Commun. Math. Phys. 73, 197–246 (1980)

Communications in Mathematical Physics

Debye Screening*

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Abstract. The existence and exponential clustering of correlation functions for a classical coulomb system at low density or high temperature are proven using methods from constructive quantum field theory, the sine gordon transformation and the Glimm, Jaffe, Spencer expansion about mean field theory. This is a vindication of a belief of long standing among physicists, known as Debye screening. That is, because of special properties of the coulomb potential, the configurations of significant probability are those in which the long range parts of r^{-1} are mostly cancelled, leaving an effective exponentially decaying potential acting between charge clouds. This paper generalizes a previous paper of one of the authors in which these results were obtained for a special lattice system. The present treatment covers the continuous mechanics situation, with essentially arbitrary short range forces and charge species. Charge symmetry is not assumed.

Introduction

In two previous papers [2], we have studied the quantum statistical mechanics of continuous systems with pair potentials such as the Yukawa $r^{-1}e^{-\alpha r}$, $\alpha>0$. Rigorous results on the existence and clustering of correlation functions were obtained using a type of cluster expansion which is convergent for a region of parameters physically associated with the plasma phase. The reason for studying such potentials is that they provide a first step towards obtaining the same type of results for the matter system, a system of positive and negative charges, one species of which is fermions, interacting by the coulomb law r^{-1}. They have in common the difficulties that arise from the singularity of the potential at the origin. Correlation functions for the matter system are the next most obvious quantities to inquire after, following the papers of Dyson and Lenard [5] on the stability of matter and Lieb and Lebowitz [12] on the existence of the thermodynamic functions. For this purpose it is necessary to control the long range part of the $\frac{1}{r}$ interaction; this was a major motivation for our present effort.

* Supported in part by N.S.F. Grant PHY 77-02187 and MCS 79-02490

0010-3616/80/0073/0197/$10.00

One of the authors has studied a discretized version of the coulomb system in classical statistical mechanics [1]. Herein two species of equal fugacities and equal opposite charges interacted by "$\frac{1}{r} = -\frac{4\pi}{\Delta_l}$", the discretized version of $\frac{1}{r}$, with Δ_l the discrete Laplacian on a lattice of side l. In a suitable range of parameters the exponential screening and the existence of correlation functions was rigorously proven.

The present paper greatly generalizes [1], to essentially arbitrary short range forces, to the continuum situation, to arbitrary charge species; charge symmetry is not assumed. In a classical pure coulomb system collapse will occur, so that the short range forces will be required to ensure the stability of the system. As well, they will have to have some exponential fall off so as not to interfere with coulomb cancellations at long distances. In the discretized version previously considered, the lattice spacing l provided a short distance cut off and ensured stability.

The mathematical development is parallel, using the sine gordon transformation [6] and the Glimm, Jaffe, Spencer expansion about mean field theory [10]. Additional difficulties with the short range potential are handled with a Mayer expansion for the short range portion of the interaction. In this connection we derive interesting new estimates for the truncated correlation functions, using expressions from [3].

The present paper rigorously proves screening in a regime corresponding to a dilute system in ionic solutions. This settles a debate as to whether indeed there is exponential screening in such a system [13, 14]. For an investigation which complements the present program by obtaining weaker results for special systems but valid for all ranges of parameters, see [7]. Our results relate to the plasma phase in the quantum statistical mechanics of matter, a high temperature, high density region. At present we do not see how to tackle the quantum mechanical system.

The thermodynamic limit is approached through a sequence of systems whose volumes increase to infinity. Conditions on the boundaries and on the fugacities are carefully tailored to avoid any difficulties with surface charge. This treatment is somewhat arbitrary and subject to some generalization, however a satisfactory study of surface phenomena is beyond our present techniques.

There is another aspect in which our treatment is not complete: the region of convergence of our expansion is not uniform in the relative sizes of the activities; thus, if one species with charge that is fractional with respect to the other species is present in much lower density than the other species the values of the constants governing convergence deteriorate. While we are still undecided whether this is an artifact of our estimates or something more physical, we are able to prove that integral charges can screen two fixed fractional charges essentially as well as they screen integral charges.

The present paper is largely self-contained, only a few specific lemmas from previous papers are used. Familiarity with the organization ideas of cluster expansions in constructive field theory is very helpful, [10, 11], however we have made an attempt in Sect. 5 to explain some of the ideas in the cluster expansion.

1. Definition of the System

The physical system we wish to study consists of s species of particles, species i having charge e_i. For simplicity and consistent with the physical situation we assume the e_i are integers. These classical particles have an interaction energy V_0, constructed from two-body translation invariant potentials, Coulomb and short range parts

$$V_0 = \frac{1}{2}\int : J(x)\frac{1}{4\pi|x-y|}J(y) : + \frac{1}{2}\sum_{i,j}\int : \sigma_i w_{ij}\sigma_j : \tag{1.1}$$

$$= U_0 + W_0 . \tag{1.2}$$

σ_i is the density of species i, it is a sum of δ-functions at the positions of particles of species i. J is the charge density

$$J = \sum e_i \sigma_i . \tag{1.3}$$

The colons in (1.1) indicate that the terms involving $w_{ij}(0)$ and "$\frac{1}{4\pi|0|}$" which arise when the definitions of J, σ are substituted into (1.1), are to be dropped. Physically, this corresponds to excluding self-interactions of point particles. The integrals in (1.1) are over $\mathbb{R}^3 \times \mathbb{R}^3$.

In terms of parameters λ, l_D, to be discussed later, we rewrite (1.1), (1.2) as

$$V_0 = U_1 + W_1 , \tag{1.4}$$

$$U_1 = \frac{1}{2}\int : J(x)\frac{1-e^{-\frac{|x-y|}{\lambda l_D}}}{4\pi|x-y|}J(y) : . \tag{1.5}$$

We have added a specific short range interaction to the Coulomb term and included a compensating term in W_1.

The statistical mechanics of this system is approached by taking grand canonical ensemble averages for a sequence of ascending volumes. However we diverge from what might be the expected procedure by taking an infinite volume limit in two stages, with the Coulomb interaction modified by boundary conditions. Let $\Lambda \subset \Lambda'$ be rectangular boxes given as a union of unit lattice cubes in $\mathbb{R}^3$. We replace U_1 by $U - d_0$, and W_1 by W.

$$V = U + W - d_0 , \tag{1.6}$$

$$U = \frac{1}{2}\int J\left(-\frac{1}{\Delta} - \frac{1}{-\Delta + \frac{1}{\lambda^2 l_D^2}}\right)J = \frac{1}{2}\int JuJ , \tag{1.7}$$

$$d_0 = \frac{1}{2}\sum_\alpha u_0(x_\alpha, x_\alpha)e_{i(\alpha)}^2 , \tag{1.8}$$

$$u = \left(\frac{1}{-\Delta} - \frac{1}{-\Delta + \frac{1}{\lambda^2 l_D^2}}\right). \tag{1.9}$$

The Laplacian, Δ, used in constructing the kernel u, and throughout the paper unless otherwise indicated, is constructed using Dirichlet boundary conditions on $\partial\Lambda$. (If the infinite volume Laplacian, Δ_0, were used U_1 would equal $U-d_0$.) u_0 is obtained by replacing Δ by Δ_0 in (1.9). The integral in U is over $\Lambda\times\Lambda$. W is defined by replacing integrals over $\mathbb{R}^3\times\mathbb{R}^3$ in W_1 by integrals over $\Lambda'\times\Lambda'$. Thus we have two volumes. Particles inside the larger one Λ' interact via a short range pair interaction W. Particles inside Λ interact in addition by a long range interaction U which in particular contains the long range part of the Coulomb interaction. The Coulomb interaction has zero boundary conditions (grounded boundary).

To study the statistical mechanics of the grand canonical ensemble of the system, we define, for A a functional of the σ_i inside Λ, $I(A)$:

$$I(A)=\sum_N \frac{z^N}{N!}\int e^{-\beta V}A. \tag{1.10}$$

Here $N!$ stands for $\prod_i (N_i!)$ and z^N stands for $\prod_i z_i^{N_i}$ where z_i is the fugacity associated to species i. The integral is over the positions of the $\sum N_i$ particles in Λ'. Since we have a fugacity for each charge, we are not enforcing neutrality, but later we will impose a condition on the fugacities that at least approximately enforces it in an average sense. We define

$$\langle A\rangle = \lim_{\Lambda'\nearrow\mathbb{R}^3} I(A)/I(1) \tag{1.11}$$

and

$$Z=\lim_{\Lambda'\nearrow\mathbb{R}^3} I(1)/Z_0\,. \tag{1.12}$$

Z_0 is $I(1)$ calculated with U set equal to zero. The existence of the Λ' limits will be discussed when we come to the Mayer expansion. Z is a normalized partition function.

Equations (1.11) and (1.12) express our strategy of taking the infinite volume limit in two stages. The easy one is $\Lambda'\nearrow\mathbb{R}^3$. We still of course have to take $\Lambda\nearrow\mathbb{R}^3$.

2. The Sine-Gordon Transformation

We construct a Gaussian measure $d\mu_0(\phi)$ on a measure space of continuous functions, $\phi(x)$, $x\in\Lambda$, with covariance u

$$\int d\mu_0(\phi)e^{i\int f\phi}=e^{-1/2\int fuf}. \tag{2.1}$$

It is then straightforward to show that

$$e^{-\beta U}=\int d\mu_0(\phi)e^{i\beta^{1/2}\sum_\alpha e_{i(\alpha)}\phi(x_\alpha)}. \tag{2.2}$$

We define

$$\tilde z_i = z_i e^{1/2\beta e_i^2 u_0(x,x)}. \tag{2.3}$$

One then has

$$Z=\int d\mu_0 Z(\phi), \tag{2.4}$$

where

$$Z(\phi)=\lim_{\Lambda'\to\mathbb{R}^3} Z_0^{-1}\sum\frac{1}{N!}\tilde{z}^N\int e^{-\beta W}\cdot e^{i\beta^{1/2}\sum e_{i(\alpha)}\phi(x_\alpha)}. \tag{2.5}$$

We have interchanged the Λ' limit and the $d\mu_0$ integral. This is justified by the Lebesgue dominated convergence theorem. The requisite bound is supplied by the Mayer expansion in the next section. We write

$$Z(\phi)=e^M, \tag{2.6}$$

where M is the Mayer expansion. For $\mathscr{A}(\phi)$ a functional of the $\phi(x)$ we define

$$\langle\mathscr{A}(\phi)\rangle=\int d\mu_0 Z(\phi)\mathscr{A}(\phi)/Z. \tag{2.7}$$

3. The Mayer Series I

We consider the relation yielding M, the Mayer series. We start with M' defined by

$$e^{M'}=Z_0^{-1}\sum\frac{1}{N!}\tilde{z}^N\int e^{-\beta W}e^{i\beta^{1/2}\sum e_{i(\alpha)}\phi(x_\alpha)}. \tag{3.1}$$

Following the notation of [3] closely, we may expand

$$M'=\sum_1^\infty K^{(N)}\tilde{z}^N-\log Z_0, \tag{3.2}$$

where the Mayer expansion is developed from

$$v_2=e_i\frac{e^{-\frac{|x-y|}{\lambda l_D}}}{4\pi|x-y|}e_j+w_{ij}(x-y) \tag{3.3}$$

and

$$v_1=-\frac{i}{\beta^{1/2}}e_i\phi(x). \tag{3.4}$$

v_2 and v_1 are the two-body and one-body potentials, respectively, used in [3]. Estimates which prove convergence of our expansions are obtained in Appendix 1. We now introduce the variables

$$\varepsilon_i(x)=e^{i\beta^{1/2}e_i\phi(x)}-1 \tag{3.5}$$

and assert that M' may be rearranged in the form

$$M'=\sum_i\int\varrho_i(x)\varepsilon_i(x)+\frac{1}{2!}\sum_{i,j}\int\varrho_{i,j}(x,y)\varepsilon_i(x)\varepsilon_j(y)+\dots, \tag{3.6}$$

where each $\varrho_{i_1,i_2,\dots,i_t}(x_1,\dots,x_t)$ is independent of $\phi(x)$. This may be understood as follows: within each term $K^{(N)}\tilde{z}^N$ in (3.2) there are factors $\tilde{z}_i'(x)=\tilde{z}_i e^{i\beta^{1/2}e_i\phi(x)}$. We expand each such factor

$$\tilde{z}_i'(x)=(\tilde{z}_i'(x)-\tilde{z}_i)+\tilde{z}_i \tag{3.7}$$

and resum (3.2) to obtain a power series in $\tilde{z}' - \tilde{z}$, i.e. (3.6). The constant term in (3.6) is missing because the normalization Z_0 has been chosen in such a way that $M' = 0$ if $\phi = 0$. Each ϱ in (3.6) depends on Λ'. We let $\Lambda' \nearrow \mathbb{R}^3$ (anticipating results in Appendix 1). M is thus the expression obtained from (3.6) by taking the limit $\Lambda' \nearrow \mathbb{R}^3$ in each ϱ. Note that the variables $\varepsilon_i(x)$ in (3.6) vanish for $x \notin \Lambda$ by our boundary conditions on u. We will require

$$\sum \varrho_i e_i = 0. \tag{3.8}$$

Here ϱ_i is the infinite volume limit of $\varrho_i(x)$, $\lim_{\Lambda' \nearrow \mathbb{R}^3} \varrho_i(x)$. Equation (3.8) is automatic in the charge symmetric situation. It is capable of some weakening, but we require some such condition, and it is the most effective condition to impose. [In Eq. (6.5) the linear terms in δ may be added because of (3.8), they are needed in later estimates.]

We would conjecture that any infinite volume equilibrium state obtained by any limiting procedure (near our range of parameters) can also be obtained by our limiting procedure, and with a choice of ϱ_i satisfying (3.8).

4. Notation and Description of Results

We begin with a discussion of the basic parameters and their dimensions. In our notation the charges, e_i, are integers and thus dimensionless. The unit of electric charge has been absorbed into β, the inverse temperature, which thus becomes a parameter with the dimensions of length. It is known as the Landau Length. The combinations

$$l_D = \left(\sum z_i e_i^2 \beta\right)^{-1/2}, \qquad \tilde{l}_D = \left(\sum \varrho_i e_i^2 \beta\right)^{-1/2} \tag{4.1}$$

have the dimensions of length. l_D is known as the Debye Length. These lengths are natural units to measure screening. $\tilde{l}_D$ can be related to l_D by using our results in Appendix 1.

In (1.5) we introduced a dimensionless parameter λ. A priori our thermodynamic limit will depend on λ. We will introduce a norm $\| \ \|_\alpha$ on the two body potential v_2 in W and require that [see (9.94)]

$$\left(\sup_i z_i\right) \beta \frac{1}{\lambda^2} \|v\|_\alpha \leqq c. \tag{4.2}$$

$\|v_2\|_\alpha$ is defined in Appendix 1, (A1.9). This dual use of λ saves the introduction of two related parameters. α is an inverse length and serves to specify an exponential fall off for v_2. We impose a condition

$$\alpha \geqq (1 - \delta_1) \frac{1}{\tilde{l}_D}, \qquad \delta_1 > 0 \tag{4.3}$$

so that the tail of v_2 does not destroy the screening properties of $\frac{1}{r}$.

The parameters β, $\tilde{l}_D$, λ, and α are external to the proofs. The proof itself involves an expansion depending on two lengths L and L'. Basically we always work in units with $\tilde{l}_D=1$, but we keep $\tilde{l}_D$ in most formulas to emphasize physical parameters. In particular "unit" cubes are of side $\tilde{l}_D$, and L, L', and $|X|$ (the volume of region X) are understood measured in units of $\tilde{l}_D$ (or $\tilde{l}_D^3$) when not appearing in dimensionless expressions. $L'/4\tilde{l}_D$ and $\tilde{l}_D/L$ are both large integers.

Our estimates are all valid provided they are preceded by the quantifiers: if L' is fixed large enough, if L is fixed small enough, and if λ is sufficiently small, then for $\beta/\tilde{l}_D$ sufficiently small depending on L, L', λ, ... We occasionally omit this qualification. c's are used for strictly positive constants. Often the same c's are used for different constants in unrelated equations.

In Sect. 9.9 we present the most general situation in which we prove Debye shielding. There, also, is the complete set of conditions on parameters and constants. We also state a result on the screening of fractional charges. In order to provide some feeling for these rather complex theorems we will present a special case which has some interest in its own right.

The System

There are two species of particles with equal fugacities, z, and equal and opposite charges $e_i=\pm 1$. There is an infinite repulsive hard core of radius R about each particle and no other short range forces.

The Observables

We consider observables of the form

$$A=\int (f(x_1,\dots,x_w)\sigma(x_1)\dots\sigma(x_w)\,. \tag{4.4}$$

$\sigma=\sigma_i(x)$ is the density of species i at x. The species indices of f, σ in (4.4) are suppressed. f is bounded and compactly supported. We say that A is supported in $X\subset\mathbb{R}^3$ if f vanishes whenever one of its arguments, $x_1,\dots,x_w$, is not in X.

Theorem 4.1. *For any given c, there is a c_1 (depending on c) such that if*

$$\xi_1\equiv\beta/R<c$$

and if

$$\xi_2\equiv zR^3<c_1$$

then the following limit exists

$$\langle A\rangle=\lim_{\Lambda\nearrow\mathbb{R}^3}\langle A\rangle_\Lambda\,, \tag{4.5}$$

where A is any observable.

Theorem 4.2. *For any $l'>l_D$ and any given c, there is a c_2 (depending on l' and c) such that if*

$$\xi_1<c$$

and

$$\xi_2 < c_2$$

then

$$|\langle AB\rangle - \langle A\rangle\langle B\rangle| \leqq c_A c_B e^{-d/l'}, \tag{4.6}$$

where d is the distance between the supports of A and B.

Remarks. 1. We have stated a theorem for a charge symmetric system (interchange of species is a symmetry) but our results in Sect. 9.9 do not require this. The pleasant feature of charge symmetric systems is that it is possible to prove that the limit in (4.5) is independent of λ (for permissible λ). In systems without charge symmetry we have imposed a neutrality condition [see (9.92)] which depends on λ.
2. Degeneracy: We let M_1 be the first term in the series (3.6) for M, linear in ε's. It is the dominant term that controls our development of the cluster expansion. In the charge symmetric situation, e^{M_1} assumes its maximum at $\phi = n\tau$, $n = 0, \pm 1, \ldots$. τ is the period of M. If on the other hand there are more than two species present, for example charges $e_i = \pm 1, \pm 2$, then e^{M_1} has local maxima at $\phi = 1/2n\tau$. If the activity of the unit charge species is very small, these secondary maxima are very nearly degenerate with the true maxima. We refer to this situation as "degeneracy". Physically one has a very low density of fractional charges with are to be screened by integral charges. Parts of our proof (particularly the ratio of Z's in Appendix 4) run into difficulties in degenerate cases and we impose condition (9.96) to avoid degeneracy. We do not know, particularly in the light of Theorem 9.15 on the screening of fractional charges, whether this reflects a physical phenomenon or a failure of our procedure.
3. The statements of Theorems 4.1 and 4.2 have been patterned on Theorem 2.1 in [1]. The transcription of the results in Sect. 9.9 to this case is straightforward, but not immediate. The principal ingredient to be supplied is a stability result for a Yukawa interaction with hard cores present, in particular the following inequality

$$\frac{1}{2}\sum_{i \neq j} e_i e_j \frac{e^{-\mu r_{ij}}}{r_{ij}} \geqq -\frac{1}{2}\sum_i \frac{e_i^2}{R}, \qquad r_{ij} \geqq 2R \quad \text{all} \quad i \neq j.$$

We use a slightly stronger form of this inequality. Let $\phi(\mathbf{r})$ be the unique continuous function satisfying

$$(-\Delta + \mu^2)\phi(\mathbf{r}) = 0, \qquad |\mathbf{r}| \neq R$$

$$\phi(\mathbf{r}) = \frac{e^{-\mu|\mathbf{r}|}}{|\mathbf{r}|}, \qquad |\mathbf{r}| \geqq R.$$

Then

$$\frac{1}{2}\sum_{i \neq j} e_i e_j \phi(r_{ij}) \geqq -\frac{1}{2}\sum_i \frac{e_i^2}{R}.$$

5. A Procedural Introduction

Much of the complexity of the present paper is due to the short range interactions, W in (1.6). In particular, if W were identically zero the paper would be vastly simplified; E would be zero. A number of improvements and simplifications over the treatment in [1] would be evident. We suggest the reader set E to zero on a first reading.

One feature of the present paper (in common with the papers of Gallavotti et al. [8]) is the use of a Gaussian process whose covariance (propagator) is the inverse of a fourth order differential operator. This has as a negative aspect the result that many theorems from constructive field theory specific to second order operators can not be employed. However the positive advantages are very pleasing.
In particular it is never necessary to normal order any expressions in ϕ!

Appendix 1 presents estimates on the fall off of the truncated correlation functions in classical statistical mechanics. These arise naturally from the development of the Mayer series given in [3]. They serve a key role in controlling the effects of the short range potentials.

Sections 1 through 6, and that portion of Sect. 7 preceding the construction of g, may be viewed as preliminary. They involve little difficulty and technicalities. Our construction of the Peierls expansion essentially is an elaboration of the approximation often used by physicists (for g^2 large)

$$e^{g^2(\cos x - 1)} \cong \sum_{n=-\infty}^{\infty} e^{-g^2/2(x-2\pi n)^2}. \tag{5.1}$$

Several of our later estimates are involved in controlling the error, justifying the value of the approximation. This is only superfically different from the use of approximate projections in [1] and [10] to derive a Peierls expansion.

The cluster expansion is detailed in Sect. 8. Most of the rest of the paper, Sect. 9 and the four Appendices, is devoted to estimates ensuring the convergence of the expansion given in Sect. 8. These estimates from the core of the research. We have used terms like "Vacuum Energy" because our methods have been taken from Field theory. However these techniques are sufficiently removed from their origin that the objectives and difficulties have changed. In particular we have no concern with "divergences" which are the central fact of Field theory. One of our principal questions is the physical meaning, if any, of some of our field theoretic concepts, especially phase boundaries, in this new context.

The remainder of this section is devoted to an informal discussion of some of the ideas behind the complicated expansion in Sect. 8.

We consider a quantity of the following type

$$L = N^{-1} \int d\phi e^{-1/2 \Sigma \phi_\alpha A_{\alpha\beta} \phi_\beta} e^{D(\phi)} P(\phi), \tag{5.2}$$

where $d\phi$ is product Lebesgue measure

$$d\phi = \prod_\alpha d\phi_\alpha. \tag{5.3}$$

$A_{\alpha\beta}$ is a positive definite matrix, $\phi = \{\phi_\alpha\}$, $\alpha \in I$. We take I to be a finite set. Except in this last restriction L resembles $I(\mathscr{A}(\phi))$. P plays the part of the observable. We take it to be a trigonometric polynomial. N is defined so that $L=1$ if $D=0$ and $P=1$.

We now suppose that I is a very large set of variables but P depends only on a small subset $I_1 \subset I$

$$I = I_1 \cup I_2, \quad \phi_1 = \{\phi_\alpha\}_{\alpha \in I_1}, \quad \phi_2 = \{\phi_\alpha\}_{\alpha \in I_2}$$
$$P(\phi) = P(\phi_1) \quad I_1 \cap I_2 = \phi. \tag{5.4}$$

We also suppose that D has a natural decomposition

$$D = D_1 + D_2 + D_{12}, \quad D_i = D_i(\phi_i), \tag{5.5}$$

where D_1 depends only on ϕ_1, D_2 on ϕ_2.

The cluster expansion is a method of studying the *approximate* factorisation

$$L = L_1 Z_2 + R, \tag{5.6}$$

where

$$L_1 = N^{-1} \int d\phi e^{-1/2 \Sigma \phi A \phi} e^{D_1} P \tag{5.7}$$

(we have suppressed α and β, but the sum is still over $\alpha, \beta \in I$)

$$Z_2 = N^{-1} \int d\phi e^{-1/2 \Sigma \phi A \phi} e^{D_2}. \tag{5.8}$$

R is a remainder which under the right circumstances will be small. Notice that if D_1, D_2, D_{12} vanish then R vanishes, $(Z_2 = 1)$. The point about the factorisation (5.6) is that L_1 will only involve a "small" number of non Gaussian variables, and so is easy to estimate, whilst Z_2 is a simpler quantity than L because P has been separated out.

We will now obtain, by a formal argument, an expression for R which puts it in the same form as L so that one can then iterate the approximate factorisation. We write L in the form

$$L = e^{1/2 \Sigma \partial_\alpha C_{\alpha\beta} \partial_\beta} e^D P|_{\phi=0}, \tag{5.9}$$

where $\partial_\alpha = \dfrac{\partial}{\partial \phi_\alpha}$ and C is the matrix inverse

$$C_{\alpha\beta} = (A^{-1})_{\alpha\beta}. \tag{5.10}$$

In (5.9), the exponential is to be expanded and ϕ set to zero after the derivatives are performed. The reader can verify that the result is a formal expansion for L. Corresponding to the decomposition, $\phi = (\phi_1, \phi_2)$, we write

$$C = \begin{pmatrix} C_1 & C_{12} \\ C_{12} & C_2 \end{pmatrix}, \quad C_0 = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}, \tag{5.11}$$

$$C(s) = sC + (1-s)C_0 = \begin{pmatrix} C_1 & sC_{12} \\ sC_{12} & C_2 \end{pmatrix}, \tag{5.12}$$

$$D(s) = D_1 + D_2 + sD_{12}. \tag{5.13}$$

C_0 is called a "diagonalised" covariance. We let

$$L^{(s)} = e^{1/2 \Sigma \partial C(s) \partial} e^{D(s)} P|_{\phi=0} \tag{5.14}$$

noting that if $s=1$, $L^{(1)}=L$ and if $s=0$

$$L^{(0)}=L_1 Z_2. \tag{5.15}$$

Therefore by the fundamental theorem of calculus

$$R=\int_0^1 \frac{d}{ds} L^{(s)} ds=\int_0^1 ds\, e^{1/2\Sigma\partial C(s)\partial} e^{D(s)} \kappa P\Big|_{\phi=0}, \tag{5.16}$$

where κ is the differential operator

$$\kappa=D_{12}+\tfrac{1}{2}\sum(\partial+\partial D(s))C_{12}(\partial+\partial D(s)). \tag{5.17}$$

By applying (5.9) backwards with C replaced by $C(s)$

$$R=\int_0^1 ds N_s^{-1}\int d\phi e^{-1/2\Sigma\phi A(s)\phi} e^{D(s)}\kappa P, \tag{5.18}$$

where $A(s)=C(s)^{-1}$.

The cluster expansion is generated by iterating this basic step. Thus we expand the κP in (5.18) into terms that depend on small subsets of the variables $\{\phi_\alpha\}$

$$\kappa P=\sum_j P_j, \qquad R=\sum_j R_j \tag{5.19}$$

and choose for each j a new division of I into a small subset and its complement. We then apply (5.6) to each R_j and so on until I is exhausted.

This describes the process by which the cluster expansion in Sect. 8 is generated up to detailing how I is to be partitioned at each stage. By referring to Sect. 8 the reader will see that the Peierls expansion controls this step.

6. The Peierls Expansion

In the expression for $Z(\phi)$

$$Z(\phi)=e^{\sum_i \varrho_i \int (e^{i\beta^{1/2} e_i \phi}-1)+E'} \tag{6.1}$$

we wish to exhibit the fact that for the portions of ϕ space that dominate the integrals over $Z(\phi)$

$$\sum_i \varrho_i (e^{i\beta^{1/2} e_i \phi}-1) \cong -\tfrac{1}{2}\sum_i \varrho_i \beta e_i^2 (\phi-n\tau)^2$$

$$\cong -\tfrac{1}{2}\frac{1}{l_D^2}(\phi-n\tau)^2 \tag{6.2}$$

for some integer n, where τ is the least common multiple of the periods of the exponentials (associated to non-zero e_i). We consider the lattice of cubes of side L, $\{\Omega_\alpha\}$, and define functions $h(x)$ that assume on each Ω_α some constant value, an integral multiple of τ. The $h(x)$ may be discontinuous at cube boundaries. We write

$$Z(\phi)=\sum_h e^{-\frac{1}{2l_D^2}\int(\phi-h)^2} e^G e^{E'}, \tag{6.3}$$

where the sum is over all possible such $h(x)$ defined on Λ. The function G is defined to absorb all the damage of approximation (6.2) and is hopefully "small". The sum over h's is the Peierls expansion. $\sum$ is the closed set along which $h(x)$ has a step discontinuity. Edges along $\partial\Lambda$ are part of $\sum$ if $h \neq 0$ in the corresponding cube along $\partial\Lambda$. (One may imagine $h=0$ outside Λ.) $\sum$ is called the Peierls contour for h. We let $\hat{\sum}$ be the set of unit lattice cubes in Λ whose distance from $\sum$ is less than L'.

It is convenient to now give an expression for G. For the lattice of cubes, $\{\Omega_\alpha\}$, of side L we define for cube Ω_α

$$A = A_\alpha = L^{-3} \int_{\Omega_\alpha} \phi(x)dx \tag{6.4}$$

and

$$\delta(x) = \phi(x) - A_\alpha(x) \quad \text{for} \quad x \in \Omega_\alpha .$$

One then has

$$e^G = \prod_\alpha r_\alpha(A_\alpha) \cdot e^{\sum \varrho_i \int \left[e^{i\beta^{1/2} e_i \delta} - 1 - i\beta^{1/2} e_i \delta + \frac{\beta}{2} e_i^2 \delta^2\right]}$$
$$\cdot e^{\sum \varrho_i \int (e^{i\beta^{1/2} e_i A} - 1)(e^{i\beta^{1/2} e_i \delta} - i\beta^{1/2} e_i \delta - 1)}, \tag{6.5}$$

$$r(A) = \frac{e^{\sum \varrho_i (e^{i\beta^{1/2} e_i A} - 1) L^3}}{\sum_n e^{-\frac{1}{2\bar{l}_D^2}(A - n\tau)^2 L^3}} . \tag{6.6}$$

7. Translation of ϕ

We study the expression for Z derived in the last section, $I(\mathscr{A}(\phi))$ may be treated in just the same way.

$$Z = \sum_h \int d\mu_0 e^{E'} e^{-\frac{1}{2\bar{l}_D^2}\int(\phi - h)^2} e^G \tag{7.1}$$

for each h we will define a $g_h(x) = g(x)$ and write

$$\phi(x) = \psi(x) + g(x) . \tag{7.2}$$

The variables $\psi(x)$ will replace $\phi(x)$. It is desired to modify the Gaussian measure μ_0 to include effects of the quadratic (ψ^2) terms from

$$-\frac{1}{2\bar{l}_D^2}(\phi - h)^2 = -\frac{1}{2\bar{l}_D^2}(\psi + g - h)^2 . \tag{7.3}$$

We also wish to include the part of E' quadratic in ψ

$$-\tfrac{1}{2}\int \psi v \psi \tag{7.4}$$

in the measure. We define

$$E = E' + \tfrac{1}{2}\int \psi v \psi .$$

See also (A1.20). We will see that G and E are "small" in regions that dominate the integral (for the choice of g that we will make). We introduce some convenient notation

$$u^{-1}+\frac{1}{l_D^2}=\lambda^2 l_D^2(-\Delta)^2+(-\Delta)+\frac{1}{l_D^2}=C_0^{-1}, \tag{7.5}$$

$$\lambda^2 l_D^2(-\Delta)^2+(-\Delta)+\frac{1}{l_D^2}+v=C^{-1}=C_0^{-1}+v, \tag{7.6}$$

$$\mathcal{L}_c=\frac{1}{l_D^2}C_0. \tag{7.7}$$

We change variables from ϕ to ψ in (7.1) and include the terms quadratic in ψ arising from (7.3) and (7.4) in the measure. The result may be written in the form

$$Z=\sum_h N\int d\mu(\psi)e^E e^G e^R, \tag{7.8}$$

where $d\mu$ is the normalized Gaussian measure defined by

$$N d\mu(\psi)=d\mu_0(\psi)e^{-\frac{1}{2l_D^2}\int\psi^2}e^{-1/2\int\psi v\psi}. \tag{7.9}$$

The covariance of $d\mu$ is C. R is given by

$$R=-\frac{1}{2l_D^2}\int(g-h)^2-\tfrac{1}{2}\int g u^{-1}g-\int\psi C_0^{-1}(g-g_c) \tag{7.10}$$

with

$$g_c=\mathcal{L}_c h. \tag{7.11}$$

Notice that if g were picked equal to g_c, the integrand in (7.8) would have no dependence on ψ outside the small terms G and E, i.e. R would vanish. However we will for later convenience define g as only approximately equal to g_c.

Analogously to in [1] we also write

$$R=-F_1-F_2 \tag{7.12}$$

with F_1 the first two terms on the right side of (7.10) and F_2 the last term (with sign changes).

In defining g we wish to satisfy four goals:

(1) $g=h$ outside $\sum^{\hat{}}$,

(2) Inside any connected component of $\sum^{\hat{}}$, g depends only on h inside the same component.

(3) g is in the domain of C_0^{-1}.

(4) The last term in (7.10) does not become too large.

For a differentiable function f to be in the domain of C_0^{-1}, it is necessary that f and $\Delta_0 f$ vanish on $\partial\Lambda$. If f satisfies these conditions we seek what conditions on ϕ insure that ϕf satisfies the boundary conditions. (ϕ is also assumed differentiable.) ϕf is automatically zero on $\partial\Lambda$. Looking at $\Delta_0(\phi f)$

$$\Delta_0(\phi f)=(\Delta_0\phi)f+(\Delta_0 f)\phi+2\nabla f\cdot\nabla\phi. \tag{7.13}$$

On the boundary $f=0$, $\Delta_0 f=0$, and ∇f is normal to the surface. Thus it is sufficient that ϕ have zero normal derivative on $\partial\Lambda$, for ϕf to be in the domain of C_0^{-1}.

We now fill Λ with a lattice of cubes of side $L'/4$. For given h, we let $\mathscr{S}$ be the set of cubes of this lattice at distance greater than $L'/4$ from $\sum$. We let $\{R_\alpha\}_{\alpha\in I}$ be the connected components of $\mathscr{S}$ and $\{\mathscr{I}_\beta\}_{\beta\in J}$ the connected components of the complement of $\mathscr{S}$. On each R_α we set $g=h$. Let $\mathscr{I}_\beta$ be a component that does not intersect $\partial\Lambda$. Let h_β^e be equal to h inside $\mathscr{I}_\beta$, be defined on $\mathbb{R}^3$, and be constant on the components of the complement of $\mathscr{I}_\beta \cap \sum$. We define

$$\tilde{g}_\beta = \frac{1}{l_D^2}\left(\lambda^2 l_D(-\Delta_0)^2 + (-\Delta_0) + \frac{1}{l_D^2}\right)^{-1} h_\beta^e. \tag{7.14}$$

If $\mathscr{I}_\beta$ intersects $\partial\Lambda$, we define h_β^e on Λ, to be equal to h inside $\mathscr{I}_\beta$, and to be constant on the components of the complement of $\mathscr{I}_\beta \cap \sum$. We define

$$\tilde{g} = \frac{1}{l_D^2} C_0 h_\beta^e. \tag{7.15}$$

g finally is to be constructed by patching together the h's on the R_α with the g_β's on the $\mathscr{I}_\beta$, with smoothing over a neighborhood of the $\partial\mathscr{I}_\beta$.

We set $B\mathscr{I}_\beta$ as the union of cubes of the unit lattice inside $\mathscr{I}_\beta$ and having non-empty intersection with $(\partial\mathscr{I}_\beta - \partial\Lambda)$. χ_β is a C^∞ function equal to zero outside $\mathscr{I}_\beta$, equal to 1 in $(\mathscr{I}_\beta - B\mathscr{I}_\beta)$, and such that

$$0 \leqq \chi_\beta \leqq 1 \tag{7.16}$$

and

$$\nabla\chi_\beta|_{\partial\Lambda} \quad \text{is normal to } \partial\Lambda \tag{7.17}$$

provided $\mathscr{I}_\beta$ intersects $\partial\Lambda$. All derivatives of the χ_β are uniformly bounded, i.e. for any derivative ∂^α (of any order)

$$|\partial^\alpha \chi_\beta(x)| \leqq b_\alpha \quad \text{all } x \text{ and } \beta. \tag{7.18}$$

These bounds do not depend on $\sum$ or Λ, they are absolute constants. We now define g

$$g = \begin{cases} h & \text{in} \quad R_\alpha \\ \tilde{g}_\beta & \text{in} \quad \mathscr{I}_\beta - B\mathscr{I}_\beta \\ \chi_\beta \tilde{g}_\beta + (1-\chi_\beta)h & \text{in} \quad B\mathscr{I}_\beta . \end{cases} \tag{7.19}$$

We observe that

$$C_0^{-1}(g - g_c) \tag{7.20}$$

the expression appearing in the last term of (7.10), is zero except in $\bigcup B\mathscr{I}_\beta$.

8. The Cluster Expansion

In writing the cluster expansion we will follow as closely as possible the development of Sects. 3.3 and 3.4 of [1], with which we assume familiarity. Aside from simple notational changes the essential change and complication we must face is due to the fact that E is a non local interaction of fields at different points. (G is also non local but it factors into a product over cubes which is sufficient locality for it to cause no problems.) We study $I(\mathscr{A}(\phi))$ where the observable, $\mathscr{A}$, is assumed to be a function of fields in a region Y_1. We also assume $\mathscr{A}$ is periodic of period τ in ϕ but this assumption is not essential, nor will it be used until the resummation described in Sect. 9. We fill $\mathbb{R}^3$ with a set of disjoint lattice cubes of side $\tilde{l}_D$. All the subsets of $\mathbb{R}^3$ in this section will be unions of these "$\tilde{l}_D$-lattice cubes". In later sections we often make a choice of units of length so that $\tilde{l}_D=1$, in which case we refer to these cubes as unit lattice cubes. We will refer to the lattice cubes in Y_1 as "distinguished cubes".

By (7.8)

$$\frac{1}{Z_0} I(\mathscr{A}(\phi)) = \int d\mu_0 Z(\phi)\mathscr{A}(\phi) \tag{8.1}$$

$$= \sum_h N \int d\mu(\psi) e^E e^G e^R \mathscr{A} . \tag{8.2}$$

The expansion we use yields for (8.2)

$$\sum_h \sum_X \mathscr{K}(X,h) N \int d\mu(\psi) e^{E(X^c)} e^{G(X^c)} e^{R(X^c)} , \tag{8.3}$$

where

$$\mathscr{K}(X,h) = \sum_X \int ds \int d\mu_s(\psi) e^{E(X,s)} \kappa(\bar{y},s) e^{G(X)} e^{R(X)} \mathscr{A} . \tag{8.4}$$

These compare quite exactly with (3.18) and (3.19) of [1]. We explain the notation above, to the extent it is not a direct translation.

We let $\tilde{Y}$ be the set whose elements are either connected components of $\sum^{\hat{}}$ or $\tilde{l}_D$-lattice cubes in $\Lambda - \sum^{\hat{}}$. $\bar{y}$ is a sequence of sets $Y_1, Y_2, \ldots, Y_n$, where each Y_i is a union of elements in $\tilde{Y}$, and the Y_i are disjoint. (In [1] each Y_i was an element of $\tilde{Y}$.) $X_1 = Y_1$, $X_i = Y_i \cup X_{i-1}$, $X_n = X$. $d\mu_s(\psi)$ is a normalized Gaussian measure with covariance $C(x,y,s) = p(x,y,s)C(x,y)$, p as given in (3.14) of [1]. For any set Y, a union of $\tilde{l}_D$-lattice cubes, in Λ, $Y^c = \Lambda - Y$, and G and R split naturally

$$G = G(Y) + G(Y^c) , \tag{8.5}$$

$$R = R(Y) + R(Y^c) . \tag{8.6}$$

We still need define $E(Y)$, $E(X,s)$, and $\kappa(\bar{y},s)$.

E may be written as a sum of terms, of which the following is a standard form

$$\frac{1}{t!} \int_{a_1} dx_1 \ldots \int_{a_t} dx_t \varrho(x_1, \ldots, x_t) \varepsilon(x_1) \ldots \varepsilon(x_t) . \tag{8.7}$$

Here the summation over species types is suppressed. Each a_i is an $\tilde{l}_D$-lattice cube. [There are also similar terms when $t=2$ containing $\psi(x_i)$ instead of $\varepsilon(x_i)$, and these will be treated identically.] $E(Y)$ is the sum of all such terms where $a_i \subset Y$ all i. $E(X,s)$ is the sum of the same terms as add to $E(X)$, where a term such as (8.7) is multiplied by $\prod_{i\in I} s_i$. Here $i\in I$ if $1\leqq i\leqq n-1$ and for some α, β, $1\leqq \alpha$, $\beta\leqq t$, $a_\alpha \subset Y_{i+1}$ and $a_\beta \subset X_i$. Clearly as s_i becomes zero the interactions in $E(X)$ between the region X_i and Y_{i+1} are shut off.

The differentiation terms, $\kappa(\bar{y}, s)$, are more complicated than in [1]. We have

$$\kappa(\bar{y}, s) = \kappa(n-1)\cdot\kappa(n-2)\dots\kappa(1), \tag{8.8}$$

where

$$\kappa(i) = \frac{d}{ds_i}E^{(i)}(X,s) + \int_{Y_{i+1}} dx \int_{\bigcup_{j\leq i} Y_j} dy \left(\frac{d}{ds_i}C(x,y,s)\right) \cdot\left[\left(\frac{\delta}{\delta\psi(x)} + \frac{\delta(E(X,s))}{\delta\psi(x)}\right)\cdot\left(\frac{\delta}{\delta\psi(y)} + \frac{\delta(E(X,s))}{\delta\psi(y)}\right)\right]^{(i)}. \tag{8.9}$$

$E^{(i)}(X,s)$ contains those terms in $E(X,s)$, multiplied by the same s's, for which

$$\left(\bigcup_{k=1}^{t} a_k\right) - X_i \mathrel{\subset\!\!\!\!\sim} Y_{i+1}. \tag{8.10}$$

By definition $X \mathrel{\subset\!\!\!\!\sim} Y$ if $X\subset Y$ and Y is the smallest union of sets from $\tilde{Y}$ that contains X.

The (i) on the brackets indicates the following restrictions on terms kept in expanding the derivatives:

1. The $\frac{\delta}{\delta\psi(x)}\frac{\delta}{\delta\psi(y)}$ term is kept only if Y_{i+1} is an element of $\tilde{Y}$.

2. The terms $\frac{\delta}{\delta\psi(x)}\frac{\delta(E(X,s))}{\delta\psi(y)}$ and $\frac{\delta(E(X,s))}{\delta\psi(x)}\frac{\delta}{\delta\psi(y)}$ include in $E(X,s)$ only those terms satisfying (8.10).

3. The term $\frac{\delta(E(X,s))}{\delta\psi(x)}\cdot\frac{\delta(E(X,s))}{\delta\psi(y)}$ includes only terms in the product such that if $a_1,\dots,a_t$ labels a term contributing to the first $E(X,s)$ and $a'_1,\dots,a'_r$ labels a term contributing to the second $E(X,s)$, then

$$\left(\bigcup_{k=1}^{t} a_k\right)\cup\left(\bigcup_{l=1}^{r} a'_l\right) - X_i \mathrel{\subset\!\!\!\!\sim} Y_{i+1}. \tag{8.11}$$

Thus in (8.9) only certain terms are kept in the expansions of $E(X,s)$ as sums of terms like (8.7). The restriction (8.10) and the three restrictions above, 1, 2, 3, each ensure that in the cluster expansion the Y_{i+1} chosen is the smallest union of sets from $\tilde{Y}$ that isolates the differentiated terms at each stage.

9. Proof of Clustering

9.0 Resummation

It is straightforward to interchange the order of summation over h and X in (8.3), restricting the sum over h to a form compatible with X. The sum over h naturally factorizes (see [1]) because of the periodicity of E, G and R. We can therefore rewrite (8.3) and (8.4) in the form

$$\frac{1}{Z_0} I(\mathscr{A}(\phi)) = \sum_X \mathscr{K}(X) Z'(\Lambda, X), \tag{9.01}$$

$$\mathscr{K}(X) = \sum_{\bar{y}} \sum_h \int ds \int d\mu_s(\psi) e^{E(X,s)} \kappa(\bar{y}, s) \cdot e^{G(X)} e^{R(X)} \mathscr{A}, \tag{9.02}$$

where

$$Z'(\Lambda, X) = \sum_h N \int d\mu(\psi) e^{E(X^c)} e^{G(X^c)} e^{R(X^c)}. \tag{9.03}$$

In (9.02) the sum over h is restricted to a form compatible with $\bar{y}$. Further details may be found in [1].

9.1. Combinatorics

The reader might be advised to omit this subsection on a first reading since it is only concerned with some of the counting involved in our convergence proof.

We are going to find an estimate for ($c'_A \geqq 0$)

$$\sum_{X, X \supset X_1} |\mathscr{K}(X) e^{c'_A |X|}|. \tag{9.11}$$

Our strategy is to list the many sums (integrals) in (9.11) and repeatedly use the elementary inequality

$$|\int dv(x) f(x)| \leqq \left(\int dv(x) \left|\frac{1}{a(x)}\right|\right) \operatorname{Sup} |a(x) f(x)| \tag{9.12}$$

to convert the sums to supremums. (This is the method of combinatoric factors used by Glimm and Jaffe.) If this inequality is used a number of times one gets an inequality of the form

$$|\int f(\ )| \leqq \left(\prod_i A_i\right) \operatorname{Sup} \left|\prod_i B_i(\ ) f(\ )\right|, \tag{9.13}$$

where at the i^{th} stage the first term on the right side of (9.12) we call A_i, and $a(x)$, B_i. We now enumerate the sums in (9.11) in the inverse order in which they are to be performed:

1. n: the length of the sequence $\bar{y}$.
2. (m_i), $i = 1, \dots, n$: Y_i is a union of m_i sets Y_{ij}.
3. (Y_{ij}): the sum over choices of sets, Y_{ij}, $i = 1, \dots, n$, $j = 1, \dots, m_i$, from $\tilde{Y}$.
4. h: the sum over all h consistent with the choice of Y_{ij}.
5. $\int ds$: the integral over interpolation parameters.
6. T: the sum over tree graphs. Suppose that $\kappa(i)$ is written as a double integral over x and y [the second term in (8.9) is in the required form already], then κ contains $2(n-1)$ integrals which we expand as follows

$$\prod_i \int_{Y_{i+1}} \int_{\bigcup_{j \leqq i} Y_j} = \sum_T \prod_i \int_{Y_{i+1}} \int_{Y_{T(i+1)}}. \tag{9.14}$$

The tree graph, T, is a map, $i \to T(i)$, such that $T(i) < i$. More details are given in [1].

7. Types of terms: $\kappa(i)$ as given in (8.9) is a sum of five types of terms. (Four are obtained by expanding the square bracket.)

8. (t): in each $\kappa(i)$ the E's are sums over $t=2,3,\ldots$, of terms as in (8.7).

9. Δ_i', Δ_i'', $i=1,\ldots,n-1$. The integrals in (9.14) are expanded into sums of cubes, $\Delta_i' \subset Y_{T(i+1)}$, $\Delta_i'' \subset Y_{i+1}$, viz,

$$\int_{Y_{i+1}} dx \int_{Y_{T(i+1)}} dy = \sum_{\Delta_i', \Delta_i''} \int_{\Delta_i''} dx \int_{\Delta_i'} dy. \tag{9.15}$$

10. (a) the cubes, $a_1, \ldots, a_t$, in (8.7) are to be summed over configurations compatible with Y_{ij}, Δ_i', Δ_i''.

Our objective is the following bound for (9.11),

$$\sup_{(\cdot)} e^{c_A F_1 + c_B|X| + \delta_2 d} \int d\mu_s |e^{E(X,s)} \kappa' e^{G(X)} e^{R(X)} \mathscr{A}|_0, \tag{9.16}$$

where c_A, c_B, δ_2 are constants, d is given by

$$d = \sum_i \operatorname{dist}(\Delta_i', \Delta_i'') \tag{9.17}$$

the supremum is over all *compatible* parameters listed under 1Σ to 10Σ, the subscript 0 on the absolute value sign means that the absolute value is to be taken inside the sum that results when all differentiations in κ' are performed and inside spatial integrals and sums over species. The definition of κ' comes after the next paragraph.

We will use $\mathscr{E}(a_i) = \mathscr{E}(a_1, \ldots, a_t)$ to denote a quantity of the form (8.7). The control of 10Σ will rely on an estimate, (A1.6), on the exponential decay of $\mathscr{E}(a_i)$ when the cubes are widely separated. In Appendix 1 we have introduced some measures of separation, $L_{\eta^A}(a_i)$. (η^A is any of a set of "augmented tree graphs".) For the purposes of this section their essential property is

$$\sum e^{-\gamma L_{\eta^A}(a_i)} \leqq c_\gamma^{t-1} \quad (\gamma > 0),$$

where the sum is over all positions of $a_1, \ldots, a_t$ with one held fixed. Equation (A1.6) can be paraphrased as follows: there exists a decomposition

$$\mathscr{E} = \sum_{\eta^A} \mathscr{E}_{\eta^A}$$

so that for constants b_{η^A} as in (A1.4)

$$|\mathscr{E}_{\eta^A}| \leqq b_{\eta^A} e^{-\alpha L_{\eta^A}(a_i)}.$$

In order to use this conveniently we introduce a formal operation denoted $e^{\gamma L_0}$, $\gamma > 0$. Given any expression containing $\mathscr{E}(a_i)$'s or their derivatives with respect to ψ, $e^{\gamma L_0}$ replaces these factors according to

$$e^{\gamma L_0} : \mathscr{E}(a_i) \to \sum_{\eta^A} e^{\gamma L_{\eta^A}(a_i)} \mathscr{E}_{\eta^A}(a_i). \tag{9.18}$$

We introduce a similar formal operator O_0 such that

$$e^{\gamma O_0} : \mathscr{E}(a_i) \to e^{\gamma t} \mathscr{E}(a_i). \tag{9.18a}$$

We can now define κ'

$$\kappa' = \kappa'(n+1) \ldots \kappa'(1), \tag{9.19}$$

where $\kappa'(i)$ is, depending on the type, (7Σ), one of the following five operators:

$$(1) \quad e^{rO_0} e^{\delta_2 L_0} \mathscr{E}(a_1, a_2, \ldots, a_t), \tag{9.110}$$

where $r>0$, $\bigcup a_j$ must be contained in X_{i+1}, contain Δ'_i, Δ''_i, and have non-zero intersection with $Y_{i+1,j}$ for $j=1,2,\ldots, m_{i+1}$.

$$(2) \quad \int_{\Delta''_i} dx \int_{\Delta'_i} dy \frac{\delta}{\delta\psi(x)} C(x,y) \frac{\delta}{\delta\psi(y)}, \tag{9.111}$$

$$(3) \quad e^{rO_0} e^{\delta_2 L_0} \int_{\Delta''_i} dx \int_{\Delta'_i} dy \frac{\delta \mathscr{E}_1}{\delta\psi(x)} C(x,y) \frac{\delta \mathscr{E}_2}{\delta\psi(y)}, \tag{9.112}$$

where

$$\mathscr{E}_1 = \mathscr{E}(a_1, \ldots, a_{t_1}), \quad \mathscr{E}_2 = \mathscr{E}(a'_1, \ldots, a'_{t_2})$$

and the union of all the a's and a''s satisfies the conditions below (9.110). (4) and (5) are obtained from the cross terms which arise by multiplying out the square bracket in (8.9) by replacing E's by $\mathscr{E}$'s and prefacing the result by

$$e^{rO_0} e^{\delta_2 L_0}.$$

Lemma 9.1. *For any* δ_2, c'_A, $c_A>0$ *there are* c_B, r *so that* (9.11) *is bounded by* (9.16) *(for* β *sufficiently small).*

By virtue of (9.13) and the remarks surrounding it we can prove this lemma by giving for each of the listed sums an A_i and a B_i satisfying

$$A_i \geqq \sum_{(\,)} B_i^{-1}(\,) \tag{9.113}$$

such that the product of the $(A_i B_i)$'s is of the form

$$e^{c_A F_1 + c_B|X| + \delta_2 d} e^{\delta_2 L_0} e^{rO_0}. \tag{9.114}$$

We will present a list of such A_i's and B_i's and prove that (9.113) holds for some of the non-trivial cases.

7Σ) $B_7 = 1, \quad A_7 = 5^{n-1}$,

6Σ) and 5Σ) $A_6 A_5 = e^{|X|}, \quad B_6 B_5 = \left(\prod_i \frac{1}{|Y_{T(i)}|}\right) \frac{1}{q(T,s)}$,

where

$$q(T,s) = \prod_{(i,j)\in T} \frac{d}{ds_{j-1}}(s_{j-1}, \ldots, s_i). \tag{9.115}$$

The validity of (9.113) for these choices for $A_6 A_5$, $B_6 B_5$ follows from (5.6) of [1] or Lemma 5.5 of [2]

4Σ) $B_4 = e^{c_4 F_1(X)}, \quad A_4 = e^{c'_4 |X|/L^3}$.

$c_4>0$ can be chosen arbitrarily. c_4' depends on c_4 and tends to zero as the period τ tends to infinity. We will prove that A_4, B_4 stand in the correct relationship, namely

$$\sum_h e^{-c_4 F_1(X)} \leqq e^{c_4'|X|/L^3} \tag{9.116}$$

giving, in the process, a simplified derivation of (5.13) of [1].

First suppose $X=Y$, an element of $\tilde{Y}$. Let $e_1, e_2, \ldots$ be a sequence of disjoint lattice cubes, of side length L, exhausting Y. We suppose the sequence selected so that each e_i shares a face with some $e_{\alpha(i)}$ with $\alpha(i)<i$. We pick e_1 to be a boundary cube so that h can be fixed in e_1 (we direct the reader to [1] to see that this is consistent with the way the sum over h factorizes). We overestimate the sum by dropping the remaining restrictions on h. Next we use

Lemma 9.2. *There is $c>0$ such that*

$$F_1(Y,h) \geqq c\sum_f |\delta h(f)|^2, \tag{9.117}$$

where $\sum_f$ is over the internal faces of the lattice cubes in Y and $\delta h(f)$ is the jump in the value of h between the two cubes joined at f.

This is identical with Lemma 5.2 of [1], if we use the fact that

$$u^{-1} \geqq -\Delta. \tag{9.118}$$

We are now reduced to proving (9.116) with the left hand side replaced by

$$\sum_h e^{-c\sum_f |\delta h(f)|^2}. \tag{9.119}$$

We sum over the values of h in e_i in inverse numerical order. (9.119) is less than

$$\sum_{i=1}^{|Y|/L^3} \left(\sum_{n=-\infty}^{n=\infty} e^{-c(n\tau)^2} \right). \tag{9.120}$$

We thus obtain

Lemma 9.3. *The sum in* (9.116) *is less than*

$$\left(1+2\frac{e^{-c\tau^2}}{1-e^{-c\tau^2}}\right)^{|Y|/L^3}. \tag{9.121}$$

This completes the proof in the case $X=Y$. In general X is a union of disjoint Y's and we prove (9.116) by taking a product. [The left hand side of (9.116) factors.]

1Σ) For any $c_1>0$ we may choose

$$A_1 = c_1' = \frac{1}{1-e^{-c_1}}, \qquad B_1 = e^{c_1|X|}.$$

2Σ) For any $c_2>0$ we may choose

$$A_2 = (c_2')^n = \left(\frac{1}{1-e^{-c_2}}\right)^n, \qquad B_2 = e^{c_2|X|}.$$

3Σ) Every Y_{ij} is constrained to be connected and to contain one of the "a" cubes mentioned in 10Σ.

$$A_3 = e^{c_3|X|}, \quad B_3 = 1.$$

9Σ) Δ'_i is contained in $Y_{T(i+1)}$. Set

$$A'_9 = \prod_{i=1}^{n-1} Y_{T(i+1)}, \quad B'_9 = 1.$$

(We are simply counting the number of cubes in $Y_{T(i+1)}$.)

For any $c_9 > 0$, let

$$A''_9 = \left(\sum_{\Delta} e^{-c_9 \operatorname{dist}(0,\Delta)}\right)^{|X|}, \quad B''_9 = e^{c_9(d+L_0)},$$

where the sum is over all unit lattice cubes in a lattice centered on the origin, 0. We now take

$$A_9 = A'_9 A''_9, \quad B_9 = B'_9 B''_9.$$

8Σ) and 10Σ) For any $c_{10} > 0$ there is a c'_{10} and c''_{10} such that we may choose

$$A_8 A_{10} = (c'_{10})^{2|X|}$$
$$B_8 B_{10} = e^{c''_{10} O_0} e^{c_{10} L_0}.$$

This concludes the proof of Lemma 9.1.

By the same methods we easily obtain the following generalization of Lemma 9.1.

Lemma 9.4. *For any $\delta_2, c'_A, c_A > 0$, there are c'_B, r such that (for β sufficiently small)*

$$\begin{aligned}\sum_{\substack{X, X \supset X_1 \\ X \cap W \neq \emptyset}} |\mathscr{K}(X)| e^{c'_A |X|} \leq \sup_{(\cdot)} & e^{c_A F_1 + c'_B |X|} \\ & \cdot e^{(1-2\delta_1+\delta_2)d} \int d\mu_s |e^{E(X,s)} \kappa'' e^{G(X)} e^{R(X)} \mathscr{A}|_0 \\ & \cdot e^{-(1-2\delta_1)\operatorname{dist}(X_1, W)}. \end{aligned} \tag{9.122}$$

κ'' is the same as κ', but with $e^{(1-2\delta_1+\delta_2)L_0}$ replacing $e^{\delta_2 L_0}$ throughout.

Remarks. Lemmas 9.1 and 9.4 are purely combinatoric in the sense that nothing we have done so far assures us that the right hand side of (9.122) is finite. In particular the sequel will show that we must have upper bounds on δ_1, δ_2, c_A and our short range forces must be such that

$$\alpha \geqq (1-\delta_1)\frac{1}{l_D}. \tag{9.123}$$

α appears in (AI.6). We will also need

$$|C(x,y)| \leqq c_u e^{-(1-\delta_1)|x-y|\frac{1}{l_D}}. \tag{9.124}$$

9.2. Hölders Inequality

We wish to study the integrals in (9.16) and (9.122). We recall the splitup of R defined in Sect. 7

$$R = -F_1 - F_2$$

and define a new splitup of G

$$G = G_1 + G_2 \qquad e^{G_1} = \prod_\alpha r_\alpha(A_\alpha) \tag{9.21}$$

following the notation of (6.4). Performing the functional derivatives in κ' or κ'', each of (9.16) and (9.122) becomes a *sum* of terms of the form

$$\int d\mu_s e^E e^{G_2} e^{-F_2} e^{-F_1} \tilde{g} \tilde{\kappa} \mathscr{A}, \tag{9.22}$$

where $\tilde{g}$ is e^{G_1} or some derivative of e^{G_1} (and integrals coupling $\tilde{\kappa}$ and $\tilde{g}$ are momentarily suppressed in the notation). We will find in Sect. 9.8 an inequality

$$|\tilde{g}\tilde{\kappa}\mathscr{A}| \leqq \int k(x_i) \prod_i |\psi(x_i)| e^{1/2\gamma\int(\psi+g-h)^2} \tag{9.23}$$

to control the last three factors in (9.22). The integrals in (9.23) are restricted to the region X, $k(x_i) \geqq 0$, and

$$\gamma < \frac{1}{\tilde{l}_D^2}. \tag{9.24}$$

Employing (9.23) we estimate (9.22) with Hölders inequality.

$$\begin{aligned} |(9.22)| &\leqq \int k(x_i) \int d\mu_s |e^E e^{G_2} e^{-F_2} e^{1/2\gamma\int(\psi+g-h)^2}| \prod_i |\psi(x_i)| \\ &\cdot e^{-F_1} \leqq \int k(x_i) \left(\int d\mu_s \left[\prod_i \psi(x_i)\right]^{p_1}\right)^{\frac{1}{p_1}} \\ &\cdot \left(\int d\mu_s e^{-p_2 F_2}\right)^{\frac{1}{p_2}} \cdot e^{-F_1} \cdot \left(\int d\mu_s e^{1/2 p_3\gamma\int(\psi+g-h)^2} e^{2p_3\int\delta^2}\right)^{\frac{1}{p_3}} \\ &\cdot \left(\int d\mu_s |e^E e^{G_2} e^{-2\int\delta^2}|^{p_4}\right)^{\frac{1}{p_4}}, \end{aligned} \tag{9.25}$$

where

$$p_i > 1, \quad \sum \frac{1}{p_i} = 1, \quad p_1 \text{ even integer} \qquad p_3\gamma < \frac{1}{\tilde{l}_D^2}. \tag{9.26}$$

The first factor on the right side of (9.25) with the polynomial in ψ's is studied in Sect. 9.3; the factor involving F_2 is studied in Sect. 9.4, the last factor but one in Sect. 9.6 and the final factor in Sect. 9.7.

9.3. *Wick's Theorem*

We choose units such that $\tilde{l}_D = 1$ and study

$$\left(\int d\mu_s \left[\prod_i \psi(x_i)\right]^{p_1}\right)^{\frac{1}{p_1}} \tag{9.31}$$

using the estimate (discussed below)

$$|C(x,y)| \leqq c_u e^{-1/2|x-y|} \tag{9.32}$$

let n_j be the number of x_i's in unit cube Δ_j. A simple generous counting of contractions in (9.31) as detailed in [11] yields

$$|(9.31)| \leqq c^{\Sigma n_j} \prod_j (n_j!) . \tag{9.33}$$

(9.32) is easy to establish when C is replaced by C_0. See for example (9.47) and the remarks below. By referring to (7.6) we see that C differs from C_0 because of the non local operator, v. Provided

$$\alpha > 1/2, \ c_2(\alpha) \text{ sufficiently small} \tag{9.34}$$

in (A1.6), we may obtain (9.32) by studying the Neuman expansion of C with v regarded as a small perturbation in a suitable norm.

9.4. $(g - g_c)$ *is Small Enough*

In this section we look at

$$(\int d\mu_s e^{-p_2 F_2})^{\frac{1}{p_2}}, \tag{9.41}$$

where, from (7.10)

$$F_2 = \int \psi(C_0^{-1})(g - g_c) . \tag{9.42}$$

(9.41) may be explicitly evaluated by completing the square in the Gaussian integral, to be given as

$$e^{1/2 p_2 \int (g-g_c) C_0^{-1} C C_0^{-1} (g-g_c)} . \tag{9.43}$$

Lemma 9.5

$$\int (g - g_c) C_0^{-1} C C_0^{-1} (g - g_c) \leqq c(L) F_1 , \tag{9.44}$$

where $c(L)$ becomes arbitrarily small as L is increased.

This lemma replaces Lemma 6.7 of [1], but since our construction of g is different, another proof is required. The remainder of this section is devoted to this proof.

We first study the covariance C_0, as defined in (7.5). With

$$r_\pm = \frac{1}{2 l_D^2 \lambda^2} \left[1 \pm \sqrt{1 - 4 \frac{\lambda^2 l_D^2}{\tilde{l}_D^2}}\right] \tag{9.45}$$

which for small λ becomes

$$r_+ \cong \frac{1}{\lambda^2 l_D^2}, \quad r_- \cong \frac{1}{l_D^2}. \tag{9.46}$$

C_0 may be expressed as

$$C_0 = \frac{1}{\lambda^2 l_D^2} \cdot \frac{1}{(r_+ - r_-)} \cdot \left[\frac{1}{(-\Delta) + r_-} - \frac{1}{(-\Delta) + r_+}\right] \tag{9.47}$$

with the small λ form

$$C_0 \cong \frac{1}{-\Delta + \frac{1}{l_D^2}} - \frac{1}{-\Delta + \frac{1}{\lambda^2 l_D^2}}. \tag{9.48}$$

From (9.47) one may read off the regularity and exponential fall off of the infinite volume limit of the covariance. The path space representation of the individual Yukawa terms in (9.47) yields the result that

$$0 \leqq C_0(x, y) \leqq C_{00}(x, y), \tag{9.49}$$

where C_{00} is the infinite volume form. [(9.49) is a pointwise estimate.] C_0 may be constructed as an infinite sum of terms each of the form of a translated C_{00}, by the method of images. This is very useful to read off estimates on the derivatives of C_0. (If we worked in more general volumes, technical results on boundary values of Green's functions would be needed.) We will use the following estimate from the above considerations

$$|D_x^\alpha C_0(x, y)| \leqq c(|\alpha|) e^{-1/2|x-y|\frac{1}{l_D}}, |x-y| > \frac{L'}{4}. \tag{9.410}$$

Of course the same estimate holds for C_{00}. In fact we will only need the estimate for derivatives of degree less than four.

We will also make use of the following simple well-known estimate.

Lemma 9.6. *Let $k(x,y)$ be a symmetrical integral kernel. Then the norm of the associated integral operator is bounded by*

$$\operatorname*{Sup}_x \int dy |k(x, y)|. \tag{9.411}$$

For the remainder of this section we set $\tilde{l}_D = 1$ for simplicity. From Lemma 9.2 we write

$$\sum_f |\delta h(f)|^2 \leqq c F_1. \tag{9.412}$$

Using Lemma 9.6 we get

$$\int (g - g_c) C_0^{-1} C C_0^{-1} (g - g_c) \leqq c \int C_0^{-1} (g - g_c) C_0^{-1} (g - g_c). \tag{9.413}$$

We complete the proof for the case where $C_0^{-1}(g - g_c) = 0$ in all but one $\mathscr{I}_\beta$, which $\mathscr{I}_\beta$ has zero intersection with $\partial\Lambda$. The extension to the general situation is straightforward and left to the reader. From the definitions in Sect. 7 and (9.410) it easily follows that

$$\int C_0^{-1}(g-g_c)C_0^{-1}(g-g_c)=c\int dx \int_{B\mathcal{I}} dy \int dz|h^e(x)-h^e(y)|e^{-1/2|x-y|} \cdot e^{-1/2|y-z|}|h^e(z)-h^e(y)|. \quad (9.414)$$

Notice in the last integral the variables $|y-z|$ and $|x-y|$ are both $\geqq \frac{L}{8}$. We define smoothings of $h^e(x)$

$$h^s(x)=\int dy\,\phi(x-y)h^e(y), \quad (9.415)$$

where $\phi(x)$ is C^∞, $\geqq 0$, of integral one, and whose support lies within distance $\frac{L}{4}$ of the origin. Then the last term in (9.414) is

$$\leqq c\int dx \int_{B\mathcal{I}} dy \int dz|h^s(x)-h^s(y)|e^{-1/2|x-y|}e^{-1/2|y-z|} |h^s(z)-h^s(y)| \quad (9.416)$$

we perform integration by parts in the radial variables $(x-y|$ and $|z-y|$ for fixed y to get

$$(9.416)\leqq c\int dx \int_{B\mathcal{I}} dy \int dz|\nabla h^s(x)|e^{-1/2|x-y|}\cdot e^{-1/2|y-z|}|\nabla h^s(z)|. \quad (9.417)$$

We use Lemma 9.6 again to arrive at

$$(9.417)\leqq c(L)\int dx|\nabla h^s(x)|^2, \quad (9.418)$$

where $c(L)$ goes to zero (exponentially) with L. The Lemma follows from the inequality

$$\int dx|\nabla h^s(x)|^2 \leqq c\sum_f |\delta h(f)|^2 \quad (9.419)$$

that follows immediately from dimensional considerations alone.

9.5. *Derivatives of* $r(A)$

We require bounds on $r(A)$ and its derivatives.

Lemma 9.7. *There are* c_1, c_2, c_3, & $\gamma<\frac{1}{l_D^2}$ *such that*

$$|D^N r(A)|e^{-(L^3/2)\gamma A^2}\leqq c_1(c_2\beta^{1/6})^N e^{c_3 N\ln N}. \quad (9.51)$$

This lemma contains all the information we will need and this section is devoted to its proof.

Taking the expression for $r(A)$ from (6.6) we replace A by a complex number $x+iy$, and easily obtain:

$$r(x+iy)=\frac{\mathrm{I}\cdot\mathrm{II}}{\mathrm{III}\cdot\mathrm{IV}} \quad (9.52)$$

where

$$\begin{aligned}
\mathrm{I} &= e^{L^3\sum \varrho_i(e^{i\beta^{1/2} e_i x} - 1)} \\
\mathrm{II} &= e^{L^3\sum \varrho_i e^{i\beta^{1/2} e_i x}(e^{-\beta^{1/2} e_i y} - 1)} \\
\mathrm{III} &= e^{\frac{L^3}{2 l_D^2} y^2} \\
\mathrm{IV} &= \sum_n e^{-\frac{L^3}{2 l_D^2}[(x - n\tau)^2 + 2iy(x - n\tau)]}.
\end{aligned} \tag{9.53}$$

We first study $r(A)$ and derivatives for $|A| \leqq \frac{1}{\beta^{1/6}}$. We consider the region

$$|x| \leqq \frac{2}{\beta^{1/6}}, \qquad |y| \leqq \frac{1}{\beta^{1/6}}. \tag{9.54}$$

In this region the following bounds hold

$$\left|\frac{\mathrm{I}}{\mathrm{IV}}\right| \leqq c, \qquad \left|\frac{\mathrm{II}}{\mathrm{III}}\right| \leqq c. \tag{9.55}$$

Terms I and II have been studied by expanding all the exponentiated exponentials in Taylor series (with easily controlled remainders). Thus

$$|\mathrm{I}| \leqq c e^{\frac{L^3}{2 l_D^2} x^2} \tag{9.56}$$

and

$$|\mathrm{II}| \leqq c e^{\frac{L^3}{2 l_D^2} y^2} \tag{9.57}$$

in this region, estimates on the remainder terms of the Taylor Series absorbed into the constants. In IV a single term $n=0$, dominates the sum. From (9.55) using Cauchy's inequality the bound of (9.51) follows (with $\gamma=0$ and $c_3=1$).

To study $r(A)$ when $|A| \geqq \frac{1}{\beta^{1/6}}$ we consider the region

$$|x| \geqq \frac{1}{2\beta^{1/6}}, \qquad |y| \leqq \frac{\varepsilon}{|x|}. \tag{9.58}$$

ε is picked sufficiently small so that the two largest terms in the sum over n in IV are within $\pi/2$ in phase. There is a $\gamma < \frac{1}{l_D^2}$ and $\varepsilon_1 > 0$ such that

$$\left|\frac{\mathrm{I}}{\mathrm{IV}}\right| \leqq c e^{(L^3/2)\gamma(1-\varepsilon_1)x^2} \tag{9.59}$$

and as in (9.55)

$$\left|\frac{\mathrm{II}}{\mathrm{III}}\right| \leqq c. \tag{9.510}$$

There follows

$$e^{-\frac{L^3}{2}\gamma A^2}(D^N r(A)) \leqq c_1 N!\left(\frac{c_2}{\beta^{1/2}}\right)^N e^{-c_3/\beta^{1/3}}$$

$$\leqq c_1 N!(c_2\beta^{1/6})^N\left(\frac{1}{\beta^{2/3}}\right)^N e^{-c_3/\beta^{1/3}}. \tag{9.511}$$

But by maximizing over values of β

$$\left(\frac{1}{\beta^{2/3}}\right)^N e^{-c_3/\beta^{1/3}} \leqq e^{c_4 N + c_5 N \ln N} \tag{9.512}$$

from which the lemma follows.

9.6. The Vacuum Energy I

Recall that $d\mu_s$ depends on a parameter λ introduced in (1.5), g as defined in Sect. 7 depends on a length L', and the fluctuation field δ [given in (6.4)] depends on L. In this section we will show that the next to last factor on the right side of (9.25) may be controlled by a proper selection of λ, L', and L.

Let

$$B = B(\psi + g - h) = \frac{1}{2}\int(\psi + g - h)^2 + \frac{2}{\gamma}\int\delta^2. \tag{9.61}$$

From the definitions of δ and ψ one sees that

$$\delta = P(\psi + g - h), \tag{9.62}$$

where P is the projection which takes a function f into its fluctuation part, i.e.

$$f(x) \to f(x) - \frac{1}{L^3}\int_{\Omega_\alpha} f(x)dx \tag{9.63}$$

with Ω_α the cube containing x.

Lemma 9.8. *Given $p\gamma < 1/\tilde{l}_D^2$, if λ and $L/\tilde{l}_D$ are sufficiently small and $L'/\tilde{l}_D$ is sufficiently large then*

$$\left(\int d\mu_s e^{p\gamma B}\right)^{1/p} \leqq e^{c|X|} e^{\delta F_1}, \tag{9.64}$$

where $\delta < 1$.

(The c in Lemma 9.8 and the similar c in Lemma 9.9 each go to infinity as $\lambda \to 0$, so as in other places, λ must be fixed small but non-zero.)

To simplify our notation we will set $\tilde{l}_D = 1$. Let $f = g - h$. Recall from Sect. 8 that the covariance, C_s, of $d\mu_s$ is a convex combination of "diagonalized" covariances, i.e.

$$C_s = \sum \lambda_i C_i, \quad \sum \lambda_i = 1, \tag{9.65}$$

where the coefficients λ_i depend on the interpolating parameters s and each C_i has the form

$$\sum \chi_j C \chi_j. \tag{9.66}$$

C was defined in (7.6) and the χ_j are characteristic functions of disjoint sets (each a union of cubes) drawn from the partition used in the cluster expansion.

We begin by proving that

$$\int d\mu_s e^{p\gamma B} \leqq \prod_i \left(\int d\mu_i e^{p\gamma B}\right)^{\lambda_i}, \tag{9.67}$$

where $d\mu_i$ has covariance C_i as in (9.65). Since $\sum \lambda_i = 1$ this inequality shows that it is sufficient to prove the lemma with $d\mu_s$ replaced by $d\mu_i$. To prove this inequality, let $d\omega(\sigma)$ be the Gaussian measure with covariance given by the positive bilinear form $2B$ so that

$$e^{p\gamma B} = \int d\omega(\sigma) e^{\sqrt{p\gamma}\int\sigma(\psi+f)}. \tag{9.68}$$

We substitute into the left hand side of (9.67) and interchange integrals to obtain

$$\int d\omega(\sigma) \int d\mu_s e^{\sqrt{p\gamma}\int\sigma(\psi+f)}. \tag{9.69}$$

We now do the integral over ψ to obtain

$$\int d\omega e^{\sqrt{p\gamma}\int\sigma f} e^{1/2 p\gamma \int \sigma C_s \sigma}. \tag{9.610}$$

Writing C_s as a convex combination of C_i's and using (9.65), we apply Hölder's inequality to obtain the upper bound

$$\prod_i \left(\int d\omega e^{\sqrt{p\gamma}\int\sigma f} e^{1/2 p\gamma \int \sigma C_i \sigma}\right)^{\lambda_i}. \tag{9.611}$$

If we reverse the steps that led from the right hand side of (9.67) to (9.610) in each factor in (9.611), we obtain the inequality (9.67).

Next, suppose that for some $\alpha > 1$ and sufficiently close to 1 we have

$$C_i \leqq \alpha C_i' \tag{9.612}$$

in the operator sense. Our choice of C_i' will be given below. We will show that

$$\int d\mu_{C_i} e^{p\gamma B} \leqq \int d\mu_{C_i'} e^{\alpha p\gamma B}. \tag{9.613}$$

We prove this by writing, as above in (9.610)

$$\begin{aligned} \int d\mu_{C_i} e^{p\gamma B} &= \int d\omega e^{\sqrt{p\gamma}\int\sigma f} e^{1/2 p\gamma \int\sigma C_i \sigma} \\ &\leqq \int d\omega e^{\sqrt{p\gamma}\int\sigma f} e^{1/2 \alpha p\gamma \int\sigma C_i' \sigma}. \end{aligned} \tag{9.614}$$

This inequality comes from (9.612),

$$\leqq \int d\omega e^{\sqrt{\alpha p\gamma}\int\sigma f} e^{1/2 \alpha p\gamma \int\sigma C_i' \sigma} \tag{9.615}$$

$$= \int d\mu_{C_i'} e^{\alpha p\gamma B}. \tag{9.616}$$

The inequality in (9.615) is because $\alpha p\gamma > p\gamma$ and for any Gaussian measure $dg(\sigma)$ with mean zero the characteristic function

$$\int dg e^{\kappa \int \sigma f} = \left(\int dg 1\right) e^{1/2 \kappa^2 \int f Q f} \tag{9.617}$$

is increasing in κ. Q is the covariance. This yields (9.613).

We will use (9.613) to control the awkward nonlocal operator v which occurs in C and thus in C_i. Choose $\alpha > 1$. We first show that if λ is sufficiently small depending on α, then

$$C \leqq \alpha C_0 . \tag{9.618}$$

Define C_{0i} by replacing C by C_0 in (9.66). Equation (9.618) implies the analogous inequality for C_i, C_{0i}. To prove (9.618) we note that

$$C^{-1} = u^{-1} + 1 + v \geqq u^{-1} + 1 - \|v\| \geqq (1 - \|v\|)(u^{-1} + 1) \tag{9.619}$$

provided $1 - \|v\| > 0$,

$$= (1 - \|v\|) C_0^{-1} . \tag{9.620}$$

Thus inverting each side yields

$$C \leqq \frac{1}{1 - \|v\|} C_0 . \tag{9.621}$$

By combining Lemma 9.6 with the bounds (9.94), (A1.6), and (A1.12) we see that $\|v\|$ tends to zero as λ tends to zero. Therefore we choose λ so small that

$$\frac{1}{1 - \|v\|} \leqq \alpha . \tag{9.622}$$

By combining (9.618) with (9.613) choosing C_i' to be C_{0i} we obtain

$$\int d\mu_i e^{p\gamma B} \leqq \int d\mu_{0i} e^{\alpha p \gamma B} , \tag{9.623}$$

where $d\mu_{0i}$ has covariance C_{0i}. This combined with (9.67) shows that to prove the lemma it is sufficient to prove that for $\alpha p \gamma < 1$,

$$\left(\int d\mu_{0i} e^{\alpha p \gamma B}\right)^{1/p} \leqq c^{|X|} e^{\delta F_1} \tag{9.624}$$

if L' is sufficiently large and L is sufficiently small.

We now start to prove (9.624). We prepare to undo the translation from ϕ to ψ by reintroducing F_1, F_2 (see Sect. 7). By the Hölder inequality the left hand side of (9.624) is less than

$$\left(\int d\mu_{0i} e^{\alpha p' \gamma B} e^{-F_1 - F_2}\right)^{1/p'} \left(\int d\mu_{0i} e^{q'/p'(F_1 + F_2)}\right)^{1/q'} , \tag{9.625}$$

where $1/p' + 1/q' = 1/p$. Choose p' so that $\alpha p' \gamma = \beta < 1$. Since the covariance C_{0i} of $d\mu_{0i}$ is a direct sum of covariances associated to disjoint regions in X, the integrals in (9.625) factor. A typical factor of the first integral has the form

$$\left(\int d\mu_{C_0} e^{\beta B} e^{-F_1 - F_2}\right)^{1/p'} , \tag{9.626}$$

where the integrals in B, F_1, F_2 are taken over the support $Y = Y_j$ of one of the χ_j's in (9.66). The covariance is C_0 instead of $\chi_j C_0 \chi_j$ because the χ_j's can be dropped since the fields in B, F_2, are localized in Y. We change variables in (9.626) by setting

$$\psi = \phi - \tilde{g} , \tag{9.627}$$

where $\tilde{g}$ is calculated from $\tilde{h}$, with

$$\tilde{h} = h - h_0 . \tag{9.628}$$

h_0 is a constant chosen so that $\tilde{h}$ vanishes near ∞. $h_0=0$ for Y that intersect $\partial\Lambda$. Note that $F_1=F_1(Y)$, $F_2(Y)$, $f|_Y$ do not change if h is changed to $\tilde{h}$ in their definitions. The result of the translation applied to (9.626) is obtained by applying formulas (7.8) to (7.10) backwards. It is

$$(N_0^{-1}\int d\mu_0(\phi)e^{-1/2\int(\phi-\tilde{h})^2}e^{\beta B})^{1/p'}, \tag{9.629}$$

where $B=B(\phi-h)$ and

$$N_0=\int d\mu_0 e^{-1/2\int\phi^2}. \tag{9.630}$$

We will prove that if L is sufficiently small,

$$(N_0^{-1}\int d\mu_0 e^{-1/2\int(\phi-\tilde{h})^2}e^{\beta B})^{1/p'}\leqq c^{1/2|Y|} \tag{9.631}$$

and if L' is sufficiently large,

$$(\int d\mu_{0i}e^{q'/p'F_2})^{1/q'}e^{1/p'F_1}\leqq c^{1/2|Y|}e^{1/pF_1}. \tag{9.632}$$

These two estimates combine to prove (9.624) and hence our lemma because by taking a product over bounds (9.631), one for each Y_j the support of χ_j in (9.66) we bound the first factor in (9.625) by $c^{1/2|X|}$. Note that $p'>p>1$, so we may take $\delta=1/p$ in our lemma. The bound (9.632) is an immediate consequence of Lemma 9.5 and the remarks preceding it.

We turn our attention to (9.631). Let $\chi(x)=1$ if $x\in Y$, 0 otherwise. Define Q to be the operator

$$Q=1-\beta\chi-\beta\frac{4}{\gamma}P\chi P, \tag{9.633}$$

where P is the projection defined in (9.63). Note that the exponents in (9.631) combine to yield

$$-1/2\int(\phi-\tilde{h})Q(\phi-\tilde{h}). \tag{9.634}$$

By completing the square, the left hand side of (9.631) becomes

$$\begin{aligned}&(N_0^{-1}\int d\mu_0(\phi)e^{-1/2\int\phi Q\phi})^{1/p'}\exp\{-1/2p'\int\tilde{h}Q\tilde{h}\\&\quad+1/2p'\int\tilde{h}Q(u^{-1}+Q)^{-1}Q\tilde{h}\}.\end{aligned} \tag{9.635}$$

Since $\tilde{h}$ is constant on the cubes Ω_α appearing in the definition of P, P annihilates $\tilde{h}$ and

$$\int\tilde{h}Q\tilde{h}=\int\tilde{h}(1-\beta\chi)\tilde{h}. \tag{9.636}$$

We prove below that if L is sufficiently small

$$u^{-1}-\beta\frac{4}{\gamma}P\chi P\geqq 0 \tag{9.637}$$

and therefore

$$(1-\beta\chi)(u^{-1}+Q)^{-1}(1-\beta\chi)\leqq 1-\beta\chi \tag{9.638}$$

which in combination with (9.636) implies that the quantity inside the curly brackets in (9.635) is negative. Therefore we have reduced the task of proving (9.631) and hence our lemma to proving (9.637) and

$$(N_0^{-1} \int d\mu_0(\phi) e^{-1/2 \int \phi Q \phi})^{1/p'} \leqq c^{1/2|Y|}. \tag{9.639}$$

In Appendix 3, we show, by explicit Gaussian integration and control of the resulting determinant, how to obtain (9.639). The bound (9.637) is implied by

$$P \leqq \frac{\gamma}{4\beta}(-\Delta) \tag{9.640}$$

because

$$\chi \leqq 1, \quad P^2 = P, \quad u^{-1} \geqq -\Delta. \tag{9.641}$$

Let $-\Delta_N$ denote the Laplacian with Neuman boundary conditions on the boundaries of all cubes Ω_α filling Λ. Since

$$-\Delta \geqq -\Delta_N \tag{9.642}$$

it is enough to prove (9.640) with Δ replaced by Δ_N, for which it is trivial because P is the projection onto the orthogonal complement of the zero eigenvectors of $-\Delta_N$. The Laplacian for a cube of size L with Neumann boundary conditions can be explicitly diagonalized and the first eigenvalue above zero is $\sim \frac{1}{L^2}$.

9.7. The Vacuum Energy II

In this section we show that the last term in our Hölder inequality (9.25) is under control provided λ is chosen sufficiently small. We summarize our result in the following lemma

Lemma 9.9. *Given $c_\lambda > 0$ and $p \geqq 1$, if λ is sufficiently small*

$$(\int d\mu_s |e^E e^{G_2} e^{-2\tilde{l}_D^2 \int \delta^2}|^p)^{1/p} \leqq e^{c|X|} e^{c_\lambda F_1}. \tag{9.71}$$

We adopt units in which $\tilde{l}_D = 1$ to simplify the notation. Our proof will use the bounds on the Mayer expansion given in Appendix 1.

From the definitions it follows that

$$|e^{G_2} e^{-2\int \delta^2}| \leqq 1 \tag{9.72}$$

by using the easy estimate

$$|e^{ix} - 1 - ix| \leqq \frac{x^2}{2}.$$

We now refer to the definition of E, (Al. 20)

$$E = \sum_{n=2}^{\infty} \frac{1}{n!} \int \varrho_{i_1,\dots,i_n} \varepsilon_{i_1}(x_1) \dots \varepsilon_{i_n}(x_n)$$
$$+ \frac{1}{2} \int \psi(x_1) v(x_1 - x_2) \psi(x_2). \tag{9.73}$$

Our integrals include summation over species indices $i_1, \ldots, i_n$. We write (9.73) in the form

$$\sum_{n=2}^{\infty} \int \varepsilon_{i_1} K_{i_1 i_2} \varepsilon_{i_2} + \frac{1}{2} \int \psi v \psi , \tag{9.74}$$

where

$$K_{i_1 i_2} = \frac{1}{n!} \int \varrho_{i_1, \ldots, i_n} \varepsilon_{i_3, \ldots,} \varepsilon_{i_n} . \tag{9.75}$$

The integration is over the coordinates and species $x_3, i_3, \ldots, x_n, i_n$. Note that K also depends on n. By taking operator norm we bound (9.74) in absolute value by

$$\left(\sum_{n=2}^{\infty} \|K\| \right) \int |\varepsilon_i|^2 + \frac{1}{2} \|v\| \int \psi^2 . \tag{9.76}$$

Next we note that since ε_i is a periodic function of ϕ

$$|\varepsilon_i(\phi)| = |\varepsilon_i(\phi - h)| \tag{9.77}$$

$$\leqq |e_i| \beta^{1/2} |\phi - h| . \tag{9.78}$$

Thus (9.76) is bounded by

$$\left(\sum_{n=2}^{\infty} \|K\| \beta \right) \left(\sum_i e_i^2 \right) \int (\phi - h)^2 + \frac{1}{2} \|v\| \int \psi^2 . \tag{9.79}$$

We bound the operator norms by Lemma 9.6 in conjunction with

$$|\varepsilon_i| \leqq 2 \tag{9.710}$$

to control the factors ε_i in K. The resulting expressions are then bounded by appealing to our estimates (A1.6), (A1.12), and (9.94). In this way we prove that

$$\sum_{n=2}^{\infty} \beta \|K\| \to 0 \quad \text{and} \quad \|v\| \to 0 \tag{9.711}$$

as $\lambda \to 0$ so that (9.79) is less than

$$1/2c'_\lambda \int (\phi - h)^2 + 1/2c'_\lambda \int \psi^2 \tag{9.712}$$

for some constant c'_λ that tends to zero as λ tends to zero.

On collecting our estimates we find that the left hand side of the bound in our lemma is less than

$$\left(\int d\mu_s e^{1/2pc'_\lambda \int (\phi - h)^2 + 1/2pc'_\lambda \int \psi^2} \right)^{1/p} . \tag{9.713}$$

Since $d\mu_s = d\mu_s(\psi)$ we substitute

$$\phi - h = \psi + g - h \equiv \psi + f \tag{9.714}$$

and complete the square in (9.713) to obtain

$$\left(\int d\mu_s e^{pc'_\lambda \int \psi^2} \right)^{1/p} e^{1/2c'^2_\lambda p \int f Q f} e^{1/2c'_\lambda \int f^2} , \tag{9.715}$$

where Q is the kernel corresponding to

$$Q=(C_s^{-1}-2c'_\lambda p)^{-1}. \tag{9.716}$$

Since C_s is a convex combination of operators bounded uniformly, C_s^{-1} is bounded below uniformly in s and λ. Therefore Q is bounded uniformly in λ if λ is sufficiently small. Our estimates on Gaussian integrals in Appendix 3 therefore imply that (9.715) is less than

$$e^{c|X|}e^{1/2c_\lambda'^2 p\|Q\| \int f^2}e^{1/2c'_\lambda \int f^2} \tag{9.717}$$

for λ sufficiently small. Since, by the definition of F_1,

$$1/2\int f^2 \leqq F_1$$

we obtain our lemma by choosing λ so small that

$$c_\lambda'^2 p\|Q\|+c'_\lambda \leqq c_\lambda. \tag{9.718}$$

9.8. Bounds on Functional Derivatives

We discuss $\tilde{g}\tilde{\kappa}\mathscr{A}$ [see (9.23)], in our Hölder inequality. For simplicity we specialize to the following form for $\mathscr{A}$

$$\mathscr{A}=\prod_{i=1}^{w} e^{i\beta^{1/2}a_i\phi(x_i)}, \tag{9.81}$$

where the a_i are integers, and such that $\mathscr{A}$ is periodic in each $\phi(x_i)$ with period τ.

We also assume

$$|a_i|\leqq A_0 \tag{9.82}$$

to simplify the form of our estimates. We choose units of length with $\tilde{l}_D=1$.

We are going to study

$$|(\kappa'' e^{G_1}e^{G_2}e^R\mathscr{A})e^{-R}e^{-G_2}e^{c_1 d}e^{c_2|X|}|_0 \tag{9.83}$$

which up to the last two numerical factors is $|\tilde{g}\tilde{\kappa}\mathscr{A}|_0$. The main result of this section is the "combinatoric bound" (9.823). It is combinatoric in the sense that no assumptions on parameters or interactions are necessary to ensure its validity. The constant c'_2 in (9.823) depends only on c_2, q, L, L'. The other result, (9.824), requires conditions (9.123), (9.124).

Each derivative $\dfrac{\delta}{\delta\psi}$ in κ'' can act on any of the following: e^{G_1}, e^{G_2}, e^{+R}, a factor in $\mathscr{A}$, one of the ε's (or ψ's if $t=2$) contained in one of $\mathscr{E}$'s which occur in the κ'' to the right of the given differentiation. Thus a convenient way of mechanizing Leibniz rule is to expand each derivative.

$$\frac{\delta}{\delta\psi}=\sum_{l'}\left(\frac{\delta}{\delta\psi}\right)_{l'}, \tag{9.84}$$

where l' is a label that specifies on which of the above factors the differentiation will act. Corresponding to (9.84), we have, by so expanding every differentiation in κ''

$$\kappa'' = \sum_l \kappa_l''. \tag{9.85}$$

Suppose κ'' specifies n_α differentiations localized in unit lattice cube, Δ_α, and m_α factors of ε and/or ψ (contained in $\mathscr{E}$'s) localized in Δ_α, then the number of terms in the expansion (9.85) for κ'' is less than

$$\prod_\alpha (m_\alpha + w_\alpha + 3)^{n_\alpha}, \tag{9.86}$$

where w_α is the number of factors of $\mathscr{A}$ with x_i [in $\phi(x_i)$] in Δ_α. This factor grows rapidly with the order of the expansion. To control it we use "exponential pinning" e.g.

Lemma 9.10. *Given $c' > 0$ there is a constant, c, such that*

$$\prod_\alpha (m_\alpha + w_\alpha + 3)^{n_\alpha} \leqq c^{\Sigma(n_\alpha + w_\alpha)} \cdot e^{c' O_0} e^{c' d}. \tag{9.87}$$

We prove this lemma in Appendix 2. The proof may be paraphrased as follows: the expansion has been constructed in such a way that if a large number, n_α, of differentiations accumulate in one cube, then there are a corresponding number of covariances, C, with large associated exponential decays. (If a large number of people in a sparsely populated region want to get together they have to travel long distances.) A similar argument applies to a large number of different $\mathscr{E}$'s having factors ε localized in a common cube.

We substitute (9.85) into (9.83) and apply the exponential pinning (9.87) to obtain the following upper bound for (9.83)

$$c^{\Sigma w_\alpha} e^{c_1 d} e^{c_2 |X|} \text{"}\sup_l\text{"} |\kappa_l'' e^{G_1} e^{G_2} e^R \mathscr{A}|_0 e^{-R} e^{-G_2}, \tag{9.88}$$

where c_1, c_2, and r, δ (in the definition of κ_l'') have been increased. The inverted commas on the supremum indicate that the supremum is to be taken after the integral (L_p norm) over the field ψ has been performed. We can do this because there is nothing to prevent us from using the conversion of a sum to a supremum after (9.83) has been integrated.

The operator $e^{c_1 d} \kappa_l''$ has the form

$$\int J(\mathbf{x}) \left[\prod_j \frac{\delta}{\delta \psi(x_j)} \right]_l \prod_k \varepsilon(x_k), \tag{9.89}$$

where $J(\mathbf{x})$ is independent of ψ and is formed by convolutions of $\varrho(x_1, \ldots, x_t)$'s from the Mayer series terms and covariances $C(x, y)$ and includes the combinatoric factors prescribed by $e^{c_1 d}$, $e^{\delta L_0}$ and $e^{r O_0}$. If κ_l'' involves factors, $\mathscr{E}$, with $t = 2$ then (9.89) must be replaced by a more complicated expression. We will ignore this because the reader will see that the estimates which follow remain correct.

We substitute (9.89) into (9.88) and obtain

$$|(9.83)| \leqq \prod_\alpha (n_\alpha!) c^{w_\alpha} e^{c_2' |X|} \text{"}\sup_l\text{"} \int |J(\mathbf{x})| \prod_{\alpha, i} |T_{\alpha i}|, \tag{9.810}$$

where $c_2' > c_2$ in (9.83), α labels a unit lattice cube, Δ_α, $T_{\alpha i}$ is a function of fields $\psi(x)$ localized in Δ_α and also may depend on some of the variables $\mathbf{x}$ localized in Δ_α. $T_{\alpha i}$ is either equal to or a derivative of one of the following types, labelled by i in (6.810):

$$
\begin{aligned}
&\text{i) } e^{G_1(\Delta_\alpha)};\\
&\text{ii) } e^{i\beta^{1/2}a_i\phi} \text{ from } \mathscr{A};\\
&\text{iii) } F_2', \text{ a numerical function, from differentiating } F_2;\\
&\text{iv) } \varepsilon(x),\ \psi(x), \text{ or } (e^{i\beta^{1/2}e_i\phi} - 1 - i\beta^{1/2}e_i\psi) \text{ from } \mathscr{E}\text{'s};\\
&\text{v) } (e^{i\beta^{1/2}e_iA} - 1) \text{ from } G_2;\\
&\text{vi) } (e^{i\beta^{1/2}e_i\delta} - 1 - i\beta^{1/2}e_i\delta) \text{ or } (e^{i\beta^{1/2}e_i\delta} - 1 - i\beta^{1/2}e_i\delta + \beta/2e_i^2\delta^2) \text{ from } G_2.
\end{aligned}
\tag{9.811}
$$

We divide the unit lattice cubes into two classes:

$$
\begin{aligned}
&\text{Class A)} \quad \text{unit cubes in which } g = h.\\
&\text{Class B)} \quad \text{unit cubes in which } g \neq h.
\end{aligned}
\tag{9.812}
$$

Thus class B) unit cubes are contained in the phase boundaries $\sum^{\wedge}$ defined in Sect. 6. We simplify the quantities $T_{\alpha i}$ by bounding them above according to the following scheme: Terms from i) are bounded by Lemma 9.7. The n^{th} derivative of a term from ii) is bounded by

$$(c\beta^{1/2})^n. \tag{9.813}$$

The n^{th} derivative of a term from v) is bounded by

$$(c\beta^{1/2})^n \cdot 2. \tag{9.814}$$

The n^{th} derivative of either term from vi) is bounded by

$$c\beta(|\delta|^{2-n} + |g-h|^{2-n}) \quad \text{if} \quad n<2, \quad (c\beta^{1/2})^n \quad \text{if} \quad n \geqq 2. \tag{9.815}$$

In class A) cubes we bound the ψ in iv) by $|\psi|$, the n^{th} derivative of the third term in iv) by

$$c\beta|\psi|^{2-n} \quad \text{if} \quad n<2, \qquad (c\beta^{1/2})^n \quad \text{if} \quad n \geqq 2. \tag{9.816}$$

The n^{th} derivative of ε for $\underline{n \geqq 1}$ is bounded by

$$(c\beta^{1/2})^n. \tag{9.817}$$

In class B) cubes (9.816) is modified for $n=0,1$ to

$$2 + c\beta^{1/2}|\psi| \quad \text{if} \quad n=0, \qquad c\beta^{1/2} \quad \text{if} \quad n=1. \tag{9.818}$$

We divide the undifferentiated ε's in iv) into two sets: distinguished ε's and undistinguished ε's. We bound the ε's by

$$2 \text{ if undistinguished,} \qquad c\beta^{1/2}|\psi(x)| \text{ if distinguished.} \tag{9.819}$$

The point is to generate sufficient factors of $\beta^{1/2}$ to control the powers of z_i contributed by factors $\varrho(x_1, \ldots, x_t)$ in J. See for example (A1.6) and (A1.12). On the other hand we must avoid creating too many factors of $|\psi|$ which will lead to uncontrollable factorials when the bound in Sect. 9.3 is used. We first select one distinguished $\varepsilon(x)$ localized in each cube of class A), wherever this is possible. From each factor $\mathscr{E}$ we select three further distinguished $\varepsilon(x)$ in class A) cubes, or if this is impossible, as many as one can select. This completes the selection of distinguished $\varepsilon(x)$. Many other selection procedures would have been possible.

The result of these estimates applied to (9.810) is a bound which is not good enough for our purposes because factorials such as $n_\alpha!$ grow too swiftly to be compensated by the powers of β available. We will now show how to use an exponential pinning lemma to include some compensating factors in our bounds. Let N_α be the number of factors of *distinguished* ε's in κ'' which depend on fields in Δ_α. In Appendix 2 we prove

Lemma 9.11. *Given $c'>0$ and q there exists c so that*

$$\prod_\alpha (n_\alpha!)^q \leqq c^{\Sigma n_\alpha} e^{c'd}, \tag{9.820}$$

$$\prod_\alpha (N_\alpha!)^q \leqq c^{\Sigma n_\alpha} e^{c'L_0} e^{c'd}. \tag{9.821}$$

By means of this lemma we can include a factor

$$\prod_\alpha (n_\alpha!)^{-q}(N_\alpha!)^{-q} \equiv f_q \tag{9.822}$$

in front of (9.88) and (9.810) at the expense of increasing c_2', c_1 and δ, r in the definition of κ_l'', J.

We apply our bounds for $T_{\alpha i}$ to (9.810) with a factor f_q included and obtain for all q and c_2' depending on q

$$\begin{aligned}|(9.83)| \leqq\, & c^{\Sigma w_\alpha} e^{c_2'|X|} f_q \text{“Sup”} \int |J| \Pi(|\psi|)\Pi(|\delta|) \\ & \cdot \Pi(|F_2'|)\Pi(|g-h|) d\mathbf{x} e^{\gamma/2\int(\psi+g-h)^2} \beta^{N/2}\beta^{M/6}.\end{aligned} \tag{9.823}$$

(Some numerical factors have already been absorbed by f_q with an unindicated index change.) Our notation for the integrand is schematic. $N/2$ is the power of β that comes from applying (9.813) to (9.819) to $T_{\alpha i}$'s. It depends on the term in the "Sup". The "Sup" is over not only l but additional parameters arising when sums like the one in (9.815) are converted to supremums. c_2' has been accordingly increased. M is the number of functional derivatives acting on e^{+G_1}. The bound (9.823) thus provides a bound of the form (9.23) for $\tilde{g}\tilde{\kappa}\mathscr{A}$. The "Sup" is not an important modification since it takes place after integrals over fields.

In order to use this bound, one must know something about the kernel J. We use (A1.6) and (A1.12) to control the factors $\varrho(x_1, \dots, x_t)$ in J. We use the exponential decay [see for example (9.32)], of the covariances. We need restrictions such as (9.123) and (9.124) so as to control the combinatoric factors $e^{c_1 d}$, $e^{\delta L_0}$, e^{rO_0}. We put this together to obtain: there is a q so that for any c_2' and $c_3>0$

$$c^{\Sigma w_\alpha} e^{-c_3 F_1} e^{c_2'|X|} f_q \beta^{N/2}\beta^{M/6} \int |J| \Pi(|F_2'|)\Pi(|g-h|) \tag{9.824}$$

tends to zero as $\beta \to 0$ uniformly in X, $|X|>1$ and all compatible h, J. The powers of β and the exponential of F_1 together control the exponential of $|X|$ because in Sect. 8 the choice of Y_{i+1} for each i is made so that every cube, $\Delta_\alpha \subset Y_{i+1}$, either contributes convergence by belonging to Σ^A which means that F_1 is large (as $\beta \to 0$) in Δ_α or it contains a term from κ'' (or κ) which contributes a power of β. The factors $g-h$, F'_2 are also controlled (in L_2 norm) by $e^{-c_3 F_1}$ and f_q.

9.9. *Conclusion of Proof*

The reader can easily check that the hypotheses of Lemmas 9.5, 9.8, 9.9 are compatible. For λ sufficiently small, L sufficiently small and L' sufficiently large, we see that the product of the second, third, fourth and fifth terms on the right of our Hölder inequality (9.25) are bounded by

$$e^{c|X|}e^{-c'F_1}, \quad c'>0, \tag{9.91}$$

where c, c' depend on γ in (9.25) but not on z_i, β. We obtain the first term in the Hölder inequality through a bound on $|\tilde{g}\tilde{\kappa}\mathscr{A}|$ by (9.823) and then use (9.824). Before we state the result, we detail conditions on our potentials and parameters that have entered our discussion.

(1) We have fixed integral charges, e_i. Our activities are constrained, from (3.8), by a neutrality condition

$$\sum e_i \varrho_i = 0. \tag{9.92}$$

(2) We assume stability of the short range potential [see (A 1.8)] in the form

$$\beta W_N \geqq -c_1 N \tag{9.93}$$

($W=V$ in Appendix 1).

(3) Referring to (A 1.9) and (A 1.10) we require smallness of the short range potential in the form

$$2e z_m \beta \|v_2\|_\alpha e^{c_1} \leqq c_2 \lambda^2 < \tfrac{1}{2}, \tag{9.94}$$

where v_2 is given by (3.3). λ first appears in (1.5).

(4) We assume

$$\alpha \geqq (1-\delta_1)\frac{1}{\tilde{l}_D}, \; |C(x,y)| \leqq c_u e^{(1-\delta_1)|x-y|/\tilde{l}_D} \tag{9.95}$$

with $\delta_1 < 1/2$, from (9.123), (9.124), and (9.34). By our remarks in Sect. 9.3 the second bound holds if λ is sufficiently small and if the first bound holds with δ_1 replaced by $\delta'_1 < \delta_1$. The second bound in (9.95) prohibits $\lambda = 0$.

(5) Non degeneracy: we limit the fugacities by the conditions

$$\frac{z_i}{z_m} \geqq c_i > 0 \tag{9.96}$$

for some constants c_i. The origin of this condition is the non-uniformity of γ in Lemma 9.7 in the relative sizes of the activities z_i. In consequence c' in (9.91) will depend on the constants c_i.

Lemma 9.12. *With the parameters* $(e_i, c_1, c_2, \lambda, \alpha, \delta_1 > 0, c_i, L, L')$ *fixed as above and with* $\mathscr{A}$ *as given in* (9.81) *supported in* X_1, *a union of* $\tilde{l}_D$*-cubes, let* c'_A *be arbitrary.*

There are c_A and c_3 independent of β, z_i such that if $\beta/\tilde{l}_D$ is sufficiently small the expansion converges in the sense

$$\sum_{\substack{X, X \supset X_1 \\ X \cap W \neq \emptyset}} |\mathscr{H}(X) e^{c'_A |X|}| \leqq c_3^w e^{c_A |X_1|} \cdot e^{-(1-2\delta_1)\operatorname{dist}(X_1, W)/\tilde{l}_D}. \tag{9.97}$$

If the sum over X is further restricted so that X strictly contains X_1 we may replace c_3^w by cc_3^w where $c \to 0$ as $\beta/\tilde{l}_D \to 0$.

Remarks. Parts of our convergence proof were given with $\tilde{l}_D = 1$. In obtaining the form of Lemma 9.12, the length scale invariance of results is useful, that is the transformation

$$\beta \to \beta/l, \quad z_i \to z_i l^3, \quad w \to lw, \quad x \to x/l,$$
$$\tilde{l}_D \to \tilde{l}_D/l, \quad \phi \to l^{1/2}\phi, \quad \lambda \to \lambda, \quad \delta_1 \to \delta_1, \quad c_1 \to c_1 \quad \text{etc.}$$

Lemma 9.12 carries forward the intentions of Lemma 9.4. There is a similar statement to correspond with Lemma 9.1.

We now consider the expansion obtained by dividing both sides of (9.01) by Z so that the left hand side becomes the finite volume expectation of $\mathscr{A}$. By combining Lemma 9.12 and results in Appendix 4 we obtain the existence of the infinite volume expectation of $\mathscr{A}$. By using a standard [1, 11] "doubling the measure" argument we also obtain exponential clustering of infinite volume correlations of $\mathscr{A}$'s.

Theorem 9.13. *With the parameters fixed as above, and with $\mathscr{A}, \mathscr{B}$ of the form* (9.81) *with $\mathscr{A}$ supported in X_1, $\mathscr{B}$ supported in X_2, products of w_1 and w_2 factors respectively, there exist c_3, c_A independent of z_i, β such that for $\beta/\tilde{l}_D$ sufficiently small, the infinite volume expectation $\langle\ \rangle$ exists and*

$$|\langle \mathscr{A}\mathscr{B} \rangle - \langle \mathscr{A} \rangle \langle \mathscr{B} \rangle| \leqq c_3^{w_1 + w_2} e^{c_A(|X_1| + |X_2|)} \cdot e^{-(1-2\delta_1)\operatorname{dist}(X_1, X_2)/\tilde{l}_D}, \tag{9.98}$$

$$|\langle \mathscr{A} \rangle| \leqq c_3^{w_1} e^{c_A |X_1|}. \tag{9.99}$$

δ_1 may be picked arbitrarily small if the first relation in (9.95) *holds by keeping all the parameters other than λ fixed and making λ small depending on δ_1 (thus forcing the short range potentials to be small by* (3)*). With δ_1 and λ fixed the above results hold for $\beta/\tilde{l}_D$ sufficiently small.*

Now we consider observables built up from particle densities $\sigma_i(x)$ (see Sect. 1). Let A be an observable of the form

$$A = \int f(x_1, \ldots, x_w) : \sigma(x_1) \ldots \sigma(x_w) : \tag{9.910}$$

where f is a bounded function supported in X_1 and species indices have been suppressed. Such observables can easily be pushed through the sine Gordon transformation and emerge as convergent sums of polynomials in $\varepsilon(x)$, $e^{ie_j\beta^{1/2}\phi}$. For example, consider the linear combination

$$\tau = i \sum_j \beta^{1/2} e_j \sigma_j (= i\beta^{1/2} J) \tag{9.911}$$

then

$$\frac{1}{Z_0} I(:\tau(x_1)\dots\tau(x_w):) = \int d\mu_0 \frac{\delta}{\delta\phi(x_1)} \cdots \frac{\delta}{\delta\phi(x_w)} Z(\phi). \tag{9.912}$$

As in Sect. 3 we write $Z(\phi)=e^{M'}$ and expand M' as in (3.6) in a convergent series. The result of performing the derivatives is a sum of terms $I(\mathscr{A}(\phi))$ with $\mathscr{A}$ a polynomial in ε, $e^{i\beta^{1/2}e_j\phi}$. The argument for an observable as in (9.910) is more complicated but no more difficult. We now prove and analogue of Theorem 9.13 for observables A of this type and deduce.

Theorem 9.14. *Let A, B be observables as in* (9.910), *supported in X_1, X_2 respectively, and with w_1, w_2 factors of σ respectively. With the parameters fixed as above the infinite volume limit exists and*

$$|\langle AB\rangle - \langle A\rangle\langle B\rangle| \leq c_A \cdot c_B \cdot e^{-(1-2\delta_1)\operatorname{dist}(X_1,X_2)/\bar{l}_D}. \tag{9.913}$$

δ_1 is as in Theorem 9.13.

Remark. In order to control the sums over observables $\mathscr{A}$ discussed above it is necessary to obtain some convergence from the ε factors. This is achieved by lumping them in with the ε's in (9.811) iv).

Next we discuss the screening of fractional charges. Suppose that in addition to species $j=1,\dots,s$ with charges e_j we have a species $j=0$ with charge e_0 not necessarily a multiple of g.c.d. $\{e_i\}$, i.e. possibly fractional. We will discuss the expectations of observables of the form

$$A = \frac{1}{z_0^w} \int f(x_1,\dots,x_w) : \sigma_0(x_1)\dots\sigma_0(x_w), \qquad f \in L_\infty \tag{9.914}$$

in the limit $z_0 \to 0$. The activity of the fractional species goes to zero. This is a way of studying fixed fractional charges in a sea of non fractional charges.

Theorem 9.15. *With the same hypotheses as in Theorem 9.14 we obtain the same conclusion for observables as in* (9.914) $(X_1 \cap X_2 = \emptyset)$.

We do not give a proof of this theorem. It is more difficult than our preceding theorems because the resummation in Sect. 9.0 is no longer valid if the observables don't have period τ as is the case after applying the Sine Gordon transformation to (9.914). However it is possible to iterate (8.3) so that the factors

$$N \int d\mu(\psi) e^{E(X^c)} e^{G(X^c)} e^{R(X^c)}$$

only appear with h constant on the boundaries of the components of X^c not containing the point at ∞. (Each such component of X^c has a single value of h over its entire boundary.) Also $h=0$ at the boundaries of the union of components of X^c containing the point at ∞. The resulting expansion can be resumed over h subject to the boundary conditions given above and yields Theorem 9.15.

Finally we state that our limits are independent of the parameter λ (if small enough) in the charge symmetric situation.

Appendix 1. The Mayer Series II

We return to the definition of the M' and the ϱ's of Sect. 3 in order to establish estimates that will justify the manipulations of Sect. 3 and also will enter into our convergence proof in Sect. 9. Recall

$$M' = \sum_i \int \varrho_i \varepsilon_i + \frac{1}{2!} \sum_{i,j} \int \varrho_{i,j} \varepsilon_i \varepsilon_j + \dots \tag{A1.1}$$

Let

$$k_s(a_i) = \int_{a_1} dx_1 \dots \int_{a_s} dx_s |\varrho_{i_1,\dots,i_s}(x_1,\dots,x_s)|, \tag{A1.2}$$

where $a_1, \dots, a_s$ is a set of unit lattice cubes. In order to examine the size of $k_s(a_i)$ when the cubes are widely separated, we define an object η^A, an augmented tree, consisting of

i) a tree η on s' vertices, $s \leqq s'$. This is a mapping $i \to \eta(i)$, $1 \leqq \eta(i) < i \leqq s'$. (Trees were defined this way in [1], unfortunately a different definition from in [3].)

ii) An augmentation mapping, A, defined on $1, \dots, s$ with range the integers. This satisfies

$$1 \leqq A(i) \leqq s'$$
$$A(i) \neq A(j) \quad \text{if} \quad i \neq j .$$

For a given η^A we define

$$L_{\eta^A}(a_i) = \inf_{\substack{x_i \\ x_{A(j)} \in a_j}} \sum_{i=2}^{s'} d(x_i, x_{\eta(i)}) . \tag{A1.3}$$

For a given $\alpha > 0$ there will be constants b_{η^A} satisfying

$$b_{\eta^A} \geqq 0, \sum_{\eta^A} b_{\eta^A} = 1 \tag{A1.4}$$

for which we define $L(a_i)$ by

$$e^{-\alpha L(a_i)} = \sum_{\eta^A} b_{\eta^A} e^{-\alpha L_{\eta^A}(a_i)} . \tag{A1.5}$$

Our basic estimate for $k_s(a_i)$ reads

$$|k_s(a_i)| \leqq c_s(\alpha) e^{-\alpha L(a_i)} . \tag{A1.6}$$

Conditions on the potentials and α that yield (A1.6) will be detailed along with an upper bound for $c_s(\alpha)$. The b_{η^A} will only be implicitly derived, for only their basic properties such as (A1.4) will be relevant. (A1.6) is uniform in Λ'.

We assume that the two body potential v_2 given in (3.3) can be decomposed into two terms

$$v_2 = w_n + w_R , \tag{A1.7}$$

where w_R is purely repulsive, i.e. $w_R \geqq 0$, and for all integers $N \geqq 0$,

$$V_n \geqq -BN \tag{A1.8}$$

on the N body subspace. V_n is the interaction constructed from the two-body potentials w_n. We define a norm $\|v_2\|_\alpha$ similarly to (A3) and [34] of [3]

$$\|v_2\|_\alpha = \sup_i \sum_j \int dx\, e^{\alpha|x|} \left[|w_n(x)| + \frac{1}{\beta} |e^{-\beta w_R(x)} - 1| \right]. \tag{A1.9}$$

The supremum and sum are over species indices i, j which have been suppressed on the w's. Our final assumption is that κ defined by

$$\kappa = 2ez_m\beta \|v_2\|_\alpha e^{\beta B} \tag{A1.10}$$

obeys $\kappa < \frac{1}{2}$. B is from (A1.8) and

$$z_m = \max_i \tilde{z}_i. \tag{A1.11}$$

With these assumptions, we will show that (A1.6) holds and $c_s(\alpha)$ is bounded by

$$c_s(\alpha) \leqq \frac{2s!}{\|v_2\|_\alpha e\beta} \kappa^s. \tag{A1.12}$$

Our arguments will assume familiarity with [3] and we will follow the notation of that reference closely. Note that (A1.12) and (A1.6) imply convergence in (A1.1) uniformly in Λ'.

We start with Eq. (49) in [3] and expand the exponential of the one-body potentials using (3.7) and then resum the series, (27) in [3], as described in Sect. 3. The result is

$$\varrho_{i_1,\ldots,i_s}(x_1,\ldots,x_s) = \sum_{t=s}^{\infty} \frac{1}{t} \sum_{\eta} \sum_{i_{s+1},\ldots,i_t} \prod_i (\tilde{z}_i) \cdot \sum_{\mathscr{S}_s} \int_{\Lambda'} (dx)^{t-s} J^{(t)}(\eta, x, x_1, \ldots, x_s), \tag{A1.13}$$

$$J^{(t)} = (-\beta)^{t-1} \int d\sigma_{t-1}\, f(\eta, \sigma_{t-1}) \prod_{r=1}^{t-1} v_2'(r+1, \eta(r+1)) \exp[-\beta W'^{(t)}(\sigma_{t-1})] \tag{A1.14}$$

where

(i) $\sum_{\mathscr{S}_s}$ is the sum over different possible s member ordered subsets of the t vertices of the tree diagram labelled by η. These are distinguished vertices which have coordinates $x_1, \ldots, x_s$. The integral is over the positions of the remaining vertices. $\sum_{i_{s+1},\ldots,i_t}$ sums over the species at the non-distinguished vertices.

(ii) The species subscripts in J, v_2', and $w'^{(t)}$ are omitted.

(iii) V_2' and $W'^{(t)}$ are constructed with the possible extension to repulsive potentials as detailed in (A4) and (A5) of [3]. A specification of a permutation of the integers $1, 2, \ldots, s$ together with a tree graph η with s distinguished vertices in (A1.13) defines an augmented tree graph η^A. By removing non-distinguished vertices that join exactly two lines, and branches of the tree containing no distinguished vertices, a unique minimal augmented tree graph $\bar{\eta}^A$ is determined. (Trim the tree and straighten its branches!) The length as defined in (A1.3) is the same for a graph and the corresponding minimal graph. These minimal graphs are the only ones that will appear in the sums in (A 1.4) and (A 1.5). It is useful for us to observe that for these minimal graphs $s' \leqq 2s-1$. We apply to (A 1.13) and (A 1.14) the estimate (A 1.8) and

$$\int (dx)^t \prod_{r=1}^{t-1} |v_2'(r+1, \eta(r+1))| \leqq \|v_2'\|_\alpha^{t-1} e^{-\alpha L_{\bar\eta}^{A}(a_i)}. \qquad \text{(A 1.15)}$$

Here the variables $x_1, \ldots, x_s$ are to be integrated over $a_1, \ldots, a_s$ and the remaining variables over Λ'. In addition we use Proposition 3 of [3]:

$$\sum_\eta \int d\sigma_{t-1} f(\eta, \sigma_{t-1}) \leqq e^{t-1}. \qquad \text{(A 1.16)}$$

In direct analogy to (56) in [3] we thus derive

$$c_s(\alpha) \leqq \sum_{t=s}^{\infty} e^{t-1} \frac{1}{t} z_m^t \beta^{t-1} \frac{t!}{(t-s)!} \|v_2\|_\alpha^{t-1} e^{t\beta B}. \qquad \text{(A 1.17)}$$

We have used the fact that there are $\dfrac{t!}{(t-s)!\,s!}$ ways of selecting a subset of s elements out of t. Thus

$$c_s(\alpha) \leqq \sum_{t=s}^{\infty} \frac{s!}{e\beta \|v_2\|_\alpha} \kappa^t \qquad \text{(A 1.18)}$$

which implies our bound (A 1.12) when $\kappa < 1/2$.

Finally, note that these inequalities and (A 1.13), (A 1.14) show that the limit $\Lambda' \nearrow \mathbb{R}^3$ exists and satisfies the same bounds.

We write

$$M = \sum_i \varrho_i \int \varepsilon_i + E' \qquad \text{(A 1.19)}$$

and

$$M = \sum_i \varrho_i \int \varepsilon_i - \tfrac{1}{2} \int \psi v \psi + E, \qquad \text{(A 1.20)}$$

where

$$v = \sum_{i,j} \beta e_i e_j \varrho_{i,j}(x, y). \qquad \text{(A 1.21)}$$

The ϱ's of this paper are the truncated correlation functions of statistical mechanics. Similar bounds to our (A 1.6) on fall off rates for these correlation functions have been obtained in [4]. Using the results of [4] our theorems may be restated with slightly different conditions on the potentials.

Appendix 2. Exponential Pinning

We begin with the proof of Lemma 9.10. For $c' > 0$ there is a constant c_r so that

$$(m_\alpha + w_\alpha + 3)^{n_\alpha} \leqq c_r^{n_\alpha + w_\alpha} n_\alpha! \, e^{c' m_\alpha}. \qquad \text{(A 2.1)}$$

This is a trivial inequality that can be obtained by considering the maximum of $N^n c^{-N}$. Next let Δ_i be unit lattice cubes packed as closely as possible about Δ_α, then for all $c'>0$ and p there is c so that

$$(n_\alpha!)^p \leqq c \prod_{i=1}^{n_\alpha} e^{c' \operatorname{dist}(\Delta_\alpha, \Delta_i)} \quad \text{if} \quad n_\alpha \neq 0. \tag{A2.2}$$

c is independent of α. The proof is easy. Now we note that the product over α of the right hand side of (A2.2) is less than

$$c^{\Sigma n_\alpha} e^{c'd} \tag{A2.3}$$

and

$$\prod_\alpha e^{c' m_\alpha} \leqq e^{c' O_0}. \tag{A2.4}$$

The proof of Lemma 9.10 follows by collecting these estimates.

We now discuss Lemma 9.11. The first inequality is immediate from (A2.2) and (A2.3). To prove the second inequality we start as in (A2.2) with

$$(N_\alpha!)^q \leqq c \prod_{i=1}^{N_\alpha} e^{c' \operatorname{dist}(\Delta_\alpha, \Delta_i)} \tag{A2.5}$$

and dominate the product over α of the right hand side by

$$c^{|X|} e^{c' L_0} e^{c'd} \tag{A2.6}$$

for some c.

Factors to be controlled have each been pinned to distant cubes by exponentials.

Appendix 3. Gaussian Integrals

We summarize some useful facts about Gaussian integrals which are in repeated use throughout the paper.

Let M be a bounded positive operator on $L^2(\Lambda)$ with $\Lambda \subset \mathbb{R}^n$ open. There exists a measure space $(\Omega, d\mu_M)$ and Gaussian random variables, symbolically denoted by

$$\int f\phi \tag{A3.1}$$

indexed by functions $f \in L^2(\Lambda)$ such that

$$\int d\mu_M e^{\int f\phi} = e^{1/2 \int fMf}. \tag{A3.2}$$

$(\Omega, d\mu_M)$ is unique up to isomorphism of measure spaces. M is called the covariance of $d\mu_M$.

The ensuing formulas are most easily understood in the light of the following heuristic representation for $d\mu_M$

$$d\mu_M = e^{-1/2 \int \phi M^{-1} \phi} \prod_{x \in \Lambda} d\phi(x) / \text{Normalization}. \tag{A3.3}$$

One may perform certain changes of variable in Gaussian integrals, e.g. the translation

$$\phi = \psi + g \tag{A3.4}$$

yields

$$d\mu(\phi) = d\mu(\psi) e^{-1/2 \int g M^{-1} g} e^{-\int \psi M^{-1} g}. \tag{A3.5}$$

This formula is valid provided g is in the domain of M^{-1} which is generally an unbounded operator on $L^2(\Lambda)$.

If M is trace class and its kernel satisfies some additional regularity properties (continuity properties on the diagonal, see for example [9]) the measure space Ω may be taken to be the space of continuous functions ϕ on Λ and (A3.1) is no longer symbolic. We are in this case in this paper (except for the measure $d\omega$ introduced briefly in Sect. 9.6) and our remarks given below will assume this additional regularity.

Let N be a bounded selfadjoint operator on $L^2(\Lambda)$. The function

$$\int \phi N \phi \tag{A3.6}$$

is a random variable (i.e. measurable). We are about to obtain conditions on N such that

$$\int d\mu_M e^{1/2 \int \phi N \phi} \tag{A3.7}$$

is finite. If M and N were matrices on a finite dimensional Hilbert space an elementary calculation involving changes of variables that diagonalize $M^{-1} - N$ and M^{-1} shows that

$$\int d\mu_M e^{1/2 \int \phi N \phi} = \det{}^{-1/2}(I - MN) \tag{A3.8}$$

$$= \det{}^{-1/2}(I - M^{1/2} N M^{1/2}) \tag{A3.9}$$

provided

$$\|M^{1/2} N M^{1/2}\| < 1 \tag{A3.10}$$

which we will henceforth assume (or establish). These formulas continue to be valid in our context.

We shall now show that

$$e^{1/2 \operatorname{tr}(MN)} \leqq \int d\mu_M e^{1/2 \int \phi N \phi} \tag{A3.11}$$

$$\leqq e^{c_N \operatorname{tr}(M|N|)}, \tag{A3.12}$$

where $|N|$ is the operator absolute value of N and

$$c_N = (1 - \|M^{1/2}|N|M^{1/2}\|)^{-1}. \tag{A3.13}$$

It is enough to find corresponding upper and lower bounds for the determinant in (A3.9).

$$\det(I - M^{1/2} N M^{1/2}) = e^{\Sigma \log(1 - \lambda_i)}, \tag{A3.14}$$

where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of the trace class operator $M^{1/2}NM^{1/2}$. We bound the log above and below by

$$\log(1-\lambda) \leqq -\lambda \tag{A3.15}$$

and

$$-\log(1-\lambda) = \log\left(\frac{1}{1-\lambda}\right) \tag{A3.16}$$

$$\leqq \frac{1}{1-\lambda} - 1 = \frac{\lambda}{1-\lambda}. \tag{A3.17}$$

In (A3.15) we take λ to be an eigenvalue λ_i and the lower bound in (A3.11) immediately follows since

$$\mathrm{tr}\, MN = \mathrm{tr}\, M^{1/2}NM^{1/2} = \sum \lambda_i. \tag{A3.18}$$

To prove the upper bound in (A3.12) we first replace N by $|N|$ which increases the integral and then take λ in (A3.17) to be in turn each of the eigenvalues of $M^{1/2}|N|M^{1/2}$ so that $(1-\lambda)^{-1} \leqq c_N$ and

$$\frac{\lambda}{1-\lambda} \leqq c_N \lambda \tag{A3.19}$$

since λ is positive. The upper bound in (A3.12) follows immediately in analogy to the lower bound.

The measure

$$d\mu_M e^{1/2 \int \phi N \phi} / \int d\mu_M e^{1/2 \int \phi N \phi} \tag{A3.20}$$

is also Gaussian and its covariance is

$$(M^{-1} - N)^{-1} = M'. \tag{A3.21}$$

This is easily seen by computing the Laplace transform, e.g.,

$$\int d\mu_M e^{1/2 \int \phi N \phi} e^{\int f \phi} = \left(\int d\mu_M e^{1/2 \int \phi N \phi}\right) e^{1/2 \int f M' f}. \tag{A3.22}$$

We have used this formula frequently in the text. It follows by using (A3.5), the formula for translation. g is chosen so that the term linear in ψ (which has been relabelled ϕ) is eliminated, i.e. we complete the square.

We now leave these general considerations and instead discuss their application to the Gaussian integrals that came up in Sects. 9.6 and 9.7. In Sect. 9.6 we have the bound (9.639) to prove. Let

$$C_0^{-1} = u^{-1} + 1. \tag{A3.23}$$

The associated Gaussian measure is $d\tilde{\mu}_0$. The left hand side of (9.639) can be written in the form

$$\left(\int d\tilde{\mu}_0 e^{1/2 \int \phi N \phi}\right)^{1/p'}, \tag{A3.24}$$

where

$$N = -(Q-1) = \beta\chi + \beta\frac{4}{\gamma} P\chi P. \tag{A3.25}$$

By virtue of our bounds (A3.11) and (A3.12) the proof of (9.639) reduces to showing that

$$\|C_0^{1/2}NC_0^{1/2}\|<1, \quad \mathrm{tr}(C_0N)\leqq c|Y|, \tag{A3.26}$$

where Y is the support of χ. Since Y is a union of cubes Ω_α, P commutes with χ and so

$$\mathrm{tr}\,C_0P\chi P=\mathrm{tr}\,C_0\chi P^2\leqq \mathrm{tr}\,C_0\chi \tag{A3.27}$$

$$\leqq c|Y|. \tag{A3.28}$$

The second inequality can be obtained from an estimate for $C_0(x,x)$. The first inequality in (A3.26) holds if L is sufficiently small because

$$\|\beta\chi\|\leqq\beta<1 \tag{A3.29}$$

and

$$\|C_0^{1/2}P\chi PC_0^{1/2}\|\leqq\|C_0^{1/2}PC_0^{1/2}\| \tag{A3.30}$$

and the right hand side goes to zero with L by (9.640). This completes the proof of (9.639).

In Sect. 9.7 we required the bound

$$\int d\mu_s e^{pc'_\lambda\int\psi^2}\leqq e^{c|X|}. \tag{A3.31}$$

This reduces by (A3.12) to showing that if λ is sufficiently small

$$\|C_s^{1/2}2pc'_\lambda\chi C_s^{1/2}\|<1, \quad \mathrm{tr}(C_s\chi)\leqq c|X|, \tag{A3.32}$$

where χ is the characteristic function of X. The first bound follows immediately from $c'_\lambda\to0$ as $\lambda\to0$ as established in Sect. 9.7. To prove the second bound recall that C_s is a convex combination of "diagonalized" covariances of the form

$$\sum\chi_iC\chi_i, \tag{A3.33}$$

where the supports of the χ_i are disjoint. This means that it is sufficient to prove the bound with C_s replaced by each of these diagonalized covariances in turn. Furthermore, if λ is small enough we have shown that

$$C\leqq 2C_0 \tag{A3.34}$$

[see (9.618)]. Therefore we are reduced to proving that

$$\mathrm{tr}\,\chi_iC_0\chi_i\chi=\mathrm{tr}\,C_0\chi_i\chi\leqq c\,\mathrm{supp}\,\chi_i\chi \tag{A3.35}$$

which again follows by estimating $C_0(x,x)$.

Appendix 4. Ratio of Partition Functions – Infinite Volume Limit

Our principle result is

Lemma A4.1. *Under the same conditions on parameters as described in Sect. 9.9 there exists a constant c independent of Λ such that*

$$\left|\frac{Z'(\Lambda,X)}{Z}\right|\leqq e^{c|X|}. \tag{A4.1}$$

Furthermore the infinite volume limit

$$\lim_{\Lambda\to\mathbb{R}^3}\frac{Z'(\Lambda,X)}{Z}$$

exists.

The proof is based on the proof of Theorem 6.1 in [11]. We write our expansion as an equation on a Banach space and solve by a Neumann expansion.

We start by recalling that in Sect. 8 $\mathbb{R}^3$ was partitioned into $\tilde{l}_D$-lattice cubes. We set $\tilde{l}_D=1$. We suppose that these cubes are assigned some arbitrary order independent of Λ. We wish to obtain an expansion for

$$Z'(\Lambda,X)=\sum_h N\int d\mu(\psi)e^{E(X^c)}e^{G(X^c)}e^{R(X^c)}. \tag{A4.2}$$

The compatibility conditions mentioned in Sect. 9.0 simply amount to having h run over all configurations for which h is constant in certain "collar" neighborhoods of width L' of connected components of X. We apply the cluster expansion of Sect. 8 to each term inside the sum in (A4.2) taking $X_1=Y_1$ equals the first lattice cube in $\Lambda\sim X=X^c$, if non-empty. After cluster expanding we resum over h as in Sect. 9.0 and the result is

$$Z'(\Lambda,X)=\sum_{X'}\mathscr{K}(X'\sim X)Z'(\Lambda,X'). \tag{A4.3}$$

$\mathscr{K}(\cdot)$ is defined in (9.02). The sum is over all X' which are unions of lattice cubes inside Λ with X' containing $X\cup$ first cube in X^c.

Define X^* by

$$X^*=X\sim\text{last cube in }X. \tag{A4.4}$$

We rewrite (A4.3) (with X replaced by X^*) in the form

$$\mathscr{K}(X\sim X^*)Z'(\Lambda,X)=Z'(\Lambda,X^*)-\sum_{\substack{X'\\ X\subsetneq X'}}\mathscr{K}(X'\sim X^*)Z'(\Lambda,X'). \tag{A4.5}$$

We divide through by $\mathscr{K}(X\sim X^*)$ and rewrite this as an equation on a Banach space. The Banach space $\mathscr{B}$ is all complex functions on subsets of Λ which are unions of lattice cubes. The norm is

$$\|\varrho\|=\sup_X|e^{-\alpha|X|}\varrho(X)| \tag{A4.6}$$

and α will be chosen below. Define an operator Q on $\mathscr{B}$ by

$$Q\varrho(X)=\frac{1}{\mathscr{K}(X\sim X^*)}\left\{\varrho(X^*)-\sum_{\substack{X'\\ X\subsetneq X'}}\mathscr{K}(X'\sim X^*)\varrho(X')\right\} \tag{A4.7}$$

provided $X\neq\emptyset$. Set $Q\varrho(\emptyset)=0$. If the sum over X' is vacuous set the sum equal to zero. Equation (A4.5) can now be written in the form

$$\varrho=Z\delta+Q\varrho, \tag{A4.8}$$

where δ is defined by

$$\delta(\emptyset)=1, \quad \delta(X)=0 \quad \text{if} \quad X\neq\emptyset. \tag{A4.9}$$

and

$$\varrho(X)=Z'(\Lambda,X), \quad \varrho(\emptyset)=Z. \tag{A4.10}$$

In order to prove the bound in the lemma it is sufficient to show that for some α

$$\|Q\|\leqq 1/2 \tag{A4.11}$$

because then (A4.8) has a unique solution

$$\varrho=(1-Q)^{-1}Z\delta=Z\sum_{n=0}^{\infty}Q^n\delta \tag{A4.12}$$

and the bound in the lemma comes from unravelling the definition of the norm in

$$\|\varrho\|\leqq Z\Sigma\|Q\|^n\leqq 2Z. \tag{A4.13}$$

We return to (A4.11). It is easy to show that

$$\|Q\|\leqq \sup|\mathscr{K}(X\sim X^*)|^{-1}\left\{e^{-\alpha}+\sup_X\sum_{X'}|\mathscr{K}(X'\sim X^*)|\cdot e^{\alpha|X'\sim X^*|}\right\}. \tag{A4.14}$$

We pick α so that

$$e^{-\alpha}\leqq\tfrac{1}{8}. \tag{A4.15}$$

We use Lemma 9.12 with X replaced by $X'\sim X^*$ and $X_1=X\sim X^*$ to bound the second term in the curly brackets by $\frac{1}{8}$. It now remains only to show that if β is sufficiently small

$$\mathscr{K}(X\sim X^*)=\mathscr{K}(\Delta)=\int d\mu(\phi)e^{E(\Delta)+G(\Delta)} \tag{A4.16}$$

is bounded below in absolute value uniformly in Δ by 1/2. First we note that

$$E(\Delta), \quad G(\Delta)\to 0 \quad \text{as} \quad \beta\to 0 \tag{A4.17}$$

so that by dominated convergence $\mathscr{K}(\Delta)$ tends to one. This argument does not quite prove the required lower bound because of the lack of uniformity in Δ. We note that if the covariance C_Λ of $d\mu$ in (A4.16) is replaced by the infinite volume covariance $C_{\mathbb{R}^3}$, uniformity follows immediately from translation invariance of $d\mu_{C_{\mathbb{R}^3}}$. Thus it suffices to prove that

$$\left(\int d\mu_{C_\Lambda}-\int d\mu_{C_{\mathbb{R}^3}}\right)(e^{E(\Delta)+G(\Delta)}) \tag{A4.18}$$

tends to zero uniformly in Δ as $\beta\to 0$. Define $d\mu^{(t)}$ to be the normalized Gaussian measure with the interpolating covariance

$$C^{(t)}=tC_\Lambda+(1-t)C_{\mathbb{R}^3}. \tag{A4.19}$$

We use the change of covariance formula to write the difference (A4.18) in the form

$$\tfrac{1}{2}\int_0^1 dt\int d\mu^{(t)}\left(\int\frac{\delta}{\delta\phi}C'\frac{\delta}{\delta\phi}\right)e^{E(\Delta)+G(\Delta)} \tag{A4.20}$$

(see Sects. 8 and 5). The prime denotes differentiation with respect to t. We bound (A4.20) in absolute value by

$$\tfrac{1}{2} \sup_{x,y\in\Delta} |C'(x,y)| \sup_{t} \int d\mu^{(t)} \left| \int \frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi} e^{E(\Delta)+G(\Delta)} \right|_0 . \tag{A4.21}$$

The 0 subscript on the absolute value sign means that the absolute value is to be taken inside the integral (over the positions of the two functional derivatives) and inside the sum obtained by performing the derivatives via Leibniz' rule. We show that the $d\mu$ integral tends to zero uniformly in Δ, Λ, t by the estimates of Sect. 9 [It is of the form (9.22) with F_1, F_2, h zero, $\mathscr{A}=1$ and $d\mu_s$ altered harmlessly.] This completes the proof of the bound in Lemma A4.1.

We now turn to the existence of the infinite volume limit. By (A4.12) it is enough to prove that $Q=Q_\Lambda$ is convergent in operator norm and uniformly bounded by 1/2. The uniform bound has just been established. By Lemma 9.12 we know that the sum over X' is uniformly convergent so norm convergence is implied by convergence of $\mathscr{K}(X)$ for each X. We refer to (8.4) to see that if Λ strictly contains X, $\mathscr{K}(X)$ depends on Λ only through the covariances

$$C=C_\Lambda$$

which occur in $d\mu_s$ and $\kappa(\bar{y},s)$. In particular $R(X)$ is independent of Λ and $E(X)$, $G(X)$ are independent of Λ because the translation g when restricted to X is independent of Λ. To analyze the Λ dependence replace C by $C^{(t)}$ defined in (A4.19) throughout $\kappa(\bar{y},s)$ and in $d\mu_s$ so that

$$d\mu_s \to d\mu_s^{(t)}, \qquad \kappa(\bar{y},s)\to\kappa^{(t)}(\bar{y},s), \qquad \mathscr{K}(X)\to\mathscr{K}^{(t)}(X). \tag{A4.22}$$

Then by the fundamental theorem of calculus it suffices to show that

$$\left| \int_0^1 \frac{d}{dt}\mathscr{K}^{(t)}(X)\,dt \right| \leqq \sup_t \left| \frac{d}{dt}\mathscr{K}^{(t)}(X) \right| \to 0 \tag{A4.23}$$

as $\Lambda\to\mathbb{R}^3$. We evaluate the derivative by Leibniz rule using the change of covariance formula

$$\frac{d}{dt}\int d\mu_s^{(t)}(\cdot) = \tfrac{1}{2}\int d\mu_s^{(t)} \left(\int \frac{\delta}{\delta\phi} C_s' \frac{\delta}{\delta\phi} \right)(\cdot). \tag{A4.24}$$

Therefore we must estimate quantities analogous to (9.22) where covariances C_Λ in $\bar{\kappa}$ and $d\mu_s$ have been replaced either by $C^{(t)}$ or C' and each term has one C' in it. The estimates of Sect. 9 are insensitive to this replacement. Furthermore, since the estimates of Sect. 9.8 involve the norm

$$\sup_{x\in X} \int |C(x,y)| e^{\bar{\alpha}|x-y|} dy \equiv \|C\|_{\bar{\alpha}} \tag{A4.25}$$

with $\bar{\alpha}<1/\tilde{l}_D$, $\|C_s^{(t)}\|_{\bar{\alpha}}$ is bounded uniformly in s, t, Λ and

$$\|C_s'\|_{\bar{\alpha}} \to 0 \quad \text{as} \quad \Lambda\to\mathbb{R}^3 \tag{A4.26}$$

(by the method of images) we obtain (A4.23) and thus complete the proof of our lemma.

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Communicated by A. Jaffe

Received September 20, 1979