Commun. math. Phys. 27, 235—239 (1972)

On the Asymptotic Behavior of Wightman Functions in Space-Like Directions* **

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Received January 3, in revised form May 9, 1972

Abstract. The asymptotic behavior of the truncated vacuum expectation value of a product of N (unbounded) quasilocal operators, $F(x) = \langle Q_1(x_1) \ldots Q_N(x_N)\rangle_T$, is investigated for some of the separations space-like. It is shown that unless all clusters $\{x_{i_1}, \ldots, x_{i_j}\}$ are partially time-like (or light-like) separated from their complements $\{x_{i_{j+1}}, \ldots, x_{i_N}\}$, $F(x)$ decreases faster than any inverse power of the diameter of the set $\{x_1, \ldots, x_N\}$.

I. Introduction

The asymptotic behavior of the vacuum expectation value (VEV) of a product of field operators, $\langle 0|\varphi(x_1)\ldots\varphi(x_N)|0\rangle$, has been studied by many authors [1–5] for some of the separations, $x_i - x_j$, space-like. Although rapid decrease of the truncated VEV (after smearing with rapidly decreasing test functions) has been proved for $x = (x_1, \ldots, x_N)$ in some regions of $\mathbb{R}^{4N}$, there does not seem to be any general statement of the space-like asymptotic behavior of this function available in the literature[1]. In this note we extend the method of Ruelle [3] to show fast decrease in a much larger region of $\mathbb{R}^{4N}$.

II. Definitions and Results

We consider a scalar Wightman field [7], $\varphi(x)$, and define the "quasi-local" operators

$$Q_i(0) = \int \left(\prod_{j=1}^{M_i} d^4 y_j\right) f_i(y_1, \ldots, y_{M_i})\, \varphi(y_1)\ldots\varphi(y_{M_i}), \tag{1}$$

for $i = 1, \ldots, N$. Here $f_i \in \mathscr{S}$, the Schwartz space of infinitely differentiable functions which decrease (along with all derivatives) faster than any

* This work was supported in part by the U.S. Atomic Energy Commission.

** Research supported in part by the National Science Foundation.

[1] We thank R. Haag for pointing out the work of H. Araki [6] whose results for the truncated VEV of *bounded* operators are essentially equivalent to our Theorem 1. We remark that Araki's proof does not generalize to unbounded operators.

inverse power of the distance. We denote the translated $Q_i(0)$ by $Q_i(x)$. In addition to the Wightman axioms [7] we assume there is a gap in the spectrum of P^2 between the vacuum and a lowest mass. The object we wish to study is the truncated VEV [8]

$$F(x) = \langle Q_1(x_1) \dots Q_N(x_N) \rangle_T, \qquad x = (x_1, \dots, x_N). \tag{2}$$

In the following the notation $A \subset B$ will mean that $A \subseteqq B$ but that A is neither empty nor equal to B. We will use a prime to designate the complement of a set.

Define the set $\mathcal{N} = \{1, \dots, N\}$. We will make use of the diameter, $D(x)$, of the set $\{x_1, \dots, x_N\}$:

$$D(x) = \operatorname{Max}\{\|x_i - x_j\| : i, j \in \mathcal{N}\}. \tag{3}$$

Here $\|z\|$ denotes the Euclidean norm of the vector z. For $\lambda > 0$ we define the region

$$\begin{aligned} T_N(\lambda) = \{x \in \mathbb{R}^{4N} &: \text{for each } X \subset \mathcal{N} \text{ there exist } i \in X, \\ &j \in X' \text{ such that } \|\boldsymbol{x}_i - \boldsymbol{x}_j\| < \lambda |t_i - t_j|\}. \end{aligned} \tag{4}$$

Thus for $\lambda = 1$, $x \in T_N(\lambda)$ means that every cluster $\{x_{i_1}, \dots, x_{i_j}\}$ contains points which are time-like separated from points in its complement. It is in the complement of $T_N(\lambda)$, $S_N(\lambda) = T'_N(\lambda)$ where we expect $F(x)$ to be rapidly decreasing. Our main result is summarized in the following theorem:

Theorem 1. *For any $\lambda > 1$, there exist constants $c_k(\lambda)$ such that for all $x \in S_N(\lambda)$ and all k*

$$|F(x)| < c_k(\lambda)\,(1 + D(x))^{-k}. \tag{5}$$

Thus as long as at least one cluster of points separates in a space-like direction from its complement (even if the time-like separation within clusters increases) the truncated function is rapidly decreasing.

It is also instructive to treat more explicitly the situation where individual clusters retain their identity as the space-like separation between them becomes large. In the case of two clusters we are thus led to consider the function

$$G(x) = \langle 0 | Q_1(x_1) \dots Q_k(x_k)\, E_0^{\perp}\, Q_{k+1}(x_{k+1}) \dots Q_N(x_N) | 0 \rangle. \tag{6}$$

Here $E_0^{\perp} = I - |0\rangle\langle 0|$ and for all $i \in \mathcal{N}_0 = \{1, \dots, k\}$ and $j \in \mathcal{N}'_0$ we have

$$\|\boldsymbol{x}_i - \boldsymbol{x}_j\| \geqq \lambda |t_i - t_j|. \tag{7}$$

The number λ is fixed and > 1. In order to see how the behavior of $G(x)$ is related to that of the truncated VEV we expand $G(x)$ in truncated

VEV's. We first introduce the abbreviation

$$\langle X \rangle_T = \langle Q_{i_1}(x_{i_1}) \dots Q_{i_j}(x_{i_j}) \rangle_T \tag{8}$$

where $X = \{i_1, \dots, i_j\}$ and it is understood that the numbers $i_1, \dots, i_j$ appear in numerical order in Eq. (8). Then we can write

$$G(x) = \sum_{\text{partitions}} \langle X_{i_1} \rangle_T \dots \langle X_{i_j} \rangle_T \tag{9}$$

where the sum is over a subset of the partitions $\{X_{i_1}, \dots, X_{i_j}\}$ of X into disjoint subsets. The partitions which do not appear in Eq. (9) are those which are subpartitions of $\{\mathcal{N}_0, \mathcal{N}_0'\}$.

Here both the limitations and strengths of Theorem 1 become apparent: Even when x satisfies condition (7), the right hand side of Eq. (9) contains factors of the form (8) with $(x_{i_1}, \dots, x_{i_j}) \in T_j(1)$, *i.e.*, in a region where all clusters are partially *time-like* separated from their complements. If $\langle X \rangle_T$ is not bounded in this region it is easy to see that $G(x)$ can grow even if the condition (7) is satisfied.

In any reasonable theory of interactions, correlations should decrease with increasing time-like separation although at a much slower rate than in space-like directions. Thus on physical grounds we are certainly justified in assuming that for all $\langle X \rangle_T$ of the form (8)

$$|\langle X \rangle_T| < \text{constant} \tag{10}$$

although it is an open question whether (10) follows from the Wightman axioms. As an easy corollary of Theorem 1 we find the following:

Theorem 2. *If* $G(x)$ *is as given in Eq.* (6) *and if* (10) *is satisfied, then with*

$$s^2 = \mathrm{Min}\{-(x_i - x_j)^2 : i \in \mathcal{N}_0, j \in \mathcal{N}_0'\} \tag{11}$$

there exist constants $c_n(\lambda)$ *such that*

$$|G(x)| < c_n(\lambda)\,(1+s)^{-n} \tag{12}$$

for all x *satisfying condition* (7).

III. Proof of Theorem 1

We first present a lemma concerning the structure of $T_N(\lambda)$. In the following a superscript c will denote closure.

Lemma 1. *If* $\lambda_1 > \lambda_2$ *then*

$$T_N^c(\lambda_2) \subset T_N(\lambda_1) \cup \{x \in \mathbb{R}^{4N} : D(x) = 0\}\,. \tag{13}$$

Proof. It is easy to see that $x \in T_N^c(\lambda_2)$ if and only if for each $X \subset \mathcal{N}$ there exist $i \in X, j \in X'$ such that

$$\|\boldsymbol{x}_i - \boldsymbol{x}_j\| \leqq \lambda_2 |t_i - t_j|\,.$$

Assuming $x \in T_N^c(\lambda_2)$ and $D(x) > 0$ we consider a subset $X \subset \mathcal{N}$. For each i we define the set

$$\mathscr{E}_i = \{j \in \mathcal{N} : x_i = x_j\} \tag{14}$$

and let $\hat{X} = \bigcup_{i \in X} \mathscr{E}_i$. Thus $\hat{X}$ is an enlarged X which contains all integers, i, whose corresponding x_i is equal to some x_j with $j \in X$.

If $\hat{X} \subset \mathcal{N}$ there exist $k \in \hat{X}, j \in \hat{X}'$ such that

$$\|\boldsymbol{x}_k - \boldsymbol{x}_j\| \leqq \lambda_2 |t_k - t_j| < \lambda_1 |t_k - t_j| . \tag{15}$$

If $\hat{X} = \mathcal{N}$, we choose any $j \in X'$ and note the existence of a $k \in \mathscr{E}_j'$ such that the inequalities (15) are satisfied. In either case there is an $i \in X$ such that $x_k = x_i$. This shows that $x \in T_N(\lambda_1)$ and hence the lemma is proved.

We now define two measures of the space-like separation between clusters in $\{x_1, \ldots, x_N\}$. Thus for $X \subset \mathcal{N}$ let

$$R_X(x) = \operatorname{Min}\{\|\boldsymbol{x}_i - \boldsymbol{x}_j\| - |t_i - t_j| : i \in X, j \in X'\} , \tag{16a}$$

$$S_X^2(x) = \operatorname{Min}\{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2 - |t_i - t_j|^2 : i \in X, j \in X'\} \tag{16b}$$

and define

$$R(x) = \operatorname{Max}\{R_X(x) : X \subset \mathcal{N}\} ,$$

$$S^2(x) = \operatorname{Max}\{S_X^2(x) : X \subset \mathcal{N}\} .$$

The next lemma summarizes the relevant relationships between $R(x)$, $D(x)$ and $S(x)$:

Lemma 2. *For $\lambda > 1$ there exists $\varepsilon_N(\lambda) > 0$ such that for all $x \in S_N(\lambda)$*

$$\varepsilon_N(\lambda)\, D(x) \leqq R(x) \leqq S(x) \leqq D(x) . \tag{18}$$

Proof. The only part of (18) which does not follow directly from the definitions is the existence of $\varepsilon_N(\lambda)$. To show this we define the compact set

$$C_N(\lambda) = S_N(\lambda) \cap \{x \in \mathbb{R}^{4N} : x_1 = 0, D(x) = 1\} . \tag{19}$$

Note that if $x \in C_N(\lambda)$, then $R(x) > 0$, for if $R(x) \leqq 0$ then by definition $x \in T_N^c(1)$ and thus by virtue of Lemma 1, $x \in T_N(\lambda)$. Since $C_N(\lambda)$ is compact and $R(x)$ is a continuous function of x we must have

$$R(x) \geqq \varepsilon_N(\lambda) > 0$$

for all $x \in C_N(\lambda)$. The translation invariance and scale invariance of our definitions then imply (18).

It is now a simple matter to prove Theorem 1 using the methods of Ruelle [3, 9] in conjunction with Lemma 2. The proof is omitted.

Acknowledgements. It is a pleasure to thank Professor E. Wichmann for discussions and Professor D. Williams for a critical reading of the manuscript.

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