f_{WW'} = f_{V_i V'_i} \circ \lambda_i$$

the function  $\Psi$  is well defined and together with  $\psi$ , satisfy all the conditions for a morphism of groupoids.

It is clear that the groupoids just used could differ from the groupoids one obtain after performing the construction defined at the beginning of this chapter, but they are respectively Morita equivalent.

On the other hand, if we are given  $\Psi : \mathcal{R} \rightarrow \mathcal{R}'$  and  $\psi : \mathcal{U} \rightarrow \mathcal{U}'$ , we can take a sufficiently small open compatible cover for  $X$  such that for  $U$  in its cover there is an open set  $U'$  of  $X'$  with the desired properties. For  $(V, G, \pi)$  and  $(V', G', \pi')$  uniformizing systems of  $U$  and  $U'$  respectively, we need to define  $f_{UU'}$ . The map between  $V$  and  $V'$  is given by  $\psi|_V$ , and the injection between  $G$  and  $G'$  is given as follows.

Let's take  $x \in V$  and  $g \in G$ . We have an automorphism of  $(V, G, \pi)$  given by the action by  $g$  on  $V$  and by conjugation on  $G$ , call this automorphism  $\lambda_g$ ; then  $[id_G, x, g]$  is an element of  $\mathcal{R}$ . Using the properties of  $\Psi$  and  $\psi$  we get that  $\Psi([id_G, x, g]) = [id_{G'}, \psi(x), g']$ , where  $g' \in G'$ . This because every automorphism of  $(V', G', \pi')$  comes from the action of an element in  $G'$  (see [29, Lemma 2.1.1]). Moreover, we have that  $g' \circ \psi(x) = \psi \circ g(x)$ . This will give us an homomorphism  $\rho_{UU'} : G \rightarrow G'$  sending  $g \mapsto g'$ , that together with  $f_{UU'}$  form the compatible system we required.  $\square$

Reduced orbifolds have the property that they can be seen as the quotient of a manifold by a Lie group. We just construct the frame bundle  $P(X)$  of  $X$ , which is a manifold, together with the natural action of  $O(n)$  as in Example 5.1.6 (cf. [1]).

*Example 5.1.8.* Let  $X$  be a  $n$ -dimensional orbifold,  $Y$  its orbifold universal cover and  $H = \pi_1^{orb}(X)$  its fundamental orbifold group and  $f : Y \rightarrow X$  the cover good map. Let  $P(Y)$  be the frame bundle of  $Y$ . By 5.1.6 we know that  $P(X)$  is a smooth manifold and it is endowed with a smooth and effective  $O(n)$  action with finite isotropy subgroups such that  $X \simeq [P(X)/O(n)]$  in the category of orbifolds (cf. [1] Prop. 2.3).

The frame bundle  $P(Y)$  is isomorphic to  $f^*P(X)$  and lifting the action of  $H$  in  $Y$  to a free action of  $H$  in  $P(Y)$  with  $P(Y)/H \simeq P(X)$  we obtain the following diagram:

$$\begin{array}{ccc} P(Y) & \xrightarrow{f'} & P(X) \\ \downarrow /O(n) & & \downarrow /O(n) \\ Y & \xrightarrow{f} & X \end{array}$$

Let's consider now the groupoids  $\mathcal{R}_Y \rightrightarrows^{s_Y, t_Y} \mathcal{U}_Y$  and  $\mathcal{R}_X \rightrightarrows^{s_X, t_X} \mathcal{U}_X$  associated to the orbifolds  $Y$  and  $X$  by using their frame bundles (i.e.  $\mathcal{R}_Y = P(Y) \times O(n)$  and  $\mathcal{U}_Y = P(Y)$  with  $s_Y$  and  $t_Y$  as in Example 4.1.3). Every  $h \in H$  induces a morphism of groupoids

$$\begin{array}{ccc} \mathcal{R}_Y & \xrightarrow{h} & \mathcal{R}_X \\ \Downarrow & & \Downarrow \\ \mathcal{U}_Y & \xrightarrow{h} & \mathcal{U}_X \end{array}$$

since the action of  $h$  in  $P(Y) \times O(n)$  commutes with the action of  $O(n)$ , for  $P(Y)$  is simply  $f^*P(X)$ .

As we are working in the reduced case, the orbifold structures of  $Y$  and  $X$  can be obtained using the frame bundles  $P(Y)$  and  $P(X)$  so we can choose a sufficiently small orbifold cover  $\{U\}$  of  $Y$ , such that for  $(V, G, \pi)$  a uniformizing system of  $U$ , and  $h \in H$ , we have an isomorphism  $\eta_h : (V, G, \pi) \cong (V', G', \pi')$ , where  $(V', G', \pi')$  is a uniformizing system for  $U' = hU$ . In other words, the map  $\eta_h$  induces a groupoid automorphism of the orbifold (a good map).

Let  $\mathcal{R}_Y \times H \rightrightarrows^{s, t} \mathcal{U}_Y$  be the groupoid defined by the following maps:

$$s(r, h) = s_Y(r), \quad t(r, h) = h(t_Y(r)), \quad e(x) = (e_Y(x), id_H),$$

$$i(r, h) = (h(i_Y(r)), h^{-1}), \quad m((r_1, h_1), (r_2, h_2)) = (m(r_1, h^{-1}(r_2)), h_2h_1),$$

then the following holds.

**Proposition 5.1.9.** *The groupoids  $\mathcal{R}_Y \times H \rightrightarrows \mathcal{U}_Y$  and  $\mathcal{R}_X \rightrightarrows \mathcal{U}_X$  are Morita equivalent.*

*Proof.* Noting that the map  $P(Y) \rightarrow P(X)$  is a surjection and recalling that  $\mathcal{R}_X = P(X) \times O(n)$ , we can see that  $s \circ \pi_2 : \mathcal{U}_Y \times_t \mathcal{R}_X \rightarrow \mathcal{U}_X$  is an étale surjection. Finally because the action of  $H$  in  $\mathcal{R}_Y$  and  $\mathcal{U}_Y$  is free and  $\mathcal{R}_Y/H \simeq \mathcal{R}_X, \mathcal{U}_Y/H \simeq \mathcal{U}_X$ , it is immediate to verify that

$$\begin{array}{ccc} \mathcal{R}_Y \times H & \xrightarrow{f' \pi_1} & \mathcal{R}_X \\ (s, t) \downarrow & & \downarrow (s_X, t_Y) \\ \mathcal{U}_Y \times \mathcal{U}_Y & \xrightarrow{h} & \mathcal{U}_X \times \mathcal{U}_X \end{array}$$

is a fibered square.  $\square$

**5.2. The category associated to a groupoid and its classifying space.** To every groupoid  $\mathcal{R} \rightrightarrows \mathcal{U}$  we can associate a category  $\mathcal{C}$  whose objects are the objects in  $\mathcal{U}$  and whose morphisms are the objects in  $\mathcal{R}$  that we have called arrows before. We can see

$$\mathcal{R}^{(n)} := \underbrace{\mathcal{R}_t \times_s \cdots \times_s \mathcal{R}_t}_n$$

as the composition of  $n$  morphisms. In the case in which  $\mathcal{R}$  is a set then  $\mathcal{R}^{(n)}$  is the set of sequences  $(\gamma_1, \gamma_2, \dots, \gamma_n)$  so that we can form the composition  $\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_n$ .

With this data we can form a simplicial set [19, 31].

**Definition 5.2.1.** A semi-simplicial set (resp. group, space, scheme)  $X_\bullet$  is a sequence of sets  $\{X_n\}_{n \in \mathbb{N}}$  (resp. groups, spaces, schemes) together with maps

$$X_0 \rightrightarrows X_1 \rightrightarrows X_2 \rightrightarrows \dots \rightrightarrows X_m \rightrightarrows \dots ,$$

$$\partial_i : X_m \rightarrow X_{m-1}, \quad s_j : X_m \rightarrow X_{m+1}, \quad 0 \leq i, j \leq m. \tag{5.2.1}$$

called boundary and degeneracy maps, satisfying

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1} \partial_i \quad \text{if } i < j, \\ s_i s_j &= s_{j+1} s_i \quad \text{if } i < j, \\ \partial_i s_j &= \begin{cases} s_{j-1} \partial_i & \text{if } i < j \\ 1 & \text{if } i = j, j + 1. \\ s_j \partial_{i-1} & \text{if } i > j + 1 \end{cases} \end{aligned}$$

The nerve of a category (see [31]) is a semi-simplicial set  $NC$ , where the objects of  $\mathcal{C}$  are the vertices, the morphisms the 1-simplices, the triangular commutative diagrams the 2-simplices, and so on. For a category coming from a groupoid then the corresponding simplicial object will satisfy  $NC_n = X_n = \mathcal{R}^{(n)}$ . We can define the boundary maps  $\partial_i : \mathcal{R}^{(n)} \rightarrow \mathcal{R}^{(n-1)}$  by:

$$\partial_i(\gamma_1, \dots, \gamma_n) = \begin{cases} (\gamma_2, \dots, \gamma_n) & \text{if } i = 0 \\ (\gamma_1, \dots, m(\gamma_i, \gamma_{i+1}), \dots, \gamma_n) & \text{if } 1 \leq i \leq n - 1 \\ (\gamma_1, \dots, \gamma_{n-1}) & \text{if } i = n \end{cases}$$

and the degeneracy maps by

$$s_j(\gamma_1, \dots, \gamma_n) = \begin{cases} (e(s(\gamma_1)), \gamma_1, \dots, \gamma_n) & \text{for } j = 0 \\ (\gamma_1, \dots, \gamma_j, e(t(\gamma_j)), \gamma_{j+1}, \dots, \gamma_n) & \text{for } j \geq 1 \end{cases}$$

We will write  $\Delta^n$  to denote the standard  $n$ -simplex in  $\mathbf{R}^n$ . Let  $\delta_i : \Delta^{n-1} \rightarrow \Delta^n$  be the linear embedding of  $\Delta^{n-1}$  into  $\Delta^n$  as the  $i^{\text{th}}$  face, and let  $\sigma_j : \Delta^{n+1} \rightarrow \Delta^n$  be the linear projection of  $\Delta^{n+1}$  onto its  $j^{\text{th}}$  face.

**Definition 5.2.2.** The geometric realization  $|X_\bullet|$  of the simplicial object  $X_\bullet$  is the space

$$|X_\bullet| = \left( \coprod_{n \in \mathbb{N}} \Delta^n \times X_n \right) / \begin{matrix} (z, \partial_i(x)) \sim (\delta_i(z), x) \\ (z, s_j(x)) \sim (\sigma_j(z), x) \end{matrix}$$

Notice that the topologies of  $X_n$  are relevant to this definition.

The semi-simplicial object  $NC$  determines  $\mathcal{C}$  and its topological realization is called  $BC$ , the classifying space of the category. Again in our case  $\mathcal{C}$  is a topological category in Segal's sense [31].

For a groupoid  $\mathcal{R} \rightrightarrows \mathcal{U}$  we will call  $B(\mathcal{R} \rightrightarrows \mathcal{U}) = BC = |NC|$  the classifying space of the groupoid.

The following proposition establishes that  $B$  is a functor from the category of groupoids to that of topological spaces. Recall that we say that two morphisms of groupoids are Morita related if the corresponding functors for the associated categories are connected by a morphism of functors.

**Proposition 5.2.3** [21] (cf. [31, Prop. 2.1]). *A morphism of groupoids  $X_1 \rightarrow X_2$  induces a continuous map  $BX_1 \rightarrow BX_2$ . Two morphisms that are Morita related will produce homotopic maps. In particular a Morita equivalence  $X_1 \sim X_2$  will induce a homotopy equivalence  $BX_1 \simeq BX_2$ . This assignment is functorial.*

*Example 5.2.4.* For the groupoid  $\bar{G} = (\star \times G \rightrightarrows \star)$  the space  $B\bar{G}$  coincides with the classifying space  $BG$  of  $G$ .

Consider now the groupoid  $X = (G \times G \rightrightarrows G)$ , where  $s(g_1, g_2) = g_1, t(g_1, g_2) = g_2$  and  $m((g_1, g_2); (g_2, g_3)) = (g_1, g_3)$ , then it is easy to see that  $BX$  is contractible and has a  $G$  action. Usually  $BX$  is written  $EG$ .

A morphism of groupoids  $X \rightarrow \bar{G}$  is the same thing as a principal  $G$  bundle over  $X$  and therefore can be written by means of a map  $G \times G \rightarrow G$ . If we choose  $(g_2, g_2) \mapsto g_1^{-1}g_2$  the induced map of classifying spaces

$$EG \longrightarrow BG$$

is the universal principal  $G$ -bundle fibration over  $BG$ .

*Example 5.2.5.* Consider a smooth manifold  $X$  and a good open cover  $\mathcal{U} = \{U_\alpha\}_\alpha$ . Consider the groupoid  $\mathcal{G} = (\mathcal{R} \rightrightarrows \mathcal{U})$ , where  $\mathcal{R}$  consists of the disjoint union of the double intersections  $U_{\alpha\beta}$ . Segal [31, Prop. 4.1] calls  $X_{\mathcal{U}}$  the corresponding topological category. There he proves that  $B\mathcal{G} = BX_{\mathcal{U}} \simeq X$ .

If we are given a principal  $G$  bundle over  $\mathcal{G}$  then we have a morphism  $\mathcal{G} \rightarrow \bar{G}$  of groupoids, that in turn induces a map  $X \rightarrow BG$ . Suppose that in the previous example we take  $G = GL_n(\mathbb{C})$ . Then we get a map  $X \rightarrow BGL_n(\mathbb{C}) \rightarrow BU$  and since  $K(X) = [X, BU]$  this is an element in  $K$ -theory.

*Example 5.2.6.* Consider a groupoid  $X$  of the form  $M \times G \rightrightarrows M$ , where  $G$  is acting on  $M$  continuously. Then  $BX \simeq EG \times_G M$  is the Borel construction for the action  $M \times G \rightarrow M$ .

**5.3. Sheaf cohomology and Deligne cohomology.** On a smooth manifold  $X$  a sheaf  $S$  can be defined as a functor from the category whose objects are open sets of  $X$ , and whose morphisms are inclusions to the category (for example) of abelian groups, and a gluing condition of the type described in the Appendix. So for every open set  $U_\alpha$  in  $X$  we have an abelian group  $S_{U_\alpha} = \Gamma_S(U_\alpha)$  called the sections of  $S$  in  $U_\alpha$ . In the representation of a smooth manifold as a groupoid  $\mathcal{R} \rightrightarrows \mathcal{U}$ , where

$$\mathcal{U} = \bigsqcup_{\alpha} U_{\alpha} \quad \mathcal{R} = \bigsqcup_{(\alpha, \beta)} U_{\alpha} \cap U_{\beta} \quad (\alpha, \beta) \neq (\beta, \alpha).$$

A sheaf can be encoded by giving a sheaf over  $\mathcal{U}$  with additional gluing conditions given by  $\mathcal{R}$ .

**Definition 5.3.1** [10]. *A sheaf  $S$  on a groupoid  $\mathcal{R} \rightrightarrows \mathcal{U}$  consists of*

- (1) *A sheaf  $S$  on  $\mathcal{U}$ .*
- (2) *A continuous (right) action of  $\mathcal{R}$  on the total space of  $S$ .*

*An action of  $\mathcal{R}$  on  $S \xrightarrow{\pi} \mathcal{U}$  is a map  $S \times_{\pi} \mathcal{R} \rightarrow S$  satisfying the obvious identities.*

The theory of sheaves over groupoids and their cohomology has been developed by Crainic and Moerdijk [10]. There is a canonical notion of morphism of sheaves. So we can define the category  $Sh(X)$  of sheaves over the groupoid  $X$ . Morita equivalent groupoids have equivalent categories of sheaves. There is a notion of sheaf cohomology of sheaves over groupoids defined in terms of resolutions. There is also a Čech version of this cohomology developed by Moerdijk and Pronk [24].

**Definition 5.3.2.** *A groupoid  $\mathcal{R} \rightrightarrows \mathcal{U}$  is called Leray if the spaces  $\mathcal{R}^{(n)}$  (see below) are diffeomorphic to a disjoint union of contractible open sets.*

*Remark 5.3.3.* The existence of such a Leray groupoid for orbifolds was proven by Moerdijk and Pronk in [24, Cor. 1.2.5].

From now on we will always take a representative of the Morita class of the groupoid that is of Leray type. The basic idea is just as in the case of a smooth manifold, an  $S$  valued  $n$  Čech cocycle is an element  $c \in \Gamma_S(\coprod U_{\alpha_1 \dots \alpha_n})$ , and in a similar fashion we can define cocycles in the groupoid  $\mathcal{R} \rightrightarrows \mathcal{U}$  in terms of the sheaf and the products  $\mathcal{R}^{(n)} := \underbrace{\mathcal{R}_t \times_s \dots \times_s \mathcal{R}_t}_n$ . Then using alternating sums of the natural collection of maps

$$\mathcal{R}^{(0)} \Leftarrow \mathcal{R}^{(1)} \Leftarrow \mathcal{R}^{(2)} \Leftarrow \mathcal{R}^{(3)} \dots ,$$

we can produce boundary homomorphisms and define the cohomology theory.

The resulting groupoid sheaf cohomology satisfies the usual long exact sequences and spectral sequences. In particular we can use the exponential sequence induced by the sequence of sheaves  $0 \rightarrow \mathbf{Z} \xrightarrow{i} \mathbf{R} \xrightarrow{\exp} U(1) \rightarrow 1$ .

In [24, 20, 10] we find a theorem that implies the following

**Theorem 5.3.4.** *For an orbifold with groupoid  $X$  and a locally constant system  $A$  of coefficients (for example  $A = \mathbf{Z}$ ) we have*

$$H^*(X, A) \cong H^*(BX, A),$$

where the left-hand side is orbifold sheaf cohomology and the right hand side is ordinary simplicial cohomology.

Moerdijk has proved that the previous theorem is true for arbitrary coefficients  $A$  [21].

Crainic and Moerdijk have also defined hypercohomology for a bounded complex of sheaves in a groupoid, and they obtained the basic spectral sequence. In [16, 17] we define *Deligne cohomology* for groupoids associated to orbifolds and also *Cheeger-Simons cohomology*.

## 6. Gerbes over Orbifolds

*6.1. Gerbes and inner local systems.* From this section on we are going to work over the groupoid associated to an orbifold. For  $\mathcal{R} \rightrightarrows \mathcal{U}$  the groupoid associated to an orbifold  $X$  defined in 5.1 we will consider the following

**Definition 6.1.1.** *A gerbe over an orbifold  $\mathcal{R} \rightrightarrows \mathcal{U}$ , is a complex line bundle  $\mathcal{L}$  over  $\mathcal{R}$  satisfying the following conditions:*

- $i^* \mathcal{L} \cong \mathcal{L}^{-1}$
- $\pi_1^* \mathcal{L} \otimes \pi_2^* \mathcal{L} \otimes m^* i^* \mathcal{L} \stackrel{\theta}{\cong} 1$
- $\theta : \mathcal{R}_t \times_s \mathcal{R} \rightarrow \mathbf{U}(1)$  is a 2-cocycle,

where  $\pi_1, \pi_2 : \mathcal{R}_t \times_s \mathcal{R} \rightarrow \mathcal{R}$  are the projections on the first and the second coordinates, and  $\theta$  is a trivialization of the line bundle.

The following proposition<sup>2</sup> shows that the analogy with a finite group can be carried through in this case.

**Proposition 6.1.2.** *To have a gerbe  $\mathcal{L}$  over a groupoid  $\mathcal{G}$  is the same thing as to have a central extension of groupoids*

$$1 \rightarrow \overline{\mathbf{U}(1)} \rightarrow \tilde{\mathcal{G}} \rightarrow \mathcal{G} \rightarrow 1.$$

**Lemma 6.1.3.** *In the case of a smooth manifold  $M$  [35, 15] we define the groupoid as in Example 4.1.2. For a line bundle  $\mathcal{L}$  over  $\mathcal{R}$  we get line bundles  $\mathcal{L}_{\alpha\beta} := \mathcal{L}|_{U_{\alpha\beta}}$  over the double intersections  $U_{\alpha\beta}$  such that  $\mathcal{L}_{\alpha\beta} \cong \mathcal{L}_{\beta\alpha}^{-1}$ , and  $\mathcal{L}_{\alpha\beta} \mathcal{L}_{\beta\gamma} \mathcal{L}_{\alpha\gamma}^{-1} \stackrel{\theta}{\cong} 1$  over the triple intersections  $U_{\alpha\beta\gamma}$ ; then we get a gerbe over the manifold as defined in Sect. 3.*

We want to relate the discrete torsions of Y. Ruan [29] over a discrete group  $G$  and the gerbes over the corresponding groupoid.

*Example 6.1.4.* Gerbes over a discrete group  $G$  are in 1-1 correspondence with the set of two-cocycles  $Z^2(G, \mathbf{U}(1))$ .

We recall that  $\overline{G}$  denotes the groupoid  $* \times G \rightrightarrows *$  the trivial maps  $s, t$  and  $i(g) = g^{-1}$  and  $m(h, g) = hg$  (clearly we can drop the  $*$  as is customary). A gerbe over  $\overline{G}$  is a line bundle  $\mathcal{L}$  over  $G$  such that, if we call  $\mathcal{L}_g$  the fiber at  $g$ ,  $\mathcal{L}_g^{-1} = \mathcal{L}_{g^{-1}}$  and  $\mathcal{L}_g \mathcal{L}_h \stackrel{\beta}{\cong} \mathcal{L}_{gh}$ . So for each  $g, h \in G$  we have a trivialization  $\beta_{g,h} \in \mathbf{U}(1)$  satisfying  $\beta_{g,h} \beta_{gh,k} = \beta_{g,hk} \beta_{h,k}$  because

$$\mathcal{L}_g \mathcal{L}_h \mathcal{L}_k \stackrel{\beta_{g,h}}{\cong} \mathcal{L}_{gh} \mathcal{L}_k \stackrel{\beta_{gh,k}}{\cong} \mathcal{L}_{ghk} \stackrel{\beta_{g,hk}^{-1}}{\cong} \mathcal{L}_g \mathcal{L}_{hk} \stackrel{\beta_{h,k}^{-1}}{\cong} \mathcal{L}_g \mathcal{L}_h \mathcal{L}_k.$$

Then  $\beta : G \times G \rightarrow \mathbf{U}(1)$  satisfies the cocycle condition and henceforth is a two-cocycle.

It is clear how to construct the gerbe over  $G$  once we have the two-cocycle.

The representations of  $L_g^\alpha : C(g) \rightarrow \mathbf{U}(1)$  defined in Sect. 2.2 for some  $\alpha \in H^2(G, \mathbf{U}(1))$  come from the fact that

$$\mathcal{L}_g \mathcal{L}_h \stackrel{\alpha_{g,h} \alpha_{h,g}^{-1}}{\cong} \mathcal{L}_h \mathcal{L}_g;$$

then  $\theta(g, h) := \alpha_{g,h} \alpha_{h,g}^{-1}$  defines a representation  $\theta(g, \cdot) : C(g) \rightarrow \mathbf{U}(1)$  and it matches the  $L_g^\alpha$  for  $\beta = \alpha$ .

<sup>2</sup> We owe this observation to I. Moerdijk.

6.2. *The characteristic class of a gerbe.* We want to classify gerbes over an orbifold. As we have pointed out before, the family of isomorphism classes of gerbes on a groupoid  $\mathcal{R} \rightrightarrows \mathcal{U}$  forms a group under the operation of the tensor product of gerbes, that we will denote as  $\text{Gb}(\mathcal{R} \rightrightarrows \mathcal{U})$ . Given an element  $[\mathcal{L}] \in \text{Gb}(\mathcal{R} \rightrightarrows \mathcal{U})$  we can choose a representative  $\mathcal{L}$  and such a representative will have an associated cocycle  $\theta : \mathcal{R}_t \times_s \mathcal{R} \rightarrow U(1)$ . Two isomorphic gerbes will differ by the co-boundary of a cocycle  $\mathcal{R} \rightarrow U(1)$ .

*Example 6.2.1.*  $\text{Gb}(\bar{G}) \cong H^2(G, U(1))$ .

Using Lemma 6.1.4 and the previous definition of the group  $\text{Gb}(\bar{G})$  we see that two isomorphic gerbes define cohomologous cycles, and vice versa.

We will call the cohomology class  $\langle \mathcal{L} \rangle \in H^2(\mathcal{R} \rightrightarrows \mathcal{U}, \mathbf{C}^*)$  of  $\theta$ , the *characteristic class* of the gerbe  $\mathcal{L}$ . As explained in Sect. 5.3 we can use the exponential sequence of sheaves to show that  $H^2(\mathcal{R} \rightrightarrows \mathcal{U}, \mathbf{C}^*) \cong H^3(\mathcal{R} \rightrightarrows \mathcal{U}, \mathbf{Z})$  and then using the isomorphism 5.3.4  $H^3(\mathcal{R} \rightrightarrows \mathcal{U}, \mathbf{Z}) \cong H^3(B(\mathcal{R} \rightrightarrows \mathcal{U}), \mathbf{Z})$  we get

**Proposition 6.2.2.** *For a groupoid  $\mathcal{R} \rightrightarrows \mathcal{U}$  we have the following isomorphism:*

$$\text{Gb}(\mathcal{R} \rightrightarrows \mathcal{U}) \cong H^3(B(\mathcal{R} \rightrightarrows \mathcal{U}), \mathbf{Z})$$

given by the map  $[\mathcal{L}] \rightarrow \langle \mathcal{L} \rangle$  that associates to a gerbe its characteristic class.

In particular using 5.2.3 we have that

**Proposition 6.2.3.** *The group  $\text{Gb}(\mathcal{R} \rightrightarrows \mathcal{U})$  is independent of the Morita class of  $\mathcal{R} \rightrightarrows \mathcal{U}$ .*

This could also have been obtained noting that a gerbe over an orbifold can be given as a sheaf of groupoids in the manner of 8.3.2.

*Example 6.2.4.* Consider an inclusion of (compact Lie) groups  $K \subset G$  and consider the groupoid  $\mathcal{G}$  given by the action of  $G$  in  $G/K$ ,

$$G/K \times G \rightrightarrows G/K.$$

Observe that the stabilizer of  $[1]$  is  $K$  and therefore we have that the following groupoid:

$$[1] \times K \rightrightarrows [1],$$

which is Morita equivalent to the one above.

From this we obtain

$$\text{Gb}(\mathcal{G}) \cong H^3(K, \mathbf{Z}).$$

As it was explained in 2.4 in the case of a smooth manifold  $M$  we have that

$$\text{Gb}(M) = [M, BBC^*],$$

where  $BBC^* = \overline{BPU(\mathcal{H})}$  for a Hilbert space  $\mathcal{H}$ .

Let us write  $\overline{PU(\mathcal{H})}$  to denote the groupoid  $\star \times \overline{PU(\mathcal{H})} \rightarrow \star$ . We have the following

**Proposition 6.2.5.** *For an orbifold  $X$  given by a groupoid  $\mathbf{X}$  we have*

$$\text{Gb}(\mathbf{X}) = [\mathbf{X}, \overline{PU(\mathcal{H})}],$$

where  $[\mathbf{X}, \overline{PU(\mathcal{H})}]$  represents the Morita equivalence classes of morphisms from  $\mathbf{X}$  to  $\overline{PU(\mathcal{H})}$



6.3. *Differential geometry of gerbes over orbifolds and the B-field.* Just as in the case of a gerbe over a smooth manifold, we can do differential geometry on gerbes over an orbifold groupoid  $\mathbf{X} = (\mathcal{R} \rightrightarrows \mathcal{U})$ . Let us define a connection over a gerbe in this context.

**Definition 6.3.1.** A connection  $(g, A, F, G)$  over a gerbe  $\mathcal{R} \rightrightarrows \mathcal{U}$  consists of a complex valued 0-form  $g \in \Omega^0(\mathcal{R}_t \times_s \mathcal{R})$ , a 1-form  $A \in \Omega^1(\mathcal{R})$ , a 2-form  $F \in \Omega^2(\mathcal{U})$  and a 3-form  $G \in \Omega^3(\mathcal{U})$  satisfying

- $G = dF$ ,
- $t^*F - s^*F = dA$  and
- $\pi_1^*A + \pi_2^*A + m^*t^*A = -\sqrt{-1}g^{-1}dg$ .

The 3-form  $G$  is called the curvature of the connection. A connection is called flat if its curvature  $G$  vanishes.

The 3-curvature  $\frac{1}{2\pi\sqrt{-1}}G$  represents the integer characteristic class of the gerbe in cohomology with real coefficients; this is the Chern-Weil theory for a gerbe over an orbifold. One can reproduce now Hitchin’s arguments in [15] *mutatis mutandis*. In particular when a connection is flat one can speak of a holonomy class in  $H^2(B\mathbf{X}, U(1))$ . Hitchin’s discussion relating a gerbe to a line bundle on the loop space has an analogue that we have studied in [18]. There, for a given groupoid  $\mathbf{X}$  we construct a groupoid  $\mathbf{LX}$  that represents the free loops on  $\mathbf{X}$ . The “coarse moduli space” or quotient space of this groupoid coincides with Chen’s definition of the loop space [7], but  $\mathbf{LX}$  has more structure. In particular if we are given a gerbe  $L$  over  $\mathbf{X}$ , using the holonomy we construct a “line bundle”  $\Lambda$  over  $\mathbf{LX}^{S^1}$ , the fixed subgroupoid under the action of  $S^1$ , by a groupoid homomorphism  $\mathbf{LX}^{S^1} \rightarrow \overline{U(1)}$ . Let us consider the groupoid  $\wedge\mathbf{X} = (\wedge\mathbf{X})_1 \rightrightarrows (\wedge\mathbf{X})_0$ , with objects  $(\wedge\mathbf{X})_0 = \{r \in \mathcal{R} | s(r) = t(r)\}$ , and arrows  $(\wedge\mathbf{X})_1 = \{\lambda \in \mathcal{R} | r_1 \xrightarrow{\lambda} r_2 \Leftrightarrow m(\lambda, r_2) = m(r_1, \lambda)\}$ . The groupoid  $\wedge\mathbf{X}$  is certainly étale, but it is not necessarily smooth. In other words the twisted sectors are an orbispace or a topological groupoid [7, 18].

**Theorem 6.3.2** [18]. *The orbifold  $\widetilde{\Sigma_1 X}$  defined in 2.2 is represented by the groupoid  $\wedge\mathbf{X}$ . There is a natural action of  $S^1$  on  $\mathbf{LX}$ . The fixed subgroupoid  $(\mathbf{LX})^{S^1}$  under this action is equal to  $\wedge\mathbf{X}$ . The holonomy line bundle  $\Lambda$  over  $\wedge\mathbf{X}$  is an inner local system as defined in 2.2.*

From this discussion we see that in orbifolds with discrete torsion as the ones considered by Witten in [35, p. 34], what corresponds to the B-field 3-form  $H$  in [35, p. 30] is the 3-form  $G$  of this section. The analogue of  $K_{[H]}$  that Witten requires in [35, p. 34] will be constructed in the next section.

Let us recall that the smooth Deligne cohomology groups of an orbifold  $\mathbf{X}$  can be defined as in Sect. 5.3. To finish this section let us state one last proposition in the orbifold case.

**Proposition 6.3.3** [17, Prop. 3.0.6.]. *The group of gerbes with connection over an orbifold  $\mathbf{X}$  are classified by the Deligne cohomology group  $H^3(\mathbf{X}, \mathbf{Z}(3)_D^\infty)$ .*

### 7. Twisted $\mathcal{L}K_{\text{gpd}}$ -Theory

7.1. *Motivation.* Think of a group  $G$  as the groupoid  $\star \times G \rightrightarrows \star$ , that is to say, a category with one object  $\star$  and an arrow from this object to itself for every element  $g \in G$ :



The theory of representations of  $G$  consists of the study of the functor  $G \mapsto R(G)$ , where  $R(G)$ , is the Grothendieck ring of representations of  $G$  with direct sum and tensor product as operations.

An  $n$ -dimensional representation  $\rho$  of  $G$  is a continuous assignment of a linear map



for every arrow  $g \in G$ . Namely a representation is encoded in a map

$$\rho: \mathcal{R} = \star \times G \rightarrow \text{GL}_n(\mathbb{C})$$

or in other words is a principal  $\text{GL}_n(\mathbb{C})$  bundle over the groupoid  $\overline{G} = (\star \times G \rightrightarrows \star)$ . For finite groups this is simply an orbifold.

For a given orbifold, the study of its  $K_{\text{orb}}$ -theory is the exact analogue to the previous situation, in other words, the representation theory of groupoids is  $K$ -theory. The analogue of a representation is an orbifold as in 4.1.6. Every arrow in the groupoid corresponds to an element in  $\text{GL}_n(\mathbb{C})$ , but now there are many objects so we get a copy of  $\mathbb{C}^n$  for every object in  $\mathcal{U}$ , namely a bundle over  $\mathcal{U}$  with gluing information.

In the case of a smooth manifold this recovers the usual  $K$ -theory.

It is clear now that we can twist  $K_{\text{gpd}}(X)$  by a gerbe  $\mathcal{L}$  over  $X$  in the very same manner in which  $R(G)$  can be twisted by an extension

$$1 \rightarrow \mathbb{C}^* \rightarrow \tilde{G} \rightarrow G \rightarrow 1.$$

Such an extension is the same thing as a gerbe over  $\overline{G} = (\star \times G \rightrightarrows \star)$ . This twisting recovers all the twistings of  $K$ -theory mentioned before in this paper.

For a moment let's restrict our attention to the groupoid  $\mathcal{G}$  associated to a smooth manifold  $X$  as defined in 4.1.2, and let's see how its  $K$ -theory can be interpreted in terms of this groupoid.

Let  $\mathcal{C}$  be the (discrete) category whose objects are finite dimensional vector spaces and whose morphisms are linear injections. Then a functor of categories

$$\mathcal{G} \longrightarrow \mathcal{C}$$

assigns to every object of  $\mathcal{G}$  a vector space and to every morphism of  $\mathcal{G}$  a linear isomorphism in a continuous fashion. If we recall that the groupoid  $\mathcal{G}$  is given by

$$\mathcal{U} = \bigsqcup_{\alpha} U_{\alpha} \quad \mathcal{R} = \bigsqcup_{(\alpha, \beta)} U_{\alpha} \cap U_{\beta} \quad (\alpha, \beta) \neq (\beta, \alpha)$$

then we realize that this is equivalent to giving a trivial vector over  $\mathcal{U}$  and linear gluing instructions, that is to say, a vector bundle over  $M$ .

It is also clear that the category  $\mathcal{C}$  is equivalent to the category with one object for every non-zero integer  $n \in \mathbb{Z}_{\geq 0}$ , and with morphisms generated by the isomorphisms

$\bigsqcup \mathrm{GL}_n(\mathbf{C})$  and the arrows  $n \rightarrow m$  whenever  $n \leq m$ . The classifying space of  $\mathcal{C}$  is  $\mathrm{Gr}(\mathbf{C}^\infty) \simeq \mathrm{BGL}_\infty(\mathbf{C}) \simeq BU$ .

In this case  $B\mathcal{G} \simeq X$  [31], and we get an element in the reduced  $K$ -theory of  $X$ ,  $[X, BU] = \tilde{K}(X)$ . This discussion is also valid in the case in which  $X$  is an orbifold and shows that our constructions do not depend on the choice of Leray étale groupoid representing the orbifold that we take. This will motivate us to *define* the  $K$ -theory of an orbifold  $X$  given by a groupoid  $\mathcal{G}$  by means of such functors  $\mathcal{G} \rightarrow \mathcal{C}$ . Actually we will use groupoid homomorphisms from  $\mathcal{G}$  to some groupoid  $\mathcal{V}$  whose classifying space is homotopic to  $\mathcal{C}$ , and this will allow us to generalize the definition to the twisted case.

Following Segal and Quillen’s ideas in algebraic  $K$ -theory we can do better in the untwisted case. Consider the category  $\widehat{\mathcal{C}}$  of virtual objects of  $\mathcal{C}$ , namely the objects of  $\widehat{\mathcal{C}}$  are pairs of vector spaces  $(V_0, V_1)$  and a morphism from  $(V_0, V_1)$  to  $(U_0, U_1)$  is an equivalence of a triple  $[W; f_0, f_1]$  where  $W$  is a vector space and  $f_i: V_i \oplus W \rightarrow U_i$  is an isomorphism. We say that  $(W; f_0, f_1) \sim (W', f'_0, f'_1)$  if and only if there is an isomorphism  $g: W \rightarrow W'$  such that  $f'_i \circ (\mathrm{id}_{V_i} \oplus g) = f_i$ .

It is a theorem of Segal that  $B\widehat{\mathcal{C}}$  is homotopy equivalent to the space of Fredholm operators  $\mathcal{F}(\mathcal{H})$ . But while for a finite group  $G$  it would be wrong to define  $K(G)$  as  $[BG, \mathcal{F}(\mathcal{H})]$  it is still correct to say that  $K(G)$  is the set of isomorphism classes of functors  $\widehat{G} \rightarrow \widehat{\mathcal{C}}$ . We can similarly define the  $K$ -theory of an orbifold given by a groupoid  $\mathcal{G}$  by functors of the form  $\mathcal{G} \rightarrow \widehat{\mathcal{C}}$ .

Let us again consider the case of a smooth manifold  $M$ . With this in mind we would like to have a group model for the space  $\mathcal{F}$  of Fredholm operators. One possible candidate is the following.

**Definition 7.1.1** [27]. *For a given Hilbert space  $\mathcal{H}$  by a polarization of  $\mathcal{H}$  we mean a decomposition*

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

where  $\mathcal{H}_+$  is a complete infinite dimensional subspace of  $\mathcal{H}$  and  $\mathcal{H}_-$  is its orthogonal complement.

We define the group  $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$  to be the subgroup of  $\mathrm{GL}(\mathcal{H})$  consisting of operators  $A$  that when written with respect to the polarization  $\mathcal{H}_+ \oplus \mathcal{H}_-$  look like

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where  $a: \mathcal{H}_+ \rightarrow \mathcal{H}_+$  and  $d: \mathcal{H}_- \rightarrow \mathcal{H}_-$  are Fredholm operators, and  $b: \mathcal{H}_- \rightarrow \mathcal{H}_+$  and  $c: \mathcal{H}_+ \rightarrow \mathcal{H}_-$  are Hilbert-Schmidt operators.

We have the following fact.

**Proposition 7.1.2.** *The map  $\mathrm{GL}_{\mathrm{res}}(\mathcal{H}) \rightarrow \mathcal{F}: A \mapsto a$  is a homotopy equivalence. Therefore  $K(M) = [M, \mathrm{GL}_{\mathrm{res}}(\mathcal{H})]$ .*

Consider a gerbe  $\mathcal{L}$  with characteristic class  $\alpha$  as a map  $M \rightarrow BBU(1) = BPU(\mathcal{H})$ , then we get a Hilbert projective bundle  $Z_\alpha(M) \rightarrow M$ . Then we form a  $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ -principal bundle over  $M$  as follows. We know [9] that polarized Hilbert bundles over  $M$  are classified by its characteristic class in  $K^1(M)$ , for in view of the Bott periodicity theorem such bundles are classified by maps  $M \rightarrow \mathrm{BGL}_{\mathrm{res}}(\mathcal{H}) = BBU = U$ , namely by elements in  $K^1(M)$ . This produces the desired map  $\mathrm{Gb}(M) = [M, BBU(1)] \rightarrow [M, U] = K^1(M)$ . In several applications it is easier to start detecting gerbes by means of their image under this map (in the smooth case, the relation to gerbes and quantum field theory of the  $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ -bundles can be found in [6]).

7.2. *The twisted theory.* In this section we are going to “twist” vector bundles via gerbes. So for the,  $\mathcal{R} \rightrightarrows \mathcal{U}$  groupoid associated to the orbifold  $X$  and  $\mathcal{L}$  a gerbe over  $\mathcal{R}$ :

**Definition 7.2.1.** *An  $n$ -dimensional  $\mathcal{L}$ -twisted bundle over  $\mathcal{R} \rightrightarrows \mathcal{U}$  is a groupoid extension of it,  $\mathcal{R} \times \mathbf{C}^n \rightrightarrows \mathcal{U} \times \mathbf{C}^n$  and a function  $\rho : \mathcal{R} \rightarrow \text{GL}_n(\mathbf{C})$  such that*

$$i^* \rho = \rho^{-1} \quad \& \quad (\pi_1^* \rho) \circ (\pi_2^* \rho) \circ ((im)^* \rho) = \theta_{\mathcal{L}} \cdot Id_{\text{GL}_n(\mathbf{C})},$$

where  $\theta_{\mathcal{L}} : \mathcal{R} \times \mathcal{R} \rightarrow \text{U}(1)$  is the trivialization of the triple intersection  $(\pi_1^* \mathcal{L} \cdot \pi_2^* \mathcal{L} \cdot (im)^* \mathcal{L}) \stackrel{\theta_{\mathcal{L}}}{\cong} 1$ ,  $Id_{\text{GL}_n(\mathbf{C})}$  is the identity of  $\text{GL}_n(\mathbf{C})$  and the functions  $\tilde{s}, \tilde{t}, \tilde{e}, \tilde{i}, \tilde{m}$  are defined in the same way as for bundles.

We have the following equivalent definition.

**Proposition 7.2.2.** *To have an  $n$ -dimensional  $\mathcal{L}$ -twisted bundle over  $\mathcal{R} \rightrightarrows \mathcal{U}$  is the same thing as to have a vector bundle  $E \rightarrow \mathcal{U}$  together with a given isomorphism*

$$\mathcal{L} \otimes t^* E \cong s^* E.$$

Notice that we then have a canonical isomorphism,

$$m^* \mathcal{L} \otimes \pi_2^* t^* E \cong \pi_1^* \mathcal{L} \otimes \pi_2^* \mathcal{L} \otimes \pi_2^* t^* E \cong \pi_1^* \mathcal{L} \otimes \pi_2^* (\mathcal{L} \otimes t^* E) \cong \pi_1^* \mathcal{L} \otimes \pi_2^* s^* E.$$

We can define the corresponding Whitney sum of  $\mathcal{L}$ -twisted bundles, so for an  $n$ -dimensional  $\mathcal{L}$ -twisted bundle with function  $\rho_1 : \mathcal{R} \rightarrow \text{GL}_n(\mathbf{C})$  and for an  $m$ -dimensional one with  $\rho_2 : \mathcal{R} \rightarrow \text{GL}_m(\mathbf{C})$ , we can define a groupoid extension  $\mathcal{R} \times \mathbf{C}^{n+m} \rightrightarrows \mathcal{U} \times \mathbf{C}^{n+m}$  with function  $\rho : \mathcal{R} \rightarrow \text{GL}_{n+m}(\mathbf{C})$  such that:

$$\rho(g) = \left( \begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array} \right) \in \text{GL}_{n+m}(\mathbf{C}).$$

**Definition 7.2.3.** *The Grothendieck group generated by the isomorphism classes of  $\mathcal{L}$  twisted bundles over the orbifold  $X$  together with the addition operation just defined is called the  $\mathcal{L}$  twisted  $K$ -theory of  $X$  and is denoted by  ${}^{\mathcal{L}}K_{\text{gpd}}(X)$ .*

Moerdijk and Pronk [24] proved that the isomorphism classes of orbifolds are in 1-1 correspondence with the classes of étale, proper groupoids up to Morita equivalence. The following is a direct consequence of the definitions.

**Lemma 7.2.4.** *The construction of  ${}^{\mathcal{L}}K_{\text{gpd}}(X)$  is independent of the groupoid that is associated to  $X$ .*

Similarly as we did with bundles over groupoids in Lemma 5.1.4, we can determine when two  $\mathcal{L}$ -twisted bundles are isomorphic.

**Proposition 7.2.5.** *An isomorphism of  $\mathcal{L}$ -twisted bundles over  $\mathcal{R} \rightrightarrows \mathcal{U}$  (with maps  $\rho_i : \mathcal{R} \rightarrow \text{GL}_n(\mathbf{C})$  for  $i = 1, 2$ ) is determined by a map  $\delta : \mathcal{R} \rightarrow \text{GL}_n(\mathbf{C})$  such that*

$$\begin{array}{ccc} \mathcal{R} \times \mathbf{C}^n & \xrightarrow{\Psi} & \mathcal{R} \times \mathbf{C}^n, & \mathcal{U} \times \mathbf{C}^n & \xrightarrow{\psi} & \mathcal{U} \times \mathbf{C}^n, \\ (r, \xi) & \mapsto & (r, \delta(r)\xi), & (u, \xi) & \mapsto & (u, \delta(e(u))\xi), \end{array}$$

satisfying  $\delta(i(r))\rho_1(r) = \rho_2(r)\delta(r)$  and  $\delta(r) = \delta(es(r))$ .

*Proof.* The proof is the same as in Lemma 5.1.4. For  $(r, r') \in \mathcal{R}_t \times_s \mathcal{R}$  we get:

$$\begin{aligned}
 \theta_{\mathcal{L}} id_{GL_n(\mathbb{C})} &= \rho_1(im(r, r'))\rho_1(r')\rho_1(r) \\
 &= \left( \delta(m(r, r'))^{-1} \rho_2(im(r, r'))\delta(im(r, r')) \right) \\
 &\quad \left( \delta(i(r'))^{-1} \rho_2(r')\delta(r') \right) \left( \delta(i(r))^{-1} \rho_2(r)\delta(r) \right) \\
 &= \delta(m(r, r'))^{-1} \left( \rho_2(m(r, r'))\rho_2(r')\rho_2(r) \right) \delta(r) \\
 &= \delta(m(r, r'))^{-1} \left( \theta_{\mathcal{L}} id_{GL_n(\mathbb{C})} \right) \delta(r) \\
 &= \theta_{\mathcal{L}} id_{GL_n(\mathbb{C})}. \quad \square
 \end{aligned}$$

Using the group structure of  $\text{Gb}(\mathcal{R} \rightrightarrows \mathcal{U})$  we can define a product between bundles twisted by different gerbes, so for  $\mathcal{L}_1$  and  $\mathcal{L}_2$  gerbes over  $X$ ,

$$\mathcal{L}_1 K_{\text{gpd}}(X) \otimes \mathcal{L}_2 K_{\text{gpd}}(X) \rightarrow \mathcal{L}_1 \otimes \mathcal{L}_2 K_{\text{gpd}}(X),$$

$$(\mathcal{R} \times \mathbf{C}^n, \rho_1) \otimes (\mathcal{R} \times \mathbf{C}^m, \rho_2) \mapsto (\mathcal{R} \times \mathbf{C}^{mn}, \rho_1 \otimes \rho_2),$$

which is well defined because

$$\begin{aligned}
 &(im)^*(\rho_1 \otimes \rho_2) \circ \pi_2^*(\rho_1 \otimes \rho_2) \circ \pi_1^*(\rho_1 \otimes \rho_2) \\
 &= ((im)^*(\mathcal{L}_1 \otimes \mathcal{L}_2) \cdot \pi_2^*(\mathcal{L}_1 \otimes \mathcal{L}_2) \cdot \pi_1^*(\mathcal{L}_1 \otimes \mathcal{L}_2)) \cdot Id_{GL_{mn}(\mathbb{C})} \\
 &= \theta_{\mathcal{L}_1 \otimes \mathcal{L}_2} Id_{GL_{mn}(\mathbb{C})} \\
 &= \theta_{\mathcal{L}_1} Id_{GL_n(\mathbb{C})} \otimes \theta_{\mathcal{L}_2} Id_{GL_m(\mathbb{C})} \\
 &= (((im)^*\mathcal{L}_1 \cdot \pi_2^*\mathcal{L}_1 \cdot \pi_1^*\mathcal{L}_1) \cdot Id_{GL_n(\mathbb{C})}) \otimes (((im)^*\mathcal{L}_2 \cdot \pi_2^*\mathcal{L}_2 \cdot \pi_1^*\mathcal{L}_2) \cdot Id_{GL_m(\mathbb{C})}) \\
 &= (((im)^*\rho_1) \circ (\pi_2^*\rho_1) \circ (\pi_1^*\rho_1)) \otimes (((im)^*\rho_2) \circ (\pi_2^*\rho_2) \circ (\pi_1^*\rho_2)),
 \end{aligned}$$

and we can define the total twisted orbifold  $K$ -theory of  $X$  as

$$T K_{\text{gpd}}(X) = \bigoplus_{\mathcal{L} \in \text{Gb}(\mathcal{R} \rightrightarrows \mathcal{U})} \mathcal{L} K_{\text{gpd}}(X).$$

This has a ring structure due to the following proposition.

**Proposition 7.2.6.** *The twisted groups  $\mathcal{L} K_{\text{gpd}}(\mathcal{G})$  satisfy the following properties:*

1. If  $\langle \mathcal{L} \rangle = 0$  then  $\mathcal{L} K_{\text{gpd}}(\mathcal{G}) = K_{\text{gpd}}(\mathcal{G})$ , in particular if  $\mathcal{G}$  represent the orbifold  $X$  then  $\mathcal{L} K_{\text{gpd}}(\mathcal{G}) = K_{\text{orb}}(X)$ .
2.  $\mathcal{L} K_{\text{gpd}}(\mathcal{G})$  is a module over  $K_{\text{gpd}}(\mathcal{G})$ .
3. If  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are two gerbes over  $\mathcal{G}$  then there is a homomorphism

$$\mathcal{L}_2 K_{\text{gpd}}(\mathcal{G}) \otimes \mathcal{L}_1 K_{\text{gpd}}(\mathcal{G}) \longrightarrow \mathcal{L}_1 \otimes \mathcal{L}_2 K_{\text{gpd}}(\mathcal{G}).$$

4. If  $\psi: \mathcal{G}_1 \longrightarrow \mathcal{G}_2$  is a groupoid homomorphism then there is an induced homomorphism

$$\mathcal{L} K_{\text{gpd}}(\mathcal{G}_2) \longrightarrow \psi^* \mathcal{L} K_{\text{gpd}}(\mathcal{G}_1).$$

*Example 7.2.7.* In the case when  $Y$  is the orbifold universal cover of  $X$  with orbifold fundamental group  $\pi_1^{orb}(X) = H$ , we can take a discrete torsion  $\alpha \in H^2(H, U(1))$  and define the twisted  $K$ -theory of  $X$  as in Definition 2.3.3. Let's associate to  $X$  the groupoid constructed in example 5.1.8; we want to construct a gerbe  $\mathcal{L}$  over  $\mathcal{R}_Y \times H \rightrightarrows \mathcal{U}_Y$  so that the twisted  $\mathcal{L}K_{\text{gpd}}(X)$  is the same as the twisted  $\alpha K_{\text{orb}}^{AR}(X)$  of section 2.3 (we added the upperscripts  $^{AR}$  to denote that is the twisted  $K$  theory defined by A. Adem and Y. Ruan [1]).

The discrete torsion  $\alpha$  defines a central extension of  $H$

$$1 \longrightarrow U(1) \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1$$

and doing the cartesian product with  $\mathcal{R}_Y$  we get a line bundle

$$\begin{array}{ccc} U(1) & \longrightarrow & \mathcal{L}_\alpha = \mathcal{R}_Y \times \tilde{H} \\ & & \downarrow \\ & & \mathcal{R}_Y \times H \end{array}$$

which, by Lemma 6.1.4 and the fact that the line bundle structure comes from the lifting of  $H$ , becomes a gerbe over  $\mathcal{R}_Y \times H \rightrightarrows \mathcal{U}_Y$ . Clearly this gerbe only depends on the one defined in Lemma 6.1.4 for the group  $H$ . For  $E \rightarrow \tilde{X}$  an  $\alpha$ -twisted bundle over  $Y$ , an element of  $\alpha K_{\text{orb}}^{AR}(X)$ , comes with an action of  $\tilde{H}$  in  $E$  such that it lifts the one of  $H$  in  $Y$ ; choosing specific lifts  $\tilde{h}, \tilde{g} \in \tilde{H}$  for every  $h, g \in H$ , and  $e \in E$ , we have  $\tilde{g}(\tilde{h}(e)) = \alpha(g, h)\tilde{g}h(e)$ . As  $E$  is a bundle over  $Y$ , it is defined by a map  $\rho : \mathcal{R}_Y \rightarrow \text{GL}_n(\mathbb{C})$ , and for  $h \in H$ , it defines an isomorphism  $\mathcal{R}_Y \times \mathbb{C}^n \xrightarrow{h} \mathcal{R}_Y \times \mathbb{C}^n$ , with  $\eta_h : \mathcal{R}_Y \rightarrow \text{GL}_n(\mathbb{C})$  such that  $(r, \xi) \mapsto (hr, \eta_h(r)\xi)$ . The  $\mathcal{L}_\alpha$ -twisted bundle over  $\mathcal{R}_Y \times H \rightrightarrows \mathcal{U}_Y$  that  $E$  determines, is given by the groupoid  $\mathcal{R}_Y \times H \times \mathbb{C}^n \rightrightarrows \mathcal{U}_Y \times \mathbb{C}^n$  and the map

$$\begin{aligned} \delta : \mathcal{R}_Y \times H &\rightarrow \text{GL}_n(\mathbb{C}), \\ (r, h) &\mapsto \rho(hr)\eta_h(r). \end{aligned}$$

Because  $h$  is an isomorphism of groupoids  $\mathcal{R}_Y \times \mathbb{C}^n \rightarrow \mathcal{R}_Y \times \mathbb{C}^n$ , it commutes with the source and target maps, and this in turn implies that for  $h \in H$  and  $r \in \mathcal{R}_Y$  we have  $\eta_h(r) = \eta_h(e_Y(s_Y(r)))$  and  $\eta_h(r)\rho(r) = \rho(hr)\eta_h(r)$ . In order to prove that this bundle is  $\mathcal{L}_\alpha$ -twisted it is enough to check that the multiplication satisfies the specified conditions.

We will make use of the following diagram in the calculation

$$\begin{array}{ccccc} x & \xrightarrow{r} & y & \xrightarrow{h} & x' & \xrightarrow{r'} & y' & \xrightarrow{j} & x'' & \xrightarrow{r''} & y'' \\ & \searrow w & \downarrow v & & \searrow w' & \downarrow v' & & & \searrow w'' & \downarrow v'' & z'' \\ & & z & & & z' & & & & & \end{array}$$

where the  $r$ 's,  $v$ 's and  $w$ 's belong to  $\mathcal{R}_Y$ , the  $x$ 's,  $y$ 's and  $z$ 's belong to  $\mathcal{U}_Y$  and  $h, j \in H$ . We have that  $m_Y(r, v) = w$  and

$$m((r, h), (v', j)) = (m(r, v), jh) = (w, jh).$$

Also

$$\begin{aligned}
 \delta(v', j)\delta(r, h) &= \rho(jv')\eta_j(v')\rho(hr)\eta_h(r) \\
 &= \eta_j(v')\rho(v')\rho(r')\eta_h(r) \\
 &= \eta_j(v')\rho(w')\eta_h(r) \\
 &= \eta_j(v')\rho(hw)\eta_h(w) \\
 &= \eta_j(v')\eta_h(w)\rho(w) \\
 &= \eta_j(v')\eta_h(r)\rho(w) \\
 &= \alpha(j, h)\eta_{jh}(w)\rho(w) \\
 &= \alpha(j, h)\delta(w, jh).
 \end{aligned}$$

The fact that  $\eta_j(v')\eta_h(r) = \alpha(j, h)\eta_{jh}(w)$  is precisely the fact that  $E$  is an  $\alpha$ -twisted bundle. This proves that  $(\mathcal{R}_Y \times H \times \mathbf{C}^n \rightrightarrows \mathcal{U}_Y \times \mathbf{C}^n, \delta)$  is endowed with the structure of a  $\mathcal{L}_\alpha$ -twisted bundle over  $\mathcal{R}_Y \times H \rightrightarrows \mathcal{U}_Y$ .

Conversely, if we have the  $\mathcal{L}_\alpha$ -twisted bundle, it is clear how to obtain the maps  $\rho$  and  $\eta_h$ . From the previous construction we can see that when  $h = id_H$ , the map  $\delta$  determines  $\rho$  (i.e.  $\delta(r, id_H) = \rho(r)$ ); hence  $\eta_h(r) = \delta(r, h)\rho(hr)^{-1}$ . Thus we can conclude,

**Theorem 7.2.8.** *In the above example  $\mathcal{L}_\alpha K_{\text{gpd}}(X) \cong {}^\alpha K_{\text{orb}}^{AR}(X)$ .*

From 3.1.1, 2.4 and 6.2.2 we have

**Proposition 7.2.9.** *If  $M$  is a smooth manifold and the characteristic class of the gerbe  $\mathcal{L}$  is the torsion element  $[H]$  in  $H^3(M, \mathbf{Z})$  then*

$$\mathcal{L}K_{\text{gpd}}(M) \cong K_{[H]}(M).$$

It remains to verify that the twisting  $\mathcal{L}K_{\text{gpd}}(M)$  coincides with the twisting  $K_\alpha(M)$  defined by 2.4.2.

**Proposition 7.2.10.** *Whenever  $\alpha = \langle \mathcal{L} \rangle$  is a torsion class and  $M$  is a smooth manifold then  $\mathcal{L}K_{\text{gpd}}(M) = K_\alpha(M)$ .*

*Proof.* We will use the following facts.

**Theorem 7.2.11** Serre [11]. *Let  $M$  be a CW-complex. If a class  $\alpha \in H^3(M, \mathbf{Z})$  is a torsion element then there exists a principal bundle  $Z \rightarrow M$  with structure group  $\mathbf{PU}(n)$  so that when seen as an element  $\beta \in [M, \mathbf{BPU}(n)] \rightarrow [M, \mathbf{BPU}] = [M, \mathbf{BBU}(1)] = [M, \mathbf{BK}(\mathbf{Z}, 2)] = [M, \mathbf{K}(\mathbf{Z}, 3)] = H^3(M, \mathbf{Z})$ , then  $\alpha = \beta$ . In other words, the image of  $[M, \mathbf{BPU}(n)] \rightarrow H^3(M, \mathbf{Z})$  is exactly the subgroup of torsion elements that are killed by multiplication by  $n$ .*

**Theorem 7.2.12** Segal [32]. *Let  $\mathcal{H}$  be a  $G$ -Hilbert space in which every irreducible representation of  $G$  appears infinitely many times. Then the equivariant index map*

$$\text{ind}_G : [Z, \mathcal{F}]_G \longrightarrow K_G(Z),$$

*is an isomorphism (where  $\mathcal{F}$  is the space of  $G$ -Fredholm operators over  $\mathcal{H}$  and  $G$  acts on  $\mathcal{F}$  by conjugation).*

**Lemma 7.2.13.** *Let  $X = Z/G$  be an orbifold where the Lie group  $G$  acts on  $Z$ . Let  $\alpha \in H^2(G, U(1))$  define a group extension  $1 \rightarrow U(1) \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ . Consider the natural homomorphism  $\psi: K_{\tilde{G}}(Z) \rightarrow K_{\tilde{G}}(G) \cong R(U(1))$  - it can also be seen as the composition of  $K_{\tilde{G}}(Z) \rightarrow K_{\tilde{G}}(*) \cong R(\tilde{G}) \rightarrow R(U(1))$ . Let  $\Delta: U(1) \rightarrow U(n)$  be the diagonal embedding representation. Then*

$${}^\alpha K_{\text{orb}}^{AR}(X) = \psi^{-1}(\Delta).$$

*Proof.* The orbifold  $X$  is represented by the groupoid  $\mathcal{G} = (Z \times G \rightrightarrows Z)$ , while the gerbe  $\mathcal{L}_\alpha$  is represented by the central extension of groupoids (Proposition 6.1.2)

$$1 \rightarrow \overline{U(1)} \rightarrow \tilde{\mathcal{G}} \rightarrow \mathcal{G} \rightarrow 1$$

where  $\tilde{\mathcal{G}} = (Z \times \tilde{G} \rightrightarrows Z)$ . Therefore, using the fact that  $K_{\text{gpd}}(\overline{U(1)}) = R(U(1))$  we get the surjective map

$$K_{\text{gpd}}(\tilde{\mathcal{G}}) \rightarrow R(U(1)),$$

using 7.2.1 and observing that  $K_{\text{gpd}}(\tilde{\mathcal{G}}) = K_{\tilde{G}}(Z)$  we get the result.  $\square$

Let us consider in the previous lemma the situation where  $X = M$  is smooth,  $Z$  is Serre's principal  $\mathbf{PU}(n)$ -bundle associated to  $\alpha = \langle \mathcal{L} \rangle$ ,  $G = \mathbf{PU}(n)$  and  $\beta$  is the class in  $H^2(\mathbf{PU}(n), U(1))$  labeling the extension

$$1 \rightarrow U(1) \rightarrow U(n) \rightarrow \mathbf{PU}(n) \rightarrow 1.$$

Then using Theorem 7.2.8 and 7.2.12 we get that

$${}^{\mathcal{L}} K_{\text{gpd}}(Z/G) = {}^\beta K_{\text{orb}}^{AR}(Z/G) = \psi^{-1}(\Delta) \subseteq K_{U(n)}(Z) = [Z, \mathcal{F}]_{U(n)}.$$

Notice that by 2.4.2  $K_\alpha(M)$  is defined as the homotopy classes of sections of the bundle  $\mathcal{F}_\alpha = Z \times_{\mathbf{PU}(n)} \mathcal{F}$ . This space of sections can readily be identified with the space  $[Z, \mathcal{F}_\alpha]_{\mathbf{PU}(n)}$  and the proposition follows from this.  $\square$

We should point out here that the theory so far described is essentially empty whenever the characteristic class  $\langle \mathcal{L} \rangle$  is a non-torsion element in  $H^3(M, \mathbf{Z})$ . The following is true.

**Proposition 7.2.14.** *If there is an  $n$ -dimensional  $\mathcal{L}$ -twisted bundle over the groupoid  $\mathcal{G}$  then  $\langle \mathcal{L} \rangle^n = 1$ .*

*Proof.* Consider the equations,

$$i^* \rho = \rho^{-1} \quad \& \quad (\pi_1^* \rho) \circ (\pi_2^* \rho) \circ ((im)^* \rho) = \theta_{\mathcal{L}} \cdot Id_{GL_n(\mathbf{C})}$$

and take determinants in both equations, we get

$$i^* \det \rho = \det \rho^{-1} \quad \& \quad (\pi_1^* \det \rho) \circ (\pi_2^* \det \rho) \circ ((im)^* \det \rho) = \det \theta_{\mathcal{L}} \cdot Id_{GL_n(\mathbf{C})}.$$

Defining  $f = \det \rho$  we have

$$i^* f = f^{-1} \quad \& \quad (\pi_1^* f) \circ (\pi_2^* f) \circ ((im)^* f) = \theta_{\mathcal{L}}^n.$$

This means that the coboundary of  $f$  is  $\theta^n$ . This concludes the proof.  $\square$



Another way to think of this is by noticing that if we restrict the central extension

$$1 \rightarrow \mathbf{U}(1) \rightarrow \mathbf{U}(n) \rightarrow \mathbf{PU}(n) \rightarrow 1$$

to the subgroup  $S\mathbf{U}(n)$  we get the  $n$ -fold covering map

$$1 \rightarrow \mathbf{Z}_n \rightarrow S\mathbf{U}(n) \rightarrow \mathbf{PU}(n) \rightarrow 1,$$

where the kernel  $\mathbf{Z}_n$  is the group of  $n$ -roots of unity.

In any case we need to consider a more general definition when the class  $\langle \mathcal{L} \rangle$  is a non-torsion class.

An obvious generalization of 2.4.2 would be to consider the class of the gerbe  $\alpha = \langle \mathcal{L} \rangle \in H^3(B\mathcal{G}, \mathbf{Z})$  and consider  $K_\alpha(B\mathcal{G})$  in the sense of 2.4.2. This works well for a manifold, but unfortunately for a finite group and the trivial gerbe  $\alpha = 1$  we have that  $K_\alpha(B\widehat{G}) = R(\widehat{G})$  and not  $R(G)$  as we should have (this is exactly the problem we encountered in the last section with  $[BG, \mathcal{F}(\mathcal{H})]$ ).

Fortunately one of the several equivalent definitions of [4] can be carefully generalized to serve our purposes. To motivate this definition consider the following situation.

Suppose first that the class  $\alpha = \langle \mathcal{L} \rangle \in H^3(B\mathcal{G}, \mathbf{Z})$  is a torsion class. Take any  $\alpha$ -twisted vector bundle  $\rho$  so that

$$i^* \rho = \rho^{-1} \quad \& \quad (\pi_1^* \rho) \circ (\pi_2^* \rho) \circ ((im)^* \rho) = \theta_{\mathcal{L}} \cdot Id_{\mathbf{U}(n)}$$

and let  $\beta: \mathcal{G} \rightarrow \mathbf{PU}(n)$  be the projectivization of  $\rho: \mathcal{G} \rightarrow \mathbf{U}(n)$ . Then  $\beta$  is a *bonafide* groupoid homomorphism, in other words

$$i^* \beta = \beta^{-1} \quad \& \quad (\pi_1^* \beta) \circ (\pi_2^* \beta) \circ ((im)^* \beta) = Id_{\mathbf{PU}(n)}$$

as equations in  $\mathbf{PU}(n)$ . Then  $\alpha$  as a map  $B\mathcal{G} \rightarrow B\mathbf{PU}(\mathcal{H})$  is simply obtained as the realization of the composition of  $\beta: \mathcal{G} \rightarrow \mathbf{PU}(n)$  with the natural inclusion  $\mathbf{PU}(n) \hookrightarrow \mathbf{PU}(\mathcal{H})$ . Fix now and for all time a  $\rho_0$  and a  $\beta_0$  constructed in this way.

Define the semidirect product  $\mathbf{U}(n) \widetilde{\times} \mathbf{PU}(n)$  as the group whose elements are the pairs  $(S, T)$ , where  $S \in \mathbf{U}(n)$  and  $T \in \mathbf{PU}(n)$  with multiplication

$$(S_1, T_1) \cdot (S_2, T_2) = (S_1 T_1 S_2 T_1^{-1}, T_1 T_2).$$

Consider the family of groupoid homomorphisms  $f: \mathcal{G} \rightarrow \mathbf{U}(n) \widetilde{\times} \mathbf{PU}(n)$  that make the following diagram commutative:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{f} & \mathbf{U}(n) \widetilde{\times} \mathbf{PU}(n) \\ & \searrow \beta_0 & \downarrow q_2 \\ & & \mathbf{PU}(n) \end{array}$$

where  $q_1$  is the projection onto  $\mathbf{U}(n)$  and  $q_2$  the projection onto  $\mathbf{PU}(n)$ .

Given a homomorphism  $f: \mathcal{G} \rightarrow \mathbf{U}(n) \widetilde{\times} \mathbf{PU}(n)$  like above we can then write  $\rho = (q_1 \circ f) \cdot \rho_0$  and verify that  $\rho$  satisfies the conditions to define a twisted vector bundle over  $\mathcal{G}$ . Conversely given a twisted vector bundle  $\rho$  we can define a homomorphism  $f$  by means of the formula

$$f(g) = (\rho(g)\rho_0(g)^{-1}, \beta_0(g)).$$

Therefore in the case of a torsion class  $\alpha$  a homomorphism  $f: \mathcal{G} \rightarrow U(n) \widetilde{\times} \mathbf{PU}(n)$  so that  $q_1 f = \beta_0$  is another way of encoding a twisted vector bundle.

In the case of a non-torsion class  $\alpha$  we need to consider infinite dimensional vector spaces. So we let  $\mathcal{K}$  be the space of compact operators of a Hilbert space  $\mathcal{H}$ . Let us write  $U_{\mathcal{K}}$  to denote the subgroup of  $U(\mathcal{H})$  consisting of unitary operators of the form  $I + K$ , where  $I$  is the identity operator and  $K$  is in  $\mathcal{K}$ . If  $h \in \mathbf{PU}(\mathcal{H})$  and  $g \in U_{\mathcal{K}}$  then  $hgh^{-1} \in U_{\mathcal{K}}$  and therefore we can define  $U_{\mathcal{K}} \widetilde{\times} \mathbf{PU}(\mathcal{H})$ . We can define now the  $K$ -theory for an orbifold  $X$  given by  $\mathcal{G}$  twisted by a gerbe  $\mathcal{L}$  with non-torsion class  $\alpha: \mathcal{G} \rightarrow \mathbf{PU}(\mathcal{H})$  (cf. 6.2.5).

**Definition 7.2.15.** *The set of isomorphism classes of groupoid homomorphisms  $f: \mathcal{G} \rightarrow U_{\mathcal{K}} \widetilde{\times} \mathbf{PU}(\mathcal{H})$  that make the following diagram commutative:*

$$\begin{array}{ccc}
 \mathcal{G} & \xrightarrow{f} & U_{\mathcal{K}} \widetilde{\times} \mathbf{PU}(\mathcal{H}) \\
 & \searrow \alpha & \downarrow q_2 \\
 & & \mathbf{PU}(\mathcal{H})
 \end{array}$$

is  ${}^{\mathcal{L}}K_{\text{gpd}}(\mathcal{G})$  the groupoid  $K$ -theory of  $\mathcal{G}$  twisted by  $\mathcal{L}$ .

This definition works for a gerbe whose class is non-torsion and has the obvious naturality conditions. In particular it becomes 2.4.2 if the groupoid represents a smooth manifold. The discussion immediately before the definition shows that this definition generalizes the one given before for  ${}^{\mathcal{L}}K_{\text{gpd}}\mathcal{G}$  when  $\langle \mathcal{L} \rangle$  was torsion. Then Proposition 7.2.6 remains valid in the non-torsion context.

In view of Theorem 6.3.2 and Theorem 7.2.8 we can reformulate Theorem 2.3.2 as follows.

**Theorem 7.2.16.** *Let  $\mathbf{X}$  be a Leray groupoid representing an orbifold  $X/\Gamma$  with  $\Gamma$  finite,  $\mathcal{L}$  a gerbe over  $\mathbf{X}$  coming from discrete torsion, and let  $\Lambda$  be the holonomy inner local system defined in 6.3.2. Then*

$${}^{\mathcal{L}}K_{\text{gpd}}(\mathbf{X}) \otimes \mathbf{C} \cong H_{\text{orb}, \Lambda}^*(\mathbf{X}; \mathbf{C}).$$

It is natural to conjecture that the previous theorem remains true even if the gerbe  $\mathcal{L}$  is arbitrary and  $\mathbf{X}$  is any proper étale groupoid. We will revisit this issue elsewhere.

The following astonishing result of Freed, Hopkins and Teleman can be written in terms of the twisting described in this section. For more on this see [18].

*Example 7.2.17* [12]. Let  $G$  be connected, simply connected and simple. Consider the groupoid  $\mathcal{G} = (G \times G \rightrightarrows G)$ , where  $G$  is acting on  $G$  by conjugation. Let  $h$  be the dual Coxeter number of  $G$ . Let  $\mathcal{L}$  be the gerbe over  $\mathcal{G}$  with characteristic class  $\dim(G) + k + h \in H_G^3(G)$ . Then

$${}^{\mathcal{L}}K_{\text{gpd}}(\mathcal{G}) \cong V_k(G),$$

where  $V_k(G)$  is the Verlinde algebra at level  $k$  of  $G$ .

7.3. *Murray's bundle gerbes.* The theory described in the previous sections is interesting even in the case in which the orbifold  $X$  is actually a smooth manifold  $M = X$ . In this case M. Murray et. al. [26, 4] have recently proposed a way to interpret the twisted  $K$ -theory  ${}^L K(M)$  in terms of bundle gerbes. Bundle gerbes are geometric objects constructed on  $M$  that give a concrete model for a gerbe over  $M$  [25]. The purpose of this section is to explain how the theory of bundle gerbes can be understood in terms of groupoids.

**Definition 7.3.1.** A bundle gerbe over  $M$  is a pair  $(L, Y)$ , where  $Y \xrightarrow{\pi} M$  is a surjective submersion and  $L \xrightarrow{p} Y \xrightarrow{\pi \times \pi} Y = Y^{[2]}$  is a line bundle satisfying

- $L_{(y,y)} \cong \mathbb{C}$
- $L_{(y_1,y_2)} \cong L^*_{(y_2,y_1)}$
- $L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \cong L_{(y_1,y_3)}$ .

We start the translation to the groupoid language with the following definition.

**Definition 7.3.2.** Given a manifold  $M$  and a surjective submersion  $Y \xrightarrow{\pi} M$  we define the groupoid  $\mathcal{G}(Y, M) = (\mathcal{R} \rightrightarrows \mathcal{U})$  by

- $\mathcal{R} = Y^{[2]} = Y \xrightarrow{\pi} \pi \times_{\pi} Y$
- $\mathcal{U} = Y$
- $s = p_1 : Y^{[2]} \rightarrow Y, s(y_1, y_2) = y_1$  and  $t = p_2 : Y^{[2]} \rightarrow Y, t(y_1, y_2) = y_2$
- $m((y_1, y_2), (y_2, y_3)) = (y_1, y_3)$ .

From Definition 6.1.1 we immediately obtain the following.

**Proposition 7.3.3.** A bundle gerbe  $(L, Y)$  over  $M$  is the same as a gerbe over the groupoid  $\mathcal{G}(Y, M)$ .

We will write  $\mathcal{L}(L, Y)$  to denote the gerbe over  $\mathcal{G}(Y, M)$  associated to the bundle gerbe  $(L, Y)$ . Notice that the groupoid  $\mathcal{G}(Y, M)$  is not necessarily étale, but it is Morita equivalent to an étale groupoid. Let  $\mathcal{M}(M, U_\alpha)$  be the étale groupoid associated to a cover  $\{U_\alpha\}$  of  $M$  as in 4.1.2.

**Proposition 7.3.4.** The groupoid  $\mathcal{G}(Y, M)$  is Morita equivalent to  $\mathcal{M}(M, U_\alpha)$  for any open cover  $\{U_\alpha\}$  of  $M$ .

*Proof.* Since all groupoids  $\mathcal{M}(M, U_\alpha)$  are Morita equivalent (for any two open covers have a common refinement) it is enough to consider the groupoid  $\mathcal{M}(M, M) = (M \rightrightarrows M)$  coming from the cover consisting of one open set. The source and target maps of  $\mathcal{M}(M, M)$  are both identity maps. Then the proposition follows from the fact that the following diagram is a fibered square

$$\begin{array}{ccc}
 Y^{[2]} & \xrightarrow{\pi s} & M \\
 (s,t) \downarrow & & \downarrow s \times t \\
 Y \times Y & \xrightarrow{\pi \times \pi} & M \times M
 \end{array}$$

□

**Corollary 7.3.5.** *The group of bundle gerbes over  $M$  is isomorphic to the group  $G\mathfrak{b}(\mathcal{M}(M, U_\alpha))$  for the Leray groupoid  $\mathcal{M}(M, U_\alpha)$  representing  $M$ . In particular there is a bundle gerbe in every Morita equivalence class of gerbes over  $M$ .*

Murray [26] defines a characteristic class for a bundle gerbe  $(L, Y)$  over  $M$  as follows.

**Definition 7.3.6.** *The Dixmier-Douady class of  $d(P) = d(P, Y) \in H^3(M, \mathbf{Z})$  is defined as follows. Choose a Leray open cover  $\{U_\alpha\}$  of  $M$ . Choose sections  $s_\alpha: U_\alpha \rightarrow Y$ , inducing  $(s_\alpha, s_\beta): U_\alpha \cap U_\beta \rightarrow Y^{[2]}$ . Choose sections  $\sigma_{\alpha\beta}$  of  $(s_\alpha, s_\beta)^{-1}(P)$  over  $U_\alpha \cap U_\beta$ . Define  $g_{\alpha\beta\gamma}: U_\alpha \cap U_\beta \cap U_\gamma \rightarrow \mathbf{C}^\times$  by*

$$\sigma_{\alpha\beta}\sigma_{\beta\gamma} = \sigma_{\alpha\gamma}g_{\alpha\beta\gamma}.$$

Then  $d(L) = [g_{\alpha\beta\gamma}] \in H^2(M, \mathbf{C}^\times) \cong H^3(M, \mathbf{Z})$ .

Then in view of Propositions 6.2.2 and 6.2.3 we have the following.

**Proposition 7.3.7.** *The Dixmier-Douady class  $d(L, Y)$  is equal to the characteristic class  $\langle \mathcal{L}(L, Y) \rangle$  defined above 6.2.2. Moreover the assignment  $(L, Y) \mapsto g_{\alpha\beta\gamma}$  realizes the isomorphism of 7.3.5.*

**Definition 7.3.8** [26]. *A bundle gerbe  $(L, Y)$  is said to be trivial whenever  $d(L, Y) = 0$ . Two bundle gerbes  $(P, Y)$  and  $(Q, Z)$  are called stably isomorphic if there are trivial bundle gerbes  $T_1$  and  $T_2$  such that*

$$P \otimes T_1 \simeq Q \otimes T_2.$$

The following is an easy consequence of 6.2.2, 6.2.3 and 7.3.7.

**Lemma 7.3.9.** *The Dixmier-Douady class is a homomorphism from the group of bundle gerbes over  $M$  with the operation of tensor product, and  $H^3(M, \mathbf{Z})$*

**Corollary 7.3.10.** *Two bundle gerbes  $(P, Y)$  and  $(Q, Z)$  are stably isomorphic if and only if  $d(P) = d(Q)$*

*Proof.* Suppose that  $(P, Y)$  and  $(Q, Z)$  are stably isomorphic. Then  $P \otimes T_1 \simeq Q \otimes T_2$ ; hence  $d(P \otimes T_1) = d(Q \otimes T_2)$ . Therefore from the previous lemma we have  $d(P) + d(T_1) = d(Q) + d(T_2)$  and by definition of trivial we get  $d(P) = d(Q)$ .

Conversely if  $d(P) = d(Q)$  then  $d(P \otimes Q^*) = 0$  and then by definition  $T_2 = P \otimes Q^*$  is trivial. Define the trivial bundle gerbe  $T_1 = Q^* \otimes Q$ . Then  $P \otimes T_1 \simeq Q \otimes T_2$  completing the proof.  $\square$

Given a bundle gerbe  $(L, Y)$  over  $M$  we will write  $\tilde{\mathcal{G}}(L, Y, M)$  to denote the  $\overline{\mathbf{U}(1)}$  central groupoid extension of  $\mathcal{G}(Y, M)$  defined by the associated gerbe, where

$$1 \rightarrow \overline{\mathbf{U}(1)} \rightarrow \tilde{\mathcal{G}}(L, Y, M) \rightarrow \mathcal{G}(Y, M) \rightarrow 1.$$

As we have explained before such extensions are classified by their class in the cohomology group  $H^3(B\mathcal{G}(Y, M), \mathbf{Z}) = H^3(M, \mathbf{Z})$ . As a consequence of this and 7.3.5 we have.

**Theorem 7.3.11.** *Two bundle gerbes  $(P, Y)$  and  $(Q, Z)$  are stably isomorphic if and only if  $\widetilde{\mathcal{G}}(P, Y, M)$  is Morita equivalent to  $\widetilde{\mathcal{G}}(Q, Z, M)$ . Therefore there is a one-to-one correspondence between stably isomorphism classes of bundle gerbes over  $M$  and classes in  $H^3(M, \mathbf{Z})$ . The category of bundle gerbes over  $M$  with stable isomorphisms is equivalent to the category of gerbes over  $M$  with Morita equivalences.*

**Definition 7.3.12.** *Let  $(L, Y)$  be a bundle gerbe over  $M$ . We call  $(E, L, Y, M)$  a bundle gerbe module if*

- $E \rightarrow Y$  is a hermitian vector bundle over  $Y$ .
- We are given an isomorphism  $\phi: L \otimes \pi_1^{-1}E \xrightarrow{\sim} \pi_2^{-1}E$ .
- The compositions  $L_{(y_1, y_2)} \otimes (L_{(y_2, y_3)} \otimes E_{y_3}) \rightarrow L_{(y_1, y_2)} \otimes E_{y_2} \rightarrow E_{y_1}$  and  $(L_{(y_1, y_2)} \otimes L_{(y_2, y_3)}) \otimes E_{y_3} \rightarrow L_{(y_1, y_3)} \otimes E_{y_3} \rightarrow E_{y_1}$  coincide.

*In this case we also say that the bundle gerbe  $(L, Y)$  acts on  $E$ . The bundle gerbe K-theory  $K_{\text{bg}}(M, L)$  is defined as the Grothendieck group associated to the semigroup of bundle gerbe modules  $(E, L, Y, M)$  for  $(L, Y, M)$  fixed.*

As a consequence of 7.3.11 and 7.2.2 we have the following fact.

**Theorem 7.3.13.** *The category of bundle gerbe modules over  $(L, Y)$  is equivalent to the category of  $\mathcal{L}(L, Y)$ -twisted vector bundles over  $\mathcal{G}(Y, M)$ . Moreover we have*

$$\mathcal{L}K_{\text{gpd}}(\mathcal{G}(Y, M)) \cong K_{\text{bg}}(M, L).$$

**Corollary 7.3.14.** *If the gerbe  $\mathcal{L}$  has a torsion class  $[H]$  then*

$$K_{[H]}(M) = K_{\text{bg}}(M, L).$$

### 8. Appendix: Stacks, Gerbes and Groupoids

We mentioned at the beginning of Sect. 4 that a stack  $\mathcal{X}$  is a space whose points can carry a “group valued” multiplicity, and that they are studied by studying the family  $\{\text{Hom}(S, \mathcal{X})\}_S$ , where  $S$  runs through all possible spaces (or schemes). In fact by Yoneda’s Lemma, as is well known, even when  $\mathcal{X}$  is an ordinary space that knowing everything for the functor  $\text{Hom}(\cdot, \mathcal{X})$  is the same thing as knowing everything about  $\mathcal{X}$ . A stack is a category fibered by groupoids, where  $\mathcal{C}_S = \text{Hom}(S, \mathcal{X})$  with an additional sheaf condition.

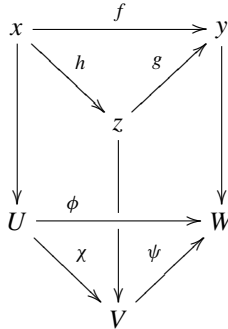
A very unfortunate confusion of terminologies occurs here. The word *groupoid* has two very standard meanings. One has been used along all the previous sections of this paper. But now we need the second meaning, namely a groupoid is a category where all morphisms have inverses. In this Appendix we use the word groupoid with both meanings and we hope that the context is enough to avoid confusion. Both concepts are, of course, very related.

*8.1. Categories fibered by groupoids.* Let  $\mathcal{C}, \mathcal{S}$  be a pair of categories and  $p: \mathcal{S} \rightarrow \mathcal{C}$  a functor. For each  $U \in \text{Ob}(\mathcal{C})$  we denote  $\mathcal{S}_U = p^{-1}(U)$ .

**Definition 8.1.1.** *The category  $\mathcal{S}$  is fibered by groupoids over  $\mathcal{C}$  if*

- For all  $\phi: U \rightarrow V$  in  $\mathcal{C}$  and  $y \in \text{Ob}(\mathcal{S}_V)$  there is a morphism  $f: x \rightarrow y$  in  $\mathcal{S}$  with  $p(f) = \phi$

- For all  $\psi : V \rightarrow W, \phi : U \rightarrow W, \chi : U \rightarrow V, f : x \rightarrow y$  and  $g : y \rightarrow z$  with  $\phi = \psi \circ \chi, p(f) = \phi$  and  $p(g) = \psi$  there is a unique  $h : x \rightarrow z$  such that  $f = g \circ h$  and  $p(h) = \chi,$



From the previous conditions we see that for  $\phi : U \rightarrow V$  and  $y \in Ob(S_V)$  there exist a unique morphism  $f : x \rightarrow y$  such that  $p(f) = \phi$ . Therefore  $x$  is defined uniquely from this information and then it will be called  $\phi^*y$ . Hence  $\phi^*$  will be a well defined functor from  $S_V$  to  $S_U$ .

8.2. Sheaves of categories.

**Definition 8.2.1.** A Grothendieck Topology (G.T.) over a category  $\mathcal{C}$  is a prescription of coverings  $\{U_\alpha \rightarrow U\}_\alpha$  such that:

- $\{U_\alpha \rightarrow U\}_\alpha$  &  $\{U_{\alpha\beta} \rightarrow U_\alpha\}_\beta$  implies  $\{U_{\alpha\beta} \rightarrow U\}_{\alpha\beta}$
- $\{U_\alpha \rightarrow U\}_\alpha$  &  $V \rightarrow U$  implies  $\{U_\alpha \times_U V \rightarrow V\}_\alpha$
- $V \xrightarrow{\cong} U$  isomorphism, implies  $\{V \rightarrow U\}$ .

A category with a Grothendieck Topology is called a Site.

This definition is easier to understand through an example.

*Example 8.2.2.* Let  $\mathcal{C}$  be *Top*, and  $\{U_\alpha \rightarrow U\}_\alpha$  will be a cover if the  $U_\alpha$ 's are homeomorphic to its image and  $U = \bigcup_\alpha im(U_\alpha)$ . The prescription is deciding whether a collection of subsets form a cover or not.

**Definition 8.2.3.** A Sheaf  $\mathcal{F}$  over a site  $\mathcal{C}$  is a functor  $p:\mathcal{F} \rightarrow \mathcal{C}$  such that

- For all  $S \in Ob(\mathcal{C}), x \in Ob(\mathcal{F}_S)$  and  $f : T \rightarrow S \in Mor(\mathcal{C})$  there exists a unique  $\phi : y \rightarrow x \in Mor(\mathcal{F})$  such that  $p(\phi) = f$
- For every cover  $\{S_\alpha \rightarrow S\}_\alpha,$  the following sequence is exact:

$$\mathcal{F}_S \rightarrow \prod \mathcal{F}_{S_\alpha} \rightrightarrows \prod \mathcal{F}_{S_\alpha \times_S S_\beta}.$$

**Definition 8.2.4.** A Stack in groupoids over  $\mathcal{C}$  is a functor  $p : \mathcal{S} \rightarrow \mathcal{C}$  such that

- $\mathcal{S}$  is fibered in groupoids over  $\mathcal{C}$
- For any  $U \in Ob(\mathcal{C})$  and  $x, y \in Ob(\mathcal{S}_U),$  the functor

$$\mathbf{U} \rightarrow \mathbf{Sets}$$

$$\phi : V \rightarrow U \mapsto Hom(\phi^*x, \phi^*y)$$

is a sheaf ( $Ob(\mathbf{U}) = \{(S, \chi) | S \in Ob(\mathcal{C}), \chi \in Hom(S, U)\}$ )

- If  $\phi_i : V_i \rightarrow U$  is a covering family in  $\mathcal{C}$ , any descent datum relative to the  $\phi_i$ 's, for objects in  $\mathcal{S}$ , is effective.

*Example 8.2.5.* For  $X$  a  $G$ -set (provided with a  $G$  action over it) let  $\mathcal{C} = Top$ , the category of topological spaces, and  $\mathcal{S} = [X/G]$  the category defined as follows:

$$Ob([X/G])_{\mathcal{S}} = \{f : E_S \rightarrow X\},$$

the set of all  $G$ -equivariant maps from principal  $G$ -bundles  $E_S$  over  $S \in Ob(Top)$ , and

$$Mor([X/G]) \subseteq Hom_{BG}(E_S, E'_S)$$

given by

$$\begin{array}{ccc} E_S & \longleftrightarrow & S \times_{S'} E_{S'} \\ \downarrow (proj, f) & & \downarrow 1 \times f' \\ S \times X & \longleftrightarrow & S \times X \end{array}$$

With the functor

$$\begin{aligned} p : [X/G] &\rightarrow Top, \\ (f : E_S \rightarrow X) &\mapsto S. \end{aligned}$$

By definition  $[X/G]$  is a category fibered by groupoids, and if the group  $G$  is finite  $[X/G]$  is a stack.

*8.3. Gerbes as stacks.* For simplicity we will start with a smooth  $X$  and we will consider a Grothendieck topology on  $X$  induced by the ordinary topology on  $X$  as in 8.2.2. We will follow Brylinski [5] very closely. The following definition is essentially due to Giraud [13].

**Definition 8.3.1.** A gerbe over  $X$  is a sheaf of categories  $\mathcal{C}$  on  $X$  so that

- The category  $\mathcal{C}_U$  is a groupoid for every open  $U$ .
- Any two objects  $Q$  and  $Q'$  of  $\mathcal{C}_U$  are locally isomorphic, namely for every  $x \in X$  there is a neighborhood of  $x$  where they are isomorphic.
- Every point  $x \in X$  has a neighborhood  $U$  so that  $\mathcal{C}_U$  is non-empty.

We will require our gerbes to have as band the sheaf  $\mathcal{A} = \mathbf{C}^*$  over  $X$ . This means that for every open  $U \subset X$  and for every object  $Q \in \mathcal{C}_U$  there is an isomorphism of sheaves  $\alpha : \underline{Aut}(Q) \rightarrow \mathcal{A}_U$ , compatible with restrictions and commuting with morphisms of  $\mathcal{C}$ . Here  $\underline{Aut}(Q)|_V$  is the group of automorphisms of  $P|_V$ .

The relation of this definition to the one we have used is given by the following

**Proposition 8.3.2.** To have a gerbe in terms of data  $(\mathcal{L}_{\alpha\beta})$  as in 3.1 is the same thing as to have a gerbe with band  $\mathbf{C}^*$  as a sheaf of categories.

*Proof.* Starting with the data in 3.1 we will construct the category  $\mathcal{C}_U$  for a small open set  $U$ . Since  $U$  is small we can trivialize the gerbe  $\mathcal{L}|_U$ . The objects of  $\mathcal{C}_U$  are the set of all possible trivializations  $(f_{\alpha\beta})$  with the obvious morphisms.

Conversely suppose that you are given a gerbe as a sheaf of categories. Then we construct a cocycle  $c_{\alpha\beta\gamma}$  as in [5, Prop. 5.2.8]. We take an object  $Q \in C_{U_\alpha}$  and an auto-morphism  $u_{\alpha\beta}: Q_\alpha|_{U_{\alpha\beta}} \rightarrow Q_\beta|_{U_{\alpha\beta}}$  and define  $h_{\alpha\beta\gamma} = u_{\alpha\gamma}^{-1}u_{\alpha\beta}u_{\beta\gamma} \in \text{Aut}(P_\gamma) = \mathbf{C}^*$  producing a Čech cocycle giving us the necessary data to construct a gerbe as in 3.1.  $\square$

**8.4. Orbifolds as stacks.** Now we can define the stack associated to an orbifold. Let  $X$  be an orbifold with  $\{(V_p, G_p, \pi_p)\}_{p \in X}$  as orbifold structure. Let  $\mathcal{C}$  be the category of all open subsets of  $X$  with the inclusions as morphisms and for  $U \subset X$ , let  $\mathcal{S}_U$  be the category of all uniformizing systems of  $U$  such that they are equivalent for every  $q \in U$  to the orbifold structure, in other words

$$\mathcal{S}_U = \{(W, H, \tau) | \forall q \in U, (V_q, G_q, \pi_q) \& (W, H, \tau) \text{ are equivalent at } q\}.$$

By Lemma 2.1.3 and the definition of orbifold structure it is clear that the category  $\mathcal{S}$  is fibered by groupoids. It is known, and this requires more work, that this system  $\mathcal{S} \rightarrow \mathcal{C}$  is also a stack, a Deligne-Mumford stack in the smooth case.

**8.5. Stacks as groupoids.** The following theorem has not been used in this paper, but it is the underlying motivation for the approach that we have followed.

**Theorem 8.5.1** [3]. *Every Deligne-Mumford stack comes from an étale groupoid scheme. Moreover, there is a functor  $\mathcal{R} \rightrightarrows \mathcal{U} \mapsto [\mathcal{R} \rightrightarrows \mathcal{U}]$  from the category of groupoids to the category of stacks inducing this realization.*

When  $s, t$  are smooth we can realize in a similar manner Artin stacks [3].

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