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A Phase Cell Cluster Expansion for a Hierarchical ϕ_3^4 Model*

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Abstract. The formalism developed in a previous paper is applied to yield a phase cell cluster expansion for a hierarchical ϕ_3^4 model. The field is expanded into modes with specific renormalization group scaling properties. The present cluster expansion for a vacuum expectation value is formally the natural factorization of each term in the perturbation expansion into the contribution of modes connected to the variables in the expectation via interactions, and that of the complementary set. The expectation value is thus realized as a sum of contributions due to *finite* subsets of the modes. We emphasize the following additional features:

1) Partitions of unity are not used.

2) There are *essentially* no cut-offs.

3) The expansion is developed directly, without an initial need to prove an ultraviolet stability bound, the most difficult part of the traditional approach.

Our main interest in the present phase cell cluster expansion is founded in the belief that it may be the right vehicle for proving the existence of a non-trivial four-dimensional field theory.

0. Introduction

Techniques developed in the study of ϕ_3^4 should eventually be useful in other directions – in statistical mechanics, in fluid mechanics, in the study of turbulence. We here restrict our sights to further applications in field theory, in particular to the construction of non-trivial four-dimensional field theories. For us the study of ϕ_3^4 is taken in this light. For convenience we divide the bulk of work on ϕ_3^4 into five tracks.

1) The first important contribution was the establishment of the ultraviolet bound by Glimm and Jaffe [9]. This most difficult paper indicated the importance

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of phase cell localization and foreshadowed (block spin) renormalization group techniques. Feldman and Osterwalder [7], and Magnen and Sénéor [12] applied cluster expansion techniques, using the results of [9], to establish field theory axioms for ϕ_3^4.

2) Gallavotti et al. introduced renormalization group (block spin) ideas to provide a new proof of the ultraviolet bound [4]. Gawedzki and Kupiainen adapted these techniques to study infrared lattice models [8]. Our first efforts in using phase cell cluster expansions were inspired by this work.

3) Brydges et al. have used random walk and correlation inequality techniques to obtain bounds on expectation values for ϕ_3^4 [5].

4) Using the results in [9] Y. M. Park has shown that the lattice approximation to ϕ_3^4 converges to the continuum theory [13]. In [1, 2] Bałaban has given a new proof of ultraviolet stability for a lattice ϕ_3^4, with bounds independent of lattice spacing. The technique used is that of block spin transformations and the renormalization group, making rigorous ideas of Wilson and Kadanoff in this setting. This effort borrows heavily from some ideas in 2).

5) Here we include our present treatment. The phase cell cluster expansion was first applied in an infrared lattice setting [6]. In [3] we applied the expansion to two dimensional models; but in fact much of the work of that paper was general, independent of spatial dimension, and immediately applicable to ϕ_3^4. (An error in this reference will be corrected herein.)

All of the above approaches except 3) use phase cell localization in an important way. Approaches 1), 2), and 4) each have as the first and most difficult step the establishment of ultraviolet bounds. In viewing extensions to further field theories, the techniques in 3) may be limited by special requirements on a theory to yield necessary correlation inequalities. Bałaban has already extended his treatment to handle gauge theories in two and three dimensions. In future papers we will extend the present work to include the actual ϕ_3^4 model, and to treat Y_2. Work is also underway to treat gauge field theories. We feel the approaches of 4) and 5) are the most promising for establishing the existence of non-trivial four-dimensional field theories. The phase cell cluster expansion is perhaps the most natural setting within which to treat non-superrenormalizable models – although difficult and still untested, it seems possible to renormalize this cluster expansion by the traditional subtraction procedures of theoretical physics. (By the use of interpolation and "removal of contours" [10] numerical factors similar to mass inserts and vertex inserts are developed inside terms in the cluster expansion. Individually some of the factors may be infinite, but the cluster expansion is organized so that only finite differences of such factors appear. Asymptotic freedom seems essential.)

We now describe the models to be considered. We will use a label k to describe our expansion functions; with the same notation as in [3, Sect. 2], so that the expansion function $u_k(x)$ and variable α_k are associated to a cube, say Δ_k. However, $u_k(x)$ need not necessarily be given by Eq. (1.4) of [3]! There is here, as there, a fixed integer s. We write

$$\phi(x) = \sum_k \alpha_k u_k(x). \tag{0.1}$$

Let $d\mu_0$ be a probability measure on continuous functions of the α_i in which the α_i are Gaussian variables distributed with covariance C_{ij}, i.e.

$$C_{ij} = \int d\mu_0 \alpha_i \alpha_j . \tag{0.2}$$

We write

$$\langle \cdot \rangle_0 = \int d\mu_0(\cdot) , \tag{0.3}$$

and set

$$\begin{aligned} I' = \lambda \int :\phi^4: + 48\lambda^2 \int dx :\phi^2(x): \int dy \langle \phi(x)\phi(y)\rangle_0^3 \\ + 12\lambda^2 \int dx \int dy \langle \phi(x)\phi(y)\rangle_0^4 \end{aligned} \tag{0.4}$$

with normal ordering defined with respect to $d\mu_0$. We write I' as

$$\begin{aligned} I' = & \sum_{1,2,3,4} w(1,2,3,4) :\alpha_1\alpha_2\alpha_3\alpha_4: \\ & + 48 \sum_{\substack{1,2,3,4,5\\1',2',3'}} w(1',2',3')\, w(1,2,3,4,5) :\alpha_4\alpha_5: C_{11'}C_{22'}C_{33'} \\ & + 12 \sum_{\substack{1,2,3,4\\1',2',3',4'}} w(1,2,3,4)\, w(1',2',3',4')\, C_{11'}C_{22'}C_{33'}C_{44'} \end{aligned} \tag{0.5}$$

with

$$w(1,2,\ldots,n) = \lambda \int u_1 u_2 \ldots u_n . \tag{0.6}$$

[We here use a formalism appropriate to $d=3$. If $d=2$ we would keep only the first term in (0.4).] It will be convenient to define for A, a subset of the k, I'^A, by restricting the sums in (0.5) to the subset A; and ϕ_A, by restricting the sum in (0.1) to the subset A.

We will be interested in two cases.

1) *The ϕ_3^4 model.* Here we pick

$$u_k = L_k \psi_k , \tag{0.7}$$

using the notation of [3] for L_k and ψ_k, and

$$C_{ij} = \frac{1}{L_i}\frac{1}{L_j} \iint \psi_i \frac{1}{-\Delta + M^2} \psi_j , \tag{0.8}$$

with

$$M \sim 1 . \tag{0.9}$$

$\left(\text{We may alternatively expand in terms of } u_k = L_k^{1-a} \frac{1}{(-\Delta+M^2)^{(1/2)a}} \psi_k \text{ with } a>0 \right.$

$\left.\text{and sufficiently small.}\right)$

2) *The hierarchical ϕ_3^4 models.* In this case we set

$$C_{ij} = \delta_{ij} . \tag{0.10}$$

Thus we may later work with the measure, proportional to μ_0,

$$\prod_i \left(\int d\alpha_i e^{-\frac{1}{2}\alpha_i^2}\right).$$

We require the u_k to satisfy the following four requirements, I–IV (stated for $d=3$).

I. *Boundedness*

$$|u_k(x)| \leqq c \frac{1}{L_k^{1/2}}, \qquad \hat{d} \leqq L_k, \tag{0.11}$$

$$|u_k(x)| \leqq c \left(\frac{L_k}{\hat{d}}\right)^{3+s} \frac{1}{L_k^{1/2}}, \qquad \hat{d} > L_k, \tag{0.12}$$

where $\hat{d}$ is the distance from x to the center of Δ_k.

II. α-*Stability*

For any set A of variables, with cardinality $|A|$

$$I'^A - \frac{\lambda}{2} \int \phi_A^4 \geqq -c|A|. \tag{0.13}$$

III. α-*Positivity*

For any $\varepsilon > 0$ there is a $c > 0$ such that

$$\int \phi_A^4 \geqq c \sum_{k \in A} L_k^{1+\varepsilon} |\alpha_k|^{4-\varepsilon}. \tag{0.14}$$

IV. α-*Renormalizability*

For any $\varepsilon > 0$

$$\begin{aligned} &|w(1,2,3,4,5)\, w(1,2,3) - w(1,2,3,4)\, w(1,2,3,5)| \\ &\leqq c\lambda^2 L_1^6 \prod_{i=1}^{3} \left(\frac{1}{L_i}\right) \frac{1}{L_4^{1/2}} \frac{1}{L_5^{1/2}} \left(\frac{L_1}{L_5}\right)^{1-\varepsilon} \prod_{j=2}^{5} h_{s/10}\left(\frac{d_{1j}}{L_j}\right), \end{aligned} \tag{0.15}$$

provided $L_1 \leqq L_2 \leqq L_3 < L_4, L_5$. The notation for h and d_{ij} is as in Sect. 3 of [3].

We will show in Sect. 1 that the choice $u_k(x) = L_k \psi_k(x)$ satisfies all these properties.

We set

$$[p]^A = \int d\mu_0 e^{-I'^A} p(\alpha), \tag{0.16}$$

with p a polynomial in the $\{\alpha_i | i \in A\}$, and

$$Z^A = [1]^A, \tag{0.17}$$

and finally

$$\langle p \rangle^A = [p]^A / Z^A. \tag{0.18}$$

We state results for ϕ_3^4 and for the hierarchical ϕ_3^4 as defined in 1) and 2).

Theorem 0.1. *There is a $\lambda_0 > 0$, such that if $0 \leqq \lambda \leqq \lambda_0$, and p is any polynomial in the variables $\{\alpha_i | i \in \mathscr{D}\}$, $\mathscr{D}$ an arbitrary finite subset of the k's, then*

$$\lim_{\substack{A \nearrow \\ A \supset \mathscr{D}}} \langle p \rangle^A \equiv \langle p \rangle \tag{0.19}$$

exists. Here the limit is over an increasing sequence of sets containing $\mathscr{D}$ and exhausting all k's.

The center of our attention is a cluster expansion. Let $\mathscr{K}$ be a finite subset of the k's. Let $\mathscr{D}$ be the set of variables in a polynomial $p(\alpha)$, the distinguished variables. (We here let $\mathscr{D}$ be a certain subset of the k's, and also the set of variables they label.) Then we write

$$\langle p \rangle^{\mathscr{K}} = \sum_{\mathscr{D} \subset A \subset \mathscr{K}} K_A(p) Z^{\mathscr{K} - A} / Z^{\mathscr{K}}. \tag{0.20}$$

Formally we expand $[p]^{\mathscr{K}}$ into a perturbation expansion in λ, evaluate the Gaussian integrals, and factor each individual term (sums in I' left undone) into the product of the contribution of variables connected to $\mathscr{D}$ by the interactions and covariances arising in the integration, and the contribution of the complementary variables. This is the usual product into a connected contribution and a disconnected contribution in these variables. Connected contributions with connected variables A are resummed into $K_A(p)$. [Any term in (0.5) is understood to couple all variables in its labelling, whether or not all the variables are present.] Of course, $K_A(p)$ may be defined (and is in the paper) without the use of a perturbation expansion. Our main result is the following:

Theorem 0.2. *Let c_1 be fixed, then there is $\lambda_0(c_1) > 0$ such that if $0 \leqq \lambda \leqq \lambda_0(c_1)$ and p is any polynomial in variables in $\mathscr{D}$, then*

$$\sum_{\mathscr{D} \subset A} |K_A(p)| e^{c_1 |A|} \leqq c(p). \tag{0.21}$$

The paper is devoted to proving this result for the hierarchical ϕ_3^4 model. Standard machinery can then be used to obtain

$$\langle p \rangle = \sum_{\mathscr{D} \subset A} K_A(p) Z^{A^c} / Z, \tag{0.22}$$

where Z^{A^c}/Z is defined as the solution of a Kirkwood-Salsburg-like equation. There are no cutoffs in (0.22). Some generalization of this may be useful in the gauge theory situation where it is hard to find gauge-invariant cutoffs.

We will prove these results for the actual ϕ_3^4 in a future paper. There is an unfortunate error in [3]: Eq. (3.7) is incorrect, and thus also α-positivity as stated there. Actually [3] provides a proof for hierarchical models as defined above (with obvious changes from $d=3$ and $d=2$). The paper should be read excluding Sects. 6 and 7, and with u_k translated by $L_k \psi_k$. Proofs of α-stability and α-positivity are as in Sect. 1 here. Reference [3] is an important paper, although flawed.

Preparatory to reading this paper it is recommended that one become familiar with portions of [3] and [6] as follows:

1) Read Sects. 3 and 4 of [6]. This provides the construction of the ψ_k in a lattice situation. Then read Sect. 2 of [3] to see the continuum ψ_k, easily derived from the lattice situation.

2) Study well factorial estimate 11.B of [6]. This easy estimate plays a very central role in the proof of convergence.

3) Read Sect. 8 and Appendices A and B of [3]. Estimates with tree graphs, along with factorial estimate 11.B of [6], handle the combinatoric aspects of the convergence proof. These estimates are the most characteristic feature of phase cell cluster expansions, and have an intrinsic beauty of their own. They should be useful in other situations in field theory and statistical mechanics.

4) Now read the rest of [3] exclusive of Sects. 6 and 7 with the understandings mentioned above.

The extension of the work of the present paper to the treatment of the actual ϕ_3^4 model represents an increase in complexity, but no fundamental innovations are required. The present paper is an interesting warm-up for ϕ_3^4, but phase cell cluster expansions with diagonal C_{ij} as here are also of importance. The Yukawa model will be treated with a diagonal boson covariance. If one could find a basis ϕ_k for $L_2(R^3)$, labelled as our expansions functions, for which $u_k \equiv \frac{1}{\sqrt{-\Delta+M^2}}\phi_k$ satisfies I–IV above then the present paper would include the ϕ_3^4 model. This although unlikely should be investigated. (Some variations on these properties are permitted.)

1. $u_k = L_k \psi_k$ Satisfy I–IV

We study these properties but not in order.

I) Boundedness is immediate for this choice of u_k. In fact the right side of (0.12) may be set equal to zero.

III) We follow [3, Sect. 6] with the formal substitutions: $j \to 0$, $r \to 4$, $D^{1-j}\psi_k \to \frac{1}{L_k}\psi_k$. In fact with these substitutions the proof is considerably simpler and there is no kinetic term.

II) We first consider the normal orderings in (0.5)

$$:\alpha_4\alpha_5: = \alpha_4\alpha_5 - \delta_{45}, \tag{1.1}$$

$$\mathrm{Sym}\ :\alpha_1\alpha_2\alpha_3\alpha_4: = \alpha_1\alpha_2\alpha_3\alpha_4 - 6\delta_{34}\alpha_1\alpha_2 + 3\delta_{12}\delta_{34}, \tag{1.2}$$

where Sym in (1.2) indicates the relationship holds when substituted into expressions symmetric in $1,2,3,4$ as (0.5). We write the three expressions on the right side of (0.5) as E_1, E_2, E_3. We first note

$$E_3 \geqq 0. \tag{1.3}$$

We write

$$E_1 - \frac{\lambda}{2}\int \phi_A^4 = \frac{\lambda}{2}\int \phi_A^4 - \lambda \int a(x)\,\phi_A^2(x) + \lambda b, \tag{1.4}$$

$$E_2 = \lambda^2 \int c(x)\,\phi_A^2(x) - \lambda^2 d. \tag{1.5}$$

Clearly

$$b \geqq 0. \tag{1.6}$$

We would be through, by completing the square in the terms involving $\phi_A(x)$, if we could show

$$|d| \leqq c|A|, \tag{1.7}$$

$$\int |a(x)|^2 \leqq c|A|, \tag{1.8}$$

$$\int |c(x)|^2 \leqq c|A|. \tag{1.9}$$

Equation (1.8) is equivalent to

$$\sum_{1,2} L_1^2 L_2^2 \int \psi_1^2 \psi_2^2 \leqq c|A|, \tag{1.10}$$

where sums are understood to be restricted to A.

One easily has

$$\sum_{\substack{2 \\ L_2 \geqq L_1}} L_1^2 L_2^2 \int \psi_1^2 \psi_2^2 \leqq c L_1^{1-\varepsilon}, \tag{1.11}$$

from which (1.10) follows. Inequality (1.7) is equivalent to

$$\sum_{1,2,3,4} L_1^2 L_2^2 L_3^2 L_4^2 \int |\psi_1 \psi_2 \psi_3| \int |\psi_1 \psi_2 \psi_3 \psi_4^2| \leqq c|A|. \tag{1.12}$$

If in (1.12) one fixes one of the indices and sums over the others, but, as in (1.11), restricts the edge lengths to be greater than or equal to that of the fixed length, the sum is $\leqq c L^{2-\varepsilon}$, where L is the fixed edge length. This establishes (1.12). Equation (1.9) is equivalent to

$$\sum_{1,2,3,4,5,6} L_1^2 L_2^2 L_3^2 L_4^2 L_5^2 L_6^2 \int |\psi_1 \psi_2 \psi_3| \int |\psi_4 \psi_5 \psi_6| \int |\psi_1 \psi_2 \psi_3 \psi_4 \psi_5 \psi_6| \leqq c|A|. \tag{1.13}$$

This is established similarly, fixing a given index and summing with like edge size restrictions. Lieb has generalized the α-stability inequalities of [3] in [11].

IV) We look at

$$|w(1,2,3,4,5)\, w(1,2,3) - w(1,2,3,4)\, w(1,2,3,5)|, \tag{1.14}$$

and note that

$$(1.14) \leqq \lambda^2 \int dx \int dy \left| \prod_{i=1}^{4} (L_i \psi_i(x)) \prod_{j=1}^{3} (L_j \psi_j(y)) \right| L_5 |\psi_5(x) - \psi_5(y)|, \tag{1.15}$$

with the restrictions $L_1 \leqq L_2 \leqq L_3 < L_4, L_5$.

We have in turn

$$(1.15) \leqq c\lambda^2 L_1^6 \prod_{i=1}^{3} \left(\frac{1}{L_i}\right) \frac{1}{L_4^{1/2}} \frac{1}{L_5^{1/2}} \left(\frac{L_1}{L_5}\right). \tag{1.16}$$

We note that in (1.15) the integrand is zero unless x and y are in the same octant of the cube associated to ψ_5, in which ψ_5 is a pure polynomial.

It is interesting that none of the counterterms are necessary for stability; that the mass counterterm and the second order energy counterterm are necessary for convergence of the cluster expansion, but not the third order energy counterterm.

Although we have not carried through the details, it should not be difficult using the results of this paper to derive a form of ultraviolet bound if we include the third order energy counterterm. One would compute the partition function including all the α_k associated to cubes with centers in a given volume V and with edge size greater than some L_0 and obtain an upper bound of the form e^{cV}, c independent of L_0.

2. Basic Expansion Scheme

We introduce a multi-index notation as in [3, Sect. 10] so that (0.5) becomes

$$\sum_{\tau:\int\tau=4} g_1(\tau):\alpha^\tau: + \sum_{\substack{\tau',\tau'':\\ \int\tau'=2,\\ \int\tau''=3}} g_2(\tau',\tau''):\alpha^{\tau'}: + \sum_{\tau:\int\tau=4} g_3(\tau), \tag{2.1}$$

with $\int\tau \equiv \sum_{k\in\mathscr{K}} \tau(k)$ and (for $M(\tau)$ as in [3])

$$g_1(\tau) \equiv \lambda M(\tau)\int u^\tau, \tag{2.2}$$

$$g_2(\tau',\tau'') \equiv 48\lambda^2 M(\tau')\,M(\tau'')\left(\int u^{\tau'+\tau''}\right)\left(\int u^{\tau''}\right), \tag{2.3}$$

$$g_3(\tau) \equiv 12\lambda^2 M(\tau)\left(\int u^\tau\right)^2. \tag{2.4}$$

As in [3] the only ingredients of our cluster expansion are interpolation ((4.1) of [3]) and integration by parts ((4.2) of [3]). For p a polynomial in the cell variables, we expand $\langle p(\alpha)\rangle$ $(=\langle p(\alpha)\rangle^{\mathscr{K}}$, for notational simplicity) by repeated use of these two operations. The variables in $p(\alpha)$ are the *distinguished vertices*, and a *move* is a choice of a right hand side term of either (4.1) or (4.2) of [3]. A *unit*, defined slightly differently from in [3], is one of the kinds of objects we encounter in a move, specifically one of the following:

$$g_1(\tau):\alpha^\tau: \text{(or derivatives)},\ \int\tau=4, \tag{2.5}$$

$$g_2(\tau',\tau''):\alpha^{\tau'}: \text{(or derivatives)},\ \int\tau'=2,\ \int\tau''=3, \tag{2.6}$$

$$g_3(\tau),\ \int\tau=4, \tag{2.7}$$

each multiplied by some monomial in the interpolation parameters. Any unit of the form (2.5) [respectively (2.6) and (2.7)] will be called a *form 1* (respectively *form 2*, *form 3*) *unit*. After a given sequence of moves applied to $\langle p\rangle$, the *interior vertices* are the distinguished vertices together with the vertices appearing in units introduced as a result of all moves; the *exterior vertices* are the complementary set.

The first term in the right hand side of (4.1) of [3] is the *decoupled term*, and an expression obtained by some sequence of moves from $\langle p\rangle$ whose last move is an interpolation choosing the decoupled term is a *completed term*. Any other allowed sequence of moves yields a *remainder term*. Moves choosing a unit give rise to a remainder term (which represents a branch point in the expansion) and calls for another move.

With $\mathscr{K}$ a finite set, our rules for moves will dictate that the iterative construction eventually terminates, and we may collect completed terms to obtain

$$\langle p(\alpha)\rangle = \sum_G \frac{Z^{\mathscr{K}\setminus A_G}}{Z^{\mathscr{K}}} K_G(p). \tag{2.8}$$

The history of moves is indexed by G, and A_G is the set of interior vertices at the end of the history. Expression (0.22) is obtained from

$$\sum_{\substack{G \\ A_G = A}} K_G(p) = K_A(p). \tag{2.9}$$

3. Interpolation of the Interaction

The interpolation of the $:\phi^4:$ term in the interaction is identical to the treatment in [3, Sect. 5]. The interpolation of the remaining two terms in (0.5) is different, to facilitate the mass renormalization cancellations. The first interpolation has $\mathscr{D}$ the set of distinguished vertices as interior vertices. The interpolated interaction is

$$\begin{aligned} & s_1 \lambda \int :\phi(x)^4: dx + (1-s_1)\left(\lambda \int :\phi_{\mathscr{D}}(x)^4: dx + \lambda \int :\phi_{\mathscr{K}\setminus\mathscr{D}}(x)^4: dx\right) \\ & \quad + 48 \sum_{1,2,3} w(1,2,3) \int dx\, u_1(x) u_2(x) u_3(x) \\ & \quad \cdot :[s_1\phi(x) + (1-s_1)(\delta_{\mathscr{D}}^{123}\phi_{\mathscr{D}}(x) + \delta_{\mathscr{K}\setminus\mathscr{D}}^{123}\phi_{\mathscr{K}\setminus\mathscr{D}}(x))]^2: \\ & \quad + 12 s_1^2 \lambda^2 \int dx \int dy \langle \phi(x)\phi(y)\rangle_0^4 + (1-s_1^2)\left(12\lambda^2 \int dx \int dy \langle \phi_{\mathscr{D}}(x)\phi_{\mathscr{D}}(y)\rangle_0^4\right. \\ & \quad \left. + 12\lambda^2 \int dx \int dy \langle \phi_{\mathscr{K}\setminus\mathscr{D}}(x)\phi_{\mathscr{K}\setminus\mathscr{D}}(y)\rangle_0^4\right), \end{aligned} \tag{3.1}$$

where

$$\phi_A(x) \equiv \sum_{k\in A} u_k(x)\alpha_k, \tag{3.2}$$

and

$$\delta_A^{123} \equiv \begin{cases} 1, & \text{if} \quad 1,2,3 \in A \\ 0, & \text{otherwise}. \end{cases} \tag{3.3}$$

The first-order term, in $:\phi^4:$, is interpolated at all stages as in [3]. We now describe the succeeding interpolations of the second-order energy counterterm. Let $W(s_1, \ldots, s_{n-1})$ be the $(n-1)^{\text{st}}$ interpolation of the energy counterterm, and $I^n(O^n)$ be the set of interior (exterior) vertices at the onset of the move corresponding to the n^{th} interpolation. Here $W(s_1, \ldots, s_{n-1})$ is a convex combination of terms of the form

$12\lambda^2 \int dx \int dy \langle \phi_A(x)\phi_A(y)\rangle_0^4$ and the n^{th} interpolation replaces $W(s_1, \ldots, s_{n-1})$ with

$$W(s_1, \ldots, s_n) \equiv s_n^2 W(s_1, \ldots, s_{n-1}) + (1-s_n^2)\hat{W}(s_1, \ldots, s_{n-1}), \tag{3.4}$$

where $\hat{W}$ is obtained from W by the replacement

$$\begin{aligned} 12\lambda^2 \int dx \int dy \langle \phi_A(x)\phi_A(y)\rangle_0^4 \mapsto & \, 12\lambda^2 \int dx \int dy \langle \phi_{A\cap I^n}(x)\phi_{A\cap I^n}(y)\rangle_0^4 \\ & + 12\lambda^2 \int dx \int dy \langle \phi_{A\cap O^n}(x)\phi_{A\cap O^n}(y)\rangle_0^4. \end{aligned}$$

The $(n-1)^{\text{st}}$ interpolation of the second-order mass counterterm has the form

$$48 \sum_{1,2,3} w(1,2,3) \int dx\, u_1(x) u_2(x) u_3(x) :\phi_{123}(x; s_1, \ldots, s_{n-1})^2:, \tag{3.5}$$

where $\phi_{123}(x;s_1,\dots,s_{n-1})$ denotes a convex combination of terms of the form $\delta_A^{123}\phi_A(x)$. The n^{th} interpolation is obtained from (3.5) by replacing $\phi_{123}(x;s_1,\dots,s_{n-1})$ with

$$\phi_{123}(x;s_1,\dots,s_n)\equiv s_n\phi_{123}(x;s_1,\dots,s_{n-1})+(1-s_n)\hat{\phi}_{123}(x;s_1,\dots,s_{n-1}), \quad (3.6)$$

where $\hat{\phi}_{123}$ is obtained from ϕ_{123} by the replacement

$$\phi_A(x)\mapsto\delta^{123}_{A\cap I^n}\phi_{A\cap I^n}(x)+\delta^{123}_{A\cap O^n}\delta_{A\cap O^n}(x).$$

It is important for us that generalized α-stability holds for the interpolated interactions. If $0\leqq\alpha_i\leqq 1$ and $\sum\alpha_i=1$, then $(\sum\alpha_i\phi_i)^2\leqq\sum_i\alpha_i\phi_i^2$. From (3.1) the term

$$(s_1\phi+(1-s_1)(\delta^{123}_{\mathscr{D}}\phi_{\mathscr{D}}+\delta^{123}_{\mathscr{K}-\mathscr{D}}\phi_{\mathscr{K}-\mathscr{D}}))^2\leqq s_1\phi^2+(1-s_1)(\delta^{123}_{\mathscr{D}}\phi_{\mathscr{D}}+\delta^{123}_{\mathscr{K}-\mathscr{D}}\phi_{\mathscr{K}-\mathscr{D}})^2,$$

which with (3.1) and results in Sect. 1 yield generalized α-stability for the first interpolated interaction. This process iterates.

4. Integration by Parts

In this section we give the rules by which we decide when to integrate by parts; this will specify allowable histories of moves. First we collect some generalized integration by parts formulas that we will treat as generating single moves. Let $P(\alpha)$ be an arbitrary polynomial in the cell variables and let $v_1,\dots,v_m$ be vertices (not necessarily distinct) such that $\alpha_1\equiv\alpha_{v_1}$ does not appear in $P(\alpha)$. Then

$$[:\alpha_1\dots\alpha_m:P(\alpha)]=\left[:\alpha_2\dots\alpha_m:P(\alpha)\frac{\partial}{\partial\alpha_1}Q(\alpha)\right], \quad (4.1)$$

where

$$Q(\alpha)\equiv-\sum_{\tau:\int\tau=4}g_1(\tau):\alpha^\tau:-\sum_{\tau',\tau'':\int\tau'=2,\int\tau''=3}g_2(\tau',\tau''):\alpha^{\tau'}:, \quad (4.2)$$

and we have suppressed the interpolation parameters that are also brought down from the exponent, which is an interpolation of the interaction. Equation (4.1) is the same kind of formula that we used in [3], and in that case this was the only integration by parts that was necessary. To control the singular behavior of the hierarchical ϕ_3^4 model, however, we will sometimes need to integrate by parts in a more complicated way. Let v_2,v_3,v_2',v_3',v_4' be vertices such that the corresponding variables $\alpha_2,\alpha_3,\alpha_2',\alpha_3',\alpha_4'$ do not appear in $P(\alpha)$. Then

$$\begin{aligned}[:\alpha_2\alpha_3::\alpha_2'\alpha_3':P(\alpha)]&=\left[:\alpha_2\alpha_3:\alpha_3'P(\alpha)\frac{\partial}{\partial\alpha_2'}Q(\alpha)\right]\\&+\left[\left(\frac{\partial}{\partial\alpha_2'}:\alpha_2\alpha_3:\right)P(\alpha)\frac{\partial}{\partial\alpha_3'}Q(\alpha)\right]\\&+\left[\left(\frac{\partial^2}{\partial\alpha_2'\partial\alpha_3'}:\alpha_2\alpha_3:\right)P(\alpha)\right], \end{aligned}\quad (4.3)$$

$$[:\alpha_2\alpha_3::\alpha_2'\alpha_3'\alpha_4':P(\alpha)]=\left[:\alpha_2\alpha_3::\alpha_3'\alpha_4':P(\alpha)\frac{\partial}{\partial\alpha_2'}Q(\alpha)\right]$$
$$+\left[\left(\frac{\partial}{\partial\alpha_2'}:\alpha_2\alpha_3:\right)\alpha_4' P(\alpha)\frac{\partial}{\partial\alpha_3'}Q(\alpha)\right]$$
$$+\left[\left(\frac{\partial^2}{\partial\alpha_2'\,\partial\alpha_3'}:\alpha_2\alpha_3:\right)\alpha_4' P(\alpha)\right], \tag{4.4}$$

$$[\alpha_2:\alpha_2'\alpha_3'\alpha_4':P(\alpha)]=\left[\alpha_2:\alpha_3'\alpha_4':P(\alpha)\frac{\partial}{\partial\alpha_2'}Q(\alpha)\right]$$
$$+\left[\left(\frac{\partial}{\partial\alpha_2'}\alpha_2\right)\alpha_4' P(\alpha)\frac{\partial}{\partial\alpha_3'}Q(\alpha)\right]. \tag{4.5}$$

When we speak of a *term* on the right hand side of any one of these integration by parts formulae, we do so with the understanding that (4.2) has been plugged in. Notice that the last term on the right hand side of either (4.3) or (4.4) $\left(\text{assuming that } \frac{\partial}{\partial\alpha_2'\,\partial\alpha_3'}:\alpha_2\alpha_3:\neq 0\right)$ is the only term that does not involve the introduction of another unit. Any move that chooses such a term will be called a *mass insertion*; any other move involving (4.3) or (4.4) or any move involving (4.5) will be called a *Class 3 (b) move,* while any move involving (4.1) will be *Class 3 (a). Class 1* and *Class 2* moves are as in Sect. 9 of [3].

Now we adopt an arbitrary but fixed scale-lexicographic order on $\mathscr{K}$; i.e., if k precedes k', then $L_{k'}\leqq L_k$. If τ is a nonnegative-integer-valued function on $\mathscr{K}$ and $\int\tau=m$, let $(\tau_1,\ldots,\tau_m)$ denote the unique m-tuple of vertices which respects the reversal of our linear ordering and whose weight function is τ. (Hence $L_{\tau_1}\leqq\ldots\leqq L_{\tau_m}$.) We are now ready for a case-by-case description of our rule. As we have already indicated in the preceding section, the first move will always involve interpolation, so assume that we have made $n-1$ moves resulting in a remainder term.

There are many cases; the concept of the "form" of a move introduced after (2.7) is important. *Loosely speaking,* one integrates by parts some vertices that are newly introduced and of a smaller scale than previously introduced variables they are coupled to in the corresponding unit. This must be done at least often enough to allow mass inserts (of small scale variables) to develop. A set of rules that works is fairly complicated, as follows:

Case 1. The $(n-1)^{\text{st}}$ move chose a form 1 unit whose weight function τ has the property that $L_{\tau_1}<L_k$ for any $k\in\operatorname{supp}\tau$ that was interior at the onset of the $(n-1)^{\text{st}}$ move. Let σ be the restriction of τ to those vertices that share this property with τ_1. In this case we require the n^{th} move to involve an application of (4.1), where $m=\int\sigma$ and $v_i=\sigma_i$.

Case 2. The $(n-1)^{\text{st}}$ move chose a form 1 unit that violates Case 1, but the $(n-2)^{\text{nd}}$ move chose a form 1 unit that satisfies Case 1 with respect to that move. Let σ' and σ be the restrictions of the weight functions of our $(n-1)^{\text{st}}$ and $(n-2)^{\text{nd}}$ units, respectively, to those vertices whose scales are strictly smaller than L_k for any k in either unit that was interior at the onset of the $(n-2)^{\mathrm{nd}}$ move. By Case 1 our remainder term has the form

$$\left[:\alpha^{\sigma-\delta_1}:\left(\frac{\partial}{\partial\alpha_1}:\alpha^{\sigma'}:\right)P(\alpha)\right]=\sigma'(\sigma_1)\left[:\alpha^{\sigma-\delta_1}::\alpha^{\sigma'-\delta_1}:P(\alpha)\right], \tag{4.6}$$

where δ_1 is the delta function located at σ_1 and $P(\alpha)$ does not depend on any of the α_k for which $k\in\operatorname{supp}\sigma\cup\operatorname{supp}\sigma'$.

Case 2a. $\int\sigma+\int\sigma'<6$. In this case we require the n^{th} move to be either Class 1 or Class 2.

Class 2b. $\int\sigma+\int\sigma'\geqq 6$ and $\sigma'_1\neq\sigma_1$. Since $\sigma_1\in\operatorname{supp}\sigma'$ and is the last vertex in $\operatorname{supp}\sigma$, it follows that $\sigma'_1\notin\operatorname{supp}\sigma$. In this case we require the n^{th} move to involve (4.1), where we absorb $:\alpha^{\sigma-\delta_1}:$ into $P(\alpha)$ and set $v_i=\sigma'_i$.

Class 2c. $\int\sigma+\int\sigma'\geqq 6$ and $\sigma'_1=\sigma_1$. Since $\int\sigma<4$, we know that $\int\sigma'\geqq 3$, and since $\int\sigma'\leqq 4$, we know that $\int\sigma\geqq 2$. If we set $v_i=\sigma_i$ and $v'_i=\sigma'_i$, then the possibilities for (4.6) are precisely the left hand side of formulae (4.3)–(4.5). In each case we require the n^{th} move to involve the appropriate formula.

Case 3. The $(n-1)^{\mathrm{st}}$ move chose a form 2 unit whose pair (τ',τ'') of weight functions has the property that $L_{\tau'_1}<L_k$ for every $k\in\operatorname{supp}\tau'\cup\operatorname{supp}\tau''$ that was interior at the onset of the $(n-1)^{\mathrm{st}}$ move and τ''_3 precedes τ'_1 with respect to our order on vertices. In this case we require the n^{th} move to involve (4.1), where $v_i=\tau'_i$ and $m=1$ or 2 $\left(\text{depending on whether } \frac{\partial}{\partial\alpha_{\tau'_2}}:\alpha^{\tau'}: \text{ or } :\alpha^{\tau'}: \text{ appears in the unit}\right)$.

Case 4. The $(n-1)^{\mathrm{st}}$ move chose a form 2 unit whose pair (τ',τ'') of weight functions violates Case 3. In this case we require the n^{th} move to be either Class 1 or Class 2.

Case 5. The $(n-1)^{\mathrm{st}}$ unit chose a form 1 unit violating Case 1 and the $(n-2)^{\mathrm{nd}}$ move chose a form 2 unit whose pair (τ',τ'') of weight functions satisfies Case 3; assume further that $L_{\tau'_2}<L_{\tau''_1}$. Let σ (respectively σ') be the restriction of τ' (respectively weight function for the form 1 unit) to the set of vertices whose scales are strictly smaller than that of every vertex in either unit that was interior at the onset of the $(n-2)^{\mathrm{nd}}$ move. By Case 3 our remainder term has the form (4.6), and we consider exactly the sub-cases that were considered in Case 2. Our instructions for Cases 5a and b are exactly what they were for Cases 2a and b, respectively. In Case 5c the only possibility is $\int\sigma=2$ and $\int\sigma'=4$, and so we require the n^{th} move to involve (4.5), where $v_i=\sigma_i$ and $v'_i=\sigma'_i$.

Case 6. The $(n-1)^{\mathrm{st}}$ move either chose a form 3 unit or was a mass insertion. In this case we require the n^{th} move to be either Class 1 or Class 2.

Case 7. The $(n-1)^{\mathrm{st}}$ move chose a form 1 unit violating Case 1 and the $(n-2)^{\mathrm{nd}}$ move;

7a. Chose a form 1 unit violating Case 1,

7b. Chose a form 2 unit for which $L_{\tau'_1}\geqq L_{\tau''_1}$ and/or Case 3 is violated,

$7c$. Either chose a form 3 unit or was a mass insertion.
In all of these cases we require the n^{th} move to be either Class 1 or Class 2.
By inspection, we see that our rules cover all possibilities.

5. Representation 1 Graphs

Having defined our cluster expansion of the expectation $\langle p(\alpha)\rangle$ of an arbitrary polynomial $p(\alpha)$ in the cell variables, we write this expansion in terms of *representation 1 graphs* as we did for our expansion of expectations in [3]. In this case, however, the expansion rules do not admit a one-to-one correspondence between representation 1 graphs and completed terms of the expansion, so the terms that are possible for a given graph must be collected.

There is also more than one kind of basic graph to consider. In this analysis the *elementary graphs* are the nonnegative-integervalued functions τ on $\mathscr{K}$ such that $\int\tau=4$ *(form 1 graphs)*, the pairs (τ',τ'') of functions where $\int\tau'=2$ and $\int\tau''=3$ *(form 2 graphs)*, and the pairs $(\tau,3)$ such that $\int\tau=4$ *(form 3 graphs)*. Obviously we intend a form i graph to label some form i unit associated with that elementary graph. (Which such unit is labeled would be determined by what differentiations, if any, fall on the undifferentiated unit when the move introducing the graph and the move following that introduction take place.)

Definition 5.1. Let $\mathscr{D}$ be the set of distinguished vertices. A *representation 1 graph rooted on* $\mathscr{D}$ is a sequence $G\equiv(G_1,\ldots,G_N)$ of elementary graphs and sequences of elementary graphs with vertices assigned to them – called the *chains of* G – such that the following properties hold:

(a) The sequence $\tilde{G}\equiv(\tilde{G}^1,\ldots,\tilde{G}^n)$ of elementary graphs induced by G if one ignores the chain structure is introduced by some allowed sequence of moves corresponding to a completed term.

(b) Every elementary graph in G and the first elementary graph in every chain of G are introduced by Class 2 moves, and no vertices are assigned to such graphs.

(c) If an elementary graph lies in a chain of G and is not the first elementary graph in the chain, then it is introduced by either a Class 3(a) move or a Class 3(b) move, and the vertex assigned to it has been integrated by parts by that move.

Remark. Although mass insertions do not introduce elementary graphs, representation 1 graphs implicitly record such moves.

The expansion is

$$\langle p(\alpha)\rangle=\sum_G\prod_i(-g(G^i))\frac{Z^{\mathscr{K}\setminus A_G}}{Z^{\mathscr{K}}}\prod_l\left(\int_0^1 ds_l\right)\prod_{k\in A_G}\left(\int_{-\infty}^{\infty}d\alpha_k\right)$$
$$\cdot\exp\left(-\tfrac{1}{2}\sum_{k\in A_G}\alpha_k^2\right)\prod_l B_G^l(\alpha)\prod_l b_G^l(s)\,p(\alpha)e^{-V_G(s,\alpha)},\tag{5.1}$$

where the summation is over all representation 1 graphs G, $g(\tilde{G}^i)$ is one of $g_1(\tau)$, $g_2(\tau',\tau'')$, or $g_3(\tau)$, $b_G^l(s)$ is the product of interpolation parameters brought down from the exponent by the move or sequence of moves introducing G_l (and respects equivalence classes of completed terms because interpolation moves are uniquely determined by the elementary graphs they introduce), $V_G(s,\alpha)$ is the form of the interaction for the completed term (and also respects equivalence classes because it has only been interpolated), and $B^l_G(\alpha)$ is the combination of all (equivalent) products and differentiated products of cell variables that are generated by the moves or sequences of moves introducing the same G_l.

The $b^l_G(s)$ and $B^l_G(\alpha)$ are implicitly defined by our expansion rules. Since most of these factors will be over-estimated in a gross manner, we will not compute the expressions for them except in those cases where they affect the combinatorics of the second-order renormalization cancellation.

6. Combinatorics of the Mass Cancellation

Before beginning our estimation of (5.1) we must group together the terms whose ultraviolet (small scale) divergences will cancel against one another. This is the combinatoric content of the second-order mass and energy renormalization.

Definition 6.1. Let G be a representation 1 graph rooted on $\mathscr{D}$. A *type 1 (respectively type 2) composite graph of* G is a pair $(\tilde{G}^i, \tilde{G}^{i+1})$ of consecutive elementary graphs in $\tilde{G}$ such that $\tilde{G}^i$ is form 1 (respectively form 2), $\tilde{G}^{i+1}$ is form 1, and the introduction of $\tilde{G}^{i+1}$ has given rise to Case 2 (respectively Case 5).

Remark. The elementary graphs in a composite graph of G clearly occur in a chain of G.

Definition 6.2. For a given representation 1 graph G rooted on $\mathscr{D}$, an *exact graph of* G is type 1 composite graph $(\tilde{G}^i, \tilde{G}^{i+1})$ of G such that $\tilde{G}^i_1 = \tilde{G}^{i+1}_1$, $\tilde{G}^i_2 = \tilde{G}^{i+1}_2$, $\tilde{G}^i_3 = \tilde{G}^{i+1}_3$, the scale of $\tilde{G}^i_3$ is strictly less than that of any vertex in $(\operatorname{supp}\tilde{G}^i \cup \operatorname{supp}\tilde{G}^{i+1}) \cap \left(\mathscr{D} \cup \bigcup_{j<i} \operatorname{supp}\tilde{G}^j\right)$, and $\tilde{G}^{i+1}$ is the last elementary graph of the chain.

Definition 6.3. A *local graph of* G is a form 2 graph $\tilde{G}^i = (\tau', \tau'')$ in $\tilde{G}$ such that τ'_1 precedes τ''_3 and $L_{\tau''_3}$ is strictly less than the scale of any vertex in $\operatorname{supp}\tilde{G}^i \cap \left(\mathscr{D} \cup \bigcup_{j<i} \operatorname{supp}\tilde{G}^j\right)$.

We now let $\hat{G}$ denote the structure obtained from G by considering the composite graphs of G as single elements. We also consider the following operations on $\{\hat{G}\}$:

(a) Replace an exact graph (τ, σ) of $\hat{G}$ with the form 2 graph (τ', τ'') such that $\tau''_1 = \tau_1$, $\tau''_2 = \tau_2$, $\tau''_3 = \tau_3$, and τ' is the weight function of the pair (τ_4, σ_4) of vertices. The resulting graph is $\widehat{G'}$, where G' is representation 1 and (τ', τ'') is a local graph of $\widehat{G'}$.

(b) Replace a local graph (τ', τ'') of $\hat{G}$ with a pair (τ, σ) of form 1 graphs such that $\tau_i = \sigma_i = \tau''_i$, $i = 1, 2, 3$, τ' is the weight function of (τ_4, σ_4), and τ_4 is an interior vertex at the onset of the move introducing (τ', τ''). The resulting graph is $\widehat{G''}$, where G'' is representation 1 and (τ, σ) is an exact graph of $\widehat{G''}$.

(c) If in case (a) [respectively case (b)] $\tau = \sigma$ (respectively $\tau' + 2\tau'' = 2\tau$) and the exact graph (respectively local graph) is the first element in a chain of $\hat{G}$ or occurs in $\hat{G}$, then replace it with the form 3 graph element $(\tau, 3)$.

(d) If a form 3 graph $(\tau, 3)$ has the same scale properties as an exact graph, then either replace it with the exact graph (τ, τ) or replace it with the corresponding local graph.

Now $\Gamma_{\hat{G}}$ is the structure obtained from $\hat{G}$ by replacing each exact, local, or case (d) form 3 graph of $\hat{G}$ with the set of all graphs that operations on $\hat{G}$ can replace it by. Such a set is called a *cancellation graph* of $\Gamma_{\hat{G}}$. We are grouping together terms that will combine to exhibit the mass renormalization cancellations. [A cancellation graph may contain one or two exact graphs and possibly a form 3 graph in addition to a local graph. If $(\tilde{G}^i, \tilde{G}^{i+1})$ is both an exact graph of $\Gamma_{\hat{G}}$ and a chain of G and $\tilde{G}_4^{i+1} \in \mathscr{D} \cup \bigcup_{j<1} \operatorname{supp} \tilde{G}^j$, then $(\tilde{G}^{i+1}, \tilde{G}^i)$ is also an exact graph of $\Gamma_{\hat{G}}$.]

Definition 6.4. A *representation* $1\frac{1}{2}$ *graph* rooted on $\mathscr{D}$ is $\Gamma_{\hat{G}}$ for some representation 1 graph G rooted on $\mathscr{D}$.

We may rewrite (5.1) as

$$\langle p(\alpha)\rangle = \sum_{\Gamma} \frac{Z^{\mathscr{K}\setminus A_\Gamma}}{Z^{\mathscr{K}}} \prod_l \left(\int_0^1 ds_l\right) \prod_{k\in A_\Gamma} \left(\int_{-\infty}^{\infty} d\alpha_k\right) \exp\left(-\tfrac{1}{2}\sum_{k\in A_\Gamma} \alpha_k^2\right)$$

$$\cdot p(\alpha) e^{-V_\Gamma(s,\alpha)} \sum_{G:\Gamma_{\hat{G}}=\Gamma} \prod_i (-g(\tilde{G}^i)) \prod_l B_G^l(\alpha) \prod_l b_G^l(s). \tag{6.1}$$

We are now ready to state the fundamental combinatoric result.

Theorem 6.5. *Let* $\Gamma \equiv (\Gamma_1, \ldots, \Gamma_N)$ *be a representation* $1\frac{1}{2}$ *graph rooted on* $\mathscr{D}$. *For* $1 \leqq l \leqq N$ *there are polynomials* $\mathscr{B}_\Gamma^l(\alpha)$ *in the* α_k *and monomials* $\mathscr{b}_\Gamma^l(s)$ *in the interpolation parameters such that*

$$\sum_{G:\Gamma_{\hat{G}}=\Gamma} \prod_i (-g(\tilde{G}^i)) \prod_l B_G^l(\alpha) \prod_l b_G^l(s) = \prod_l \mathscr{B}_\Gamma^l(\alpha) \prod_l \mathscr{b}_\Gamma^l(s) \prod_\mu \mathscr{I}(\tilde{\Gamma}^\mu), \tag{6.2}$$

where $\tilde{\Gamma}$ *is the sequence of elementary graphs, composite graphs, and cancellation graphs induced by* Γ *if the chain structure is ignored, and*

$$\mathscr{I}(\tilde{\Gamma}^\mu) = \begin{cases} -g(\tilde{\Gamma}^\mu), & \tilde{\Gamma}^\mu \text{ is an elementary graph}, \\ g(\tilde{G}^i)\, g(\tilde{G}^{i+1}), & \tilde{\Gamma}^\mu \text{ is a composite graph } (\tilde{G}^i, \tilde{G}^{i+1}) \text{ of } \Gamma, \\ \lambda^2 \int u^\sigma \int u^\tau - \lambda^2 \int u^{\tau'+\tau''} \int u^{\tau''}, & \tilde{\Gamma}^\mu \text{ is a cancellation graph of } \Gamma. \end{cases} \tag{6.3}$$

Remark. In the definition of $\mathscr{I}(\tilde{\Gamma}^\mu)$ it is understood that (τ', τ'') is the local graph and that (τ, σ) [respectively (τ, σ) and (σ, τ)] is (respectively are) the corresponding exact graph (respectively exact graphs).

The proof of this theorem is the content of Appendix A, and a crucial role is played by the way we have chosen to interpolate the mass counterterm. Suppose that we are building a completed term for whose representation 1 graph G we have $\Gamma_{\hat{G}} = \Gamma$; suppose that we have done $n-1$ interpolation moves and are confronted with the n^{th} interpolation. The set of interior vertices at the onset of this interpolation is given by

$$I^n = \mathscr{D} \cup \bigcup_{l=1}^{n-1} \operatorname{supp} G_l = \mathscr{D} \cup \bigcup_{l=1}^{n-1} \operatorname{supp} \Gamma_l \tag{6.4}$$

while a form 2 unit $g_2(\tau', \tau'') : \alpha^{\tau'}$: in the exponent is multiplied by some product of the interpolation parameters $s_1, \ldots, s_{n-1}$. By inspection of (3.3), (3.6), and (3.7), we see that (assuming that $\operatorname{supp}\tau'' \subset O^n$)

(a) if $\operatorname{supp}\tau' \subset I^n$, then the n^{th} interpolation introduces an s_n^2 factor for the unit,

(b) if $\operatorname{supp}\tau'$ meets both I^n and O^n, then the n^{th} interpolation introduces an s_n factor,

(c) if $\operatorname{supp}\tau' \subset O^n$, then the n^{th} interpolation introduces no factor.

We are assuming that $\operatorname{supp}\tau'' \subset O^n$ because this is the only case in which (τ', τ'') can possibly be a local graph of G. It is obvious from (6.3) that there is no combinatoric work to do unless there is a cancellation graph involved at a given stage.

7. α-Stability, α-Positivity, and the Tree-Graph Estimation

Combining (6.1) with (6.2), we obtain

$$\langle p(\alpha) \rangle = \sum_{\Gamma} \prod_{\mu} \mathscr{I}(\tilde{\Gamma}^{\mu}) \frac{Z^{\mathscr{K}-A}}{Z^{\mathscr{K}}} H_{\Gamma}(p), \tag{7.1}$$

where the sum is over all representation $1\frac{1}{2}$ graphs rooted on $\mathscr{D}$, $\mathscr{I}(\tilde{\Gamma}^{\mu})$ is defined by (6.3), and

$$H_{\Gamma}(p) \equiv \prod_{l} \left(\int_0^1 ds_l \right) \prod_{k \in A_{\Gamma}} \left(\int_{-\infty}^{\infty} d\alpha_k \right) \exp\left(-\tfrac{1}{2} \sum_{k \in A_{\Gamma}} \alpha_k^2 \right) \cdot e^{-V_{\Gamma}(s,\alpha)} p(\alpha) \prod_{l} \mathscr{B}_{\Gamma}^{l}(\alpha) \prod_{l} \mathscr{C}_{\Gamma}^{l}(s). \tag{7.2}$$

The basic idea is to apply this expansion to the problem of showing that for sufficiently small λ the expectation $\langle p(\alpha) \rangle$ converges as the cut-off in volume and scale represented by $\mathscr{K}$ is removed. As we pointed out in [3], the crux of such a problem is to prove:

Theorem 7.1. *There is an $\varepsilon > 0$ and a $\lambda_0 > 0$ such that for $0 \leqq \lambda \leqq \lambda_0$,*

$$\sum_{\Gamma} \prod_{\mu} |\mathscr{I}(\tilde{\Gamma}^{\mu})|^{1-\varepsilon} |H_{\Gamma}(p)| \leqq c.$$

The proof of this theorem is the content of the sequel. As in [3], the preliminary step is to apply α-stability and α-positivity, which are preserved under interpolation. Combining this with the Schwarz inequality, we see that

$$|H_{\Gamma}(p)| \leqq \langle p(\alpha)^2 \rangle_0^{1/2} e^{c|A_{\Gamma}|} \left[\prod_{l} \mathscr{B}_{\Gamma}^{l}(\alpha)^2 \right]_{A_{\Gamma}}^{1/2} \prod_{l} \left(\int_0^1 ds_l \right) \prod_{l} \mathscr{C}_{\Gamma}^{l}(s), \tag{7.3}$$

where $\langle \ \rangle_0$ is the free ($\lambda = 0$) expectation and $[\ ]_A$ is the expectation with respect to the measure

$$\exp\left(-\tfrac{1}{4} \sum_{k \in A} \alpha_k^2 - c\lambda \sum_{k \in A} L_k^{1+\varepsilon'} |\alpha_k|^{4-\varepsilon'} \right) \prod_{k \in A} d\alpha_k. \tag{7.4}$$

Now we define the nonnegative-integer-valued function $\hat{\Gamma}^\mu$ by

$$\hat{\Gamma}^\mu = \begin{cases} \tilde{\Gamma}^\mu, & \tilde{\Gamma}^\mu \text{ is a form 1 graph} \\ \tau', & \tilde{\Gamma}^\mu \text{ is a form 2 graph } (\tau', \tau''), \\ 0, & \tilde{\Gamma}^\mu \text{ is a form 3 graph,} \\ \tau', & \tilde{\Gamma}^\mu \text{ is a cancellation graph of } \Gamma \text{ with } (\tau', \tau'') \text{ as local graph,} \\ \tau+\sigma, & \tilde{\Gamma}^\mu \text{ is a type 1 composite graph } (\tau, \sigma), \\ \tau'+\sigma, & \tilde{\Gamma}^\mu \text{ is a type 2 composite graph } ((\tau', \tau''), \sigma). \end{cases} \tag{7.5}$$

It follows from inspection of cases that

$$\prod_l \mathscr{B}_\Gamma^l(\alpha)^2 \leqq c^{n_\Gamma} \hat{\alpha}^{2\sum_\mu \hat{\Gamma}^\mu}, \tag{7.6}$$

where

$$\hat{\alpha}_k \equiv \max\{1, |\alpha_k|\} \tag{7.7}$$

and n_Γ is the length of the sequence $\tilde{\Gamma}$. Hence

$$|H_\Gamma(p)| \leqq \langle p(\alpha)^2 \rangle_0^{1/2} e^{cn_\Gamma} \left[\hat{\alpha}^{2\sum_\mu \hat{\Gamma}^\mu}\right]_{A_\Gamma}^{1/2} \prod_l \left(\int_0^1 ds_l\right) \prod_l \mathscr{C}_\Gamma^l(s) \tag{7.8}$$

because $|A_\Gamma| \leqq |\mathscr{D}| + 8n_\Gamma$.

The next step involves the general tree graph identity that was stated and proven in [3]. The point is that

$$b_\Gamma^l(s) \leqq s_{\eta_\Gamma(l)} \dots s_{l-1}, \tag{7.9}$$

where η_Γ is the tree graph associated with the sequence $(\operatorname{supp}\Gamma_1, \dots, \operatorname{supp}\Gamma_N)$ rooted on $\mathscr{D}$. This inequality follows from throwing away the unnecessary factors in the monomial $b_\Gamma^l(s)$ via the inequality $0 \leqq s_i \leqq 1$.

Definition 7.2. A *representation 2 graph* rooted on $\mathscr{D}$ is the set of elements occurring in some representation $1\frac{1}{2}$ graph rooted on $\mathscr{D}$.

For a given representation $1\frac{1}{2}$ graph Γ we will denote the associated representation 2 graph by T_Γ; it follows from Theorem B.3 of [3] that for a given representation 2 graph T rooted on $\mathscr{D}$,

$$\sum_{\Gamma: T_\Gamma = T} \prod_l \left(\int_0^1 ds_l\right) \prod_l (s_{\eta_\Gamma(l)} \dots s_{l-1}) \leqq 1. \tag{7.10}$$

We have an inequality instead of an identity because of the restrictions on possible graphs imposed by our rules for integrating by parts. Now, in view of (7.8)–(7.10) we see that Theorem 7.1 will follow from:

Lemma 7.3. *There is an* $\varepsilon > 0$ *and a* $\lambda_0 > 0$ *such that for* $0 \leqq \lambda \leqq \lambda_0$,

$$\sum_T \prod_\mu |\mathscr{I}(\tilde{\Gamma}_T^\mu)|^{1-\varepsilon} e^{cn_T} \left[\hat{\alpha}^{2\sum_\mu \hat{\Gamma}^\mu}\right]_{A_T}^{1/2} \leqq c,$$

where the sum is over all representation 2 graphs rooted on $\mathscr{D}$, Γ_T *is an arbitrary but fixed representative of* T, *and* $A_T \equiv A_{\Gamma_T}$, $n_T \equiv n_{\Gamma_T}$.

8. Attachments and "New Variables"

Throughout our proof of Lemma 7.3 our arbitrary choice Γ_T for each representation 2 graph T will remain fixed. For a given element $\Gamma_{T,l}$ of T we define the *attachment of $\Gamma_{T,l}$ (relative to T)* as the last vertex in $\operatorname{supp}\Gamma_{T,l}\cap\left(\mathscr{D}\cup\bigcup_{j<l}\operatorname{supp}\Gamma_{T,j}\right)$ with respect to our scale-lexicographic order on the vertices, unless $\Gamma_{T,l}$ is a chain for which (with $\tilde{\Gamma}_T^\mu$ the last element in the chain) the set

$$\operatorname{supp}\tilde{\Gamma}_T^\mu\cap\left(\mathscr{D}\cup\bigcup_{j<l}\operatorname{supp}\Gamma_{T,j}\right) \tag{8.1}$$

is non-empty. In this case the attachment is the last vertex in

$$(\operatorname{supp}\tilde{\Gamma}_T^{\mu-1}\cup\operatorname{supp}\tilde{\Gamma}_T^\mu)\cap\left(\mathscr{D}\cup\bigcup_{j<l}\operatorname{supp}\Gamma_{T,j}\right).$$

Chains with non-empty set (8.1) are called "extraordinary," other chains "ordinary."

Definition 8.1. For a given representation 2 graph T rooted on $\mathscr{D}$, let $(\tilde{\Gamma}_T^{m+1},\ldots,\tilde{\Gamma}_T^{m+n})$ be a chain of Γ_T and let v be the vertex to which this element of T is attached relative to T; set

$$t=\max\{j\,|\,m<j\leqq m+n,\, v\in\operatorname{supp}\tilde{\Gamma}_T^j\}.$$

Then $\tilde{\Gamma}_T^t$ is the *attachment graph* of the chain.

Our notion of attachment graph is identical to that used in [3]; we must also introduce a notion of *new variable* as we did in [3] – i.e., nonnegative-integer-valued functions σ_T^μ for $1\leqq\mu\leqq n_T$. If $\tilde{\Gamma}_T^\mu$ is the attachment graph or the last element of a chain $\Gamma_{T,l}$ of Γ_T, then

$$\sigma_T^\mu(k)\equiv\begin{cases}\hat{\Gamma}_T^\mu(k), & k\notin\mathscr{D}\cup\bigcup_{j<l}\operatorname{supp}\Gamma_{T,j},\\ 0, & \text{otherwise}.\end{cases} \tag{8.2}$$

Otherwise,

$$\sigma_T^\mu(k)\equiv\begin{cases}\hat{\Gamma}_T^\mu(k), & k\notin\mathscr{D}\cup\bigcup_{j<\mu-1}\operatorname{supp}\tilde{\Gamma}^j,\\ 0, & \text{otherwise}.\end{cases} \tag{8.3}$$

Old variables are the complementary set to the new variables.

Lemma 8.2. *For a given representation 2 graph T rooted on $\mathscr{D}$,*

$$\sum_{\mu=1}^{n_T}\sigma_T^\mu(k)\leqq c, \tag{8.4}$$

$$\int(\hat{\Gamma}_T^\mu-\sigma_T^\mu)\leqq\begin{cases}6, & \tilde{\Gamma}_T^\mu \text{ is composite}\\ 3, & \text{otherwise}.\end{cases} \tag{8.5}$$

Since the rules for integrating by parts are more complicated here than in [3], we will find it convenient to supplement the notion of "new variable" represented by σ_T^μ with an additional nonnegative-integer-valued function χ_T^μ defined as follows: if $\tilde{\Gamma}_T^\mu$ either occurs in Γ_T or occurs as the first element of a chain of Γ_T, then $\chi_T^\mu \equiv 0$, and otherwise, (with $\Gamma_{T,l}$ as the chain)

$$\chi^\mu(k) \equiv \begin{cases} \hat{\Gamma}_T^\mu(k) - \sigma_T^\mu(k), & k \notin \mathscr{D} \cup \bigcup_{m<l} \operatorname{supp} \Gamma_{T,m} \text{ and } L_k \text{ is strictly less} \\ & \text{than the scale of the vertex that was integrated by parts} \\ & \text{by the move introducing } \tilde{\Gamma}_T^\mu, \\ 0, & \text{otherwise}. \end{cases}$$

Lemma 8.3. *For a given representation 2 graph T rooted on $\mathscr{D}$,*

$$\sum_{\mu=1}^{n_T} \chi_T^\mu(k) \leqq c. \tag{8.6}$$

Let

$$\Omega_T(k) \equiv \sum_{\mu=1}^{n_T} [\hat{\Gamma}_T(k) - \sigma_T^\mu(k) - \chi_T^\mu(k)]\,; \tag{8.7}$$

it follows from (8.5) that

$$\int \Omega_T \leqq c n_T. \tag{8.8}$$

As in [3], we apply the Schwarz inequality to estimate out the integration variables contributed by the weights subtracted out in (8.7) with the free part of the interaction represented by the product measure $[\ ]_{A_T}$ and estimate out the other variables with the other part of the exponent. By (7.4), (8.4), (8.6), and (8.7), we have

$$\begin{aligned} \sum_T \prod_\mu |\mathscr{I}(\tilde{\Gamma}_T^\mu)|^{1-\varepsilon} e^{cn_T} [\hat{\alpha}^{2\sum_\mu \hat{\Gamma}_T^\mu}]_{A_T}^{1/2} \leqq \sum_T c^{n_T} \prod_\mu |\mathscr{I}(\tilde{\Gamma}_T^\mu)|^{1-\varepsilon} \\ \cdot \prod_k [(\Omega_T(k)!)^\beta \, (L_k^{-\beta'} \lambda^{-\beta'})^{\Omega_T(k)}], \end{aligned} \tag{8.9}$$

where $\beta' \equiv \dfrac{1+\varepsilon'}{4-\varepsilon'}$, $\beta \equiv \dfrac{1}{4-\varepsilon'}$. The right hand side is dominated by

$$\sum_T c^{n_T} \lambda^{n_T \varepsilon''} \prod_\mu \mathscr{L}(\tilde{\Gamma}_T^\mu)^{1-\varepsilon} \prod_k [L_k^{-\beta' \Omega_T(k)} (\Omega_T(k)!)^\beta], \tag{8.10}$$

where $\mathscr{L}(\tilde{\Gamma}_T^\mu)$ denotes

(a) the bound (3.6) of [3] on $\int \mu^\tau$ if $\tilde{\Gamma}_T^\mu$ is a form 1 graph τ,

(b) the corresponding bound on $\mu^{\tau'+\tau''} \int \mu^{\tau''}$ if $\tilde{\Gamma}_T^\mu$ is a form 2 graph (τ', τ''),

(c) $\mathscr{L}(\tau)^2$ if $\tilde{\Gamma}_T^\mu$ is a form 3 graph (τ, τ),

(d) the bound (0.15) on $\int \mu^\tau \int \mu^\sigma - \int \mu^{\tau'+\tau''} \int \mu^{\tau''}$ if $\tilde{\Gamma}_T^\mu$ is a cancellation graph with local graph (τ', τ'') and exact graph (τ, σ) and/or (σ, τ).

(e) $\mathscr{L}(\tilde{G}^1)\mathscr{L}(\tilde{G}^2)$ if $\tilde{\Gamma}_T^\mu$ is a composite graph $(\tilde{G}^1, \tilde{G}^2)$. The powercounting for λ has worked as follows: if $\tilde{\Gamma}_T^\mu$ is not a composite graph, then $\int (\hat{\Gamma}_T^\mu - \sigma_T^\mu - \chi_T^\mu) \leqq 3$ by (8.5), so the contribution to the power of λ is $\lambda^{1-\varepsilon}\lambda^{-3\beta'}$; if $\tilde{\Gamma}_T^\mu$ is a composite graph, then (8.5) gives only $\int (\hat{\Gamma}_T^\mu - \sigma_T^\mu - \chi_T^\mu) \leqq 6$, but $\mathscr{I}(\tilde{\Gamma}_T^\mu)$ contributes λ^2, so the power of λ is $\lambda^{2-2\varepsilon}\lambda^{-6\beta'}$ in this case.

We have reduced the problem of proving Lemma 7.3 to that of bounding (8.10) with a constant independent of the cut-off in scale and volume. Now for a given representation 2 graph T rooted on $\mathscr{D}$, consider the sequence of attachments relative to T induced by $\Gamma_T \equiv (\Gamma_{T,1}, \ldots, \Gamma_{T,N})$ and extract the subsequence of first appearances. For each such vertex list every element of T attached to that vertex in some arbitrary order and adjoin the sequences in the order that their attachments appear in the subsequence. Such a re-ordering of $(\Gamma_{T,1}, \ldots, \Gamma_{T,N})$ is a *representation 3 graph rooted on* $\mathscr{D}$. (As in [6], we could have defined a representation 3 graph as a set of sequences instead of an adjunction of them, because the ordering of the elements of T attached to a given vertex is the ordering that we really wish to introduce. The "first appearance" ordering is just a convenience, as it was in [3].)

For a given representation 3 graph J rooted on $\mathscr{D}$, let T_J denote the unique representation 2 graph associated with it and let $r_J(k)$ denote the number of elements of T_J attached to k relative to T_J. Obviously, $\prod_k r_J(k)!$ is the number of representation 3 graphs that T_J is associated with, so we may trivially rewrite (8.10) as

$$\sum_J c^{n_J} \frac{1}{\prod_k r_J(k)!} \lambda^{n_J \varepsilon''} \prod_\mu \mathscr{L}(\tilde{J}^\mu)^{1-\varepsilon} \prod_k [L_k^{-\beta' \Omega_J(k)} (\Omega_J(k)!)^\beta], \tag{8.11}$$

where $n_J \equiv n_{T_J}$, $\Omega_J \equiv \Omega_{T_J}$, and $\tilde{J}$ is the sequence of elementary graphs, composite graphs, and cancellation graphs induced by J if we ignore the chain structure (and is therefore a permutation of $\tilde{\Gamma}_{T_J}$).

9. Counting (Including the Number Divergence)

In this section we carry through the final numerical estimations, quite as in Sects. 16 and 17 of [3]. The nomenclature we have introduced in this process (attachment, pinning, new variable, etc.) is quite natural; we feel anyone studying the "topological" description of terms in the cluster expansion would create similar concepts.

9.1. Pinning and the Hook

In this subsection we define "pinning" and the "hook"; the first of these is a generalization of "pinning" as in [3], the second a new complication we did not need to consider there.

We here fix a representation 3 graph J. Let $(\tau_J^1, \ldots, \tau_J^{n_J})$ [respectively $(\varrho_J^1, \ldots, \varrho_J^{n_J})$] be the permutation of $(\hat{\Gamma}_{T_J}^1, \ldots, \hat{\Gamma}_{T_J}^{n_J})$ [respectively $(\sigma_{T_J}^1 + \chi_{T_J}^1, \ldots, \sigma_{T_J}^{n_J} + \chi_{T_J}^{n_J})$] whose action takes $\tilde{\Gamma}_{T_J}$ to $\tilde{J}$. We first define the "hook":

Definition 9.1.1. Let J be a representation 3 graph rooted on the set $\mathscr{D}$ of distinguished vertices and consider a chain $(\tilde{J}^{m+1}, \ldots, \tilde{J}^{m+n}) = (\tilde{\Gamma}_{T_J}^{s+1}, \ldots, \tilde{\Gamma}_{T_J}^{s+n})$ of J (i.e., of Γ_{T_J}). The *hook* of the chain is the last vertex in $\operatorname{supp} \tilde{J}^{m+n} \cap \bigcup_{\mu=m+1}^{m+n-1} \operatorname{supp} \tilde{J}^\mu$ with respect to the scale-lexicographic order on vertices. If v is the hook of the chain and

$$t = \min\{\mu \mid m < \mu < m+n, v \in \operatorname{supp} \tilde{J}^\mu\},$$

then $\tilde{J}^t$ is the *hook graph* of the chain.

We define the *pinning* of $\tilde{J}^\mu$ as the vertex $p_\mu \in \operatorname{supp} \tilde{J}^\mu$ such that

(I) If $\tilde{J}^\mu$ occurs in J, then p_μ is the attachment of $\tilde{J}^\mu$.

(II) If $\tilde{J}^\mu$ occurs in a chain of J and

(IIa) if $\tilde{J}^\mu$ is the attachment graph of the chain, then p_μ is the attachment of the chain.

(IIb) If $\tilde{J}^\mu$ is the hook graph in a chain when the last graph in the chain is the attachment graph, then p_μ is the hook of the chain.

(IIc) If $\tilde{J}^\mu$ violates (IIb) and if $\tilde{J}^\mu$ occurs before the attachment and/or after the hook graph, but is not the last graph, then p_μ is the vertex that was integrated by parts giving rise to $\tilde{J}^{\mu+1}$.

(IId) If $\tilde{J}^\mu$ is the last element of the chain and is *not* the attachment graph of the chain, then p_μ is the hook of the chain.

(IIe) $\tilde{J}^\mu$ violates (IIa)–(IIc), then p_μ is the vertex in $\tilde{J}^{\mu-1}$ that was integrated by parts giving rise to $\tilde{J}^\mu$.

We also define the *bottom* of $\tilde{J}^\mu$ as the last vertex in $\operatorname{supp} \tilde{J}^\mu$ with respect to the scale lexicographic ordering on vertices.

Lemma 9.1.2. *No vertex can be the pinning of more than one case* (IIb)–(IIe) *graph. Moreover no vertex can be the bottom of more than one composite graph.*

Lemma 9.1.3. L_{p_μ} *is less than or equal to the scale size of every vertex in* $\operatorname{supp}(\tau_J^\mu - \varrho_J^\mu)$.

9.2. The Number Divergence – An Abstract Discussion

For a fixed representation 3 graph J we assume a set of positive numerical factors $f(k,j)$, $j, k \in \mathscr{K}$ satisfying $\sum_k f(k,j) \leq c$. Let $(k_1, \dots, k_N)$ be a sequence of vertices such that the first $\int(\tau_J^1 - \varrho_J^1)$ vertices have weight function $\tau_J^1 - \varrho_J^1$, the next $\int(\tau_J^2 - \varrho_J^2)$ vertices have weight function $\tau_J^2 - \varrho_J^2$, etc. Each "occurrence" of a vertex in this sequence is said to "occur" in $\tilde{J}^\mu$ if the occurrence is in the segment associated with $\tau_J^\mu - \varrho_J^\mu$. We have a mapping $\mathrm{NDF}: \{1, \dots, N\} \to (0, \infty)$ of a special nature so that any occurrence of vertex j has as an image an element of the set

$$S_j \equiv \{f(k,j) \mid k \in \mathscr{K}\} \cup \{f(k,j)^{1/2} f(k',j)^{1/2} \mid k, k' \in \mathscr{K}\}.$$

An occurrence of vertex j is said to be *singly bound* to k if its image is $f(k,j)$ and *jointly bound* to k and k' if its image is $f(k,j)^{1/2} f(k',j)^{1/2}$. The *weight* of a binding is 1 for single bindings and 1/2 for joint bindings. Referring to (8.7), (8.11), and [3] or [6], we see the numerical factor

$$\prod_i (\mathrm{NDF}(i))^{\frac{1}{4}+} L_{j(i)}^{\frac{1}{4}+}, \tag{9.2.1}$$

with the i^{th} occurrence an occurrence of vertex $j(i)$, is sufficient to control the number divergence provided that the sum of the weights of all bindings to vertex k is dominated by

$$4^- r_J(k) + c, \quad \text{with} \quad \tfrac{1}{4}^+ \cdot 4^- = 1, \quad \text{for all } k.$$

Remark 1. In [3] all bindings were single.

Remark 2. As we find numerical factors to serve as our $f(k,j)$, they will arise as (with $\bar{s}=cs$)

$$\left(\frac{L_k}{L_j}\right)^{3^+} h_{\bar{s}}\left(\frac{1}{L_j}d_{k,j}\right), \quad L_k \leqq L_j,$$

$$L_j^{0^+} h_{\bar{s}}\left(\frac{1}{L_k}d_{k,j}\right), \quad L_k > L_j.$$

If the application of the ideas in this section to [3] is understood (with trivial modification due to the change in dimension) the application to the hierarchical ϕ_3^4 is immediate.

9.3. *Assignment of Numerical Factors*

From the consideration of the last section we need only specify the nature of the bindings to see what numerical factors (NF) we are using to control the number divergence. We give rules NF 1) and NF 2) to specify the bindings.

NF 1) An occurrence of a vertex in a graph, that is in a chain but neither

a) the attachment graph

nor

b) either of the last two graphs in an extraordinary chain, is bound to one of the new variables in the same graph of highest scale size.

NF 2) Let S be either

a) a graph not in a chain

or

b) the two last graphs in an extraordinary chain

(Essentially the two last graphs in such a chain are here being treated as a single graph.)

or

c) an attachment graph of an ordinary chain.

Let v_1 be the attachment and v_2 one of the largest scale new vertices in S. Let v be the largest scale vertex of v_1 and v_2 (or v_2 if they are the same scale). Any occurrence of a vertex in S is bound singly or jointly to v_1 and v_2. This is done so that the sum of the weights of the bindings to v (of such occurrences inside S) is as large as possible, subject to the limitation that if $v=v_1$ the sum must be $\leqq 3.5$.

The assignment of numerical factors by rules NF 1) and NF 2) ensure that the restriction on the sum of weights of bindings to each vertex given in the last section is satisfied. We now have knowledge of the numerical factors we will use to control the number divergence. As in [3] and [6] we divide numerical factors into two parts, one for counting, and one for controlling the number divergence. Factors of h and λ will never be a difficulty; we *will only keep track of factors of L's.*

9.4. *Numerical Factors Needed for Counting*

Counting estimates are performed by the usual sums into sups procedure. Similar to [3] and [6] we sum in an iterative manner over:

a) the number of attachments to a vertex,

b) whether an attachment is a chain or a graph not in a chain,

c) the length of a chain,

d) which graph in a chain is the attachment graph and which the hook graph,

e) the case and subcase of moves that gave rise to a graph,

f) the bottom of a graph once its pinning is known,

g) the remaining vertices (with multiplicities) in a graph once its bottom is known,

h) for each vertex in a graph, whether it is a new variable, a hook, or a pinning of another graph in the same chain.

The sums in a)–e) and h) may all be handled by a factor of λ^{0^+} borrowed from each unit. The sum in g) may be compensated by a factor of $L_B^{0^+}$, where L_B is the edge size of the bottom. The sum in f) may be controlled by a factor

$$\left(\frac{L_B}{L_p}\right)^{3^+},$$

where L_B and L_p are respectively the edge sizes of the bottom and pinning. Thus to control counting we need a factor

$$\left(\frac{L_{B_i}}{L_{p_i}}\right)^{3^+} L_{B_i}^{0^+},$$

for each graph, an overall factor

$$\prod_i \left(\frac{L_{B_i}}{L_{p_i}}\right)^{3^+} L_{B_i}^{0^+}, \tag{9.4.1}$$

where the product is over all graphs.

Here as in Sect. 1 simple geometric estimates of sums and integrals replace the usual power counting techniques. Thus in summing over possible bottoms of edge size L_B for a given pinning of edge size L_p we are concerned with compensating a factor of $\left(\frac{L_p}{L_B}\right)^3$, the number of cubes of side L_B in a cube of side L_p. $\left[\text{Including summing over size of } L_B \text{ we have } c\left(\frac{L_p}{L_B}\right)^{3^+}.\right]$ Such considerations yield (9.4.1).

9.5. *Finding the Numerical Factors*

We must now show that the numerical factors generated in the cluster expansion provide sufficient powers of L to do the counting, (9.4.1), and to handle the number divergence, (9.2.1). The basic results we have for graphs will now be presented. The bounds will involve a parameter $\varepsilon > 0$ that may be picked arbitrarily small [by adjusting ε' in (7.4)]. We will present bounds for classes of graphs, where in each case we have already divided out the numerical factors assigned to the number divergence in Sects. 9.2 and 9.3 for occurrences inside the given graphs. These bounds are derived by a case by case study; each case is easy to analyze but there are many cases. Inequality (0.15) is used to study cancellation graphs. Appendix B contains a representative derivation of one of our bounds.

9.6. *Bounds on Graphs After Dividing Out Factors for Number Divergence*

We separate our graphs into a number of types. To a graph we associate a number, d_1, with $d_1 = 1$ if the graph is form 1, and $d_1 = 2$ otherwise. If a vertex was integrated by parts to introduce a graph we will call it the graph's "entering" vertex, and similarly a graph's "exiting" vertex is the entering vertex of the very next graph. For our first four bounds below, Bound 1–4, we let L_p, L_B, L_1, and L_2 be the edge sizes of the pinning, bottom, entering vertex, and exiting vertex, respectively.

Bound 1. A graph that is not in a chain

$$L_p^{(1/4)}\left(\frac{L_B}{L_p}\right)^3 L_B^{\varepsilon} L_p^{-5\varepsilon}. \tag{9.6.1}$$

Bound 2. The last graph in an ordinary chain

$$L_1^{d_1}\left(\frac{L_B}{L_p}\right)^3 L_B^{\varepsilon} L_1^{-5\varepsilon}. \tag{9.6.2}$$

Bound 3. Not one of the last two graphs in an extraordinary chain, and either

a) before the attachment graph

or

b) after both the hook graph and the attachment graph but not the last graph in the chain

$$L_2^{(5/4)d_1-1}\left(\frac{L_B}{L_2}\right)^3 L_B^{\varepsilon} L_2^{-5\varepsilon}. \tag{9.6.3}$$

Bound 4. Not one of the last two graphs in an extraordinary chain and neither

a) before the attachment graph

nor

b) after both the attachment graph and the hook graph

$$L_p^{(1/4)+(5/4)(d_1-1)}\left(\frac{L_p}{L_2}\right)^s \left(\frac{L_B}{L_p}\right)^3 L_B^{\varepsilon} L_2^{-5\varepsilon}, \tag{9.6.4}$$

where

$$s=\begin{cases} 3/2 & \text{if graph form } 1, \\ 1 & \text{if graph form } 2, \\ 1/2 & \text{otherwise}. \end{cases}$$

For extraordinary chains we must consider the last two graphs in the chain in greater detail. Let T be the last graph and N the next to last graph in the chain. Let L_{B_1}, L_{B_2}, L, L_H, and L_0 designate the edge sizes of the bottom of N, the bottom of T, the entering vertex of T, the hook, and the attachment, respectively.

Let d and $\bar{d}$ be the d_1 values of N and T, respectively, S_1 and S_2, the number of bound occurrences in N and T, respectively.

Bound 5. N, if $L \leqq L_0$

$$\frac{L^{\gamma}}{L_0^{(3/4)}} \cdot \left(\frac{L_{B_1}}{L}\right)^3 \cdot L_{B_1}^{\varepsilon} \cdot L^{-5\varepsilon} \cdot \begin{cases} \left(\dfrac{L^{(1/2)}}{L_0^{(3/4)}}\right)^{S_1-1}, & S_1 \geqq 1 \\ 1, & S_1=0, \end{cases} \tag{9.6.5}$$

where $\gamma = d$ if graph is not the first graph in the chain, and $\gamma = 3 - s$ if the graph is first in the chain, with s as in Bound 4. If $L > L_0$, we use (9.5.3) with L_2, L_B, d_1 replaced by L, L_{B_1}, d.

Bound 6. T

$$L^{\bar{d}}\left(\frac{L_{B_2}^3}{L^{(3/2)}L_M^{(3/2)}}\right)L_{B_2}^{\varepsilon}L_M^{-5\varepsilon} \cdot \begin{cases} \left(\dfrac{L^{(1/2)}}{L_0^{(3/4)}}\right)^{S_2}, & L_0 \geqq L, S_2 + S_1 < 4 \\ \left(\dfrac{L^{(1/2)}}{L_0^{(3/4)}}\right)^{3.5 - S_1}\left(\dfrac{1}{L^{(1/4)}}\right)^{S_1 + S_2 - 3.5}, & L_0 \geqq L, S_1 + S_2 \geqq 4 \\ \left(\dfrac{1}{L_0^{(1/4)}}\right)^{S_2}, & L_0 \leqq L \end{cases} \tag{9.6.6}$$

with L_M the minimum of L_0 and L_H.

Once the tedious task of compiling the table of values (9.6.1)–(9.6.6) is behind us, verifying that one has sufficient numerical factors to control the counting is not difficult. The product of factors from the table, for any given chain, is smaller than the product of factors from (9.4.1) contributed by the same chain. [For a graph not in a chain, the contribution from the table, (9.6.1), is trivially smaller than the factor for this graph in (9.4.1).] As we proceed to verify this, for a given chain, careful attention will have to be paid to the positions of the attachment graph and the hook graph in the chain.

We first treat an ordinary chain. We divide the graphs of the chain into a number of segments, S1–S4, some of which may be empty:

S1, graphs that precede the attachment graph

S2, graphs that precede the last graph in the chain, but follow both the attachment graph and the hook graph

S3, the attachment graph if the hook graph does not follow the attachment graph. If the hook graph follows the attachment graph, then S3 is the hook graph, the attachment graph, and all graphs between these two

S4, the last graph in the chain.

For each of these four sets we compute bounds for the "excess" factor. This will be defined as the quotient of the product of factors from the table for each graph in the set by the product of factors from (9.4.1). Provided the product of the excess factors for S1, S2, S3, S4 is less than c the numerical factors are under control.

For the sets S1 and S2 the excess factors are easily seen to be $\leqq c$. If d' is the d_1 value for the last graph in the chain, and L' is the edge size of the entering vertex of the last graph in the chain, then the excess factor for S4 is $\leqq L'^{d'^-}$. We let the last graph in the set S3 have exiting vertex of edge size L. We find the following bound for the excess factor for the set S3:

$$\frac{1}{L^{s^+}},$$

where s is function defined after (9.6.4) for the last graph in the set S3.

Notice that $L' \leqq L$. We see from these bounds that the product of excess factors is $\leqq c$ except possibly when the last graph in S3 is form 1 or form 2. The last two graphs in a chain cannot be a form 1 or form 2 graph followed by a form 1 graph. This implies that in the situation we are worried about S2 must contain at least one $d_1 = 2$ graph. This will supply [by (9.6.3)] a necessary power of L. This completes the study of ordinary chains.

Our last task is the treatment of an extraordinary chain. We must look at the products of bounds for N and T from our table of bounds. The case $L_0 < L$ is not very troublesome. Our considerations for $L_0 \geqq L$ may easily be adopted to this case. We now assume $L_0 \geqq L$. If T is the attachment we can bound the product of our estimates for N and T [(9.6.5) and (9.6.6)] by (with L' = entering vertex of N)

$$\left(\frac{1}{L'^{(3/8)}}\right)^r L^{(13/8)} \left(\frac{L_{B_1}^3}{L^3}\right)\left(\frac{L_{B_2}^3}{L_0^3}\right)\frac{1}{L_H^{(3/2)}} L_{B_1}^{\varepsilon} L_{B_2}^{\varepsilon} \frac{1}{L^{5\varepsilon}}\frac{1}{L_H^{5\varepsilon}}, \tag{9.6.7}$$

where $r = 0$ if N is the first graph in the chain, and $r = 1$ otherwise. If N is the attachment, our bound for the product is

$$\left(\frac{1}{L'^{(3/8)}}\right)^r L^{(1/8)} \left(\frac{L_{B_1}^3}{L_0^3}\right)\left(\frac{L_{B_2}^3}{L_H^3}\right) L_H^{(1/8)} L_{B_1}^{\varepsilon} L_{B_2}^{\varepsilon} \frac{1}{L^{5\varepsilon}}\frac{1}{L_H^{5\varepsilon}}. \tag{9.6.8}$$

If N is the first graph in the chain we rewrite (9.6.8) as follows

$$\frac{L_H^{(3/2)}}{L^{(11/8)}}\left(\frac{L_{B_1}^3}{L_H^3}\right)\left(\frac{L_{B_2}^3}{L_0^3}\right) L_{B_1}^{\varepsilon} L_{B_2}^{\varepsilon} \frac{1}{L^{5\varepsilon}}\frac{1}{L_H^{5\varepsilon}}. \tag{9.6.9}$$

Clearly the excess factor for (9.6.8) and (9.6.9) are $\leqq c$ and so the counting is under control whenever N is the first element in a chain. Now suppose N is not the first element of the chain. If N is the attachment, we need only know that the excess factor for that portion of the chain preceding N is $\leqq (L')^{(1/8)}$. This follows from (9.6.3) and (9.4.1). The case when T is the attachment graph and N not the first element in the chain is similar to the cases we have just considered.

Appendix A. Proof of Combinatoric Theorem

In this appendix we give the essential features of the proof of Theorem 6.3. Clearly, the conversion of the left hand side of (6.2) into the right hand side is done inductively with respect to the sequence $(\Gamma_1, \ldots, \Gamma_N)$. Now, by (6.4) and the nature of interpolation, $b^l_{(\Gamma_1, \ldots, \Gamma_{n-1}, G_n, \ldots, G_N)}(s)$ is well-defined for $l \geqq n$; moreover, it follows from inspection of our integration by parts rules that $B^l_G(\alpha)$ depends only on the chain or elementary graph G_l. Hence, in combining the terms of the multiple sum, we may work "from the outside in."

We assume that the left hand side of (6.2) has been transformed into

$$\prod_{l=1}^{n-1} \mathscr{B}^l_\Gamma(\alpha) \prod_{l=1}^{n-1} \mathscr{b}^l_\Gamma(s) \prod_{\mu=1}^{\widetilde{n-1}} \mathscr{I}(\tilde{\Gamma}^\mu) \sum_{G' : \Gamma_{G'} = (\Gamma_n, \ldots, \Gamma_N)} \prod_i (-g(\tilde{G}'^i))$$
$$\cdot \prod_{l'} B^{l'}_{G'}(\alpha) \prod_{l=n}^{N} b^l_{(\Gamma_1, \ldots, \Gamma_{n-1}) \vee G'}(s),$$

where $(\tilde{\Gamma}^1, \ldots, \tilde{\Gamma}^{n-1})$ is the sequence of elementary graphs, composite graphs, and cancellation graphs induced by $(\Gamma_1, \ldots, \Gamma_{n-1})$. By the remark above, the remaining multiple sum factors into [n^{th} sum depending on $\Gamma_1, \ldots, \Gamma_n$]

$$\sum_{G' : \Gamma_{G'} = (\Gamma_{n+1}, \ldots, \Gamma_N)} \prod_i (-g(\tilde{G}'^i)) \prod_{l'} B_{G'}^{l'}(\alpha) \prod_{l=n+1}^{N} b^l_{(\Gamma_1, \ldots, \Gamma_n) \vee G'}(s),$$

and the essential work is to examine this n^{th} sum for each case that must be considered for Γ_n. If Γ_n is either an elementary graph, a composite graph, or a chain containing no cancellation graph, then the sum involves one term and we need only set

$$\mathscr{B}^n_\Gamma(\alpha) = B^1_{G'}(\alpha),$$

$$\mathscr{b}^n_\Gamma(s) = b^n_{(\Gamma_1, \ldots, \Gamma_{n-1}) \vee G'}(s).$$

The nontrivial cases in our combinatoric proof are:

Case 1. Γ_n is a cancellation graph.

Case 2. Γ_n is a chain terminated by a cancellation graph. *We stipulate that the vertices of any form 1 graph we consider in this argument have no multiplicity;* the reader can easily check our proof against the other possibilities. The interpolation defined in Sect. 3 splits Case 1 into three sub-cases.

Case 1(a). $\Gamma_n = \{(\tau, \sigma), (\tau', \tau'')\}$, where $\tau \neq \sigma$ are form 1 and (τ', τ'') is form 2. In this case τ_4 lies in $\mathscr{D} \bigcup \bigcup_{l=1}^{n-1} \operatorname{supp} \Gamma_l$ and σ_4 does not [otherwise (σ, τ) would be included in Γ_n]; since τ' is the weight function of (τ_4, σ_4), it follows from our remarks at the end of Sect. 6 (and our multiplicity assumption applied to the integration by parts) that the n^{th} sum is

$$(-g_1(\tau))(-g_1(\sigma)) s_\eta \ldots s_{n-1} \alpha_{\tau_4} \alpha_{\sigma_4} - g_2(\tau', \tau'') s_\eta \ldots s_{n-1} \alpha_{\tau_4} \alpha_{\sigma_4}, \tag{A.1}$$

where η is the smallest of all integers l such that $\tau_4 \in \operatorname{supp} \Gamma_{l-1}$, with the convention $\operatorname{supp} \Gamma_0 \equiv \mathscr{D}$. The first term in (A.1) indicates the development of the mass insertion that terminates the two-element chain (τ, σ) and the Wick ordering in the second term has been dropped because $\tau_4 \neq \sigma_4$. By (2.2) and (2.3), (A.1) reduces to

$$[M(\tau) M(\sigma) \textstyle\int u^\tau \int u^\sigma - 48 M(\tau') M(\tau'') \int u^{\tau' + \tau''} \int u^{\tau''}] \lambda^2 s_\eta \ldots s_{n-1} \alpha_{\tau_4} \alpha_{\sigma_4}.$$

Since $\tau_i'' = \sigma_i$, $i = 1, 2, 3$, it follows from our multiplicity assumption that $M(\tau) = M(\sigma) = 4!$ and $M(\tau'') = 3!$; moreover, $M(\tau') = 2$ because $\tau_4 \neq \sigma_4$. Thus the combinatoric factors of the two terms match, and we have the desired result if we set $\mathscr{B}^n_\Gamma(\alpha) = (4!)^2 \alpha_{\tau_4} \alpha_{\sigma_4}$ and $\mathscr{b}^n_\Gamma(s) = s_\eta \ldots s_{n-1}$.

Case 1(b). $\Gamma_n = \{(\tau, \sigma), (\sigma, \tau), (\tau', \tau'')\}$, where $\tau \neq \sigma$ are form 1 and (τ', τ'') is form 2. In this case $\tau_4, \sigma_4 \in \mathscr{D} \bigcup \bigcup_{l=1}^{n-1} \operatorname{supp} \Gamma_l$, so $\operatorname{supp} \tau'$ is contained in this set. By (a) of the concluding remarks of Sect. 6, the n^{th} interpolation introduces an s_n^2 factor for the unit $g_2(\tau', \tau'') : \alpha^{\tau'}$: in the exponent. The n^{th} sum is

$$\begin{aligned} &2(-g_1(\tau))(-g_1(\sigma))(s_\eta \ldots s_{n-1})(s_\varepsilon \ldots s_n) \alpha_{\tau_4} \alpha_{\sigma_4} \\ &\quad - g_2(\tau', \tau'')(2 s_n s_{n-1}^2 s_{n-2}^2 \ldots s_\varepsilon^2 s_{\varepsilon-1} s_{\varepsilon-2} \ldots s_n) \alpha_{\tau_4} \alpha_{\sigma_4}, \end{aligned} \tag{A.2}$$

where η (respectively ε) is the smallest of all integers l such that τ_4 (respectively σ_4) lies in $\mathrm{supp}\,\Gamma_{l-1}$ and we have assumed $\eta \leqq \varepsilon$ for definiteness. As before the Wick ordering has been dropped from the second term because $\tau_4 \neq \sigma_4$; the factor of 2 in the first term arises from having a mass insertion for (σ, τ) as well as for (τ, σ), while the factor of $2s_n$ arises from the differentiation of s_n^2 associated with the interpolation move that brings the form 2 unit down from the exponent. Once again we see that the combinatoric factors, the interpolation parameters, and the cell variables match for the two terms, and we have the desired result if we set $\mathscr{B}_\Gamma^n(\alpha) = 2(4!)^2 \alpha_{\tau_4}\alpha_{\sigma_4}$ and $\mathscr{E}_\Gamma^n(s) = s_\eta \ldots s_{\varepsilon-1} s_\varepsilon^2 \ldots s_{n-1}^2 s_n$.

Case 1(c). $\Gamma_n = \{(\tau, \tau), (\tau', \tau''), (\tau, 3)\}$, where τ is form 1 and (τ', τ'') is form 2. As in the preceding case, the n^{th} interpolation introduces an s_n^2 factor for the form 2 unit in the exponent. In this case, however,

$$:\alpha^{\tau'}: = :\alpha_{\tau_4}^2: = \alpha_{\tau_4}^2 - 1\,;$$

moreover, since representation one does not distinguish between a form 3 graph and a two-element chain consisting of identical form 1 graphs, there are two terms associated with (τ, τ). The n^{th} sum is

$$\begin{aligned} &(-g_1(\tau))^2 (s_\eta \ldots s_{n-1})(s_\eta \ldots s_n)\alpha_{\tau_4}^2 - g_3(\tau)(2s_n s_{n-1}^2 s_{n-2}^2 \ldots s_\eta^2) \\ &\quad - g_2(\tau', \tau'')(2s_n s_{n-1}^2 s_{n-2}^2 \ldots s_\eta^2)(\alpha_{\tau_4}^2 - 1)\,, \end{aligned} \tag{A.3}$$

where η is defined as before. By (2.2)–(2.4), (A.3) reduces to

$$\begin{aligned} &[M(\tau)^2 (\textstyle\int u^\tau)^2 - 96 M(\tau'') M(\tau') \int u^{\tau' + \tau''} \int u^{\tau''}] \lambda^2 s_\eta^2 \ldots s_{n-1}^2 s_n \alpha_{\tau_4}^2 \\ &\quad - 24 M(\tau)(\textstyle\int u^\tau)^2 - 96 M(\tau') M(\tau'') \int u^{\tau' + \tau''} \int u^{\tau''}] \lambda^2 s_\eta^2 \ldots s_{n-1}^2 s_n\,. \end{aligned}$$

Once again, by our multiplicity assumption, $M(\tau) = 4!$ and $M(\tau'') = 3!$; however, $M(\tau') = 1$ in this case, so the combinatoric factors match, and we have desired result if we set $\mathscr{B}_\Gamma^n(\alpha) = (4!)^2(\alpha_{\tau_4}^2 - 1)$ and $\mathscr{E}_\Gamma^n(s) = s_\eta^2 \ldots s_{n-1}^2 s_n$.

The nature of our expansion procedure splits Case 2 into three sub-cases as well.

Case 2(a). Γ_n is a chain terminated by a cancellation graph $\{(\tau, \sigma), (\tau', \tau'')\}$, where $\tau \neq \sigma$ are form 1 and (τ', τ'') is form 2. The n^{th} sum factors into

$$\begin{aligned} &[(-g_1(\tau))(-g_1(\sigma)) s_\varepsilon \ldots s_n \alpha_{\sigma_4} - g_2(\tau', \tau'') s_\varepsilon \ldots s_n \alpha_{\sigma_4}] b(s_1, \ldots, s_n) P(\alpha) \\ &\quad \cdot \prod_{\mu = \widetilde{n-1}+1}^{\tilde{n}-1} \mathscr{I}(\tilde{\Gamma}^\mu)\,, \end{aligned} \tag{A.4}$$

where ε is defined as in Case 1(b), $b(s_1, \ldots, s_n)$ is the total monomial of interpolation parameters brought down by the preceding moves in the chain, and $P(\alpha)$ is the total polynomial of cell variables brought down by those moves. The interpolation parameters in the second term have no multiplicity because $\tau_4 \notin \mathscr{D} \cup \bigcup_{l=1}^{n-1} \mathrm{supp}\,\Gamma_l$ and τ' is the weight function of (τ_4, σ_4). In both terms α_{τ_4} has been differentiated out because it is the variable with respect to which one integrates by parts to introduce either τ or (τ', τ''). In the case where $\eta = n+1$, how does one rule out the interpretation of (τ', τ'') as the term where α_{σ_4} is differentiated out? The answer is the content of the following lemma:

Lemma A.1. *Let G be representation one graph rooted on $\mathscr{D}$ and (τ', τ'') a local graph of G that terminates a chain of G. Let (τ, σ) be a type 1 composite graph such that the replacement of (τ', τ'') with (τ, σ) yields a representation 1 graph. Thus τ' is the weight function of (τ_4, σ_4) and the form 2 unit $g_2(\tau', \tau'')\alpha_{\sigma_4}$ is associated with (τ', τ''). Here (τ', τ'') includes the form 2 unit $g_2(\tau', \tau'')\alpha_{\tau_4}$ if and only if the replacement of (τ', τ'') with (σ, τ) also yields a representation 1 graph.*

The proof is a matter of inspecting the rules for integrating by parts. Applying this lemma to the case at hand, we see that the form 2 unit $g_2(\tau', \tau'')\alpha_{\tau_4}$ is ruled out because our specification of the cancellation graph implicitly rules out the (σ, τ) possibility. We have the desired result in this case because the combinatoric factors are exactly as in Case 1(a).

Case 2(b). Γ_n is a chain terminated by a cancellation graph $\{(\tau, \sigma), (\sigma, \tau), (\tau', \tau'')\}$, where $\tau \neq \sigma$ are form 1 and (τ', τ'') is form 2. The n^{th} sum factors into

$$[(-g_1(\tau)(-g_1(\sigma))\alpha_{\sigma_4} + (-g_1(\tau)) - g_1(\sigma))\alpha_{\tau 4} - g_2(\tau', \tau'')\alpha_{\sigma_4} - g_2(\tau', \tau'')\alpha_{\tau_4}]$$
$$\cdot b(s_1, \ldots, s_n) P(\alpha) \prod_{\mu = \widetilde{n-1}+1}^{\tilde{n}-1} \mathscr{I}(\tilde{\Gamma}^\mu), \tag{A.5}$$

where the terms in the brackets are dictated by Lemma A.1. The combinatoric factors are exactly as in the preceding case, so we have the desired result if we set $\mathscr{B}^n_\Gamma(\alpha) = (4!)^2(\alpha_{\sigma_4} + \alpha_{\tau_4})P(\alpha)$ and $\mathscr{E}^n_\Gamma(s) = b(s_1, \ldots, s_n)$.

Case 2(c). Γ_n is a chain terminated by a cancellation graph $\{(\tau, \tau), (\tau', \tau'')\}$, where τ is form 1 and (τ', τ'') is form 2. The n^{th} sum factors into

$$[(-g_1(\tau))^2\alpha_{\tau_4} - g_2(\tau', \tau'')(2\alpha_{\tau_4})] b(s_1, \ldots, s_n) P(\alpha) \prod_{\mu = \widetilde{n-1}+1}^{\tilde{n}-1} \mathscr{I}(\tilde{\Gamma}^\mu), \tag{A.6}$$

where the factor $2\alpha_{\tau_4}$ in the second term arises from the differentiation of $:\alpha^2_{\tau_4}: = \alpha^2_{\tau_4} - 1$. As in Case 1(c), $M(\tau') = 1$ in this case, so the combinatoric factors still match.

Appendix B. Representative Derivation

We derive Bound 2 here to illustrate the case-by-case reasoning involved in the proofs of our estimates. Since Bound 2 deals with the last graph in an ordinary chain, there are no occurrences in this situation, so nothing is extracted from $\mathscr{L}(\tilde{J}^\mu)^{1-\varepsilon}$ for the number divergence cancellation.

Case 1. $\tilde{J}^\mu$ is a form 1 graph τ. The scale expression is $(L_B^3 L^{-(1/2)\tau})^{1-\varepsilon}$, which is clearly dominated by $L_B^{1^-}$. But our integration by parts rules imply that $L_p = L_H = L_B$ in this case, so $L_B^{1^-} \leqq (9.6.2)$. $\left(\text{We have naturally written } L^\tau = \prod_k L_k^{\tau(k)}.\right)$

Case 2. $\tilde{J}^\mu$ is a form 2 graph (τ', τ''). The scale expression in this case is $(L_B^3 L_{\tau_1''}^3 L^{-\tau''-(1/2)\tau'})^{1-\varepsilon}$. Our rules in Sect. 4 imply that $L_{\tau_1'} \leqq L_{\tau_3''}$ and that $L_p = L_H \leqq L_{\tau_1'}$, so

$$L_B^3 L_{\tau_1''}^3 L^{-\tau''-(1/2)\tau'} \leqq L_B^3 L_{\tau_1''} L_p^{-2}.$$

Now, if $L_{\tau_1''}=L_B$, we obviously have the upper bound $\left(\frac{L_B}{L_p}\right)^{4^-} L_p^{2^-}$. If $L_{\tau_1''}>L_B$, then $L_{\tau_1'}=L_B$, so $L_B=L_H$ and we use

$$L_B^3 L_{\tau_1''}^3 L^{-\tau''-(1/2)\tau'} \leqq L_B^3 L^{-(1/2)\tau'}$$

instead to obtain $L_p^{2^-}$ as an upper bound.

Remark. A form 3 graph cannot occur in a chain.

Case 3. $\tilde{J}^\mu$ is a cancellation graph whose local graph is (τ', τ''). The expression is $(L_B^{7^-} L^{-\tau''-(1/2)\tau'} L_{\tau_2'}^{-1^-})^{1-\varepsilon}$. Since $L_p=L_H \leqq L_{\tau_2'}$,

$$L_B^{7^-} L^{-\tau''-(1/2)\tau'} L_{\tau_2'}^{-1^-} \leqq L_B^{(7/2)^-} L_p^{-(3/2)^-},$$

so we have the bound $\left(\frac{L_B}{L_p}\right)^{(7/2)^-} L_p^{2^-}$.

Case 4. $\tilde{J}^\mu$ is a type 1 composite graph (τ, σ). We have

$$L_B^3 L_{\tau_1}^3 L^{-(1/2)\sigma-(1/2)\tau} \leqq L_B^{(3/2)} L_{\tau_1}^{(1/2)}.$$

If $L_{\tau_1}>L_B$, then the integration by parts rules dictate $L_p=L_H=L_B$, and the numerical expression is dominated by $L_p^{2^-}$. If $L_{\tau_1}=L_B$, then our rules dictate that the restriction of $\sigma+\tau$ to vertices whose scales are strictly smaller than $L_p=L_H$ has weight $\leqq 5$, so we use

$$L_B^3 L_{\tau_1}^3 L^{-(1/2)\sigma-(1/2)\tau} \leqq L_B^{(7/2)} L_p^{-(3/2)}$$

instead to obtain the bound $\left(\frac{L_B}{L_p}\right)^{(7/2)^-} L_p^{2^-}$.

Case 5. $\tilde{J}^\mu$ is a type 2 composite graph $((\tau', \tau''), \sigma)$ with $L_{\tau_1'} \leqq L_{\tau_1''}$.

$$L_B^3 L_{\tau_1'}^3 L_{\tau_1''}^3 L^{-\tau''-(1/2)\tau'-(1/2)\sigma} \leqq L_B^3 L_{\tau_1'}^3 L^{-(1/2)\tau'-(1/2)\sigma}.$$

If $L_p=L_H<L_{\tau_1'}$, then $L_B=L_p$, and we have the bound $L_p^{(3/2)^-} L_{\tau_1'}^{(3/2)^-}$ because the part of $\tau'+\sigma$ with scale $\geqq L_{\tau_1'}$ is at least 3 in weight. On the other hand, if $L_p \geqq L_{\tau_1'}$, then our rules dictate $L_B=L_{\tau_1'}$, in which case we estimate

$$L_B^3 L_{\tau_1'}^3 L^{-(1/2)\tau'-(1/2)\sigma} \leqq L_B^{(7/2)} L_p^{-(1/2)},$$

because $L_p=L_H<L_1=L_{\tau_2'}$; in this situation our bound is $\left(\frac{L_B}{L_p}\right)^{(7/2)^-} L_p^{3^-}$.

Case 6. $\tilde{J}^\mu$ is a type 2 composite graph $((\tau', \tau''), \sigma)$ with $L_{\tau_1'}>L_{\tau_1''}$:

$$L_{\tau_1''}^6 L_{\sigma_1}^3 L^{-\tau''-(1/2)\tau'-(1/2)\sigma} \leqq L_{\tau_1''}^4 L_{\sigma_1}^{(3/2)} L_{\tau_1'}^{-2} L_{\tau_2'}^{-(1/2)},$$

because $L_{\tau_1'} \leqq L_{\tau_3''}$, as our rules imply. If $L_{\sigma_1}<L_{\tau_1''}$, then $L_B=L_{\sigma_1}=L_H=L_p$ and we obtain the bound $L^{(3/2)^-} L_{\tau_1'}^{(3/2)^-}$. If $L_{\sigma_1} \geqq L_{\tau_1''}$, then $L_B=L_{\tau_1''}$ and we estimate

$$L_{\tau''_1}^4 L_{\sigma_1}^{(3/2)} L_{\tau_1'}^{-2} L_{\tau_2'}^{-(1/2)} \leqq L_B^{(7/2)} L_p^{-(1/2)},$$

because $L_{\sigma_1} \leqq L_{\tau_1'}$ and $L_p \leqq L_{\tau_2'}$. In this case we have the bound

$$\left(\frac{L_B}{L_p}\right)^{(7/2)^-} L_p^{3^-}.$$

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