Invent. math. 102, 663–690 (1990)

Inventiones mathematicae

Twisted vertex representations of quantum affine algebras

Naihuan Jing ⋆
School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA

Oblatum 15-XI-1989

1. Introduction

Recent interests in quantum groups are stimulated by their marvelous relations with quantum Yang-Baxter equations, conformal field theory, invariants of links and knots, and q-hypergeometric series. Besides understanding the reason of the appearance of quantum groups in both mathematics and theoretical physics there is a natural problem of finding q-deformations or quantum analogues of known structures.

Quantum groups were first defined by Drinfeld [2] and Jimbo [9] (also see [4]) as a q-deformation of the universal enveloping algebras of the Kac-Moody algebras in the work of trigonometric solutions of Yang-Baxter equations. In the same spirit it was shown in [13], [14], that there exists a $1-1$ correspondence between the integrable highest weight representations of symmetrizable Kac-Moody algebras and those of the corresponding quantum groups, where both spaces have the same dimension in the case of generic q (i.e. q is not a root of unity). Moreover, one can be very explicit in the case of quantum $\mathfrak{gl}(n)$ to write down the irreducible highest weight representations.

Quantum affine algebras are the quantum groups associated to affine Lie algebras. Following Drinfeld's realization [3] of q-analog of loop algebras, the vertex representation of untwisted simply laced quantum affine algebras was constructed in Frenkel-Jing [6], which is a q-deformation of Frenkel-Kac [7] and Segal [15] construction in the theory of affine Lie algebras. Subsequently, the same was done for the quantum affine algebra of type B in [1].

In the present work we construct vertex representations of quantum affine algebras twisted by an automorphism of the Dynkin diagram, which generalizes certain important cases in the ordinary twisted vertex operator calculus [5,

⋆ Research supported by NSF Grant No. DMS 8610730 through Institute for Advanced Study. This work is reported at the AMS meeting No. 857 on Classical and Quantum Groups, held at Pennsylvania State University on April 7–8, 1990

Address after September 1, 1990: Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

11]. We start from the twisted quantum Heisenberg algebras of Drinfeld [3] to build the whole representation of the quantum affine algebras. The Fock space is generated, as usual, by half of the generators of the Heisenberg algebra.

Our approach is a q-deformed vertex operator calculus, of which the ordinary counterpart already shows its power in the theory of affine Lie algebras [cf, 8, 12]. Especially the formal power series approach turns out to be a useful tool in proving various relations of quantum affine algebras. The new feature is that we have only first order derivatives of δ-functions, more explicitly the q-differences of δ-functions, while in the ordinary case we have higher order derivatives. This is an implicit consequence of defining relations of quantum affine algebra, which deforms the singularity of order s at $q=1$ to several simple poles at $q, q^2, \dots, q^s$. Another novelty is that the Serre relations in this case are more subtle. The proof of it involves decomposing formal power series into a sum of simpler series, the symmetrization of it will have a cancellation which establish the Serre relations in various cases.

The paper is organized as follows. In Sect. 2 we recall the notion of quantum groups and Drinfeld's realization. Then in Sect. 3 the twisted quantum vertex operators are constructed. Finally, in Sect. 4 we prove that this construction gives the basic representation of twisted quantum affine algebras.

The q-deformed vertex operators provide an important example of the so-called q-analogue of the bosonic field theory and can be used as a testing ground for the q-analogue of the conformal field theory. The further study of q-vertex operators is believed to offer new perspectives of many diverse problems.

We remark that a version of this paper in the case of untwisted quantum affine algebras was contained in [10].

Acknowledgement. I am grateful to J. Lepowsky for sending me his unpublished notes on twisted vertex operator algebras (announced in [11]). I thank I.B. Frenkel for stimulating discussions.

2. Twisted quantum affine algebras

The q-analogue of the Kac-Moody algebra is defined for any generalized symmetrizable Cartan matrix. Following [2] and [9], let $A=(A_{ij})_{l\times l}$ be a generalized Cartan matrix, i.e., an integral matrix with $A_{ij}\leqq 0 (i\neq j)$, $A_{ii}=2$ and $A_{ij}=0$ if an only if $A_{ji}=0$. Moreover, there exists an l-tuple of integers $(d_1, d_2, \dots, d_l)$ such that $d_i A_{ij}=d_j A_{ji}$, and the greatest common divisor of d_i's is 1.

Let $q\neq \pm 1$ be a complex number. The quantum group or q-deformed Kac-Moody algebra $\mathscr{U}_q(\mathfrak{g}(A))$ is an associative algebra with unit 1 over $\mathbb{C}$ generated by elements $x_i^+, x_i^-, k_i, k_i^{-1}, i=1, \dots, l$ satisfying relations

(2.1) $$k_i k_i^{-1}=k_i^{-1} k_i=1, \quad [k_i, k_j]=0,$$

(2.2) $$k_i x_j^{\pm} k_i^{-1}=q_i^{\pm d_i A_{ij}} x_j^{\pm},$$

(2.3) $$[x_i^+, x_j^-]=\delta_{ij}\frac{k_i-k_i^{-1}}{q_i-q_i^{-1}}$$

(2.4) $$\sum_{r=0}^{1-A_{ij}} (-1)^r \begin{bmatrix} 1-A_{ij} \\ r \end{bmatrix}_{q_i} (x_i^{\pm})^{1-A_{ij}-r} x_j^{\pm} (x_i^{\pm})^r=0 \quad (i\neq j)$$

where $q_i = q^{d_i}$ and we introduced the notation

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{(q^m - q^{-m}) \dots (q^{m-n+1} - q^{-m+n-1})}{(q^n - q^{-n}) \dots (q - q^{-1})} \qquad \text{for } n \leqq m.$$

Let $A = (A_{ij})$ be a Cartan matrix of the affine type, then we call $\mathscr{U}_q(\mathfrak{g}(A))$ a *quantum affine algebra* associated to A. To our purpose we are only concerned about simply laced quantum affine algebras, i.e. A is one of the following types: $A_N^{(1)} (N \geqq 1)$, $D_N^{(1)} (N \geqq 4)$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$; $A_N^{(2)} (N \geqq 2)$, $D_N^{(2)} (N \geqq 3)$, $E_6^{(2)}$; and $D_4^{(3)}$, where the upper indices refer to the orders of automorphisms of the Dynkin diagrams (we will come back to it later), and we will refer to them as type 1–3 respectively.

For type 1 Cartan matrix, $(d_1, d_2, \dots, d_{N+1}) = (1, 1, \dots, 1)$. For the other cases $(d_1, d_2, \dots, d_l)$ are given by the following table in Fig. 1.

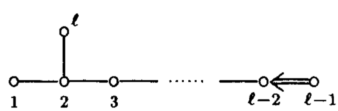
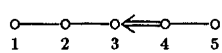
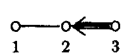
A	Dynkin diagram	$(d_1, d_2, \dots, d_\ell)$
$A_2^{(2)}$		$(1,4)$
$A_{2\ell}^{(2)} (\ell \geq 2)$		$(1,2,\dots,2,4)$
$A_{2\ell-1}^{(2)} (\ell \geq 3)$		$(1,\dots,1,2,1)$
$D_{\ell+1}^{(2)} (\ell \geq 2)$		$(1,2,2,\dots,2,1)$
$E_6^{(2)}$		$(1,1,1,2,2)$
$D_4^{(3)}$		$(1,1,3)$

Fig. 1

For type 1 matrix A, the Serre relation (2.4) will be

(2.5) $$[x_i^\pm, x_j^\pm] = 0 \qquad \text{for } A_{ij} = 0$$

(2.6) $$x_i^\pm x_i^\pm x_j^\pm - (q + q^{-1}) x_i^\pm x_j^\pm x_i^\pm + x_j^\pm x_i^\pm x_i^\pm = 0$$

(2.7) $$x_i^\pm x_i^\pm x_i^\pm x_j^\pm - (q^2 + 1 + q^{-2}) x_i^\pm x_i^\pm x_j^\pm x_i^\pm + (q^2 + 1 + q^{-2}) x_i^\pm x_j^\pm x_i^\pm x_i^\pm - x_j^\pm x_i^\pm x_i^\pm x_i^\pm = 0 \qquad \text{for } A_{ij} = -2$$

where (2.7) is not present except for $A_1^{(1)}$.

Remark. Let $k_i = q^{d_i h_i}$ formally and $q \to 1$ in (2.1–4), one obtains the defining relations of Chevalley generators for the Kac-Moody algebra associated to the Cartan matrix A.

In the development of the theory of affine Lie algebras, the realization of loop algebras plays an important role. The q-deformation of loop algebras was constructed by Drinfeld [3], and was written in terms of components. We will give a slightly modified form as follows.

Let Γ be the Dynkin diagram of the finite dimensional simple Lie algebra $\mathfrak{g}$ of simply laced type (i.e. ADE types), and σ an automorphism of the Dynkin diagram with order $k \geqq 1$. Then $k=1$, 2 or 3, and the order 3 only appears in the case of D_4. The twisted affine algebra $\hat{\mathfrak{g}}_\sigma$ is given by

$$\hat{\mathfrak{g}}_\sigma = \bigoplus_{n \in \mathbb{Z}} (t^n \otimes \mathfrak{g}_{n \bmod k}) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

where $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}/k\mathbb{Z}} \mathfrak{g}_n$ under the diagonal action of σ.

For the q-analogue of twisted loop algebras, it is better to represent $\hat{\mathfrak{g}}_\sigma$ as $L(t, t^{-1}) \otimes \mathfrak{g}$ and then impose the periodic relations under σ.

Let Δ and $\hat{\Delta}$ be the root system of Γ and $\hat{\Gamma} = \Gamma \cup \{0\}$, the extended Dynkin diagram associated to $\hat{\mathfrak{g}}$. Then Δ has a basis consisting of simple roots $\alpha_1, \alpha_2, \ldots, \alpha_l$, and $\hat{\Delta}$ contains $\{\alpha_i\}_{1 \leqq i \leqq l}$ with an additional imaginary root $\alpha_0 = c - \alpha_{\max}$, where $\alpha_{\max}$ is the maximal root of $\mathfrak{g}$. There exists a symmetric form $\langle \, , \rangle$ on $\mathfrak{g}$ such that

$$\langle \alpha_{\max}, \alpha_{\max} \rangle = 2 \tag{2.8}$$

For $\alpha, \beta \in \Delta$, we set

$$g_{\alpha\beta}(z) = \sum_{n \in \mathbb{Z}_+} c_{\alpha\beta n} z^n$$

to be a formal series in z, and $c_{\alpha\beta n}$ are determined from the Taylor series expansion at $\xi = 0 \in \mathbb{C}$ of the function

$$f_{\alpha\beta}(\xi) = \frac{G_{\alpha\beta}(\xi, 1)}{F_{\alpha\beta}(\xi, 1)} = \sum_{n \in \mathbb{Z}_+} c_{\alpha\beta n} \xi^n \tag{2.9}$$

where

$$F_{\alpha\beta}(u, v) = \prod_{r \in \mathbb{Z}/k\mathbb{Z}} (u - \omega^r v q^{\langle \alpha, \sigma^r \beta \rangle}) \tag{2.10}$$

$$G_{\alpha\beta}(u, v) = \prod_{r \in \mathbb{Z}/k\mathbb{Z}} (u q^{\langle \alpha, \sigma^r \beta \rangle} - \omega^r v) \tag{2.11}$$

and ω is the primitive k-th root $e^{\frac{2\pi i}{k}}$.

We remark that $f_{\alpha\beta}(\xi^{-1}) = f_{\alpha\beta}(\xi)^{-1}$, which is not true for $g_{\alpha\beta}(z)$, since the latter is only a formal series. We also denote $g_{\alpha_i \alpha_j}(z)$ simply by $g_{ij}(z)$, and $F_{ij}^{\pm}(u, v) = F_{\pm\alpha_i, \alpha_j}(u, v)$, $G_{ij}^{\pm}(u, v) = G_{\pm\alpha_i, \alpha_j}(u, v)$, $i, j = 1, \ldots, l$. We introduce a formal power series

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n \tag{2.12}$$

which is crucial in the formal variable approach to vertex operator algebras considered in [8]. A characteristic property of $\delta(z)$ is the following: if $f(z_1, z_2)$ is a formal power series in two variables, then

$$f(z_1, z_2)\,\delta\left(\frac{z_1}{a}\right) = f(a, z_2)\,\delta\left(\frac{z_1}{a}\right), \qquad a \in \mathbb{C}. \tag{2.13}$$

or if $f(z, z)$ exists,

$$f(z_1, z_2)\,\delta\left(\frac{z_2}{z_1}\right) = f(z_1, z_1)\,\delta\left(\frac{z_2}{z_1}\right),$$

which are verified directly.

Now we can recall the following

(2.14) **Theorem** (Drinfeld [3]). *The twisted quantum affine algebra $\mathscr{U}_q(\hat{\mathfrak{g}}_\sigma)$ is a complex associative algebra with unit 1 and the generators*

$$\{x_{ik}^{\pm}, \phi_{im}, \psi_{in}, \gamma^{\frac{1}{2}}, \gamma^{-\frac{1}{2}} \mid i = 1, \ldots, l;\ k \in \mathbb{Z}, m \in -\mathbb{Z}_+, n \in \mathbb{Z}_+\} \tag{2.15}$$

subject to the relations below written in terms of generating functions in z:

$$x_i^{\pm}(z) = \sum_{k \in \mathbb{Z}} x_{ik}^{\pm} z^{-k}, \tag{2.16}$$

$$\phi_i(z) = \sum_{-m=0}^{+\infty} \phi_{im} z^{-m}, \tag{2.17}$$

$$\psi_i(z) = \sum_{n=0}^{+\infty} \psi_{in} z^{-n}, \tag{2.18}$$

The relations are

$$x_{\sigma(i),n}^{\pm} = \omega^n x_{in}^{\pm}, \qquad \varphi_{\sigma(i),m} = \omega^m \varphi_{im}, \qquad \varphi_{\sigma(i),n} = \omega^n \psi_{in}. \tag{2.19}$$

$$\gamma^{\frac{1}{2}}\gamma^{-\frac{1}{2}} = \gamma^{-\frac{1}{2}}\gamma^{\frac{1}{2}} = 1 \qquad \gamma^{\pm\frac{1}{2}} \text{ are central,} \tag{2.20}$$

$$\phi_{i0}\,\psi_{i0} = \psi_{i0}\,\phi_{i0} = 1 \tag{2.21}$$

$$[\phi_i(z), \phi_j(w)] = [\psi_i(z), \psi_j(w)] = 0, \tag{2.22}$$

$$\phi_i(z)\,\psi_j(w)\,\phi_i(z)^{-1}\,\varphi_j(w)^{-1} = g_{ij}(zw^{-1}\gamma^{-1})/g_{ij}(zw^{-1}\gamma), \tag{2.23}$$

$$\phi_i(z)\,x_j^{\pm}(w)\,\phi_i(z)^{-1} = g_{ij}(zw^{-1}\gamma^{\mp\frac{1}{2}})^{\pm 1}\,x_j^{\pm}(w), \tag{2.24}$$

$$\psi_i(z)\,x_j^{\pm}(w)\,\psi_i(z)^{-1} = g_{ij}(z^{-1}w\gamma^{\mp 1/2})^{\mp 1}\,x_j^{\pm}(w), \tag{2.25}$$

$$F_{ij}^{\pm}(z, w)\,x_i^{\pm}(z)\,x_j^{\pm}(w) = G_{ij}^{\pm}(z, w)\,x_j^{\pm}(w)\,x_i^{\pm}(z) \tag{2.26}$$

(2.27)

$$[x_i^+(z), x_j^-(w)] = \frac{1}{q - q^{-1}} \sum_{r=0}^{k-1} \delta_{\sigma^r(i),j} \left\{\psi_i(z\gamma^{-\frac{1}{2}})\,\delta\left(\frac{z}{w}\omega^r\gamma^{-1}\right) - \varphi_i(z\gamma^{\frac{1}{2}})\,\delta\left(\frac{z}{w}\omega^r\gamma\right)\right\}$$

(2.28)

$$\operatorname{Sym}\left\{P_{ij}^{\pm}(z_1, z_2) \sum_{r=0}^{2} (-1)^r \begin{bmatrix} 2 \\ r \end{bmatrix}_{q^{d_{ij}}} x_i^{\pm}(z_1) \ldots x_i^{\pm}(z_r)\,x_j^{\pm}(w)\,x_i^{\pm}(z_{r+1}) \ldots x_i^{\pm}(z_2)\right\} = 0$$

for $A_{ij} = -1$ *and* $\sigma(i) \neq j$.

(2.29)

$$\operatorname{Sym}\{(q^{\mp 3k/4} z_1 - (q^{k/4} + q^{-k/4})\,z_2 + q^{\pm 3k/4} z_3)\,x_i^{\pm}(z_1)\,x_i^{\pm}(z_2)\,x_i^{\pm}(z_3)\} = 0$$

for $A_{i,\sigma(i)} = -1$

where the Sym *means the symmetrization over* z_i, $P_{ij}^{\pm}(z,w)$ *and* d_{ij} *are defined as follows*:

If $\sigma(i)=i$, *then* $P_{ij}^{\pm}(z,w)=1$ *and* $d_{ij}=k$.

If $A_{i,\sigma(i)}=0$ *and* $\sigma(j)=j$, *then* $P_{ij}^{\pm}(z,w)=\dfrac{z^k q^{\pm 2k}-w^k}{zq^{\pm 2}-w}$ *and* $d_{ij}=k$.

If $A_{i,\sigma(i)}=0$ *and* $\sigma(j)\neq j$, *then* $P_{ij}^{\pm}(z,w)=1$ *and* $d_{ij}=\frac{1}{2}$.

If $A_{i,\sigma(i)}=-1$, *then* $P_{ij}^{\pm}(z,w)=zq^{\pm k/2}+w$ *and* $d_{ij}=\dfrac{k}{4}$.

The explicit isomorphism of this realization to definition (2.1–2.4) was given in [3], and is not needed in the sequel.

For the case of the identity automorphism, i.e. $\sigma=1$, (2.28–2.29) is reduced to the following relations:

$$[x_i^{\pm}(z), x_j^{\pm}(w)]=0 \qquad \text{for } A_{ij}=0 \tag{2.30}$$

(2.31)
$$\{x_i^{\pm}(z_1)x_i^{\pm}(z_2)x_j^{\pm}(w)-(q+q^{-1})x_i^{\pm}(z_1)x_j^{\pm}(w)x_i^{\pm}(z_2)+x_j^{\pm}(w)x_i^{\pm}(z_1)x_i^{\pm}(z_2)\}$$
$$+\{z_1\leftrightarrow z_2\}=0 \qquad \text{for } A_{ij}=-1.$$

In the case of $A_1^{(1)}$, (2.30–2.31) are replaced by the following relation

$$(z-q^{\pm 2}w)\,x_1^{\pm}(z)\,x_1^{\pm}(w)=x_1^{\pm}(w)\,x_1^{\pm}(z)\,(q^{\pm 2}w-z). \tag{2.32}$$

3. Construction of twisted q-vertex operators

Assume from now on that q is generic, i.e. not a root of unit, and we choose t with $q=e^{t/2}$.

We shall employ the q-analogue of an integer n by

$$[n]_q=\frac{q^n-q^{-n}}{t};$$
$$[n]_q!=[n]_q[n-1]_q\cdots[1]_q.$$

Thus the q-binomial coefficients defined in (2.4) can be expressed as

$$\begin{bmatrix} m \\ n \end{bmatrix}_q=\frac{[m]_q!}{[n]_q!\,[m-n]_q!} \qquad \text{for } n\leq m.$$

We remark that in the standard combinatoric literature the q-integer n is defined as $[n]_q=\dfrac{q^n-q^{-n}}{q-q^{-1}}$, which is essentially the same as ours if one observes that $t\sim q-q^{-1}$.

By a representation of $\mathcal{U}_q(\mathfrak{g}(A))$ or a $\mathcal{U}_q(\mathfrak{g}(A))$-module we mean a left $\mathcal{U}_q(\mathfrak{g}(A))$-module considering of $\mathcal{U}_q(\mathfrak{g}(A))$ as an algebra. Our paper concerns the basic representation of twisted quantum algebras, which is an important non-trivial example of highest weight representations of $\mathcal{U}_q(\hat{\mathfrak{g}}_\sigma)$ (cf. [13]).

The starting point of vertex operator representations of quantum affine algebras is the q-deformation of Heisenberg algebras. It is generated by ϕ_{im}, ψ_{in}, and $\gamma^{\pm 1}(m \in -\mathbb{N}, n \in \mathbb{N})$, and was obtained by Drinfeld. To see this clearer, we define elements α_{in}, $i=1, \ldots, l$, $n \in \mathbb{Z} \backslash \{0\}$, from the relations

$$\phi_i(z) = \phi_{i0} \exp\{-t \sum_{m \in -\mathbb{N}} \alpha_{im} z^{-m}\} = \sum_{m \in -\mathbb{Z}_+} \phi_{im} z^{-m} \tag{3.1}$$

$$\psi_i(z) = \psi_{i0} \exp\{t \sum_{n \in \mathbb{N}} \alpha_{in} z^{-n}\} = \sum_{n \in \mathbb{Z}_+} \psi_{in} z^{-n} \tag{3.2}$$

and define α_{i0} by $\phi_{i0} = e^{-t\frac{\alpha_{i0}}{2}}$ or $\psi_{i0} = e^{t\frac{\alpha_{i0}}{2}}$.

These α_{in} can be thought as the q-analogue of the affinization of the root $\alpha_i \in \Delta$. In fact we have

(3.3) **Proposition.** *The twisted quantum Heisenberg algebra $\mathcal{U}_q(\hat{\mathfrak{h}}_\sigma)$ is an associative algebra generated by α_{in}, $i=1, \ldots, l$, $n \in \mathbb{Z} \backslash 0$ and $\gamma^{\pm 1}$ satisfying the following relations*

$$\alpha_{\sigma(i), n} = \omega^n \alpha_{in} \tag{3.4}$$

$$[\alpha_{in}, \gamma] = [\alpha_{in}, \gamma^{-1}] = 0, \qquad \gamma\gamma^{-1} = \gamma^{-1}\gamma = 1 \tag{3.5}$$

$$[\alpha_{im}, \alpha_{in}] = \delta_{m,-n} \frac{1}{m} \sum_{r=0}^{k-1} \omega^{mr} [\langle \alpha_i, \sigma^r \alpha_j \rangle m]_q [m]_\gamma \tag{3.6}$$

where we used

$$[n]_\gamma = \frac{\gamma^n - \gamma^{-n}}{t}$$

and treated the center element γ as numbers.

Proof. The elements α_{in} are determined inductively in terms of ϕ_{im}, ψ_{in} from the relations (3.1–2). The relations (3.4–5) are immediate by definition. As for (3.6) we substitute (3.1–2) into (2.23). Then formally taking its logarithm and using the identity $e^A e^B = e^B e^A e^{[A,B]}$, we obtain the required relations.

Another way to see it is that we can show the relations between $\phi_i(z)$ and $\psi_j(w)$ in terms of (3.6). In fact we have

$$\begin{aligned} &\phi_i(z)\,\psi_j(w) \\ &= \psi_j(w)\,\phi_i(z) \exp\{t^2 [\sum_{n>0} -\alpha_i(-n) z^n, \sum_{n>0} \alpha_j(n) w^{-n}]\} \\ &= \psi_j(w)\,\phi_j(z) \exp\left\{ \sum_{\substack{n=1 \\ r \in \mathbb{Z}/k\mathbb{Z}}}^{+\infty} \frac{1}{n} \omega^{nr} (q^{\langle \alpha_i, \sigma^r \alpha_j \rangle n} - q^{-\langle \alpha_i, \sigma^r \alpha_j \rangle n}) \right. \\ &\quad \left. \cdot (\gamma^n - \gamma^{-n}) \left(\frac{z}{w}\right)^n \right\}. \end{aligned}$$

Using the formal identity

$$\log(1-z) = -\sum_{n=1}^{+\infty} \frac{z^n}{n},$$

we arrive at

$$\begin{aligned}
&\phi_i(z)\,\psi_j(w)\\
&=\psi_j(w)\,\phi_i(z)\prod_{r\in\mathbb{Z}/k\mathbb{Z}}\frac{1-\omega^r q^{\langle\alpha_i,\sigma^r\alpha_j\rangle}\gamma^{-1}\frac{z}{w}}{1-\omega^r q^{\langle\alpha_i,\sigma^r\alpha_j\rangle}\gamma\frac{z}{w}}\\
&\quad\cdot\frac{1-\omega^r q^{-\langle\alpha_i,\sigma^r\alpha_j\rangle}\gamma\frac{z}{w}}{1-\omega^r q^{-\langle\alpha_i,\sigma^r\alpha_j\rangle}\gamma^{-1}\frac{z}{w}}\\
&=\psi_j(w)\,\phi_i(z)\,g_{ij}(zw^{-1}\gamma^{-1})/g_{ij}(zw^{-1}\gamma)
\end{aligned}$$

which is (2.23).

Remark. Note that (3.6) can also be written as

$$[\alpha_{im},\alpha_{in}]=\delta_{m,-n}\frac{1}{m}\sum_{r=0}^{k-1}\omega^{mr}[A_{i,\sigma^r j}\,m]_q\,[m]_\gamma. \tag{3.7}$$

By an analogous argument of (3.3) one can show the following

(3.8) **Proposition.** *For $m\neq 0$, we have*

$$[\alpha_{im},x_j^{\pm}(z)]=\pm\frac{z^m}{m}\sum_{r=0}^{k-1}\omega^{mr}[A_{i,\sigma^r j}\,m]_q\,\gamma^{\mp|m|/2}x_j^{\pm}(z).$$

Let $\mathcal{U}_q(\hat{\mathfrak{h}}_\sigma^+)$ ($\mathcal{U}_q(\mathfrak{h}_\sigma^-)$) be the subalgebras of $\mathcal{U}_q(\hat{\mathfrak{h}}_\sigma)$ generated by α_{in}, $\gamma^{\pm 1}$, $n\in\mathbb{N}$ (α_{in}, $n\in-\mathbb{N}$). Then it is clear that $\mathcal{U}_q(\hat{\mathfrak{h}}_\sigma^+)$ and $\mathcal{U}_q(\hat{\mathfrak{h}}_\sigma^-)$ are commutative subalgebras. Moreover, we have

$$\mathcal{U}_q(\hat{\mathfrak{h}}_\sigma)=\mathcal{U}_q(\hat{\mathfrak{h}}_\sigma^+)\,\mathcal{U}_q(\mathfrak{h}_\sigma^-).$$

This suggests that $\mathcal{U}_q(\hat{\mathfrak{h}}_\sigma)$ has a canonical representation constructed as follows. Let $S(\mathfrak{h}_\sigma^-)$ be the symmetric algebra generated by α_{in}, $n\in-\mathbb{N}$, then as associative algebras

$$S(\mathfrak{h}_\sigma^-)\simeq\mathcal{U}_q(\mathfrak{h}_\sigma^-). \tag{3.9}$$

We define $\mathcal{U}_q(\hat{\mathfrak{h}}_\sigma)$-action on $S(\mathfrak{h}_\sigma^-)$ by the following rules.

$$\begin{aligned}
&\gamma.1=q,\qquad \gamma^{-1}.1=q^{-1}\\
&\alpha_{in}=\text{multiplication operator},\qquad n\in-\mathbb{N}\\
&\alpha_{in}=\text{annihilation operator subject to (3.7)},\qquad n\in\mathbb{N}.
\end{aligned}\tag{3.10}$$

It is easy to see that this defines an irreducible representation of $\mathcal{U}_q(\hat{\mathfrak{h}}_\sigma)$ on $S(\mathfrak{h}_\sigma^-)$. We shall use $\alpha_i(n)$ to specify the corresponding operator of α_{in} on

$S(\mathfrak{h}_\sigma^-)$. Thus the corresponding operators $\Phi_i(m)$ and $\Psi_i(n)$ for ϕ_{im}, ψ_{in} are defined by

(3.11) $$\Phi_i(z)=\sum_{m\in -\mathbb{Z}_+}\Phi_i(m)\,z^{-m}=\exp\left\{-t\left(\frac{\alpha_i(0)}{2}+\sum_{n=1}^{+\infty}\alpha_i(-n)\,z^n\right)\right\}$$

(3.12) $$\Psi_i(z)=\sum_{n\in\mathbb{Z}_+}\Psi_i(n)\,z^{-n}=\exp\left\{t\left(\frac{\alpha_i(0)}{2}+\sum_{n=1}^{+\infty}\alpha_i(n)\,z^{-n}\right)\right\}$$

where the action of $\alpha_i(0)$ is defined below (cf. (3.24)).

Another way to define the representation $\mathcal{U}_q(\hat{\mathfrak{h}}_\sigma)$ is that it is actually the induced representation

(3.13) $$S(\hat{\mathfrak{h}}_\sigma)\simeq\mathcal{U}_q(\hat{\mathfrak{h}}_\sigma)\otimes_{\mathcal{U}_q(\mathfrak{h}_\sigma^+)}\mathbb{C}1_S$$

where $\mathbb{C}1_S$ is the one-dimensional representation of $\mathcal{U}_q(\hat{\mathfrak{h}}_\sigma)$ such that γ and γ^{-1} act as multiplications by q and q^{-1} respectively, and all the other generators act trivially.

Now we can set out to define twisted vertex operators. Let Q be the root lattice of the simple Lie algebra $\mathfrak{g}$ (ADE type), and set $\omega_0=(-1)^k\omega$. Following [11], there exists a unique central extension

(3.14) $$1\to\langle\omega_0\rangle\to\hat{Q}\xrightarrow{\ \bar{}\ }Q\to 1$$

of Q by the cyclic group $\langle\omega_0\rangle$ with the commutator map C defined below,

(3.15) $$\begin{aligned} aba^{-1}b^{-1}&=C(\bar{a},\bar{b})\qquad\text{for } a,b\in\hat{Q}\\ &=\prod_{r\in\mathbb{Z}/k\mathbb{Z}}(-\omega^r)^{\langle\bar{a},\sigma^r\bar{b}\rangle}\end{aligned}$$

We remark that the commutator map C has the following property:

(3.16) $$\begin{aligned} C(\alpha+\beta,\gamma)&=C(\alpha,\gamma)\,C(\beta,\gamma)\\ C(\alpha,\beta+\gamma)&=C(\alpha,\beta)\,C(\alpha,\gamma)\end{aligned}$$

(3.17) $$\begin{aligned} C(\alpha,\alpha)&=1\\ C(\sigma\alpha,\sigma\beta)&=C(\alpha,\beta)\end{aligned}$$

for $\alpha,\beta,\gamma\in Q$. Also $C(\alpha,\beta)=C(\beta,\alpha)^{-1}$.

The automorphism σ can be lifted to an automorphism $\hat{\sigma}$ of the extension of $\hat{Q}$ of Q such that

(3.18) $$\begin{aligned} (\hat{\sigma}a)^-&=\sigma\bar{a}\qquad\text{for all } a\in\hat{Q}\\ \hat{\sigma}a&=a\qquad\text{whenever } \sigma\bar{a}=\bar{a}.\end{aligned}$$

We make a conventional choice $\bar{a}_i=\alpha_i$, $i=1,\dots,l$. Under the action of σ, the vertices of the Dynkin diagram Γ decomposes into $k\sigma$-orbits, i.e.

(3.19) $$\Gamma=\Gamma_0\cup\Gamma_1\cup\dots\cup\Gamma_{k-1}$$

Then we denote by

(3.20) $$N=\{\alpha\in Q\,|\,\langle\alpha,\alpha_i\rangle=0,\, i\in\Gamma_0\}.$$

By $\hat{N}$ we mean the pullback of N to $\hat{Q}$.

From [11, Sect. 6] there exists an irreducible finite-dimensional $\hat{N}$-module T on which ω_0 acts as the multiplication by itself and for each $a\in\hat{Q}$, the element $a\hat{\sigma}a^{-1}\in\hat{Q}$ acts as the following scalar multiplication

(3.21) $$a\hat{\sigma}a^{-1}\mapsto\omega^{-\langle\bar{a},\Sigma\sigma^r\bar{a}\rangle/2}.$$

Consider the induced $\hat{Q}$-module

(3.22) $$U=\mathbb{C}[\hat{Q}]\otimes_{\mathbb{C}[\hat{N}]}T$$

on which the action of $\hat{Q}$ and $Q_0=\{\alpha\in Q\,|\,\sigma(\alpha)=\alpha\}$ are as follows:

(3.23) $$\begin{aligned} &a.b\otimes t=ab\otimes t \qquad && a,b\in\hat{Q}, \quad t\in T\\ &\alpha.b\otimes t=\langle\alpha,\bar{b}\rangle\, b\otimes t && \alpha\in Q_0.\end{aligned}$$

We define $z^{\alpha(0)}(\alpha\in Q)$ as an operator on U via

(3.24) $$z^{\alpha(0)}.b\otimes t=z^{\langle\alpha,\bar{b}\rangle}b\otimes t$$

and the operator $\omega^{\alpha(0)}$ on U by $\omega^{\alpha(0)}.b\otimes t=\omega^{\langle\alpha,\bar{b}\rangle}b\otimes t$.

It is not difficult to check that as operators on U

(3.25) $$z^{\alpha(0)}a=a\,z^{\alpha(0)+\langle\alpha,\bar{a}\rangle}$$

(3.26) $$\omega^{\alpha(0)}a=a\,\omega^{\alpha(0)+\langle\alpha,\hat{a}\rangle}$$

(3.27) $$\hat{\sigma}\,a=a\,\omega^{-\Sigma\sigma^r\bar{a}(0)-\langle\bar{a},\Sigma\sigma^r\bar{a}\rangle/2}.$$

Motivated by (3.8) we introduce the following:

(3.28) **Definition.** *The space of vertex operators is formed as*

$$V=S(\mathfrak{h}_\sigma^-)\otimes U.$$

Set for $\alpha\in\Delta$, the four exponential operators $E_-^{\pm}(\alpha,z)$, $E_-^{-}(\alpha,z)$; $E_+^{+}(\alpha,z)$, $E_-^{\pm}(\alpha,z)$ on V via

(3.29) $$E_-^{\pm}(\alpha,z)=\exp\left(\pm\sum_{n=1}^{+\infty}\frac{q^{\mp n/2}}{[n]}\alpha(-n)\,z^n\right)$$

(3.30) $$E_+^{\pm}(\alpha,z)=\exp\left(\mp\sum_{n=1}^{+\infty}\frac{q^{\mp n/2}}{[n]}\alpha(n)\,z^{-n}\right).$$

Then the vertex operators $X_i^+(z)$ and $X_i^-(z)$ on V are defined as

$$X_i^{\pm}(z)=E_-^{\pm}(\pm\alpha_i,z)\,E_+^{\pm}(\pm\alpha_i,z)\,a_i^{\pm 1}\,z^{\pm\Sigma\sigma^r\bar{a}_i(0)+\Sigma\langle\bar{a}_i,\sigma^r\bar{a}_i\rangle/2}$$
$$=\sum_{n\in\mathbb{Z}}X_i^{\pm}(n)\,z^{-n} \tag{3.31}$$

Remark 1. Although the vertex operators $X_i^{\pm}(z)$ map V into the formal power series $V\{z\}$, their components $X_i^{\pm}(n)$ are well-defined operators on V.

Remark 2. If $\sigma=1$, the above construction of $X_i^{\pm}(z)$ is the untwisted vertex representation of $\mathcal{U}_q(\hat{\mathfrak{g}})$ announced in [6] and proved in [10].

If we specialize $q=1$, the above construction is an important case studied first by Frenkel [5] for $k=2$ and then by Lepowsky [11] (cf. [12]) for general k. In fact, let $\gamma=q^c$ in (3.6) and taking limit $q\to 1$ we have

$$[\alpha_{im},\alpha_{in}]=\delta_{m,-n}\left(\sum_{\gamma=0}^{k-1}\langle\alpha_i,\sigma^r\alpha_j\rangle\,\omega^{mr}\right)mc. \tag{3.32}$$

On the other hand in the context of [11], let

$$x_{(n)}=\frac{1}{k}\sum_{r\in\mathbb{Z}/k\mathbb{Z}}\omega^{-ni}\sigma^i x \qquad \text{for } x\in\mathfrak{h} \tag{3.33}$$

then $\sigma x_{(n)}=\omega^n x_{(n)}$.

Recall further that the twisted Heisenberg algebra is

$$\hat{\mathfrak{h}}_\sigma=\bigoplus_{n\in\mathbb{Z}}\mathfrak{h}_{(n)}\otimes t^n\oplus\mathbb{C}c \tag{3.34}$$

where $\mathfrak{h}_{(n)}=\{x\in\mathfrak{h}\,|\,\sigma x=\omega^n x\}$ and the relations are:

$$[x\otimes t^m,y\otimes t^n]=\frac{1}{k}\langle x,y\rangle\,m\,\delta_{m,-n}\,c. \tag{3.35}$$

Set $\alpha_{in}=k(\alpha_i)_{(n)}\otimes t^n$, then from (3.35) it follows that

$$\begin{aligned}[\alpha_{im},\alpha_{in}]&=\delta_{m,-n}\,km\langle(\alpha_i)_{(m)},(\alpha_i)_{(-m)}\rangle\,c\\&=\delta_{m,-n}\frac{m}{k}\sum_{r,p\in\mathbb{Z}/k\mathbb{Z}}\langle\omega^{-mr}\sigma^r\alpha_i,\omega^{mp}\sigma^p\alpha_j\rangle\,c\\&=\delta_{m,-n}\left(\sum_{r=0}^{k-1}\langle\alpha_i,\sigma^r\alpha_j\rangle\,\omega^{mr}\right)mc\end{aligned}$$

which is exactly the same of (3.32). In other words, our construction is specialized to that of [11].

We now state the main result of our paper.

(3.36) **Theorem.** *The basic representations of the twisted quantum affine algebras $\mathcal{U}_q(\hat{\mathfrak{g}}_\sigma)$ are given by the twisted vertex operators defined above in the following way*

$$\gamma \mapsto q (c \mapsto 1)$$
$$\phi_{in} \mapsto \Phi_i(n), \qquad \psi_{im} \mapsto \Psi_i(m)$$
$$x_{in}^{\pm} \mapsto Y_i^{\pm}(n)$$

where $Y_i^{\pm}(n)$ are defined by $Y_i^{\pm}(z) = \sum_{n\in\mathbb{Z}} Y_i^{\pm}(n) z^{-n} = \varepsilon(\alpha_i)\, X_i^{\pm}(z)$, and $\varepsilon(\alpha_i)$ is a constant (cf. (4.18–19)).

4. The proof of the theorem

We are going to prove Theorem (3.36) by the so-called formal series calculus in the theory of the ordinary vertex operator algebras, which is developed in several papers and culminated in the monograph [8].

The counterpart approach in the quantum case enjoys the same setting, thus we shall only emphasize those phenomena which are characteristic in the case of quantum affine algebras. We remark that a parallel method – contour integrals can be written down word for word according to what follows (cf. Appendix in [8]).

The idea of the proof is that starting from the relations of twisted Heisenberg algebras we shall show that the operators $X_i^{\pm}(z)$, $\Phi_i(z)$, and $\Psi_i(z)$ satisfy all the relations of Drinfeld's realization with $\gamma = q$. More explicitly, we want to show that $X_i^{\pm}(z)$, $\Phi_i(z)$ and $\Psi_i(z)$ satisfy relations (2.19–2.29) in terms of (3.6) and our constructions of vertex operators. We are going to prove it relation by relation, and divide the proof into several steps.

Recalling the proof of (3.3) we already prove the relations (2.23). As for the relation (2.20–22), they are proved either by definition or by relations (3.6).

Relations (2.24)–(2.25) are the following

(4.1) **Lemma.** *For $i, j \in \Gamma$, we have*

$$\Phi_i(z)\, X_j^{\pm}(w)\, \Phi_i(z)^{-1} = g_{ij}\left(\frac{z}{w} q^{\mp\frac{1}{2}}\right)^{\pm 1} X_j^{\pm}(w),$$

$$\Psi_i(z)\, X_j^{\pm}(w)\, \Psi_i(z)^{-1} = g_{ij}\left(\frac{z}{w} q^{\mp\frac{1}{2}}\right)^{\mp 1} X_j^{\pm}(w).$$

Proof. From (3.11–12) and (3.29–30) we compute that

$$\begin{aligned}
&\Phi_i(z)\, E_{\mp}^{\pm}(\alpha_j, w)\\
&= E_{\mp}^{\pm}(\alpha_j, w)\, \Phi_i(z) \exp\left\{ \pm t \sum_{n=1}^{\infty} \frac{q^{\mp\frac{n}{2}}}{[n]} [\alpha_i(-n), \alpha_j(n)] \left(\frac{z}{w}\right)^n \right\}\\
&= E_{\mp}^{\pm}(\alpha_j, w)\, \Phi_i(z) \exp\left\{ \mp \sum_{\substack{n\geq 1\\ r\in\mathbb{Z}/k\mathbb{Z}}} \frac{\omega^{nr}}{n} \left(q^{n\langle\alpha_i, \sigma^r\alpha_j\rangle} - q^{-n\langle\alpha_i, \sigma^r\alpha_j\rangle}\right) \cdot q^{\mp\frac{n}{2}} \left(\frac{z}{w}\right)^n \right\}\\
&= E_{\mp}^{\pm}(\alpha_j, w)\, \Phi_i(z) \left(\prod_{r\in\mathbb{Z}/k\mathbb{Z}} \frac{1 - \omega^r q^{\langle\alpha_i, \sigma^r\alpha_j\rangle \mp \frac{1}{2}} z w^{-1}}{1 - \omega^r q^{-\langle\alpha_i, \sigma^r\alpha_j\rangle \mp \frac{1}{2}} z w^{-1}} \right)^{\pm 1}\\
&= E_{\mp}^{\pm}(\alpha_j, w)\, \Phi_i(z)\, q^{\pm\sum_r \langle\alpha_i, \sigma^r\alpha_j\rangle} g_{ij}\left(\frac{z}{w} q^{\mp\frac{1}{2}}\right)^{\pm 1}.
\end{aligned}$$

An analogous computation will give

$$\Psi_i(z)\, E^{\pm}_{-}(\alpha_j, w) = E^{\pm}_{-}(\alpha_j, w)\, \Psi_i(z)\, q^{\mp\sum_r \langle \alpha_i, \sigma^r \alpha_j \rangle}\, g_{ij}\left(\frac{z}{w} q^{\mp\frac{1}{2}}\right)^{\mp 1}.$$

Hence the lemma is proved once we observe the following identity about the middle term of $X_i^{\pm}(z)$:

$$e^{t\alpha_i(0)/2}\, a_j^{\pm 1}\, z^{\pm\sum_r \sigma^r \alpha_j(0)} = q^{\pm\sum_r \langle \alpha_i, \sigma^r \alpha_j \rangle}\, a_j^{\pm 1}\, z^{\pm\sum_r \sigma^r \alpha_j(0)}\, e^{t\alpha_i(0)/2}$$

which is an immediate consequence of (3.25), since $e^{t\alpha_i(0)/2} = q^{\alpha_i(0)}$.

To prove the remaining relations we need to introduce a useful notion – *normal ordering*, which plays an important role in the theory of ordinary vertex operator calculus. The effect of normal ordering is to move annihilation operators to the right of multiplication operators. In our case the operators $\alpha_i(n)$ are annihilation operators for $n \in \mathbb{Z}_+$ and multiplication operators otherwise. Thus we have

(4.2) $$:\alpha_i(n)\, \alpha_j(-n): = :\alpha_j(-n)\, \alpha_i(n): = \alpha_j(-n)\, \alpha_i(n), \qquad n > 0$$

(4.3) $$:\alpha_i(0)\, a_j: = :a_j\, \alpha_i(0): = \tfrac{1}{2}(\alpha_i(0)\, a_j + a_j\, \alpha_i(0))$$

and all the others are trivial normal ordering, i.e. the same as the usual products.

We can extend the notion to the vertex operators. For example we define

(4.4) $$\begin{aligned} :X_i^{\pm}(z)\, X_j^{\pm}(w): &= E^{\pm}_{-}(\alpha_i, z)\, E^{\pm}_{-}(\alpha_j, w)\, E^{\pm}_{+}(\alpha_i, z)\, E^{\pm}_{+}(\alpha_j, w) \\ &\cdot a_i^{\pm}\, b_j^{\pm 1}\, z^{\pm\sum_r \sigma^r \alpha_i(0) + \langle \sum_r \sigma^r \alpha_i, \alpha_i + \alpha_j \rangle}\, w^{\pm\sum_r \sigma^r \alpha_j(0) + \langle \sum_r \sigma^r \alpha_j, \alpha_j + \alpha_i \rangle} \end{aligned}$$

(4.5) $$\begin{aligned} :X_i^{\pm}(z)\, X_j^{\mp}(w): &= E^{\pm}_{-}(\alpha_i, z)\, E^{\mp}_{-}(\alpha_j, w)\, E^{\pm}_{+}(\alpha_i, z)\, E^{\mp}_{+}(\alpha_j, w) \\ &\cdot a_i^{\pm 1}\, b_j^{\mp 1}\, z^{\pm\sum_r \sigma^r \alpha_i(0) + \langle \sum_r \sigma^r \alpha_i, \alpha_i - \alpha_j \rangle}\, w^{\mp\sum_r \sigma^r \alpha_j(0) + \langle \sum_r \sigma^r \alpha_j, \alpha_j - \alpha_i \rangle} \end{aligned}$$

and we can use Wick's theorem (cf. [8]) to extend the normal ordering to products of several vertex operators.

We notice that the normal orderings satisfy the following relations:

(4.6) $$:X_i^{\pm}(z)\, X_j^{\pm}(w): = C(\alpha_i, \alpha_j):X_j^{\pm}(w)\, X_i^{\pm}(z):$$

(4.7) $$:X_i^{\pm}(z)\, X_j^{\mp}(w): = C(\alpha_i, -\alpha_j):X_j^{\mp}(w)\, X_i^{\pm}(z):$$

where $C(\alpha, \beta) = \prod_{r \in \mathbb{Z}/k\mathbb{Z}} (-\omega^r)^{\langle \alpha, \sigma^r \beta \rangle}$ for the central extension $\hat{Q}$ of Q.

We also need the following standard notations.

(4.8) $$(a; q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1})$$

(4.9) $$(a; q)_\infty = \lim_{n \to \infty} (a; q)_n = \prod_{n \geq 0} (1 - aq^n).$$

Thus we may define $(a; q)_n$ by

(4.10) $$(a; q)_n = (a; q)_\infty / (aq^n; q)_\infty$$

Moreover, we have the following useful product.

(4.11)
$$(1-q^{m-1}\,a)(1-q^{m-3}\,a)\dots(1-q^{-m+1}\,a)=\frac{(q^{-m+1}\,a;q^2)_\infty}{(q^{m+1}\,a;q^2)_\infty},\qquad \text{for } m\in\mathbb{N}$$

where m specifies the number of factors on the left hand side.

(4.12) **Lemma.** *For $i,j\in\Gamma$, we have*

$$X_i^{\pm}(z)\,X_j^{\mp}(w)=:X_i^{\pm}(z)\,X_j^{\mp}(w):\,\left(\frac{z}{w}\right)^{-\langle\alpha_i,\Sigma\sigma^r\alpha_j\rangle/2}\cdot\prod_{r\in\mathbb{Z}/k\mathbb{Z}}\frac{(q^{-\langle\alpha_i,\sigma^r\alpha_j\rangle+1}\,\omega^r\,w/z;q^2)_\infty}{(q^{\langle\alpha_i,\sigma^r\alpha_j\rangle+1}\,\omega^r\,w/z;q^2)_\infty}$$

$$X_i^{\pm}(z)\,X_j^{\pm}(w)=:X_i^{\pm}(z)\,X_j^{\pm}(w):\,\left(\frac{z}{w}\right)^{\langle\alpha_i,\Sigma\sigma^r\alpha_j\rangle/2}\cdot\prod_{r\in\mathbb{Z}/k\mathbb{Z}}\frac{(q^{\langle\alpha_i,\sigma^r\alpha_j\rangle+1}\,q^{\mp1}\,\omega^r\,w/z;q^2)_\infty}{(q^{-\langle\alpha_i,\sigma^r\alpha_j\rangle+1}\,q^{\mp1}\,\omega^r\,w/z;q^2)_\infty}$$

where the two factors are called contraction factors, denoted as $\underline{X_i^{\pm}(z)\,X_j^{\mp}}(w)$ *and* $\underline{X_i^{\pm}(z)\,X_j^{\pm}}(w)$ *respectively in the sequel.*

Proof. The vertex operator is a product of two exponential operators and a middle term operator. So we consider

$$\begin{aligned}&E_+^{\pm}(\alpha_i,z)\,E_-^{\mp}(-\alpha_j,w)\\&=E_-^{\mp}(-\alpha_j,w)\,E_+^{\pm}(\alpha_i,z)\exp\left\{-\sum_{n=1}^{\infty}\frac{1}{[n]_q^2}\,[\alpha_i(n),\alpha_j(-n)]\left(\frac{w}{z}\right)^n\right\}\\&=E_-^{\mp}(-\alpha_j,w)\,E_+^{\pm}(\alpha_i,z)\exp\\&\quad\cdot\left\{\sum_{\substack{n\geq1\\r\in\mathbb{Z}/k\mathbb{Z}}}\frac{q^{-n\langle\alpha_i,\sigma^r\alpha_j\rangle}-q^{n\langle\alpha_i,\sigma^r\alpha_j\rangle}}{n(q^n-q^{-n})}\left(\omega^r\,\frac{w}{z}\right)^n\right\}\end{aligned}$$

$$\begin{aligned}&E_+^{\pm}(\alpha_i,z)\,E_-^{\pm}(\alpha_j,w)\\&\quad=E_-^{\pm}(\alpha_j,w)\,E_+^{\pm}(\alpha_i,z)\exp\\&\quad\cdot\left\{\sum_{\substack{n\geq1\\r\in\mathbb{Z}/k\mathbb{Z}}}\frac{q^{n\langle\alpha_i,\sigma^r\alpha_j\rangle}-a^{-n\langle\alpha_i,\sigma^r\alpha_j\rangle}}{n(q^n-q^{-n})}\left(\omega^r\,q^{\mp1}\,\frac{w}{z}\right)^n\right\}.\end{aligned}$$

For $m\in\mathbb{N}$ and a with $|a|<1$ we compute that

(4.13)
$$\begin{aligned}&\exp\left\{\sum_{n=1}^{\infty}\frac{q^{mn}-q^{-mn}}{n(q^n-q^{-n})}\,a^n\right\}\\&=\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}\,(q^{(m-1)n}+q^{(m-3)n}+\dots q^{-(m-1)n})\,a^n\right\}\\&=(1-q^{m-1}\,a)^{-1}\,(1-q^{m-3}\,a)^{-1}\dots(1-q^{-(m-1)}\,a)^{-1}\\&=\frac{(q^{m+1}\,a;q^2)_\infty}{(q^{-m+1}\,a;q^2)_\infty}\end{aligned}$$

where we have used the formula: $\log(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}$.

Observe that the extremes of (4.13) is also true for any integer m, i.e. we obtain an invariant formula for the exponential with respect to $m\in\mathbb{Z}$. Hence we have

$$\begin{aligned}E_+^{\pm}(\alpha_i,z)\,E_-^{\mp}(-\alpha_j,w)&=E_-^{\mp}(-\alpha_j,w)\,E_+^{\pm}(\alpha_i,z)\\&\quad\cdot\prod_{r\in\mathbb{Z}/k\mathbb{Z}}\frac{(q^{-\langle\alpha_i,\sigma^r\alpha_j\rangle+1}\omega^r w/z;q^2)_\infty}{(q^{\langle\alpha_i,\sigma^r\alpha_j\rangle+1}\omega^r w/z;q^2)_\infty}\\E_+^{\pm}(\alpha_i,z)\,E_-^{\pm}(\alpha_j,w)&=E_-^{\pm}(\alpha_j,w)\,E_+^{\pm}(\alpha_i,z)\\&\quad\cdot\prod_{r\in\mathbb{Z}/k\mathbb{Z}}\frac{(q^{\langle\alpha_i,\sigma^r\alpha_j\rangle+1}q^{\mp1}\omega^r w/z;q^2)_\infty}{(q^{-\langle\alpha_i,\sigma^r\alpha_j\rangle+1}q^{\mp1}\omega^r w/z;q^2)_\infty}.\end{aligned}$$

Finally as for the middle term we have from (3.25)

$$\begin{aligned}&a\,z^{\Sigma\sigma^r\alpha+\langle\Sigma\sigma^r\alpha,\alpha\rangle/2}\,b\,w^{\Sigma\sigma^r\beta+\langle\Sigma\sigma^r\beta,\beta\rangle/2}\\&=ab\,z^{\Sigma\sigma^r\alpha+\langle\Sigma\sigma^r\alpha,\alpha\rangle/2+\langle\Sigma\sigma^r\alpha,\beta\rangle}\,w^{\Sigma\sigma^r\beta+\langle\Sigma\sigma^r\beta,\beta\rangle/2}\\&=ab\,z^{\Sigma\sigma^r\alpha+\langle\Sigma\sigma^r\alpha,\alpha+\beta\rangle/2}\,w^{\Sigma\sigma^r\beta+\langle\Sigma\sigma^r\beta,\beta+\alpha\rangle/2}\left(\frac{z}{w}\right)^{\langle\Sigma\sigma^r\alpha,\beta\rangle/2}\end{aligned}$$

by using $\langle\sum\sigma^r\alpha,\beta\rangle=\langle\sum\sigma^r\beta,\alpha\rangle$. Here $a,b\in\hat{Q}$, and $\bar{a}=\alpha$, $\bar{b}=\beta$.

Combining the above identities and the definition of: $X_i^{\pm}(z)\,X_j^{\pm}(w)$: and : $X_i^{\pm}(z)\,X_j^{\mp}(w)$:, we derive the required relations in (4.12).

Remark. Let $I_n=I_n(\alpha,\beta)=\{r\in\mathbb{Z}/k\mathbb{Z}\,|\,\langle\alpha,\sigma^r\beta\rangle=n\}$. Then we have

$$\begin{aligned}&\prod_{r\in\mathbb{Z}/k\mathbb{Z}}\frac{(q^{-\langle\alpha_i,\sigma^r\alpha_j\rangle+1}\omega^r w/z;q^2)_\infty}{(q^{\langle\alpha_i,\sigma^r\alpha_j\rangle+1}\omega^r w/z;q^2)_\infty}\\&=\prod_{r\in I_{-1}}\left(1-\omega^r\frac{w}{z}\right)\prod_{r\in I_2}\left(1-\omega^r\frac{w}{z}q\right)^{-1}\left(1-\omega^r\frac{w}{z}q^{-1}\right)^{-1}\\&\prod_{r\in\mathbb{Z}/k\mathbb{Z}}\frac{(q^{\langle\alpha_i,\sigma^r\alpha_j\rangle+1}q^{\mp1}\omega^r w/z;q^2)_\infty}{(q^{-\langle\alpha_i,\sigma^r\alpha_j\rangle+1}q^{\mp1}\omega^r w/z;q^2)_\infty}\\&=\prod_{r\in I_{-1}}\left(1-\omega^r\frac{w}{z}q^{\mp1}\right)^{-1}\prod_{r\in I_2}\left(1-\omega^r\frac{w}{z}\right)\left(1-\omega^r\frac{w}{z}q^{\mp2}\right)\end{aligned}$$

where $I_n=I_n(\alpha_i,\alpha_j)$.

Then we can prove the relation (2.26) as follows.

(4.14) **Lemma.** *The vertex operators satisfy the following*

$$F_{ij}^{\pm}(z,w)\,X_i^{\pm}(z)\,X_j^{\pm}(w)=G_{ij}^{\pm}(z,w)\,X_j^{\pm}(w)\,X_i^{\pm}(z)$$

which is understood as an identity of formal series, and $F_{ij}^{\pm}$ and $G_{ij}^{\pm}$ are defined in the remark after (2.10–11).

Proof. Formally we have

$$\begin{aligned}\frac{F_{ij}^{\pm}(z,w)}{G_{ij}^{\pm}(z,w)}&=\prod_{r\in\mathbb{Z}/k\mathbb{Z}}\frac{(z-\omega^r w q^{\pm\langle\alpha_i,\sigma^r\alpha_j\rangle})}{(zq^{\pm\langle\alpha_i,\sigma^r\alpha_j\rangle}-\omega^r w)}\\
&=\prod_{r\in I_{-1}}\frac{z-\omega^{-r}wq^{\mp 1}}{zq^{\mp 1}-\omega^r w}\prod_{r\in I_2}\frac{(z-\omega^r w q^{\pm 2})(z-\omega^r w)}{(zq^{\pm 2}-\omega^{-r}w)(z-\omega^{-r}w)}\\
&=\prod_{r\in\mathbb{Z}/z\mathbb{Z}}(-\omega^r)^{\langle\alpha_i,\sigma^r\alpha_j\rangle}\prod_{r\in I_{-1}}\frac{w-\omega^r z q^{\pm 1}}{z-\omega^r w q^{\pm 1}}\prod_{r\in I_2}\frac{(z-\omega^r w q^{\pm 2})(z-\omega^r w)}{(w-\omega^r z q^{\pm 2})(w-\omega^r z)}\\
&=C(\alpha_i,\alpha_j)\left(\frac{z}{w}\right)^{-\Sigma\langle\alpha_i,\sigma^r\alpha_j\rangle}\prod_{r\in\mathbb{Z}/k\mathbb{Z}}\frac{(q^{\langle\alpha_i,\sigma^r\alpha_j\rangle+1}q^{\mp 1}\omega^r w/z;q^2)_\infty}{(q^{-\langle\alpha_i,\sigma^r\alpha_j\rangle+1}q^{\mp 1}\omega^r w/z;q^2)_\infty}\\
&\quad\cdot\prod_{r\in\mathbb{Z}/k\mathbb{Z}}\frac{(q^{-\langle\alpha_j,\sigma^r\alpha_i\rangle+1}q^{\mp 1}\omega^r z/w;q^2)_\infty}{(q^{\langle\alpha_j,\sigma^r\alpha_i\rangle+1}q^{\mp 1}\omega^r z/w;q^2)_\infty}\end{aligned}$$

Notice that: $X_i^{\pm}(z)X_j^{\pm}(w){:=}C(\alpha_i,\alpha_j){:}\ X_j^{\pm}(w)X_i^{\pm}(z){:}$ and combining with Lemma (4.12) we know that (4.14) is true.

Let us pause to talk about the automorphisms of the Dynkin diagrams of simply laced types. It is well known that the automorphism σ can be described by the Fig. 2

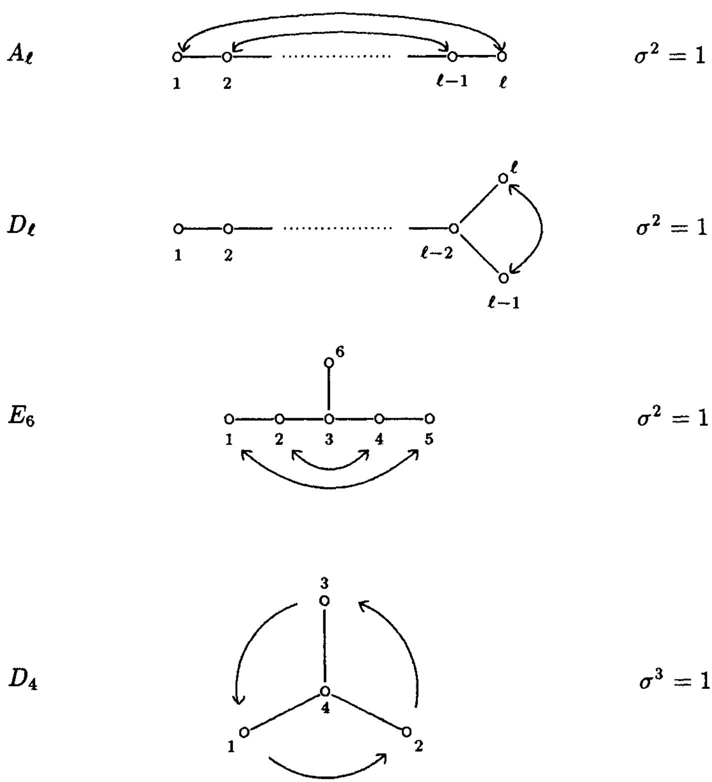


Fig. 2

where σ fixes the vertices which are not pointed by any arrows, and σ sends the vertex i to the vertex j if there is an arrow from i to j.

Then we can show

(4.15) **Proposition.** *For $i, j \in \Gamma$, one has*

$$[Y_i^+(z), Y_j^-(w)]$$
$$=\frac{1}{q-q^{-1}}\sum_{r=0}^{k-1}\delta_{\sigma^r(i),j}\left\{\Psi_i(zq^{-\frac{1}{2}})\,\delta\left(\frac{z}{w}\omega^r q^{-1}\right)-\Phi_i(zq^{\frac{1}{2}})\,\delta\left(\frac{z}{w}\omega^r q\right)\right\}.$$

Proof. For $i, j \in \Gamma$ if the automorphism σ^r has not the property $\alpha_i=\sigma^r\alpha_j$, then $I_2(\alpha_i,\alpha_j)=\{r\in\mathbb{Z}/k\mathbb{Z}\,|\,\langle\alpha_i,\sigma^r\alpha_j\rangle=2\}=\emptyset$. From the remark after (4.12) it follows that both contraction factors $\underbracket{X_i^+(z)\,X_j^-(w)}$ and $\underbracket{X_j^-(w)\,X_i^+(z)}$ are polynomials in $\frac{w}{z}$ and $\frac{z}{w}$ respectively. Furthermore, we observe that

$$\underbracket{X_i^+(z)\,X_j^-(w)}=\underbracket{X_j^-(w)\,X_i^+(z)}\,C(\alpha_i,-\alpha_j)$$

Thus in the case we have that

$$[X_i^+(z), X_j^-(w)]=:X_i^+(z)\,X_j^-(w):\ (\underbracket{X_i^+(z)\,X_j^-(w)}-\underbracket{X_j^-(w)\,X_i^+(z)}\,C(-\alpha_j,\alpha_i))$$
$$=0$$

Observe that $X^\pm_{\sigma(i)}(z)=X_i^\pm(\omega z)$, then if $I_2(\alpha_i,\alpha_j)\neq\emptyset$, we are essentially led to the following four cases.

Case i). $\sigma^2=1$, $\alpha_i=\alpha_j$ and $\langle\alpha_i,\sigma(\alpha_j)\rangle=0$. In this case one has

$$X_i^+(z)\,X_j^-(w)=:X_i^+(z)\,X_j^-(w):\left(\frac{w}{z}\right)\left(1-q^{-1}\frac{w}{z}\right)^{-1}\left(1-q\frac{w}{z}\right)^{-1}$$
$$C(\alpha_i,-\alpha_j)=1.$$

Then by the partial fractions we obtain that

$$\text{(4.16)}\quad \frac{1}{\left(1-q^{-1}\frac{w}{z}\right)\left(1-q\frac{w}{z}\right)}=\frac{1}{q-q^{-1}}\ \frac{q}{1-q\frac{w}{z}}-\frac{q^{-1}}{1-q^{-1}\frac{w}{z}}$$
$$=\frac{1}{q-q^{-1}}\left(\sum_{n\geq 0}q^{n+1}\left(\frac{w}{z}\right)^n-\sum_{n\geq 0}q^{-n-1}\left(\frac{w}{z}\right)^n\right)$$
$$=\sum_{n\geq 0}\frac{q^{n+1}-q^{-n-1}}{q-q^{-1}}\left(\frac{w}{z}\right)^n.$$

Therefore, the bracket will be

$$
\begin{aligned}
&[X_i^+(z), X_j^-(w)]\\
&=:X_i^+(z)\,X_i^-(w)\!:\left(\sum_{n\geqq 0}\frac{q^{n+1}-q^{-n-1}}{q-q^{-1}}\left(\frac{w}{z}\right)^{n+1}\right.\\
&\left.-\sum_{n\geqq 0}\frac{q^{n+1}-q^{-n-1}}{q-q^{-1}}\left(\frac{z}{w}\right)^{n+1}\right)\\
&=:X_i^+(z)\,X_i^-(w)\!:\left(\sum_{n\in\mathbf{Z}}\left(q^{-1}\frac{z}{w}\right)^n-\sum_{n\in\mathbf{Z}}\left(q\frac{z}{w}\right)^n\right)\Big/(q-q^{-1})\\
&=:X_i^+(z)\,X_i^-(w)\!:\left(\delta\left(\frac{z}{w}q^{-1}\right)-\delta\left(\frac{z}{w}q\right)\right)\Big/(q-q^{-1}).
\end{aligned}
$$

Hence the required relation is proved in the case, provided one has the following important identities:

(4.17) **Lemma.** *As operators on V we have*

$$
\begin{aligned}
:X_i^+(zq^{-1})\,X_i^-(z)\!:&=\Phi_i(zq^{-\frac{1}{2}})\\
:X_i^+(zq)\,X_i^-(z)\!:&=\Psi_i(zq^{\frac{1}{2}})
\end{aligned}
$$

which can be easily verified directly.

Case ii). $\sigma^2=1$, $\alpha_i=\alpha_j$, $\langle\sigma(\alpha_j),\alpha_i\rangle=-1$. We then have

$$
X_i^+(z)\,X_j^-(w)=:X_i^+(z)\,X_j^-(w)\!:\left(\frac{w}{z}\right)^{\frac{1}{2}}\left(1+\frac{w}{z}\right)\left(1-q^{-1}\frac{w}{z}\right)^{-1}\left(1-q\frac{w}{z}\right)^{-1}
$$

$$
C(\alpha_i,-\alpha_j)=1.
$$

As in the Case i) we have

$$
\begin{aligned}
&[X_i^+(z), X_j^-(w)]\\
&=:X_i^+(z)\,X_i^-(w)\!:\ (zw)^{-\frac{1}{2}}(z+w)\left(\left(\frac{w}{z}\right)\left(1-q\frac{w}{z}\right)^{-1}\left(1-q^{-1}\frac{w}{z}\right)^{-1}\right.\\
&\qquad\left.-\left(\frac{z}{w}\right)\left(1-q\frac{z}{w}\right)^{-1}\left(1-q^{-1}\frac{z}{w}\right)^{-1}\right)\\
&=:X_i^+(z)\,X_i^-(w)\!:\ (zw)^{-\frac{1}{2}}(z+w)\left(\delta\left(\frac{z}{w}q^{-1}\right)-\delta\left(\frac{z}{w}q\right)\right)\Big/(q-q^{-1})\\
&=\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{q-q^{-1}}\left\{\Psi_i(zq^{-\frac{1}{2}})\,\delta\left(\frac{z}{w}q^{-1}\right)-\Phi_i(zq^{\frac{1}{2}})\,\delta\left(\frac{z}{w}q\right)\right\}.
\end{aligned}
$$

Case iii). $\sigma^k=1$, $\sigma(\alpha_i)=\alpha_i$. Then

$$X_i^+(z)\,X_i^-(w)=:X_i^+(z)\,X_i^-(w):\left(\frac{w}{z}\right)^k\prod_{r\in\mathbb{Z}/k\mathbb{Z}}\left(1-\omega^r q\frac{w}{z}\right)^{-1}\left(1-\omega^r q^{-1}\frac{w}{z}\right)^{-1}$$

$$C(\alpha_i,-\alpha_i)=1.$$

It is well-known that

$$\frac{1}{\prod_{r\in\mathbb{Z}/k\mathbb{Z}}(1-\omega^r x)}=\frac{1}{k}\sum_{r\in\mathbb{Z}/k\mathbb{Z}}\frac{1}{1-\omega^r x}.$$

Using (4.16) it follows that

$$\frac{1}{\prod_{r\in\mathbb{Z}/k\mathbb{Z}}\left(1-\omega^r q\frac{w}{z}\right)\left(1-\omega^r q^{-1}\frac{w}{z}\right)}
=\frac{1}{k(q^k-q^{-k})}\sum_{r\in\mathbb{Z}/k\mathbb{Z}}\left(\frac{q^k}{1-\omega^r q\frac{w}{z}}-\frac{q^{-k}}{1-\omega^r q^{-1}\frac{w}{z}}\right).$$

Then we arrive at

$$[X_i^+(z),X_i^-(w)]
=\frac{1}{k(q^k-q^{-k})}\sum_{r\in\mathbb{Z}/k\mathbb{Z}}\left\{\Psi_i(zq^{-\frac{1}{2}})\,\delta\left(\frac{z}{w}\omega^r q^{-1}\right)-\Phi_i(zq^{\frac{1}{2}})\,\delta\left(\frac{z}{w}\omega^r q\right)\right\}.$$

Case iv). $\sigma^3=1$, and $i=\sigma^{r_0}(j)$, $1\leqq i,j\leqq 3$ as in the Fig. 2. Then

$$X_i^+(z)\,X_j^-(w)
=:X_j^+(z)\,X_j^-(w):\left(1-\omega^{r_0}\frac{w}{z}q\right)^{-1}\left(1-\omega^{r_0}\frac{w}{z}q^{-1}\right)
\cdot\left(1-\omega^{r_0+1}\frac{w}{z}\right)\left(1-\omega^{r_0+2}\frac{w}{z}\right)$$

$$X_j^-(w)\,X_i^+(z)
=:X_j^-(w)\,X_i^+(z):\left(1-\omega^{-r_0}\frac{z}{w}q\right)^{-1}\left(1-\omega^{-r_0}\frac{z}{w}q^{-1}\right)^{-1}
\cdot\left(1-\frac{z}{w}\omega^{-r_0-1}\right)\left(1-\frac{z}{w}\omega^{-r_0-2}\right)$$

$$C(\alpha_i,-\alpha_j)
=(-\omega^{r_0})^{-2}(-\omega^{r_0+1})(-\omega^{r_0+2})=1.$$

Using the same partial fraction as in the case i), we have

$$
\begin{aligned}
&[X_i^+(z), X_j^-(w)] \\
&=: X_i^+(z) X_j^-(w): \frac{(z-\omega^{r_0+1} w)(z-\omega^{r_0+2} w)}{\omega^{r_0} z w} \\
&\quad \cdot \left(\delta\left(\frac{z}{w}\omega^{-r_0} q^{-1}\right) - \delta\left(\frac{z}{w}\omega^{-r_0} q\right)\right) \Big/ (q-q^{-1}) \\
&= \frac{(q+1+q^{-1})}{q-q^{-1}}: X_i^+(z) X_j^-(w): \left(\delta\left(\frac{z}{w}\omega^{-r_0} q^{-1}\right) - \delta\left(\frac{z}{w}\omega^{-r_0} q\right)\right) \\
&= \frac{(q+1+q^{-1})}{q-q^{-1}} \left(\Psi_i(zq^{-\frac{1}{2}})\,\delta\left(\frac{z}{w}\omega^{-r_0} q^{-1}\right) - \Phi_i(zq^{\frac{1}{2}})\,\delta\left(\frac{z}{w}\omega^{-r_0} q\right)\right)
\end{aligned}
$$

where we have used $X_j^\pm(z) = X_{\sigma(j)}^\pm(\omega z)$.

From the above discussion it is clear how to normalize the vertex operators such that they will satisfy the relation (2.27). In fact, we define $\varepsilon: \Gamma \to \mathbb{C}$ via

$$
(4.18) \qquad \varepsilon(\alpha) = \begin{cases} 1 & \text{if } \sigma^2 = 1, \langle \alpha, \sigma(\alpha)\rangle = 0; \\ (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{-\frac{1}{2}} & \text{if } \sigma^2 = 1, \langle \alpha, \sigma(\alpha)\rangle = -1; \\ \sqrt{k}(q^{k-1} + \dots + q^{-k+1})^{\frac{1}{2}} & \text{if } \sigma(\alpha) = \alpha, \sigma^k = 1. \\ (q+1+q^{-1})^{-\frac{1}{2}} & \text{if } \sigma^3 = 1, \sigma(\alpha) \neq \alpha. \end{cases}
$$

With ε we rescale the vertex operators $X_i^\pm(z)$ by

$$
(4.19) \qquad Y_i^\pm(z) = \varepsilon(\alpha_i)\, X_i^\pm(z).
$$

Then the vertex operators $Y_i^\pm(z)$ will satisfy the relation (4.15).

Finally let us prove the cubic Serre relations (2.28–29). Since the vertex operators $Y_i^\pm(z)$ are multiples of the vertex operators $X_i^\pm(z)$, we can just show that $X_i^\pm(z)$ satisfy the cubic Serre relations.

Although the proof is a little bit lengthy, the idea is very simple that almost all the cases of twisted cubic Serre relations can be derived from that of untwisted case. Moreover, the proof will explain why one needs to present the cubic Serre relations in five subcases (see (2.28–20)).

Now we start to show the relation (2.28) for $k=1$, i.e.

$$
\begin{aligned}
(4.20) \qquad & X_j^\pm(w) X_i^\pm(z_1) X_i^\pm(z_2) - (q+q^{-1}) X_i^\pm(z_1) X_j^\pm(w) X_i^\pm(z_2) \\
& + X_i^\pm(z_1) X_i^\pm(z_2) X_j^\pm(w) + \{z_1 \leftrightarrow z_2\} = 0 \qquad \text{for } A_{ij} = -1.
\end{aligned}
$$

We only prove the " + " case as follows. From (4.12) it follows that

$$X_j^+(w)X_i^+(z) = :X_j^+(w)X_i^+(z):\left(\frac{z}{w}\right)^{\frac{1}{2}}\left(1-\frac{z}{w}q^{-1}\right)^{-1} \tag{4.21}$$

$$X_i^+(z_1)X_i^+(z_2) = :X_i^+(z_1)X_i^+(z_2):\left(\frac{z_2}{z_1}\right)^{-1}\left(1-\frac{z_2}{z_1}\right)\left(1-\frac{z_2}{z_1}q^{-2}\right) \tag{4.22}$$

By the properties of normal orderings we then have

$$\begin{aligned}&X_j^+(w)X_i^+(z_1)X_i^+(z_2)\\ &= :X_j^+(w)X_i^+(z_2)X_i^+(z_2):\ \frac{z_1}{w}\left(\frac{z_2}{z_1}\right)^{-\frac{1}{2}}\frac{\left(1-\frac{z_2}{z_1}\right)\left(1-\frac{z_2}{z_1}q^{-2}\right)}{\left(1-\frac{z_1}{w}q^{-1}\right)\left(1-\frac{z_2}{w}q^{-1}\right)}\end{aligned} \tag{4.23}$$

$$\begin{aligned}&X_i^+(z_1)X_j^+(w)X_i^+(z_2)\\ &= :X_i^+(z_1)X_j^+(w)X_i^+(z_2):\ \left(\frac{z_2}{z_1}\right)^{-\frac{1}{2}}\frac{\left(1-\frac{z_2}{z_1}\right)\left(1-\frac{z_2}{z_1}q^{-2}\right)}{\left(1-\frac{z_2}{w}q^{-1}\right)\left(1-\frac{w}{z_1}q^{-1}\right)}\end{aligned} \tag{4.24}$$

$$\begin{aligned}&X_i^+(z_1)X_i^+(z_2)X_j^+(w)\\ &= :X_i^+(z_1)X_i^+(z_2)X_j^+(w):\ \frac{w}{z_2}\left(\frac{z_2}{z_1}\right)^{-\frac{1}{2}}\frac{\left(1-\frac{z_2}{z_1}\right)\left(1-\frac{z_2}{z_1}q^{-2}\right)}{\left(1-\frac{w}{z_1}q^{-1}\right)\left(1-\frac{w}{z_2}q^{-1}\right)}\end{aligned} \tag{4.25}$$

Since $C(\alpha_i, \alpha_i) = -C(\alpha_i, \alpha_j) = 1$, we have

$$:X_j^+(w)X_i^+(z_1)X_i^+(z_2): = -:X_i^+(z_1)X_j^+(w)X_i^+(z_2): = :X_i^+(z_1)X_i^+(z_2)X_j^+(w):$$

Then one might want to substitute all the above expressions into the right hand side of (4.20). After pulling out the factor: $X_j^+(w)\ X_i^+(z_1)\ X_i^+(z_2)$:, we can verify that the remaining rational functions sum up to zero. Thus one might want to conclude that the relation is proved. Unfortunately, this is not correct due to the fact that all the contraction factors are formal power series and one can not manipulate them simply as rational functions.

Let us proceed to show this formal series identity. Both as formal series and fractions, we can express the contraction factors as follows.

$$\frac{(z_1-z_2)\left(1-\frac{z_2}{z_1}q^{-2}\right)}{\left(1-\frac{z_1}{w}q^{-1}\right)\left(1-\frac{z_2}{w}q^{-1}\right)} = w\left(1-\frac{z_2}{z_1}q^{-2}\right)\left(\frac{q}{1-\frac{z_1}{w}q^{-1}}-\frac{q}{1-\frac{z_2}{w}q^{-1}}\right) \tag{4.26}$$

$$\frac{(z_1-z_2)\left(1-\frac{z_2}{z_1}q^{-2}\right)}{\left(1-\frac{w}{z_1}q^{-1}\right)\left(1-\frac{z_2}{w}q^{-1}\right)} = q^{-1}w(z_1-z_2)\left(\frac{1}{z_1\left(1-\frac{w}{z_1}q^{-1}\right)}+\frac{q}{w\left(1-\frac{z_2}{w}q^{-1}\right)}\right) \tag{4.27}$$

$$\frac{(z_1-z_2)\left(1-\frac{z_2}{z_1}q^{-2}\right)}{\left(1-\frac{w}{z_1}q^{-1}\right)\left(1-\frac{w}{z_2}q^{-1}\right)} = \left(1-\frac{z_2}{z_1}q^{-2}\right)\left(\frac{z_1}{1-\frac{w}{z_2}q^{-1}}-\frac{z_2}{1-\frac{w}{z_1}q^{-1}}\right) \tag{4.28}$$

Changing the positions of z_1 and z_2, we obtain the other three expressions for the part $\{z_1 \leftrightarrow z_2\}$ in the cubic Serre relation. Substitute the above expressions into the right hand side of (4.20) and factoring out the normal ordering product, we obtain that

$$\begin{aligned}(z_1 z_2)^{-\frac{1}{2}}\Bigg\{&(z_1-z_2q^{-2})\left(\frac{q}{1-\frac{z_1}{w}q^{-1}}-\frac{q}{1-\frac{z_2}{w}q^{-1}}\right)\\ &+(1+q^{-2})\,w(z_1-z_2)\left(\frac{1}{z\left(1-\frac{w}{z_1}q^{-1}\right)}+\frac{q}{w\left(1-\frac{z_2}{w}q^{-1}\right)}\right)\\ &+\frac{w}{z_2}\left(1-\frac{z_2}{z_1}q^{-2}\right)\left(\frac{z_1}{1-\frac{w}{z_2}q^{-1}}-\frac{z_2}{1-\frac{w}{z_1}q^{-1}}\right)+(z_1\leftrightarrow z_2)\Bigg\}\end{aligned}$$

where $\dfrac{1}{1-\frac{z_1}{w}q^{-1}}$ stands for $\sum_{n=0}^{\infty} q^{-n}\left(\frac{z_1}{w}\right)^n$ and similar fractions for other formal power series.

Collecting the factors $\frac{1}{1-q^{-1}z_1/w}$, $\frac{1}{1-q^{-1}z_2/w}$, $\frac{1}{1-q^{-1}w/z_1}$, $\frac{1}{1-q^{-1}w/z_2}$, we then have

$$\begin{aligned}&\frac{1}{1-q^{-1}z_1/w}(q(z_1-q^{-2}z_2)-q(z_2-q^{-2}z_1)+(q+q^{-1})(z_2-z_1))\\&\quad+\frac{1}{1-q^{-1}z_2/w}(-q(z_1-q^{-2}z_2)+(q+q^{-1})(z_1-z_2)+q(z_2-q^{-2}z_1))\\&\quad+\frac{1}{1-q^{-1}w/z_1}\left((1+q^{-2})w\left(1-\frac{z_2}{z_1}\right)-w\left(1-q^{-2}\frac{z_2}{z_1}\right)+w\left(\frac{z_2}{z_1}-q^{-2}\right)\right)\\&\quad+\frac{1}{1-q^{-1}w/z_w}\left(w\left(\frac{z_1}{z_2}-q^{-2}\right)+(1+q^{-2})\left(1-\frac{z_1}{z_2}\right)w-w\left(1-\frac{z_1}{z_2}q^{-2}\right)\right)\\&=0\end{aligned}$$

where actually each term is zero. Hence we finish the proof of the untwisted cubic Serre relation.

By the properties of the automorphism σ of the Dynkin diagram we consider the twisted cubic Serre relations in the following five cases.

Case i). $A_{ij}=-1$, $\sigma(i)\neq j$, and $\sigma(i)=i$. We have

(4.29)

$$\begin{aligned}X_j^{\pm}(w)X_i^{\pm}(z)&=:X_j^{\pm}(w)X_i^{\pm}(z):\left(\frac{z}{w}\right)^{-\langle\Sigma\sigma^r\alpha_j,\alpha_i\rangle/2}\prod_{r\in I_{-1}}\left(1-\omega^r q^{\mp 1}\frac{w}{z}\right)^{-1}\\&\quad\cdot\prod_{r\in I_2}\left(1-\omega^r\frac{w}{z}\right)\left(1-\omega^r\frac{w}{z}q^{\mp 2}\right)\\&=:X_j^{\pm}(w)X_i^{\pm}(z):\left(\frac{z}{w}\right)^{k/2}\left(1-q^{\mp k}\frac{w^k}{z^k}\right)^{-1}\left(1-q^{\mp 2k}\frac{w^k}{z^k}\right)\end{aligned}$$

Similarly,

$$X_i^{\pm}(z_1)X_i^{\pm}(z_2)=:X_i^{\pm}(z_1)X_i^{\pm}(z_2):\left(\frac{z_2}{z_1}\right)^{-k}\left(1-\frac{z_2^k}{z_1^k}\right)\left(1-\frac{z_2^k}{z_1^k}q^{\mp 2k}\right). \tag{4.30}$$

The cubic Serre relation in the case is

$$\begin{aligned}X_j^{\pm}(w)X_i^{\pm}(z_1)X_i^{\pm}(z_2)&-(q^k+q^{-k})X_i^{\pm}(z_1)X_j^{\pm}(w)X_i^{\pm}(z_2)\\&+X_i^{\pm}(z_1)X_i^{\pm}(z_2)X_j^{\pm}(w)+\{z_1\leftrightarrow z_2\}=0\end{aligned}\tag{4.31}$$

which tells us that the cubic Serre relation in this case is true and the proof is just a mimic of that of the untwisted one provided that we change z_1, z_2, w and q by z_1^k, z_2^k, w^k and q^k respectively.

Case ii). $A_{ij}=-1$, $\sigma(i)\neq j$ and $\langle\alpha_i,\sigma\alpha_i\rangle=0$, $\sigma(j)=j$. The cubic Serre relation in the case is

$$(4.32)\ \frac{z_1^k q^{\pm 2k}-z_2^k}{z_1 q^{\pm 2}-z_2}\{X_j^\pm(w)X_i^\pm(z_1)X_i^\pm(z_2)-(q^k+q^{-k})X_i^\pm(z_1)X_j^\pm(w)X_i^\pm(z_2)$$
$$+X_i^\pm(z_1)X_i^\pm(z_2)X_j^\pm(w)\}+\frac{z_2^k q^{\pm 2k}-z_1^k}{z_2 q^{\pm 2}-z_1}\{z_1\leftrightarrow z_2\}=0$$

It is easy to see that $C(\alpha_i,\alpha_i)=1$, $C(\alpha_i,\alpha_j)=(-1)^k\omega^{\frac{-k(k-1)}{2}}=-1$ and

$$(4.33)\qquad X_i^\pm(z_1)X_i^\pm(z_2)=:X_i^\pm(z_1)X_i^\pm(z_2):\left(\frac{z_2}{z_1}\right)^{-1}\left(1-\frac{z_2}{z_1}\right)\left(1-\frac{z_2}{z_1}q^{\mp 2}\right)$$

$$(4.34)\qquad X_j^\pm(w)X_i^\pm(z)=:X_j^\pm(w)X_i^\pm(z):\left(\frac{z}{w}\right)^{\frac{k}{2}}\left(1-\frac{z^k}{w^k}q^{\mp k}\right)^{-1}$$

Multiplying the polynomial $\sum_{m+n=k-1} z_1^m z_2^m=\frac{z_1^k-z_2^k}{z_1-z_2}$ besides the factor $\frac{z_1^k q^{\pm 2k}-z_2^k}{z_1 q^{\pm 2}-z_2}$, we obtain that

$$\frac{z_1^k-z_2^k}{z_1-z_2}\,\frac{z_1^k q^{\pm 2k}-z_2^k}{z_1 q^{\pm 2}-z_2}X_j^\pm(w)X_i^\pm(z_1)X_i^\pm(z_2)$$
$$=:X_j^\pm(w)X_i^\pm(z_1)X_i^\pm(z_2):\ q^{\pm 2(k-1)}(z_1z_2)^{k-1}$$
$$\cdot\frac{z_1^k}{w^k}\left(\frac{z_2}{z_1}\right)^{-\frac{k}{2}}\frac{(1-z_2^k/z_1^k)(1-q^{\mp 2k}z_2^k/z_1^k)}{(1-q^{\mp k}z_1^k/w^k)(1-q^{\mp k}z_2^k/w^k)},$$

$$\frac{z_1^k-z_2^k}{z_1-z_2}\,\frac{z_1^k q^{\pm 2k}-z_2^k}{z_1 q^{\pm 2}-z_2}X_i^\pm(z_1)X_j^\pm(w)X_i^\pm(z_2)$$
$$=:X_i^\pm(z_1)X_j^\pm(w)X_i^\pm(z_2):\ q^{\pm 2(k-1)}(z_1z_2)^{k-1}$$
$$\cdot\left(\frac{z_2}{z_1}\right)^{-\frac{k}{2}}\frac{(1-z_2^k/z_1^k)(1-q^{\mp 2k}z_2^k/z_1^k)}{(1-q^{\mp z}z_2^k/w^k)(1-q^{\mp k}w^k/z_1^k)},$$

$$\frac{z_1^k-z_2^k}{z_1-z_2}\,\frac{z_1^k q^{\pm 2k}-z_2^k}{z_1 q^{\pm 2}-z_2}X_i^\pm(z_1)X_i^\pm(z_2)X_j^\pm(w)$$
$$=:X_i^\pm(z_1)X_i^\pm(z_2)X_j^\pm(w):\ q^{\pm 2(k-1)}(z_1z_2)^{k-1}$$
$$\cdot\frac{w^k}{z_2^k}\left(\frac{z_2}{z_1}\right)^{-\frac{k}{2}}\frac{(1-z_2^k/z_1^k)(1-q^{\mp 2k}z_2^k/z_1^k)}{(1-q^{\mp k}w^k/z_1^k)(1-q^{\mp k}w^k/z_2^k)}.$$

Observe that the contraction factors are $q^{\pm 2(k-1)}(z_1z_2)^{k-1}$ times of those of the Case ii), thus the cubic Serre relation is proved by repeating the argument of Case ii).

Case iii). $A_{ij} = -1$, $\sigma(i) \neq j$ and $\langle \alpha_i, \sigma\alpha_i \rangle = 0$, $\sigma(j) \neq j$, $\sigma(i) \neq j$.

It is clear that this can only happen when $k=2$. So the cubic Serre relation is

$$\{X_j^{\pm}(w)\, X_i^{\pm}(z_1)\, X_i^{\pm}(z_2) - (q+q^{-1})\, X_i^{\pm}(z_1)\, X_j^{\pm}(w)\, X_i^{\pm}(z_2) + X_i^{\pm}(z_1)\, X_i^{\pm}(z_2)\, X_j^{\pm}(w)\} + \{z_1 \leftrightarrow z_2\} = 0. \tag{4.35}$$

Since $A_{ij} = -1$, the vertex i is connected to the vertex j in the Dynkin diagram. Then i being a fixed vertex under σ will exclude Γ to be types A, D and E_6. Hence $\langle \alpha_i, \sigma\alpha_j \rangle = 0$.

Therefore we have that

$$X_i^{\pm}(z_1)\, X_i^{\pm}(z_2) =: X_i^{\pm}(z_1)\, X_i^{\pm}(z_2) \colon \left(\frac{z_2}{z_1}\right)^{-1} \left(1 - \frac{z_2}{z_1}\right)\left(1 - q^{\mp 2}\frac{z_2}{z_1}\right) \tag{4.36}$$

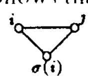
$$X_j^{\pm}(w)\, X_i^{\pm}(z) =: X_j^{\pm}(w)\, X_i^{\pm}(z) \colon \left(\frac{z}{w}\right)^{\frac{1}{2}} \left(1 - \frac{z}{w} q^{\mp 1}\right) \quad \text{(cf. (4.21–22))}. \tag{4.37}$$

Thus this is exactly the same as the untwisted case, and the relation is proved.

Case iv). $A_{ij} = -1$, $\sigma(i) \neq j$ and $\langle \alpha_i, \sigma(\alpha_i) \rangle = -1$.

We remark that this case exists only for type $A_{2l} (k=2)$. The cubic Serre relation is

$$(q^{\pm \frac{k}{2}} z_1 + z_2)\, \{X_j^{\pm}(w)\, X_i^{\pm}(z_1)\, X_i^{\pm}(z_2) - (q^{\frac{k}{2}} + q^{-\frac{k}{2}})\, X_i^{\pm}(z_1)\, X_j^{\pm}(w)\, X_i^{\pm}(z_2) + X_i^{\pm}(z_1)\, X_i^{\pm}(z_2)\, X_j^{\pm}(w)\} + (q^{\pm \frac{k}{2}} z_2 + z_1)\, \{z_1 \leftrightarrow z_2\} = 0. \tag{4.38}$$

By the assumption it follows that the vertices i and $\sigma(j)$ must be disconnected, otherwise there is a loop 

in the Dynkin Diagram. That is, $\langle \alpha_i, \sigma(\alpha_j) \rangle = 0$. Then we have

$$X_i^{\pm}(z_1)\, X_i^{\pm}(z_2) =: X_i^{\pm}(z_1)\, X_i^{\pm}(z_2) \colon \left(\frac{z_2}{z_1}\right)^{-\frac{1}{2}} \left(1 - \frac{z_2}{z_1}\right)\left(1 - \frac{z_2}{z_1} q^{\mp 2}\right)\left(1 + \frac{z_2}{z_1} q^{\mp 1}\right)^{-1} \tag{4.39}$$

$$X_j^{\pm}(w)\, X_i^{\pm}(z) =: X_j^{\pm}(w)\, X_i^{\pm}(z) \colon \left(\frac{z}{w}\right)^{\frac{1}{2}} \left(1 - \frac{w}{z} q^{\mp 1}\right)^{-1}. \tag{4.40}$$

Thus the contraction factors multiplied by $(q^{\pm 1} z_1 + z_2)$ will be the same as for the untwisted case (cf. (4.21–22)). Hence the cubic Serre relation is proved in this case.

Case v). $\langle\alpha_i, \sigma(\alpha_i)\rangle = -1$. The proof is different from the previous four cases. The relation we want to prove is

$$\text{(4.41)}\qquad \mathrm{Sym}\{(q^{\mp 1} z_1 - (1+q^{\pm 1}) z_2 + q^{\pm 2} z_3) X_i^{\pm}(z_1) X_i^{\pm}(z_2) X_i^{\pm}(z_3)\} = 0$$

where Sym means the symmetrization on z_1, z_2, z_3, and we used the remark in case iv) that $\langle\alpha_i, \sigma(\alpha_i)\rangle = -1$ only exists in the type A_{2l}, i.e. $\sigma^2 = 1$.

By (4.12) it follows that

$$\text{(4.42)}\qquad X_i^{\pm}(z) X_i^{\pm}(w) = :X_i^{\pm}(z) X_i^{\pm}(w):\ (zw)^{-\frac{1}{2}} \frac{(z-w)(z-q^{\mp 2} w)}{z+q^{\mp 1} w}$$

It is easy to see $C(\alpha_i, \alpha_i) = 1$. We have

$$\text{(4.43)}\qquad X_i^{\pm}(z_1) X_i^{\pm}(z_2) X_i^{\pm}(z_3) = :X_i^{\pm}(z_1) X_i^{\pm}(z_2) X_i^{\pm}(z_3):\ (z_1 z_2 z_3)^{-1} \prod_{i<j} \cdot \frac{(z_i - z_j)(z_i - q^{\mp 2} z_j)}{z_i + q^{\mp 1} z_j}.$$

Thus (4.41) is equivalent to the following formal identity.

$$\text{(4.44)}\qquad \sum_{a\in\mathfrak{S}_3} a.\left\{(q^{\mp 1} z_1 - (1+q^{\pm 1}) z_2 + q^{\pm 2} z_3) \prod_{i<j} \frac{(z_i - z_j)(z_i - q^{\mp 2} z_j)}{z_i + q^{-1} z_j}\right\} = 0$$

where $(z_i + q^{\mp 1} z_j)^{-1} = \sum_{n\geq 0} z_i^{-1} q^{\mp n} \left(-\frac{z_j}{z_i}\right)^n$ as formal series. The $\mathfrak{S}_3$-action on z_i is defined via

$$a.z_i = z_{ai}, \qquad a \in \mathfrak{S}_3$$

and extended linearly.

For simplicity let us consider the "+" case only. We have the following similar decompositions as (4.26–28).

$$\text{(4.45)}\qquad \frac{z_1 - q^{-2} z_3}{(z_1 + q^{-1} z_2)(z_2 + q^{-1} z_3)} = \frac{1}{z_2 + q^{-1} z_3} - \frac{q^{-1}}{z_1 + q^{-1} z_2}$$

$$\text{(4.46)}\qquad \frac{z_1 - z_2}{(z_1 + q^{-1} z_3)(z_2 + q^{-1} z_3)} = \frac{1}{z_2 + q^{-1} z_3} - \frac{1}{z_1 + q^{-1} z_3}$$

$$\text{(4.47)}\qquad \frac{z_2 - z_3}{(z_1 + q^{-1} z_2)(z_1 + q^{-1} z_3)} = \frac{q}{z_1 + q^{-1} z_3} - \frac{q}{z_1 + q^{-1} z_2}$$

which can be considered as formal series identities.

By (4.45–47) it follows that

$$\text{(4.48)}\qquad \prod_{i<j} \frac{(z_i - z_j)(z_i - q^{-2} z_j)}{(z_i + q^{-1} z_j)} = (z_1 - z_3)(z_1 - q^{-2} z_2)(z_2 - q^{-2} z_3) \cdot \left(\frac{z_1 - z_2}{z_1 + q^{-1} z_2} - \frac{z_1 - z_3}{z_1 + q^{-1} z_3} + \frac{z_2 - z_3}{z_2 + q^{-1} z_3}\right).$$

Plugging (4.48) into the left hand side of (4.44), we see that only the action of the permutations 1, (23) and (132) will produce the terms containing the series $(z_1+q^{-1}z_2)^{-1}$. In other words, we can write (4.44) in the following way.

(4.49)

$$\sum_{a\in\mathfrak{S}_3} a.\left\{\frac{z_1-z_2}{z_1+q^{-1}z_2}(1-(23)+(132)).((z_1-z_3)(z_1-q^{-2}z_2)(z_2-q^{-2}z_3)\right.$$
$$\left.\cdot(q^{-1}z_1-(1+q)z_2+q^2z_3))\right\}=0.$$

We can verify that

(4.50)

$$(1-(23)+(132)).[(z_1-z_3)(z_1-q^{-2}z_2)(z_2-q^{-2}z_3)(q^{-1}z_1-(1+q)z_2+q^2z_3)]$$
$$=(q^{-2}-q^{-1})(z_1+q^{-1}z_2)(z_1-z_3)(z_2-z_3)(q^{-1}z_1-(1+q)z_3+q^2z_2)$$

Substitute (4.50) into the left hand side of (4.49), we then have

$$\sum_{a\in\mathfrak{S}_3} a.\{(q^{-2}-q^{-1})\prod_{i<j}(z_i-z_j)\cdot(q^{-1}z_1-(1+q)z_3+q^2z_2)\}$$
$$=\prod_{i<j}(z_i-z_j)(q^{-2}-q^{-1})\sum_{a\in\mathfrak{S}_3}(-1)^{l(a)}a.\{q^{-1}z_1-(1+q)z_3+q^2z_2\}$$
$$=0.$$

Hence we finished the proof of the theorem.

References

1. Bernard, D.: Vertex operator representations of the quantum affine algebra $\mathscr{U}_q(B_r^{(1)})$. Lett. Math. Phys. **17**, 239–245 (1989)
2. Drinfeld, V.G.: Quantum groups. In: Gleason, A.N. (ed.) Proceedings of the international congress of mathematicians, Berkeley, California 1986. (pp. 798–820) American Mathematical Society 1987
3. Drinfeld, V.G.: New realizations of Yangians and quantum affine algebras. Sov. Math. Dokl. **36**, 212–216 (1988)
4. Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: Quantization of Lie groups and Lie algebras. In: Kashiwara, M., Kawai, T. (eds.) Algebraic analysis. I. pp. 129–139. New York: Academic Press 1988
5. Frenkel, I.B.: Two constructions of affine Lie algebras representations and boson-fermion correspondence in quantum field theory. J. Funct. Anal. **44**, 259–327 (1981)
6. Frenkel, I.B., Jing, N.H.: Vertex representations of quantum affine algebras. Proc. Natl. Acad. Sci. USA **85**, 9373–9377 (1988)
7. Frenkel, I.B., Kac, V.G.: Basic representations of affine Lie algebras and duel resonance models. Invent. Math. **62**, 23–66 (1980)
8. Frenkel, I.B., Lepowsky, J., Meurman, A.: A vertex operator algebras and the Monster. New York: Academic Press 1988
9. Jimbo, M.: A q-difference analogue of $\mathscr{U}(\mathfrak{g})$ and the Yang-Baxter equation. Lett. Math. Phys. **10**, 63–69 (1985)
10. Jing, N.H.: Vertex operators, symmetric functions and their q-deformations. Ph.D. Dissertation, Yale University 1989
11. Lepowsky, J.: Calculus of twisted vertex operators. Proc. Natl. Acad. Sci. USA **82**, 8295–8299 (1985)

12. Lepowsky, J., Wilson, R.L.: The structure of standard modules. I., Invent. Math. **77**, 199–290 (1984). II., Invent. Math. **79**, 417–442 (1985)
13. Lusztig, G.: Quantum deformations of certain simple modules over enveloping algebras. Adv. Math. **70**, 237–249 (1988)
14. Rosso, M.: Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra. Commun. Math. Phys. **117**, 581–593 (1988)
15. Segal, G.: Unitary representations of some infinite-dimensional groups. Commun. Math. Phys. **80**, 301–342 (1981)