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A Mass Zero Cluster Expansion

Part 2. Convergence*

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Abstract. Convergence is proven for the mass zero cluster expansion presented in Part 1 of this paper. An indication is given of changes necessary to treat the more difficult $\lambda(\vec{\nabla}\phi)^4$ model and the lattice dipole gas.

9. Counting I

In these sections we will be concerned with enumerating the terms in the cluster expansion in a way suitable for estimation. The complexities are largely notational, and due to the need to consider a number of different cases, there are no real difficulties. Then too, this is a new type of cluster expansion, and its "standard tricks" have to be invented. We will try to give motivation for a number of the avenues taken.

9.A. Representation 1

We here present the representation (labelling) of a single term in the cluster expansion, basically as developed in Sect. 8. In later subsections we will find alternate representations more useful for computation.

A term in the cluster expansion is determined by giving

1) A finite sequence

$$(i_1,x_1),\ (i_2,x_2),\ldots$$

with

$$(i_s,x_s) < (i_{s+1},x_{s+1}) \tag{9.A.1}$$

in the order (8.A.1) of [2].

2) A mapping

$$(i_s,x_s) \to (\alpha_1(s),\alpha_2(s),\alpha_3(s),\alpha_4(s)). \tag{9.A.2}$$

Clearly the elements in the sequence are just the (i,x) not mapped into N by T, and (9.A.2) is just the mapping T. There are compatibility conditions we are omitting so that not all terms we have specified are nonzero. The order of the four

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α's in (9.A.2) is so far arbitrary, so we introduce a requirement (later we will introduce others).

R.1) $\alpha_1(s)$ is an interior variable at the onset of step s.

We now begin to introduce some useful language. x_s is "*attached*" to $\alpha_1(s)$ at step s. $\alpha_2(s), \alpha_3(s), \alpha_4(s)$ are "*tied*" to x_s at step s. If after step s, $\{\alpha_2, \alpha_3, \alpha_4\}$ are part of a tadpole, we will say this tadpole is "attached" to $\alpha_1(s)$ and "tied" to x_s at this step. If $\alpha_i(s)(i = 2, 3, \text{ or } 4)$ was an exterior variable at the onset of step s, we say $\alpha_i(s)$ was "*generated*" at step s. Each $\alpha_i(s)$ is "*connected*" to each $\alpha_j(s)$ at step s.

The present cluster expansion has several features one should become aware of, they are somewhat alien to one experienced in standard cluster expansions. First we observe that if the mapping T, defined in Sect. 8.A, maps a complete sequence of (i,x) (a sequence of length p^{3N}) into N, then the corresponding term vanishes if any later pairs (j,y) are not mapped into N. One has completely decoupled the interaction, and later differentiations yield zero. This will be used in realizing Representation 1 from Representation 2. Secondly, we will later use the fact that terms in which a tadpole is completely decoupled vanish. This follows from the symmetry of the interaction under $\phi \to -\phi$.

The third and final feature we now call attention to, is also elementary, but very easy to overlook, leading to great confusion. It is essential to our reconstruction of Representation 1 from Representation 2, given later. To emphasize it we give it a catchy name:

Attachment Urgency. If α is an interior variable at the onset of step s, and step s maps (i,x) into N, then if any later step $(i',x), i' > i$, involves an attachment to α, the term vanishes.

9.B. Representation 2

We now present an alternate representation of a term in the cluster expansion. It does not uniquely represent a single term as we later discuss. It is much closer than Representation 1 to a form suitable for estimates.

A term in the cluster expansion is determined (not quite uniquely) by giving

1) A subset of the α's, $\{\alpha_k\}_{k\in I}$.
2) A mapping for each α_k *in* $\{\alpha_k\}_{k\in I}$

$$\alpha_k \to x_1(k), \ldots, x_{n(k)}(k), \tag{9.B.1}$$

where the set of sites on the right may be empty, allows repetitions, but is unordered.

3) For each x appearing in (9.B.1) a number $d(x)$, the "*degree*" of x, of mappings

$$x \to (\alpha_2(x), \alpha_3(x), \alpha_4(x))_r,\ r = 1, 2, \ldots, d(x). \tag{9.B.2}$$

The "degree" of x is exactly the total number of times, counting multiplicities, that x appears as an image of the α_k's.

This is derived from Representation 1 in a natural way. The set of 1) is just the collection of interior variables. The mapping (9.B.1) gives just the set of x's attached to α_k through all the steps of the cluster expansion, counting repetitions. (9.B.2) gives the set of α's tied to x at the r^{th} time interpolation takes place at x.

The term is constructed iteratively from the information in 1), 2), 3). If the interpolation step (i,x) has just taken place (using the notation of Representation 1), one seeks the first (j,y) such that

$$(j,y) > (i,x),$$

and y is one of the sites attached to an α that is an interior variable after step (i,x), and with this attached y not yet used, Remember attachment urgency! 2) enables one to read off these attachments, one has checked off ones already used. The α's tied to y at this step are read from 3), again one picks the lowest r value in (9.B.2) whose mapping has not yet been used. Thus step (j,y) is determined. But careful—there we have the non-uniqueness—there may have been several α's with an attachment at y to choose from! This leads to a non-uniqueness throughout the process overestimated as $\prod_x (d(x)!)$. This is handled by the counting rule.

C.1) The sum in absolute value of all terms associated to a single specification of Representation 2 is less than $\prod_x (d(x)!)$ times the supremum of the absolute value of any single such term.

The $\prod(d!)$ will be controlled similarly to the way the number divergence is handled in usual cluster expansions.

The basic strategy in combinatoric estimates for cluster expansions is to use repeatedly the simple inequality

$$\sum_i \sum_j |A_{ij}| \leqq \left(\sum_i |B_i|\right) \operatorname{Sup}_i \left(\sum_j \frac{1}{|B_i|} |A_{ij}|\right). \tag{9.B.3}$$

If one tries straightforwardly to use (9.B.3) to control the sum over terms in the cluster expansion as labelled by Representation 2), using in a simple way numerical factors in the interaction term, one has some success. The mappings of 3) are easily dominated by (9.B.3), as is the number, $n(k)$, of x's attached to any α_k by 2). However the sum over possible values for $x_i(k)$ is not so easily dealt with. It is to control these sums that tadpoles were introduced! Representation 3 will be designed to take advantage of the tadpole device to control x site sums. We will first need some new ways of viewing the connectivity properties of terms, and more language.

We will call the r value (see Sect. 4 [2] of an α its "*level*" and thus α_a will be lower level than α_b if its r value is smaller.

9.C. *Painting*

We introduce the concept of "*painting*" the interpolation steps that take place in a particular term in the cluster expansion. They will be "painted" in stages, painting a chunk of steps each stage. At the same time one paints an interpolation step, say step s, one paints the corresponding α's, $\alpha_1(s)$, $\alpha_2(s)$, $\alpha_3(s)$, and $\alpha_4(s)$. Some of them may already be painted, repainting has no effect. At any stage of the painting, the painted subset of α's is a union of connected components (via the interactions), each containing a distinguished variable. The order of painting will be the order in which sums are changed into sups in (9.B.3). At a given stage of the painting, we view as painted all elements (steps and α's) painted at the stage and at all previous stages. Note that the order in which steps will be painted, to be specified later, will not necessarily correspond to their ordering in (9.A.1).

9.D. Solid Attachments

At a given stage of the painting we define the idea that step s, connecting $\alpha_1(s)$, $\alpha_2(s)$, $\alpha_3(s)$, $\alpha_4(s)$ be a "*solid attachment*". (At this step some of the painted α's will be interior, some exterior.) It is "*solid*" if the level of α_1 is not greater than the level of all of the three other variables; and $\alpha_1(s)$ is painted, but not step s. It is also said to be "*solid*" if at least two of the α's are painted with no conditions on the levels, but step s is not yet painted.

9.E. Binding of Tadpoles

Again we are at a certain stage of the painting. We are given throughout this subsection a fixed set of painted α's, and proceed through the steps of the cluster expansion, as given in Representation 1, say.

Two tadpoles "*bind*" at the step when α_a is connected to α_b if

1) α_a and α_b are each in tadpoles containing no painted elements at the onset of the step.

2) Let t_a be the largest tadpole set containing α_a and containing no painted elements, and t_b be the largest tadpole set containing α_b and containing no painted elements; then t_a and t_b are disjoint.

3) Let t_a be attached to α_A and t_b attached to α_B, then the level $\alpha_A \leqq$ level α_B and α_A is painted. (It is notation only to interchange the roles of a and b.) Then we say t_a and t_b bind at this step, the "*hit*" step.

t_a was attached to α_A at step s_a, by tying it to x_{s_a}; t_b attached to α_B at step s_b, by x_{s_b}. s_a is the "*root step*" for the binding, s_b the "*target step*", x_{s_a}, the "*root site for binding*", and x_{s_b}, the "*target site for binding.*"

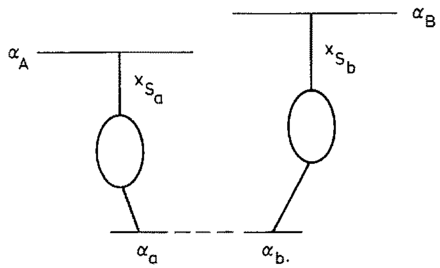

Fig. 1.

9.F. Seating of Tadpoles

Again throughout this entire subsection we are at a fixed stage in the painting.

A tadpole is "*seated*" at the step when α_a is connected to α_b if

1) α_a Is painted;

2) α_b is in a tadpole t_b, the largest tadpole containing α_b, and no painted elements. t_b is attached to α_B, at step s_b.

3) level $\alpha_a \leqq$ level α_B.

x_{s_b} is a "*root site for seating*". If the hit step was s_h, the x_{s_h} is a "*hit site for seating*".

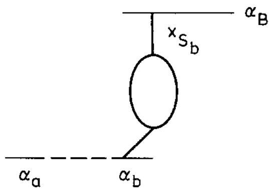

Fig. 2

9.G. *Taming of Tadpoles*

The tadpole t_a (with no painted elements), attached to painted α_A, at step s_A, the "*root step for taming*" at site x_A, the "*root site for taming*" is tamed at step s_t, the "*taming step*", if one of the $\alpha_i(s_t)$, say $\alpha_1(s_t)$, is in t_a, and another of the $\alpha_i(s_t)$, say $\alpha_2(s_t)$, satisfies level $\alpha_2(s_t) \geqq$ level α_A.

9.H. *Bridges*

An ascending "*ordered bridge*" joining the s_I^{st} interpolation step to the later s_F^{st} interpolation step ($s_F > s_I$) is a sequence of s_i

$$s_I < s_1 < s_2 \ldots < s_N < s_F, \tag{9.H.1}$$

and the corresponding α's such that

$$\begin{aligned} \alpha_1(s_1) \quad &= \alpha_2(s_I) \\ \alpha_1(s_{i+1}) &= \alpha_2(s_i) \\ \alpha_1(s_F) \quad &= \alpha_2(s_N), \end{aligned} \tag{9.H.2}$$

and such that each of these α_2's is generated at the corresponding interpolation step. Variables 2, 3, and 4 may have to be interchanged (relabelled) to achieve an ordered bridge. Descending "ordered bridges" are defined analogously when $s_F < s_I$. A "*bridge*" is obtained from an ordered bridge by keeping only the information in the sets

$$(x_s, \alpha_1(s), \alpha_2(s), \alpha_3(s), \alpha_4(s)) \tag{9.H.3}$$

for $s = s_I, s_F, s_i, i = 1, \ldots, N$.

9.I. *Order of Painting*

The order in which the elements (α's and steps) are painted corresponding to a given term in the cluster expansion is, naturally enough, defined inductively. One first paints the distinguished variables. Now suppose one knows the order of painting to some stage. Run through the steps in the interpolation procedure with all the steps and α's painted for which the order of painting is now known. Stop the first time one reaches, at an unpainted step, either

1) a solid attachment

or

2) a hit step for binding

or

3) a hit step for seating

or

4) a taming step.

In the first case, paint the elements (α's and the step) of the solid attachment at the next stage. In the second case, first build an ascending bridge from the root step to the hit step and then a descending bridge from the hit step to the target step (we view this as a single bridge). In this case one paints all the elements of the bridge at the next stage. In the third case, build a descending bridge from the hit step to the root step. One paints all the elements of the bridge at the next stage. In the last case, 4), build an ascending bridge from the root step to the taming step, and paint all elements of the bridge at the next stage.

If the same step may realize more than one process (a seating and a taming, or two different binding processes, for example) we arbitrarily choose one of them to determine the painting at this stage.

One should convince oneself that this process does paint all the interpolation steps of the given term in the cluster expansion, as one proceeds through all the stages in the painting process.

10. Counting II and Estimating

Section 9 was mainly an exercise in learning a new language. In this section we will present Representation 3, and use it to estimate and control the sum over terms in the cluster expansion.

10.A Representation 3

We now present a new description of our given term in the cluster-expansion. Basically we construct a term inductively, introducing interpolation steps (and variables α) in a way related to the process of painting in the last section. In the painting process, steps were introduced in chunks, either elements of a bridge, or a solid attachment. In the present representation steps and α's will be introduced as members of the same chunks, but the chunks will not necessarily be introduced in the order in which they were painted.

We need to present one new term, "pinning", which will be analogous to attaching, but not necessarily the same. We will use the term in the following cases

1) In a solid attachment at step s in which only $\alpha_1(s)$ is painted, we say x_s is "*pinned*" *to* $\alpha_1(s)$.

2) In a solid attachment at step s in which at least two of the α's involved are

painted, we will "*pin*" x_s to the painted α of lowest level (or one of the painted α's of lowest level if there is more than one.)

3) In a binding process, we say the root site for binding is "*pinned*" to the same α to which it is attached.

4) In a seating process, using the language of Sect. 9.F, x_{s_h} is "*pinned*" to $\alpha_{a'}$.

5) In a taming process, the root site for taming is "*pinned*" to the α to which it is attached.

Representation 3 constructs the cluster expansion term inductively, starting at step 1 with the distinguished variables. It presents inductively

1) For each α introduced, the x's pinned to this α, counting repetitions, and taken unordered,

2) For any x pinned to any α, the type of pinning taking place at the pinning step; i.e. one of 1) through 5) above, and the four α's connected in a solid attachment, or the bridge in the other cases.

It has been a long road but this is our final description of the cluster expansion term. Counting estimates are fairly trivial in this representation. At the end of the inductive process one has a collection of sets each containing an x, and 4 α's. Each is thus of the form

$$(x_i, \alpha_1(x_i), \alpha_2(x_i), \alpha_3(x_i), \alpha_4(x_i)) \tag{10.A.1}$$

(A solid attachment clearly specifies this information, a bridge contains a finite number of such sets making up the bridge.) A specification of which $\alpha_i(x_i)\, x_i$ is attached to, and an ordering of the like x's in 1) of Representation 3, would clearly give us Representation 2! But there are at most $4^{|T|} \prod_x (d(x)!)$ possibilities. Finally then, a Representation 3 presentation is associated to at most $4^{|T|} \prod_x (d(x)!)$ Representation 2 presentations, and $4^{|T|} \prod_x (d(x)!)^2$ Representation 1 presentations. (Many different Representation 3 presentations may be associated to the same Representation 1 term, and many Representation 3 presentations we will count correspond to no term—we do not impose all compatibility conditions—thus over counting.)

10.B Numerical Factors I.

When we look at $(\Delta\phi(x))^4$, breaking this up into a sum over monomials in the α_k, we clearly have associated to the monomial

$$\alpha_1 \alpha_2 \alpha_3 \alpha_4, \tag{10.B.1}$$

a numerical factor

$$\lambda \prod_{i=1}^{4} \left(\Delta \frac{1}{\sqrt{-\Delta}} \psi_i \right)(x). \tag{10.B.2}$$

It is natural to define

$$f(\alpha_k, x) = \lambda^{1/4} \left| \left(\Delta \frac{1}{\sqrt{-\Delta}} \psi_k \right)(x) \right|. \tag{10.B.3}$$

We split f into factors

$$(f^{\varepsilon})^{v} f^{1-\varepsilon v}, \tag{10.B.4}$$

for v a positive integer to be picked, and ε a small number.

We state some estimates each under the conditions (of course using the notation of [2]): if M is large enough, $M \geqq M_0(\varepsilon)$, and λ is small enough, $0 \leqq \lambda < \lambda_0(p, M, \varepsilon)$:

Estimate 10.B.1. $\sum_{\alpha} f^{\varepsilon}(\alpha, x) \leqq 1.$ (10.B.5)

Estimate 10.B.2. $\prod_{i=1}^{n} f^{\varepsilon}(\alpha_{k(i)}, x) \leqq \frac{1}{n!},$ (10.B.6)

where $\alpha_{k(i)} \neq \alpha_{k(j)}, i \neq j$.

Estimate 10.B.3. If

$$v\varepsilon \leqq \frac{1}{12}, \tag{10.B.7}$$

then

$$f^{1-\varepsilon v}(\bar{\alpha}, y) \sum_{x} f^{1-\varepsilon v}(\alpha, x) \leqq L_{\alpha}^{-3/5}, \tag{10.B.8}$$

if level $\bar{\alpha} \geqq$ level α (where L_{α} is the edge length of the cube to which α is associated, as in [2]).

Estimate 10.B.4. If

$$v\varepsilon \leqq \frac{1}{12},$$

then

$$\sum_{x} (f^{1-\varepsilon v})(\alpha, x) \leqq L_{\alpha}^{3/4}. \tag{10.B.9}$$

The choice of the powers in (10.B.7), (10.B.8), (10.B.9), $(\frac{1}{12}, \frac{3}{5}, \frac{3}{4})$ is subject to large possible variation. If one were working with models $\lambda(|\Delta|^{\alpha}\phi)^4$ instead of $\lambda(\Delta\phi)^4$, other choices would be made. These largely arbitrary choices follow through in a number of places.
These follow from Estimates 7.1 and 7.2 of [2].

10.C. The Counting Theorem

Associated to a term in the cluster expansion we will have a numerical factor

$$N = \prod_{s} \prod_{i=1}^{4} f(\alpha_i(s), x_s). \tag{10.C.1}$$

We will factor N into

$$N = N_1 \cdot N_2, \tag{10.C.2}$$

where we will use N_2 to control the number divergence in Sect. 11 and N_1 to control the counting process in this section. We proceed to define N_1. There is an integer $v_1 < v$ to be fixed. Again we must consider several cases.

1) A solid attachment. Let the pinning take place onto α_1 and α_2 have the highest level of $\alpha_2, \alpha_3, \alpha_4$. Then the contribution of this interaction term to N_1 is

$$\prod_{i=1}^{4} f_i^{v_1 \varepsilon} f_1^{1-v\varepsilon} \frac{1}{L_1^{3/4}}. \tag{10.C.3}$$

Here f_i is shorthand for $f(\alpha_i, x)$.

2) Each other situation involves the contribution of a bridge. Let the bridge be

$$(x_i, \alpha_1(i), \alpha_2(i), \alpha_3(i), \alpha_4(i)) \quad i = 1, \ldots, n, \tag{10.C.4}$$

with

$$\alpha_1(i+1) = \alpha_2(i) \tag{10.C.5}$$

and be pinned onto $\alpha_1(1)$. Then the contribution to N_1 is

$$L_M^{3/5} \left(\prod_{j=1}^{4} \prod_{i=1}^{n} f_{j,i}^{v_1 \varepsilon} \right) \cdot f_1^{1-v\varepsilon} \cdot \frac{1}{L_1^{3/4}} \cdot \prod_{i=1}^{n-1} (f_{1,i+1} f_{2,i})^{1-v\varepsilon}. \tag{10.C.6}$$

Here $f_{j,i}$ is short for $f(\alpha_j(i), x_i)$. We assume $\alpha_2(n)$ has the highest level of $\alpha_2(n)$, $\alpha_3(n)$, $\alpha_4(n)$. $L_M = \sup_{1 \leqq i \leqq n-1} L_{\alpha_2}(i)$.

The factors $(f_{1,i+1} f_{2,i})^{1-v\varepsilon}$ and $\left(f_1^{1-v\varepsilon} \frac{1}{L_1^{3/4}} \right)$ control sums over sites (by Estimates 10.B.3 and 10.B.4) in the Sums into Sups process. $L_M^{3/5}$ is "available" by Estimate 10.B.3, as are the f's in numerical factors from the interaction. The $L_1^{3/4}$ will be reabsorbed in the estimates of Sect. 11.C. For this purpose it is essential that the bridge is constructed (as it must be) with level $\alpha_1(1) \leqq$ level $\alpha_2(n)$.

We will present now a theorem that will include as a corollary the Main Theorem as stated at the end of [2]. It has two parts, one part proved in this section, one part in the next section. Now our double theorem

Counting and Number Divergence Theorem *There is an $M_0 > 0$, a v_1 and v, $0 < v_1 < v$, and an $\varepsilon > 0$, $(v\varepsilon \leqq 1/12)$, such that for any fixed p, and any fixed c (see Main Theorem), if $M \geqq M_0$, and $\lambda \leqq \lambda_0(M, c, p)$ then*

if
$$\mathscr{A} = \alpha_1^{\ell_1} \ldots \alpha_t^{\ell_t} \tag{10.C.7}$$

$$\sum_T |K_T(\mathscr{A})| e^{c|T|} \leqq 2^t \operatorname{Sup}_T N_2(T) \prod_i ((\ell_i + m_i)!)^{1/2} \prod_i (p_i!)^{-1} \tag{10.C.8}$$

and
$$\operatorname{Sup}_T N_2(T) \prod_i ((\ell_i + m_i)!)^{1/2} \prod_i (p_i!)^{-1} \leqq c(\ell_i). \tag{10.C.9}$$

(10.C.8) is the result of this section, and (10.C.9) the result of the next section. $N_2(T)$ is the N_2 factor of (10.C.2) for the given term T. p_i is the number of x's pinned to α_i. m_i is the number of times α_i is differentiated down in the interpolation process, $\ell_i = 0$ if $i > t$. (The labelling of the α_i is special to the theorem). We will not detail the form of the function $c(\ell_i)$.

10.D. Sums into Sups

(10.C.8) is shown by iteratively converting sums into sups using (9.B.3), by now a standard procedure in cluster expansion estimates. It is tedious but rather trivial in our case. Representation 3 is most suitable for our purposes. We restrict ourselves to several examples.

Example 1. Given α_1, we wish to sum over sites x for the pinning of a solid attachment on α, and possible $\alpha_2, \alpha_3, \alpha_4$. We look at

$$\sum_x \sum_{\alpha_2} \sum_{\alpha_3} \sum_{\alpha_4} G(x, \alpha_2, \alpha_3, \alpha_4), \tag{10.D.1}$$

where G contains all later dependences on this process. We use (9.B.3) to find (10.D.1) satisfies

$$\begin{aligned} |(10.D.1)| \leq & \left(\sum_x \sum_{\alpha_2} \sum_{\alpha_3} \sum_{\alpha_4} \left(f_1^{1-\nu\varepsilon}(x) \frac{1}{L_1^{3/4}} \right) f_2^{\varepsilon} f_3^{\varepsilon} f_4^{\varepsilon} \right) \\ & \cdot \sup_{x, \alpha_2, \alpha_3, \alpha_4} \left(L_1^{3/4} \frac{1}{f_1^{1-\nu\varepsilon}} \frac{1}{f_2^{\varepsilon}} \frac{1}{f_3^{\varepsilon}} \frac{1}{f_4^{\varepsilon}} G \right). \end{aligned} \tag{10.D.2}$$

The sums are now controlled by Estimates 10.B.1 and 10.B.4. The numerical factors needed in the sup are taken from N_1.

Example 2. We first note that if d interpolations take place at a site x, then at least $d/2$ different α's are differentiated down by interpolations at x. (The tadpole process allows the number to be less by $\sim 50\%$ from the usual number $d + 1$.) Thus using Estimate 10.B.2

$$\prod_s \prod_{i=1}^{4} f^{\varepsilon}(\alpha_i(s), x_s) \leq \prod ((d(x)/2)!)^{-1}. \tag{10.D.3}$$

This can be used to control the $(\prod d(x)!)^2$ factor that arises from the transition from Representation 3 to Representation 1 presentations—among other places. The f^{ε} factors again can be borrowed from N_1 (or N_2 for uses in controlling number divergences).

Example 3. If n x's are pinned to α there are 5^n different types of actions that may take place at the sites.

Example 4. Let us sum over the number of x's, n, that are pinned to an α. And let

$$f^{\varepsilon}(\alpha, x) < \varepsilon_1, \quad \text{all} \quad \alpha, x. \tag{10.D.4}$$

Then

$$\sum_{n=0}^{\infty} G(n) \tag{10.D.5}$$

is estimated as

$$\left|\sum_{n=0}^{\infty} G(n)\right| \leqq \left(\sum_{n=0}^{\infty}\left(\prod_{j=1}^{n} f_j^{\varepsilon}\right)\right) \operatorname*{Sup}_n\left(\prod_{j=1}^{n}((f_j^{\varepsilon})^{-1})|G(n)|\right)$$
$$\leqq \left(\frac{1}{1-\varepsilon_1}\right) \operatorname*{Sup}_n\left(\prod_{j=1}^{n}((f_j^{\varepsilon})^{-1})|G(n)|\right). \qquad (10.D.6)$$

Here f_j^{ε} is a small factor borrowed at the j^{th} pinning from N_1.

Example 5. In summing over the positions of the p_i x's pinned to α_i, as in Example 1, one will naturally count any given configuration $p_i!$ times, as the order of selecting each x_j and the objects pinned to it is irrelevant to the Representation 1 presentations associated to the Representation 3 presentation.

11. The Number Divergence

The "number divergence" is the numerical factor that arises in integrating a polynomial (of high degree) over the measure, in our case integrating over α, a variable occurring to the ℓ^{th} power in the distinguished variables and differentiated down m times, one gets

$$\int d\alpha e^{-\alpha^2/2}|\alpha|^{\ell+m}, \qquad (11.1)$$

which we estimate as

$$\sim c^{\ell+m}\cdot\left(\frac{\ell+m}{2}\right)!. \qquad (11.2)$$

One must control the factor

$$\prod_i\left(\frac{\ell_i+m_i}{2}\right)!. \qquad (11.3)$$

This section deals with dominating this factor by

$$N_2(T)\prod_i\frac{1}{p_i!}. \qquad (11.4)$$

11.A Numerical Factors II

We consider a bridge as defined by equations (10.C.4) and (10.C.5). Denoting this bridge as B we associate to B a numerical factor $F_B^{\varepsilon}(\alpha_1(1),\alpha_2(n))$ (suppressing the dependence on other variables).

$$F_B^{\varepsilon}(\alpha_1(1),\alpha_2(n)) = f^{\varepsilon}(\bar{\alpha}_1, x_1)\cdot f^{\varepsilon}(\bar{\alpha}_2, x_1)\cdot f^{\varepsilon}(\bar{\alpha}_2, x_2)\cdot$$
$$\cdots f^{\varepsilon}(\bar{\alpha}_n, x_{n-1})\cdot f^{\varepsilon}(\bar{\alpha}_n, x_n) f^{\varepsilon}(\bar{\alpha}_{n+1}, x_n) \qquad (11.A.1)$$
$$\cdot L_M^{-3/5}\cdot f^{1-\nu\varepsilon}(\bar{\alpha}_{n+1}, x_n)$$

where here

$$\begin{aligned} \bar{\alpha}_1 &= \alpha_1(1) \\ \bar{\alpha}_i &= \alpha_1(i) \quad i = 2, \dots, n, \\ \bar{\alpha}_{n+1} &= \alpha_2(n) \end{aligned} \tag{11.A.2}$$

(See (10.C.6) for definition of L_M) and we use

$$\begin{aligned} &\text{level } \bar{\alpha}_i < \text{level } \bar{\alpha}_{n+1} \quad i = 2, \dots, n \\ &\text{level } \bar{\alpha}_1 \leqq \text{level } \bar{\alpha}_{n+1} \end{aligned} \tag{11.A.3}$$

(one or the other end of a bridge has the highest level). We use the conditions listed before Estimate 10.B.1, in stating:

Bridge Estimate. Let $F^\varepsilon(\alpha_a, \alpha_b)$ be the sup of $F^\varepsilon_B(\alpha_a, \alpha_b)$ over all bridges and x_i connecting α_a and α_b, for all n, but of course satisfying (11.A.2) and (11.A.3). Then

$$\sum_{\alpha_a} (F^\varepsilon(\alpha_a, \alpha_b))^2 L_a^{3/2} \leqq 1, \tag{11.A.4}$$

where L_a, L_b are the edge sizes of α_a, α_b.

This estimate is a natural geometric reflection of the scaling properties of the $f(\alpha, x)$ under level changes in the α. The heart of the result is the statement that

$$\sum_{\alpha_a} (\hat{F}^\varepsilon(\alpha_a, \alpha_b))^2 \left(\frac{L_a}{L_b}\right)^3 \leqq 1, \tag{11.A.5}$$

where $\hat{F}^\varepsilon$ is calculated as F^ε, except that the last two factors $L_M^{-3/5} \cdot f^{1-\nu\varepsilon}$ are omitted from (11.A.1). Notice that if $\text{Max}(L_m, L_a)$ were always greater than $(L_b)^{1/2}$, (11.A.5) would imply (11.A.4)! The content of (11.A.5) can be appreciated by viewing $(f(\alpha, x)$ as approximately (for L the edge size of α)

$$\cong \begin{cases} \varepsilon_1 L^{-\varepsilon'} & \text{if } x \text{ is in the cube} \\ & \text{associated to } \alpha \\ 0 & \text{otherwise} \end{cases} \tag{11.A.6}$$

in which putative situation, $\hat{F}^\varepsilon(\alpha_a, \alpha_b)$ would be dominated by

$$L_a^{-\varepsilon'} e^{-\gamma(1/L_b)\text{dist}(\Delta_a, \Delta_b)}, \tag{11.A.7}$$

where Δ_a and Δ_b are the cubes associated to α_a and α_b. The Bridge Estimate is proven in Appendix A.

11.B A Factorial Estimate

We now state a most useful estimate for controlling factorial factors:

Factorial Estimate 11.B. *Let $s_i \geqq 0$, $g_i \geqq 0$, and $\sum g_i \leqq 1$ (i ranging over any finite indexing set) then*

$$(\textstyle\sum s_i)^{\Sigma s_i} \prod (g_i^{s_i} s_i^{-s_i}) \leqq 1. \tag{11.B.1}$$

Proof. The inequality is equivalent to

$$\sigma \ln \sigma + \textstyle\sum s_i \ln (g_i s_i^{-1}) \tag{11.B.2}$$

being less than or equal to zero, where

$$\sigma = \textstyle\sum s_i. \tag{11.B.3}$$

We use Lagrange multipliers to find the value of (11.B.2) at its stationary point, keeping σ fixed.

$$\ln \sigma + \ln g_i + \lambda = \ln s_i \tag{11.B.4}$$

$$s_i = e^{\lambda} g_i \sigma = b g_i \sigma. \tag{11.B.5}$$

Here λ is the Lagrange multiplier and $b = e^{\lambda}$. Substituting these values for the s_i into (11.B.2) we find

$$-\sum b g_i \sigma \ln b, \tag{11.B.6}$$

where we have used

$$\sum s_i = \sigma = b\sigma \sum g_i. \tag{11.B.7}$$

Thus $b \geqq 1$ suffices to ensure (11.B.2) $\leqq 0$ at the (only) stationary point. This is exactly $\Sigma g_i \leqq 1$. If we now consider the values of (11.B.2) along its boundary, i.e. some of the s_i are zero, we find our procedure for a smaller index set ensures that (11.B.2) $\leqq 0$ also on its boundary. This proves Estimate 11.B. This estimate also follows from the arithmetic mean geometric mean inequality.

11.C The Final Reckoning

We view occurrences of a given α, as a distinguished variable from $\mathscr{A}$, or as differentiated down; there again are different cases.

1) Powers of α arising from $\mathscr{A}$ have no numerical factors associated with them, say these contribute a power

$$\alpha^{\ell}. \tag{11.C.1}$$

2) Powers of α arising from bridging elements of a bridge (an $\alpha_1(i)$, $i = 2, \ldots, n$ or $\alpha_2(i)$, $i = 1, \ldots, n-1$ in (10.C.4)). There are at most two such α's. These contribute a power

$$\alpha^{\beta}, \beta = 0 \text{ or } 1 \text{ or } 2 \tag{11.C.2}$$

associated to numerical factors

$$\varepsilon_1^{\beta} \tag{11.C.3}$$

(see (10.D.4)).

3) All powers of α not included in 1), 2) or 4) below, say s_i from x_i. These contribute a power

$$\alpha^{\Sigma s_i} \tag{11.C.4}$$

and may be associated to a numerical factor from N_2

$$\prod_i \left(((f^{1-v\varepsilon}(\alpha, x_i))^{s_i} \left(\frac{1}{s_i}\right)^{s_i/2} \varepsilon_1^{s_i} \right). \tag{11.C.5}$$

4) Powers of α arising as $\alpha_1(1)$ or $\alpha_2(n)$ in (10.C.4) or α_1 or α_2 in (10.C.3). These each may be associated to an $\alpha_i(\alpha_1(1)$ to itself, $\alpha_2(n)$ to $\alpha_1(1)$, α_1 to itself, α_2 to $\alpha_1)$, t_i to α_i. We get a power

$$\alpha^{\Sigma t_i} \tag{11.C.6}$$

and associated numerical factors (for v large enough)

$$\prod_i \left(((\bar{F}(\alpha_i, \alpha))^{t_i} \left(\frac{1}{t_i}\right)^{t_i/2} \varepsilon_1^{t_i} \right), \tag{11.C.7}$$

where

$$\sum_{\alpha_i} \bar{F}^2(\alpha_i, \alpha) \leqq 1/2, \tag{11.C.8}$$

for λ small enough (primarily the Bridge Inequality).

In this paragraph we derive the factors $\bar{F}(\alpha_i, \alpha)$ by detailing numerical factors assigned to the four sources of α's in 4), $\alpha_1(1)$, $\alpha_2(n)$, α_1, α_2 (using the generic labelling). The factors are given in the table

α	*numerical factor*
$\alpha_1(1)$	ε_1
α_1	ε_1
$\alpha_2(n)$	$\varepsilon_1 F^\varepsilon(\alpha_1(1), \alpha_2(n)) L_1^{3/4}$
α_2	$\varepsilon_1 f^\varepsilon(\alpha_1, x) L_{1'}^{3/4} f^{1-v\varepsilon}(\alpha_2, x)$.

Here L_1 and $L_{1'}$ are the edge sizes of $\alpha_1(1)$ and α_1 respectively. It is clear from (10.C.3) and (10.C.6) that the numerical factors in the table are "available" in N_2. From the table we see we may choose

$$\begin{aligned} \bar{F}(\alpha_A, \alpha_B) = \operatorname{Max} \{ & \varepsilon_1 \delta_{\alpha_A, \alpha_B}, \operatorname{Sup}_x (\varepsilon_1 f^\varepsilon(\alpha_A, x) L^{3/4} f^{1-v\varepsilon}(\alpha_B, x)), \\ & (\varepsilon_1 F^\varepsilon(\alpha_A, \alpha_B) L^{3/4}) \}, \end{aligned} \tag{11.C.9}$$

where L is the edge size of α_A.

Using (for λ small enough)

$$\sum_{\alpha_i} \bar{F}^2(\alpha_i, \alpha) + \sum_{x_i} (f^{1-v\varepsilon}(\alpha, x_i))^2 \leqq 1 \tag{11.C.10}$$

and the Factorial Estimate 11.B, the *square* of the overall estimate we get for the integral over α is

$$\frac{(\sum s_i + \sum t_i + \ell + \beta)!}{(\sum s_i + \sum t_i)!} \varepsilon_2^{(\Sigma t_i + \Sigma s_i + \beta)}, \tag{11.C.11}$$

where ε_2 can be made arbitrarily small as $\lambda \to 0$.

We observe finally that

$$(11.C.11) \leqq c(\ell), \tag{11.C.12}$$

where $\quad c(0) \leqq 1.$

The $\left(\frac{1}{s_i}\right)^{s_i/2}$ in (11.C.5) arise as in Example 2 of 10.D. The $\left(\frac{1}{t_i}\right)^{t_i/2}$ in (11.C.7) arise from the $\left(\frac{1}{p_i}\right)!$ in (11.4). It is almost miraculous how cleverly the numerical factors contrive to satisfy our requirements.

12. Notes on Extension to $\lambda(\vec{\nabla}\phi)^4$ and Dipole Gas

There are two essential difficulties in extending the present program to the more complicated $\lambda(\vec{\nabla}\phi)^4$ and dipole gas models. We will give a brief discussion of these difficulties, and indicate the alterations in the expansion necessary to treat them. Details are left to a later paper. Models such as the classical Heisenberg model would require even further ideas, and we have not considered them at all.

The first problem is to control the number divergence for an α_k corresponding to a large cube. When there are many interior variable α's close to α_k, in particular α's corresponding to small cubes inside or close to the big cube, the controlling factors (not as effective as those in Sect. 11) become insufficient. At some stage one adds all the α's close enough to α_k into the set of interior variables. This is an old idea going back to an unpublished preprint of *T.* Spencer (see [3] also). One key to controlling estimates resulting from this process is the inequality of [1].

The second problem is that our tadpole procedure is not sufficient to control the numerical factors leading to convergence. It must be generalized in a straightforward way to tadpoles of even total weight. In this new situation, expectations of tadpoles—isolated in the interpolation procedure—may be nonzero. One uses the fact that these expectations are nearly independent of the position of the tadpole, so when the position is summed over one is roughly doing a numerical integral of the dependence of numerical factors on the position. This in useful places contributes small factors arising from the estimate

$$\left|\int\left(\vec{\nabla}\frac{1}{\sqrt{-\Delta}}\psi_{k_1}\right)\left(\vec{\nabla}\frac{1}{\sqrt{-\Delta}}\psi_{k_2}\right)\right| \leqq c\frac{L_{k_2}^{3/2}}{L_{k_1}^{3/2}}\left[\frac{L_{k_2}}{L_{k_1}}\ln^2(L_{k_1})+\frac{L_{k_2}\ln(\hat{d})}{\hat{d}}\right]. \tag{12.1}$$

Here we assume $L_{k_1} > L_{k_2}$ and $\hat{d}$ is the distance of the cube corresponding to k_2 from $\hat{\partial}(r_{k_1}, \gamma_{k_1})$. (See Sect. 7 and in particular Estimate 7.2.)

Appendix A — The Bridge Estimate

Extensions of the Bridge Estimate may be needed in later work, the form of the result may be modified in a number of directions. With the observations of Sect. 11.A, the proof is straightforward. It follows, with a sequence of small observations, from the following simple estimate.

Estimate A. Let $0 \leqq \delta \leqq 1$, and $t > 0$, then

$$\sup_{N\in Z^+}\ \sup_{\substack{x_i \geqq 0\\ \sum_{i=1}^N x_i = x}} \prod_{i=1}^{N}\left(\frac{\delta}{x_i^t+1}\right) \leqq \operatorname{Max}\left(\delta^{x/2}, \frac{2\delta}{\left(\frac{x}{2}\right)^t+1}\right). \tag{A.1}$$

Proof of Estimate A. We first maximize $\frac{1}{x_i^t+1}$ by $\ell(x_i)$,

$$\ell(x_i)=\begin{cases}1 & x_i \leqq 1\\ 1/x_i^t & x_i > 1\end{cases}$$

and seek the sup of $\pi(\delta\ell(x_i))$ under the same restrictions as the sup in (A.1).

We split the set of x_i that achieve this Sup into two sets, $\{x_i \leqq 1\}$ and $\{x_i > 1\}$. Let y be the sum of the x_i in the first set. Then the number of such x_i is $\geqq y$, and each such x_i contributes a factor δ to the product; so a total factor of $\leqq \delta^y$ arises from these x_i's. Let there be n_2 x_i's in the second set, with sum 2. Using Lagrange multipliers we may see the sup of $\prod\left(\frac{\delta}{x_i^t}\right)$ would be achieved with all x_i equal, for fixed n_2 and z. Looking at $\left(\frac{\delta}{\left(\frac{z}{n_2}\right)^t}\right)^{n_2}$ we see the sup of this expression in the range $1 \leqq n_2 \leqq z$ is obtained either at $n_2 = 1$ or $n_2 = z$, letting n_2 be nonintegral. (Notice its logarithm has second derivative positive in the variable n_2.) Thus

$$\left(\frac{\delta}{\left(\frac{z}{n_2}\right)^t}\right)^{n_2} \leqq \mathrm{Max}\left\{\delta^z, \frac{\delta}{z^t}\right\}. \tag{A.2}$$

Noting that either z or y is $\geqq x/2$, and $z \geqq 1$, the Estimate A easily follows.

Step 1. For any $\varepsilon > 0, t > 0, \delta > 0$ there is an $\bar{M}_0$ such that for $M \geqq \bar{M}_0$ and $\lambda \leqq \bar{\lambda}_0(M, p)$ one has

$$f^\varepsilon(\alpha, x) f^\varepsilon(\alpha, y) \leqq \frac{\delta}{\left(\frac{|x-y|}{L_\alpha}\right)^t + 1} \tag{A.3}$$

and

$$f^\varepsilon(\alpha, x) \leqq \frac{\delta}{\left(\frac{|x-x_\alpha|}{L_\alpha}\right)^t + 1} \cdot \frac{1}{L_\alpha^{\varepsilon'}}, \tag{A.4}$$

where x_α is the center of cube α, and $\varepsilon' > 0$.

Step 2. We pick $t > 3$ and then (11.A.5) follows from Step 1 and Estimate A. Thus if Max (L_a, L_M) were always greater than $L_b^{1/2}$ the Bridge Estimate would be proven. The situation here is essentially as in the simplified picture given at the end of Sect. 11.A.

Step 3. For those bridges with $\mathrm{Max}(L_a, L_M) < L_b^{1/2}$, we use the factor $(f^{1-v\varepsilon}(\bar{\alpha}_{n+1}, x_n))^2$ to replace the factor $\frac{1}{L_b^3}$ in (11.A.5). (We do not seek help from factors of L_M or L_a as we do then $\mathrm{Max}(L_a, L_M) \geqq L_b^{1/2}$.) f, as given by (10.B.3), is estimated in (7.4). If there were only the first term on the right side of (7.4), we could deduce the Bridge Estimate directly from (11.A.5), in all cases (and without help from L_a and L_M factors). In the actual situation we must deal with the factor $\frac{1}{\hat{d}}$ in the last term of (7.4). For those x_n with $\hat{d} > L_b^{1/2}$, this factor easily yields the Bridge Estimate from (11.A.5). (It is sufficient to prove the Bridge Estimate for the three classes of bridges separately: $\mathrm{Max}(L_a, L_M) \geqq L_b^{1/2}$; $\mathrm{Max}(L_a, L_M) < L_b^{1/2}, \hat{d} \geqq L_b^{1/2}$; $\mathrm{Max}(L_a, L_M) < L_b^{1/2}, \hat{d} < L_b^{1/2}$.) For the final case we note from Estimate A, Step 1, and the definition of $\hat{d}$, that α_a tries to live within distance $L_b^{1/2}$ from $\hat{\partial}(r_b, \gamma_b)$ (see above equation (7.2)). This leads to control of the sum in (11.A.4).

Appendix B—Two Point Fall Off Estimate

In this appendix we prove the Clustering Theorem of Sect. 8.F. This is mainly a book-keeping exercise we perform in a sequence of steps.

Step 1. Choice of parameters. We need two more "small" factors f^ε than in the proof of the Main Theorem. Therefore we reconsider the proof of the paper with "large" factors $f^{1-(v+2)\varepsilon}$ and "small" factors f^ε. This will put more stringent conditions on $M, p, \lambda, \varepsilon$. We let M, p, ε be values of these parameters for which the Counting and Number Divergence Theorem holds for λ sufficiently small—with $v+2$ instead of v in the definition of the "large" factors—and which in addition satisfy

$$2 \cdot \tfrac{3}{2}(1-(v+2)\varepsilon) \geqq \gamma . \tag{B.1}$$

(v is a fixed integer, γ is given.) (B.1) and considerations in Sect. 10.B require ε be small enough. M and p then may be chosen, sufficiently large (to fulfill conditions in Sect. 10.B and Appendix A). The values of λ for which the Main Theorem holds may have to be further restricted; λ may have to be smaller, $\leqq \lambda_\gamma$, for the Clustering Theorem.)

Step 2. $x-y$ Bridges. In Eq. (8.F.3) to a $k=k'$ term, we associate a degenerate (empty) $x-y$ bridge, and a numerical factor

$$f^\varepsilon(\alpha_k, x) f^\varepsilon(\alpha_k, y). \tag{B.2}$$

f is defined generalizing (10.B.3)

$$f(\alpha, x) = \lambda^{1/4} \left| \left(|\Delta|^\alpha \frac{1}{\sqrt{-\Delta}} \psi \right)(x) \right| . \tag{B.3}$$

Corresponding to the k, k' term in (8.F.3), we define an $x-y$ bridge as a sequence of sets (associated to interpolation steps)

$$\begin{gathered}(x_s, \alpha_1(s), \alpha_2(s), \alpha_3(s), \alpha_4(s)) \\ s = s_1, s_2, \ldots, s_n,\end{gathered} \tag{B.4}$$

for some $n \geqq 1$, as in Sect. 9.H. For a suitable ordering of the α's, we want

$$\begin{aligned} \alpha_1(s_1) &= \alpha_k \\ \alpha_2(s_n) &= \alpha_{k'} \\ \alpha_1(s_{i+1}) &= \alpha_2(s_i) \quad i = 1, \ldots, n-1. \end{aligned} \tag{B.5}$$

The associated numerical factor is

$$f^{\varepsilon}(\alpha_k, x) \prod_{i=1}^{n} (f^{\varepsilon}(\alpha_1(s_i), x_i) f^{\varepsilon}(\alpha_2(s_i), x_i)) f^{\varepsilon}(\alpha_{k'}, y). \tag{B.6}$$

For an $x - y$ bridge, B, degenerate or not, we define $L_{xy}(B)$ by

$$\operatorname{Max}\left(L_{\alpha_k}, L_{\alpha_{k'}}, \bigcup_i L_{\alpha_1}(s_i), \bigcup_i L_{\alpha_2}(s_i) \right). \tag{B.7}$$

Step 3. Idea of proof. For a given term in cluster expansion for (8.F.3), let L_0 be the smallest value of $L_{xy}(B)$ for $x - y$ bridges associated to this term. By Estimate A, the numerical factor (B.2) or (B.6) associated to the bridge with $L_{xy} = L_0$ is $\leqq$

$$c \operatorname{Max}\left(e^{-c_1|x-y|/L_0}, \frac{c_2}{\left(\frac{|x-y|}{L_0}\right)^t + 1} \right). \tag{B.8}$$

We are going to find an additional numerical factor

$$\frac{c}{L_0^{\gamma}}. \tag{B.9}$$

Since $\gamma < 3 \leqq t$, the product of (B.8) and (B.9) will be bounded by

$$\frac{c}{|x-y|^{\gamma}}.$$

Both factors (B.8) and (B.9) will be "available", otherwise unused, factors in our estimate of the cluster expansion term (for λ small enough). This yields the theorem. The factor (B.8) clearly is present, as we've kept extra factors of f^{ε} in Step 1. Our entire burden is to find the factor (B.9).

Step 4. Order of painting. Unfortunately, we must modify the order of painting as given in Sect. 9.I. We start the painting process as in Sect. 9.I. We paint chunk by chunk. At a given stage of the painting, the painted elements form either a single connected set (in the obvious sense); or two connected sets, each containing either α_k or $\alpha_{k'}$. We deviate from the order of painting in Sect. 9.I, if at some stage before the painted elements form a single connected set, one of the connected painted subsets, say the one containing α_k (connected to x), contains some α with $L_{\alpha} \geqq L_0$. Having used one of the new factors f^{ε} in (B.2) and (B.6) we use the other new small factor to enable us to take a sup over cluster expansion terms with fixed value of L_0.

We will paint chunks of elements that connect to the subset connected to y until either one of the painted elements in the subset connected to y has edge length $\geqq L_0$ or until the painted elements form a single connected set. Thereafter the painting proceeds as it would in 9.I—the inductive construction is as in 9.I, the sets themselves may be different. We must explain the painting process in the intermediate region where we do not follow the construction of 9.I.

In this region, we suppose we know the painted elements and interaction steps to some stage. We then, as in 9.I, run through the steps in the interpolation procedure with all steps and α's painted for which the order of painting is now known.

We stop the first time one reaches at an unpainted step either (compare these directions to those in 9.I):

1) a solid attachment, attached to one of the painted elements connected to y.

or

2) a hit step for binding, with both tadpoles attached to painted elements connected to y

or

3) a hit step for seating, with both α_a and α_B (see Sect. 9.F) painted elements connected to y

4) a taming step, with the tadpole attached to one of the painted elements connected to y

or

5) a step in which an element connected to y is connected to an element connected to x, painted or not.

For alternatives 1) through 4) the painting is as in 9.I.

In situation 5), there are two possibilities. (In Example 3 of Sect. 10.D there are now 7 actions not 5, this at most requires a smaller value of λ.) With the interaction steps up to this point we build either a bridge associated to a tamed tadpole (rearranging the order of steps performed so far, a tadpole taming may be constructed) or a bridge associated to binding of tadpoles, one connected to x, the other connected to y (again by rearranging the order of steps such a binding may be realized). In the second case the bridging elements may be required to be of edge size $\leqq L_0$, and one of the tadpoles, attached to a painted element of edge length $\leqq L_0$ connected to x. Rearranging order of steps does not effect our counting process. The restrictions on edge lengths in the second case follow from the requirement that L_0 be the minimum of $L_{xy}(B)$, and our construction.

Step 5. Extraction of $L_0^{-\gamma}$. We will find a numerical factor $\leqq L_0^{-\gamma/2}$ associated to the painted elements connected to each of x and y, at the stage when the painted elements connected to each of x and y first include an element of edge size $\geqq L_0$. We consider the y contribution, the x contribution is entirely similar. At the initial step we borrow from (8.F.3) $f^{1-(v+2)\varepsilon}(\alpha_{k'}, y)$. If we assume that previous to the painting of a given chunk (connected to y) the largest edge size cube connected to y is L_1 and after the painting of the chunk the largest is L_2, we will find a factor $\leqq \left(\frac{L_1}{L_2}\right)^{\gamma/2}$. This will complete the proof. Referring to Sect. 11.C, the element α_M, with edge length L_2, that we are considering, cannot be a bridging element. If it arises as an α included in the 3) case of Sect. 11.C, see Eq. (11.C.5), we extract a factor $f^{1-(v+2)\varepsilon}(\alpha_M, x_i)$. If α_M arises as a 4) case of Sect. 11.C (including the two new types of bridges developed in Step 4), we borrow a factor of $f^{1-(v+2)\varepsilon}(\bar{\alpha}_{n+1}, x_n)L_1^{\gamma/2}$ from (11.A.1) or the analogous $f^{1-(v+2)\varepsilon}(\alpha_2, x_n)L_1^{\gamma/2}$ from the table in 11.C. We have thus found numerical factors giving us (B.9), we must know removing these numerical factors does not destroy numerical estimates of Sect. 11. But the result of our extractions adds to a number of α's one or two additional powers of α with no numerical factors associated to them—case 1) of 11.C, in this situation each associated to small factors, f^{ε}—which does not upset the estimates of 11.C.

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