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Technical Report

ERRORS IN ATMOSPHERIC TEMPERATURE STRUCTURE SOLUTIONS FROM REMOTE RADIOMETRIC MEASUREMENTS

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ORA Project 05863

under contract with:

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
CONTRACT NO. NASr-54(03)
WASHINGTON, D.C.

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

December 1963

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ABSTRACT

Calculations have been made which show that the library method of solution of the 'Kaplan' experiment yields temperature structure solutions which are dominated by the random errors in measurement and computation of radiation. A least squares method of solution is developed and modified by consideration of the eigenvalues and eigenvecters of the matrix equation. It exhibits some of the limitations of the experiment and shows that, using radiometric measurements, certain quite different profiles are essentially indistinguishable from one another.

INTRODUCTION

An earlier report[1] stressed the importance of determining errors in the calculation and measurement of the intensity of infrared radiation in the Kaplan experiment. Workers in the field well know that the problem of inverting the radiation equation to deduce the temperature structure of the atmosphere is complicated by the introduction of quite small errors arising from either physical measurements or inaccuracies in calculations. Kaplan[2] showed that the effect of systematic errors was small but stated that random errors introduced much more serious discrepencies. Unfortunately, his paper gave no details. In order to investigate such solutions, the coefficients given by Kaplan were used to make a preliminary analysis on which to base more accurate and detailed calculations.

METHOD OF CALCULATION

Kaplan's method follows, briefly: The intensity of radiation at frequency ν, $I(\nu)$ was calculated for a number of atmospheric models, which were determined by the temperature at a number of pressure levels in the atmosphere (50, 100, 200, 300, 400, 700, and 1000 mb). The intensity of radiation $I_o(\nu)$ was determined for some fixed model, and the value of $I(\nu)$ for a slightly different model is assumed to be expanded in the form

$$\frac{I(\nu)-I_o(\nu)}{I_o(\nu)} = \sum_i C_i \Delta T_i + \sum_{i,j} C_{ij} \Delta T_i \Delta T_j + \ldots \quad (1)$$

where ΔT_i is the difference in temperature from the fixed model at the level i. The coefficients C_i and C_{ij} were determined from calculations of $I(\nu)$ for various models.

In the following analysis we shall consider only the first order coefficients; second order terms should be included for an accurate solution, but the essential characteristics and behavior of solutions can be demonstrated without them. Table 1 (reproduced from Kaplan's paper) gives the coefficients C_i for the "middle route" for each of the nine channels calculated. Table 2 (also reproduced) gives details of the "middle route."

Kaplan chose the first seven frequencies (675, 685, 695, 700, 705, 710, and 730 cm^{-1}) to make his calculations, and neglected the two remaining channels (745 and 760 cm^{-1}). To find the value of ΔT_i one must invert the matrix formed by the first seven columns of Table 1. The inverted matrix is shown in Table 3. The temperature differences induced by a 1/3% systematic error may be calculated by putting $\frac{I(\nu)-I_o(\nu)}{I_o(\nu)}$ = 1/300 for each of the

TABLE 1

$C_i = \partial(\log_e I)/\partial T_i(^\circ C)$ FOR MIDDLE ROUTE

$\nu(cm^{-1})$	675	685	695	700	705	710	730	745	760
(cgs Units Per Steradian)	185.0	182.4	199.8	227.8	264.7	301.9	356.8	391.6	444.5
Pressure (mb)									
50	.0145	.0132	.0068	.0038	.0021	.0011	.0005	.0002	0
100	.0057	.0062	.0049	.0030	.0016	.0009	.0004	.0001	0
200	.0018	.0026	.0048	.0037	.0022	.0013	.0006	.0002	0
300	.0001	.0003	.0024	.0029	.0024	.0016	.0009	.0004	0
400	0	0	.0010	.0030	.0042	.0040	.0027	.0017	.0004
700	0	0	.0001	.0009	.0025	.0037	.0040	.0031	.0010
1000	0	0	0	0	.0003	.0015	.0055	.0086	.0135

TABLE 2

COMPARISON OF COMPUTED WITH ACTUAL TEMPERATURES (°K) WITHOUT AND WITH SYSTEMATIC ERRORS. BASIC SIGNAL $\sim 3\times10^{-5}$ WATTS/STERADIAN

Pressure (mb)	50	100	200	300	400	700	1000
Middle Route	210.0	210.0	220.0	230.0	250.0	260.0	270.0
$+10^{-7}$ watts/ster	210.2	210.5	220.0	230.5	250.2	260.3	270.1
$+10^{-6}$ watts/ster	211.7	214.8	219.8	235.1	251.5	262.9	270.8

TABLE 3

INVERSE OF MATRIX FORMED BY FIRST 7 COLUMNS OF TABLE 1
DETERMINANT OF COEFFICIENT MATRIX = -4.05×10^{-20}

4.91×10^{2}	-1.05×10^{3}	-1.61×10^{2}	1.45×10^{3}	-9.84×10^{2}	6.83×10^{2}	-2.02×10^{2}
2.24×10^{2}	-1.95×10^{3}	4.81×10^{3}	-7.91×10^{3}	4.20×10^{3}	-2.73×10^{3}	8.17×10^{2}
-4.24×10^{3}	1.89×10^{4}	-2.78×10^{4}	3.63×10^{4}	-1.78×10^{4}	1.13×10^{4}	-3.40×10^{3}
1.02×10^{4}	-4.54×10^{4}	6.60×10^{4}	-8.40×10^{4}	4.02×10^{4}	-2.54×10^{4}	7.68×10^{3}
-1.29×10^{4}	5.72×10^{4}	-8.29×10^{4}	1.05×10^{5}	-4.90×10^{4}	3.05×10^{4}	-9.18×10^{3}
7.92×10^{3}	-3.51×10^{4}	5.08×10^{4}	-6.41×10^{4}	2.97×10^{4}	-1.79×10^{4}	5.23×10^{3}
-1.46×10^{3}	6.45×10^{3}	-9.34×10^{3}	1.18×10^{4}	-5.43×10^{3}	3.23×10^{3}	-7.43×10^{2}

seven frequencies ν in question. This yields

$$\Delta T_{50} = +0.9°K,\ \Delta T_{100} = -3.0°K,\ \Delta T_{200} = +4.7°K,\ \Delta T_{300} = -5.4°K$$

$$\Delta T_{400} = +2.9°K,\ \Delta T_{700} = -1.3°K,\ \Delta T_{1000} = +0.7°K$$

The most surprising aspect of the matrix inversion was the magnitude of the ΔT_i for non-systematic errors in radiation. For +1, -1, +1, -1, +1, -1, +1% error respectively for ν = 675, 685, 695, 700, 705, 710, 730 cm^{-1}, the ΔT_i were -365°K, +1640°K, -2420°K, +3100°K, -1470°K, +918°K, -273°K. It should be emphasized that, with ΔT_i of this magnitude, Eq. (1) is not applicable. Nevertheless, it does give some indication that the temperature errors will be inconveniently large.

The reason for the large ΔT_i is easy to see by examining the inverse matrix. The coefficients are large in absolute value, but their algebraic sums, taken by columns, are small. It is not surprising that Kaplan failed to get convergent solutions in a number of cases.

From a purely physical viewpoint one should not expect the first seven channels to give the best results. Table 1 shows that nearly all the radiation comes from ground level at 760 cm^{-1}, so this channel should certainly be included for an accurate determination of T_{1000}. A number of coefficient matrices were inverted, and the systematic and maximum "random" errors were calculated. The results are summarized in Table 4. The determinant of the coefficient matrix provides a rough guide to the best choice of channels. The effect of 1% random noise is still very severe and quite unacceptable, even in the most favorable cases. With values of ΔT_i so large, it is clear that the use of Eq. (1) is not justified, and that any physical result obtained from it will be meaningless.

A least squares method can be used for cases in which the number of observations exceeds the number of unknowns. The theory of this method is developed in the Appendix, which should be consulted for details. The method was applied to Eq. (1), using seven temperature levels and observations on nine channels. The maximum values of the $|\Delta T_i|$ were very close to the best result using seven channels and offer very little improvement. In addition, the systematic errors were not markedly different.

The number of unknown temperatures was reduced to four, by two different sets of assumptions:

Case I

$$T_{50} = T_{100}$$
$$T_{200} = 1/2(T_{100} + T_{300})$$
$$T_{700} = 1/2(T_{400} + T_{1000})$$

Case II

$$T_{200} = 1/2(T_{100} + T_{300})$$
$$T_{400} = T_{300} + 20°K$$
$$T_{700} = T_{1000} - 10°K$$

TABLE 4

SUMMARY OF EFFECT OF ERRORS USING SELECTED CHANNELS

Omitting Channels Centered at		Maximum Value of $\lvert \Delta T_i \rvert$ From Linear Approximation With Absolute Value of Error $\leq$ 1% (°K)							Determinant of Coefficient Matrix	Remarks
cm^{-1}	cm^{-1}	50	100	200	300	400	700	1000		
745	760	365	1640	2420	3100	1470	918	273	4.05×10^{-20}	Used by Kaplan
705	745	39	147	171	167	50	15	3	4.57×10^{-18}	
700	745	40	135	170	150	40	17	3	3.82×10^{-18}	
695	745	55	220	260	220	55	18	3	1.69×10^{-18}	
None		35	133	151	175	56	20	2	---	Least Squares

In Case I, the ΔT_i were also calculated from the four channels 675, 700, 710, 760 cm^{-1} subject to errors of +1, -1, +1, -1% respectively, giving

$$\Delta T_{100} = +.9°K, \ \Delta T_{300} = -8.5°K, \ \Delta T_{400} = +5.2°K, \ \Delta T_{1000} = -1.1°K$$

The matrix $(A'A)^{-1}A'$ was found for both cases. The maximum temperature differences subject to 1% errors:

Case I

$$|\Delta T_{100}| = .9°K, |\Delta T_{300}| + 6.4°K, |\Delta T_{400}| + 4.8°K, |\Delta T_{1000}| = 1.4°K$$

Case II

$$|\Delta T_{50}| = 3.9°K, |\Delta T_{100}| + 8.2°K, |\Delta T_{300}| = 3.8°K, |\Delta T_{1000}| = 1.2°K$$

In Case I the least squares method yields a rather better solution. In fact it is slightly better that the figures indicate, since the $|\Delta T_i|$ will not all achieve their maximum for the same error distribution. In addition, one is much less likely to have all the errors go 'the wrong way' over nine channels as compared with a four channel system. One can therefore conclude that the least squares method will yield a useful but limited improvement over a plain solution of linear equations.

LIMITATIONS

This section will attempt to give a mathematical explanation for the large errors. Physically they arise from the fact that two quite different atmospheric structures may produce an almost equal radiation intensity over the whole of the 15μ CO_2 band under consideration.

Consider an integral equation of the first kind,

$$h(x) = \int_a^b f(x,y)g(y)dy \qquad (2)$$

The problem is to invert this equation, solving for g(y) assuming that f and h are given functions. In practice, sets of numerical values will be given for f and h, solving Eq. (2) by reducing it to a set of simultaneous linear equations.

Now replace the left-hand side of Eq. (2) by $h(x) + \epsilon(x)$, where $\epsilon(x)$ is a small error function. We are interested in knowing its effect on the solution g(y). Unfortunately, a small $\epsilon(x)$ may produce a large change in g(y). Phillips[3] gives an example, showing that

$$\int f(x,y)\sin(my)dy \to 0 \text{ as } m \to \infty$$

for any integrable kernel f. Thus there is a basic instability in Eq. (2); any solution obtained from it may be correct from a mathematic standpoint, but physically meaningless. For this reason a least squares solution will, in general, provide only limited aid.

Phillips goes on to develop a method whereby a smoothness constraint placed on the solution eliminates these unwanted oscillations. The technique has been generalized and modified by Twomey[4] to a form more convenient for numerical solution.

A basic difficulty is introduced: how smooth shall the solution be? If an insufficient degree of smoothness is introduced, oscillations will still predominate. On the other hand, too much smoothing will destroy the essential physical characteristics in which we are most interested.

The reduction of the number of unknown temperatures in the previous sections operates as a smoothing constraint in that it prevents fluctuation of intermediate temperatures. Approximation by a polynomial of low degree acts in the same way.

So far we have made little or no use of known properties of the atmospheric temperature structure. Because we have some idea of its form, it is possible to seek a solution which best approximates (in some sense to be defined) a standard structure.

In the next section, we shall again return to the least squares method of solving the matrix equations, examining closely which components contribute most to the large errors.

EIGEN VECTOR SOLUTIONS OF THE MATRIX EQUATION

In the appendix, a least squares method is developed to replace the system of equations

$$A(\Delta T) = (\Delta I) \qquad (3)$$

where A is a n x m matrix, $n \geq m$, by the system

$$(A'A)(\Delta T) = A(\Delta I) \qquad (4)$$

If the rank of A is m, the matrix A'A is non-singular, and Eq. (4) may be written

$$(\Delta T) = (A'A)^{-1} A'(\Delta I) \qquad (5)$$

We have shown that the application of Eq. (5) produces physically unrealistic solutions, arising from the fact that small errors in (ΔI) may produce large errors in (ΔT), so much so that these errors dominate the solution. Let us consider how they arise by looking at the eigen values of the matrix A'A. Suppose these are $\lambda_1, \lambda_2, \ldots, \lambda_m$, ordered in such a way that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$, with a corresponding orthonormal system of eigenvectors $v_1, \ldots, v_m$.
Then

$$(A'A)v_i = \lambda_i v_i \qquad i = 1, \ldots, m$$

Suppose λ_i is large. Then putting $\Delta T = v_i$ in Eq. (4) we see that

$$\lambda_i v_i = A'(\Delta I)$$

and that such a ΔT will produce a comparatively large change in radiation intensity, i.e., a large $A'(\Delta I)$. However, when λ_i is small, the opposite will occur; putting $\Delta T = v_i$ produces a small change in the radiation intensity. Now any (ΔT) can be written uniquely in the form

$$(\Delta T) = \sum_{i=1}^{m} \beta_i v_i$$

and

$$(A'A)\ (\Delta T) = \sum_{i=1}^{m} \beta_i \lambda_i v_i \quad .$$

Thus it is now clear that the v_i corresponding to small λ_i are precisely those components of (ΔT) which have little effect on the radiation intensity, and which cannot be determined from radiometric measurements.

Let us consider the effects of small errors on the right side of Eq. (2). If $(A'A)(\Delta T) = v_i$, then $(\Delta T) = \frac{1}{\lambda_i} v_i$. The roles are now reversed: for small λ_i, the addition of a small error in the v_i component of $A'(\Delta I)$ will produce a large error in ΔT, while for large λ_i the effect of such an error in $A'(\Delta I)$ will be small.

To summarize, the v_i corresponding to small λ_i have little effect on the radiation intensity, while at the same time their inclusion in Eq. (3) produces large temperature errors. The obvious solution is to ignore these components. Since (ΔT) can be written

$$(\Delta T) = \sum_{i=1}^{m} \beta_i v_i \quad ,$$

instead of $\sum_{i=1}^{m} \beta_i v_i$ we write $\sum_{i=1}^{p} \beta_i v_i$ with $p < m$.

The value of p chosen depends on the values of λ_p and on the expected errors in radiation intensity. So $(\Delta T) = V'\beta$ where V is the m x p matrix of the eigenvector columns $v_i, \ldots, v_p$, and β is a p x 1 matrix. Eq. (3) now becomes

$$(AV')\,(\beta) = (\Delta I), \text{ with } V'(\beta) = (\Delta T)$$

Solving this by the least squares method

$$(\beta) = [(AV')'(AV')]^{-1}(AV')'(\Delta I)$$

or

$$(\Delta T) = V'(\beta) = V'[(AV')'(AV')]^{-1}(AV')'(\Delta I) = C(\Delta I) \text{ say.} \qquad (6)$$

This method lends itself well to computer evaluation. After finding the eigenvalues and vectors the program may be written in order these in the manner indicated. The solution C may now be evaluated for all values of p from 1 to m and the results compared (p = m reduces to Eq. (15)).

It should be noted that although Eq. (6) gives ΔT_i for i = 1,..., m we do not get m independent pieces of information, but only p. Dependence was introduced by assuming that (ΔT) could be written as a linear combination of $v_1, \ldots, v_p$. It has the implication that the finer details of the atmospheric structure are obscured. But it is precisely these details that we cannot expect ot obtain using radiometric techniques.

Previously the number of variables was reduced by assuming a relation between ΔT_i in adjacent layers; the eigenvector method may be considered as the optimum way of choosing a relationship between the various ΔT_i.

The eigenvalues of the matrix AA' are given in Table 5a. It can be seen that λ_1 is approximately 10^5 times greater than λ_7, showing the relative small influence of the v_7 component of (ΔT) on the intensity of radiation. Table 5b shows the eigenvectors; it is interesting to note that as

TABLE 5a

EIGENVALUES OF MATRIX A'A

$\lambda_1 = 5.98\times10^{-4}$ $\lambda_5 = 1.99\times10^{-6}$

$\lambda_2 = 3.17\times10^{-4}$ $\lambda_6 = 1.52\times10^{-7}$

$\lambda_3 = 8.91\times10^{-5}$ $\lambda_7 = 7.61\times10^{-9}$

$\lambda_4 = 1.95\times10^{-5}$

TABLE 5b

EIGENVECTORS OF MATRIX A'A

v_1	v_2	v_3	v_4	v_5	v_6	v_7
8.576×10^{-1}	-1.064×10^{-1}	-2.555×10^{-1}	-2.747×10^{-1}	-1.773×10^{-1}	2.569×10^{-1}	1.224×10^{-1}
4.197×10^{-1}	-0.336×10^{-1}	0.576×10^{-1}	2.448×10^{-1}	2.934×10^{-1}	-6.631×10^{-1}	-4.835×10^{-1}
2.443×10^{-1}	0.203×10^{-1}	3.431×10^{-1}	6.049×10^{-1}	3.094×10^{-1}	1.137×10^{-1}	5.896×10^{-1}
1.027×10^{-1}	0.587×10^{-1}	3.961×10^{-1}	3.228×10^{-1}	-2.118×10^{-1}	5.620×10^{-1}	-6.035×10^{-1}
1.020×10^{-1}	1.881×10^{-1}	6.258×10^{-1}	-2.195×10^{-1}	-5.848×10^{-1}	-3.692×10^{-1}	1.904×10^{-1}
0.617×10^{-1}	2.668×10^{-1}	4.161×10^{-1}	-5.689×10^{-1}	6.291×10^{-1}	1.710×10^{-1}	-0.563×10^{-1}
0.627×10^{-1}	9.365×10^{-1}	-3.034×10^{-1}	1.505×10^{-1}	-0.648×10^{-1}	-0.069×10^{-1}	-0.006×10^{-1}

λ_i decreases, the number of changes in sign of the v_i increases. This is the reason why the "straight" solution gives ΔT_i which are alternatively positive and negative.

Calculations of the matrix C were made for all values of p from p = 2 to p = 7 in order to compare the solutions obtained from various sets of assumptions.

Table 6 shows the rapid increase with p of the maximum values of ΔT_i with $|\Delta I_i| = 1\%$, obtained by taking the sum of the absolute values of the rows of the matrix C. For p = 4 all ΔT_i are less than 4°K, while for p = 5 they rise to above 13°K.

TABLE 6

MAXIMUM VALUE OF $|\Delta T_i|$ FOR p = 2,..,7 WITH $|\Delta I_i|$ = 1%

	p = 2	p = 3	p = 4	p = 5	p = 6	p = 7
ΔT_{50}	.8°K	1.0°K	1.9°K	1.8°K	16°K	35°K
ΔT_{100}	.4	.5	1.4	5.6	40	133
ΔT_{200}	.2	1.0	3.5	5.9	11	151
ΔT_{300}	.2	1.1	2.0	4.3	35	175
ΔT_{400}	.3	1.8	2.0	11.7	29	56
ΔT_{700}	.3	1.3	3.7	13.1	18	20
ΔT_{1000}	1.1	1.4	1.4	1.7	2	2

It is important to know how well a given temperature structure can be reproduced. The values of ΔI were calculated for each of the nine frequencies on the assumption that ΔT_i = +1°K for every i. Equation (6) was solved using these ΔI, giving the results in Table 7a; p = 2 gives a poor approximation, expecially for the middle of the atmosphere. However, p = 3 is much improved;

TABLE 7a

SOLUTION OF EQ. (6) USING VALUES OF ΔI CALCULATED FROM EQ. (3) WITH ΔT_i = +1°K ALL i

	p = 2	p = 3	p = 4	p = 5	p = 6	p = 7
ΔT_{50}	1.44°K	1.12°K	1.05°K	1.01°K	1.03°K	1.00°K
ΔT_{100}	.74	.81	.87	.93	.88	1.00
ΔT_{200}	.48	.92	1.08	1.14	1.14	1.00
ΔT_{300}	.27	.77	.86	.82	.85	1.00
ΔT_{400}	.44	1.24	1.18	1.07	1.04	1.00
ΔT_{700}	.47	.80	.85	.97	.99	1.00
ΔT_{1000}	1.01	.97	1.01	1.00	1.00	1.00

TABLE 7b

ΔI_2-ΔI_1*

ν	p = 2	p = 3	p = 4	p = 5	p = 6	p = 7	ΔI_1
675	+.39%	+.05%	+.01%	.00%	.00%	.00%	2.21%
685	+.26	+.01	.00	.00	.00	.00	2.23
695	-.31	-.08	-.01	.00	-.01	.00	2.00
700	-.53	-.06	+.01	.00	.00	.00	1.73
705	-.61	-.03	+.01	.00	.00	.00	1.53
710	-.58	-.03	.00	.00	.00	.00	1.41
730	-.44	-.06	-.02	.00	.00	.00	1.46
745	-.28	-.06	-.01	.00	.00	.00	1.43
760	-.06	-.05	+.01	.00	.00	.00	1.49

*ΔI_1 calculated from ΔT_i = 1°K all i
ΔI_2 calculated from ΔT_i in Table 7a

this improvement continues gradually for p = 4, 5, 6, giving successively better approximations. If we calculate the difference between the ΔI for these temperatures and the ΔI for $\Delta T = +1°K$ all i, we find that even for p = 3 this difference is less than 0.1% in absolute value and does not exceed 0.02% for p = 4 (Table 3b) both of these being much less than the expected experimental error. Thus the experiment cannot distinguish between any of the solution profiles for $p \geq 3$.

The limitations of the technique are illustrated by performing the same computations with $\Delta T_{300} = +5°K$ and all other $\Delta T_i = 0°K$. Even for p = 6 the errors in ΔT_i are almost as high at 2°K, while for p = 4, ΔT_{300} is 1.37°K instead of 5°K (Table 8a). However, the difference in radiation intensity is less than 0.1% for each of the nine channels for p = 4 and rather less for p = 5 and 6. It is precisely these sharp-peaked variations that the technique is not able to resolve; instead we obtain a smoothed solution, the degree of smoothing depending on how low a value of p is selected.

Lastly, $\Delta I = +1\%$ over all nine channels was used to obtain values of ΔT_i. The result was somewhat surprising: For values of p from p = 3 to p = 6, for each pressure level the value of ΔT differed by less than 0.1°K, but differed greatly from the solution with p = 7 (Table 9a). Again, the difference in radiation intensity is less than .03% for $3 \leq p \leq 7$, showing the essentially indistinguishable nature of the various solutions.

We should like to be able to conclude which value of p gives the best results. However, this is not a question to which an absolute answer can be given for it depends on what is meant by a "best" solution. If the expected error lies around ±1%, then Table 2 shows that $p \geq 5$ must be eliminated. Likewise, p = 2 must be discarded on the grounds that it does not give enough detail in the middle atmosphere. From the limited calculations made, it

TABLE 8a

SAME AS TABLE 7a WITH ΔT_{300} = +5°K, ALL OTHER ΔT_i = 0

	p = 2	p = 3	p = 4	p = 5	p = 6	p = 7
ΔT_{50}	.41°K	- .10°K	- .54°K	- .35°K	.37°K	0°K
ΔT_{100}	.20	.32	.71	.40	-1.45	0
ΔT_{200}	.12	.81	1.79	1.46	1.78	0
ΔT_{300}	.07	.85	1.37	1.60	3.18	5.00
ΔT_{400}	.11	1.35	.99	1.61	.57	0
ΔT_{700}	.11	.82	.01	- .65	- .17	0
ΔT_{1000}	.31	-1.11	- .05	.02	.00	0

TABLE 8b

$\Delta I_2 - \Delta I_1$*

ν	p = 2	p = 3	p = 4	p = 5	p = 6	p = 7	ΔI_1
675	+.68%	+ .14%	-.09%	-.05%	+.01%	.00%	.05%
685	+.55	+ .15	+.08	+.06	.00	.00	.15
695	-.74	- .38	+.07	.00	+.01	.00	1.20
700	-1.13	- .37	-.08	-.03	-.01	.00	1.45
705	- .96	- .05	-.06	+.01	.00	.00	1.20
710	- .62	+ .14	+.05	+.05	+.02	.00	.80
730	- .17	- .24	+.03	+.02	-.01	.00	.45
745	+ .13	- .62	+.02	-.02	.00	.00	.20
760	+ .43	-1.36	-.03	+.03	+.01	.00	.00

*ΔI_1 calculated from ΔT_{300} = +5°K, all other ΔT_i = 0
ΔI_2 calculated from ΔT_i in Table 8a

TABLE 9a

SOLUTION WITH ΔI = +1% FOR EACH CHANNEL

	p = 2	p = 3	p = 4	p = 5	p = 6	p = 7
ΔT_{50}	.69°C	.44°C	.45°C	.47°C	.47°C	.81°C
ΔT_{100}	.36	.41	.41	.37	.37	- .95
ΔT_{200}	.23	.59	.57	.53	.53	2.16
ΔT_{300}	.15	.55	.54	.56	.56	-1.07
ΔT_{400}	.27	.90	.91	.98	.98	1.50
ΔT_{700}	.30	.73	.75	.67	.67	.53
ΔT_{1000}	.95	.66	.66	.66	.66	.66

TABLE 9b

VALUES OF ΔI OBTAINED BY SUBSTITUTING VALUES
OF ΔT_i IN TABLE 9a

ν	p = 2	p = 3	p = 4	p = 5	p = 6	p = 7
675 cm^{-1}	1.25%	.98%	.99%	.99%	.99%	1.01%
685	1.20	1.00	1.01	1.00	1.00	1.01
695	.82	1.01	1.03	.99	.99	1.02
700	.61	1.00	1.00	1.00	1.00	1.01
705	.51	1.00	1.00	1.03	1.03	1.02
710	.52	.98	.97	.98	.98	1.01
730	.79	1.02	1.03	1.02	1.02	1.02
745	.98	.99	1.00	.99	.99	.99
760	1.32	1.00	1.00	1.00	1.00	1.00

would appear that p = 4 is the most suitable choice, giving a significantly better result than p = 3 in Tables 4a and 4b. The only way to come to a valid decision is to make a statistical analysis based on real atmospheric temperature soundings. The model chosen by Kaplan does not sufficiently approximate any part of the earth's atmosphere to justify such an analysis.

The method outlined has a number of advantages:

a. It used a library method, making calculation of results comparatively simple. The bulk of the calculations can be made in advance, so that even when large amounts of data are obtained interpretation can be made at once.

b. Known physical characteristics of the atmosphere are used in taking the basic model to be that of a standard atmosphere, the choice depending on the latitude and season of the observation.

c. The essential limitations of the technique are recognized, and precisely those features which we cannot hope to measure and which contribute so greatly to unwanted oscillations in the solution are eliminated.

d. It lends itself to a statistical analysis based on the deviation of temperature structures from the standard atmospheres.

CONCLUSIONS

The method described appears to be promising enough to justify further development. A series of calculations will shortly be made with the following features:

1. New and more accurate transmission functions.
2. The SIRS channel frequencies and their triangular response functions will be used to calculate radiation intensities.
3. A flexible program to allow radiation from any temperature structure to be readily calculated.
4. A statistical analysis leading to a choice of p.
5. A detailed analysis of the effect of intensity errors.
6. Influence of cloudy or partly cloudy conditions on solutions.

ACKNOWLEDGMENTS

I am very grateful to C. L. Mateer, Department of Meteorology and Oceanography, The University of Michigan, who developed the eigenvector technique for atmospheric ozone soundings.

In addition, I should like to thank members of the project for useful discussions and for checking calculations.

REFERENCES

1. Drayson, S. R., A survey of progress and problems in the 'Kaplan' Experiment. The University of Michigan, Technical Report No. 05863-2-T, July 1963.

2. Kaplan, L. D., The spectroscope as a tool for atmospheric sounding by satellites, Instr. Soc. Am., Conf. Preprint No. 9-NY 60. Published in J. Quart. Spectry. Radiative Transfer, 1, 98 (1961).

3. Phillips, D. L., A technique for the numerical solution of certain integral equations of the first kind, J. Assn. for Comput. Mach., 9, 84 (1962).

4. Twomey, S., On the numerical solution of Fredholm integral equations of the first kind by the inversion of the linear system produced by quadrature, J. Assn. for Comput. Mach., 10, 97 (1963).

APPENDIX

LEAST SQUARES SOLUTIONS

Suppose we have n equations in m unknowns

$$f_i(x_1,x_2,\ldots,x_m) - \beta_i = 0 \qquad i = 1,2,\ldots n \qquad \text{(A1)}$$

with $n > m$. In general, these will be mutually inconsistent and have no solution. However, we can look for a solution $(x_1,x_2,\ldots,x_m)$ for which the sum of the squares of the left-hand side of Eq. (A1) is a minimum, i.e.

$$F(x) = \sum_{i=1}^{n} [f_i(x_1,\ldots,x_m) - \beta_i]^2 \text{ is a minimum.}$$

A necessary, but not normally sufficient condition, is

$$\frac{\partial F}{\partial x_j} = 0 \qquad j = 1,\ldots,m$$

i.e.

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j} [f_i(x_1,\ldots,x_m) - \beta_i] = 0 \qquad j = 1,\ldots,m \qquad \text{(A2)}$$

Suppose now that f_i are linear forms in $x_1,\ldots,x_m$ for $i = 1,\ldots,n$ say

$$f\ (x_1,\ldots,x_m) = \sum_{k=1}^{m} \alpha_{ik}\, x_k \qquad i = 1,\ldots,n$$

Then Eq. (A2) reduces to

$$\sum_{i=1}^{n} \alpha_{ij} \sum_{k=1}^{m} (\alpha_{ih}\, x_h = \beta_i) = 0 \qquad j = 1,\ldots,m \qquad \text{(A3)}$$

This may conveniently be written in matrix form

$$A'A(x) = A'(\beta) \tag{A4}$$

where $A = (\alpha_{ij})$, $x = (x_1,\ldots,x_m)'$, $(\beta) = (\beta_1,\ldots,\beta_n)'$, the ' denoting matrix transposition.

Equation (A4) will yield a unique solution (x) provided that the matrix $A'A$ is nonsingular. It will be assumed that rank $A = m$. If this is not the case, there exist less than m linearly independent forms $\sum_{i=1}^{m} \alpha_{ij} x_j$ and m may be reduced to m-1.

Because rank $A = m$, there exist nonsingular matrices P, Q such that

$$A = P\begin{pmatrix} I_m \\ 0 \end{pmatrix}Q \quad .$$

P is n x n matrix, Q is m x m, I_m is the m x m identity matrix, and 0 is the (n-m)x(n-m) null matrix.

Then

$$A'A = Q'(I_m 0)P'P\begin{pmatrix} I_m \\ 0 \end{pmatrix}Q \quad .$$

Since P is nonsingular, P'P belongs to a positive definite quadratic form, which implies that every principal minor of P'P is positive. Hence the m x m matrix B in the top left corner of P'P is nonsingular. But $(I_m 0)(P'P)\begin{pmatrix} I_m \\ 0 \end{pmatrix} = B$,

and therefore

$$A'A = Q'BQ$$

is nonsingular, being the product of nonsingular matrices.

Equation (A4) may now be rewritten

$$(x) = (A'A)^{-1} A'(\beta) \qquad \text{(A5)}$$

which is in a form convenient for computer evaluation. In the special case where m = n, Eq. (A5) reduces to $(x) = A^{-1} (\beta)$, which is the original equation.