

Poincaré series and holomorphic averaging

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Summary. We provide an alternate proof of McMullen’s theorem on contractive properties of the Poincaré series operator in the special case of the universal covering. This case includes in particular Kra’s Theta Conjecture.

Let X be a Riemann surface and let $p: \tilde{X} \rightarrow X$ be its universal covering. By a *holomorphic averaging sequence* for X we shall mean a sequence g_j of holomorphic functions on $\tilde{X} \setminus p^{-1}(E)$, E a discrete subset of X , such that the sums

$$\delta_j(z) = \sum_{w \in p^{-1}(z)} |g_j(w)|$$

and

$$v_j(z) = \sum_{w \in p^{-1}(z)} g_j(w)$$

tend to 1 uniformly on compact subsets of $X \setminus E$. E will be called the *exceptional set*.

Our principle result is that holomorphic averaging sequences are rare.

Theorem 1 *If X admits a holomorphic averaging sequence then the fundamental group $\pi_1(X)$ is abelian.*

The abelian case is discussed in Remark 1 below.

Our interest in Theorem 1 stems in part from its connection with Poincaré series. Recall that a quadratic differential on a Riemann surface X is a section $\varphi = \varphi(z)dz^2$ of the square of the holomorphic cotangent bundle. $|\varphi| = |\varphi(z)| |dz|^2$ is a measure, so the L^1 norm $\|\varphi\| = \int_X |\varphi|$ is well-defined. Let $Q(X)$ denote the space of integrable holomorphic quadratic differentials on X , let $B(X)$ denote the unit ball $\{\varphi \in Q(X): \|\varphi\| < 1\}$, and let $S(X) = \partial B(X) = \{\varphi \in Q(X): \|\varphi\| = 1\}$. If $\psi: Y \rightarrow X$ is a covering map of Riemann surfaces then the Poincaré series operator $\Theta_\psi: Q(Y) \rightarrow Q(X)$ is defined by summing over fibers. It is clear that Θ_ψ maps $B(Y)$ into $B(X)$.

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Let Δ denote the unit disk in \mathbb{C} .

Theorem 2 *If $\psi: \Delta \rightarrow X$ is a covering map and if $\overline{\Theta_\psi(B(\Delta))} \cap S(X) \neq \emptyset$ then X admits a holomorphic averaging sequence.*

Corollary. *If $\psi: \Delta \rightarrow X$ is a covering map and if $\overline{\Theta_\psi(B(\Delta))} \cap S(X) \neq \emptyset$ then $\pi_1(X)$ is abelian.*

Corollary (Kra's Theta Conjecture) *If X is a compact Riemann surface minus a finite number of points and if X admits a covering map $\psi: \Delta \rightarrow X$ then the operator norm $\|\Theta_\psi\|$ is < 1 .*

Proof. This follows from the preceding corollary together with the observations that $Q(X)$ must be finite dimensional and $\pi_1(X)$ must be non-abelian. \square

See [M] for a discussion of the history and significance of this conjecture. In [M] McMullen in fact proves the following stronger result.

Theorem 3 *If $\psi: Y \rightarrow X$ is a covering map of hyperbolic Riemann surfaces then $\overline{\Theta_\psi(B(Y))} \cap S(X) \neq \emptyset$ if and only if ψ is amenable. If ψ is amenable then in fact $\Theta_\psi(B(Y)) = B(X)$.*

See [M] for the definition of an amenable covering. It is well-known that if Y is simply-connected then ψ is amenable if and only if $\pi_1(X)$ is abelian.

Theorem 1 is a direct consequence of the following two theorems.

Theorem 4 *Suppose that X admits a holomorphic averaging sequence. Let $A \rightarrow X$ be an affine bundle, and let f be a holomorphic section of the bundle $p^*A \rightarrow \tilde{X}$ which is bounded on $p^{-1}(K)$ for all compact $K \subset X$. Then there exists a (single-valued) holomorphic section \hat{f} of $A \rightarrow X$ such that $\hat{f}(z)$ lies in the closed convex hull of the set $f(p^{-1}(z))$ for all $z \in X$.*

The condition that f be bounded on $p^{-1}(K)$ means that there is a compact subset L of A such that the graph of f over $p^{-1}(K)$ is contained in $p_A^{-1}(L)$, where p_A is the induced map $p^*A \rightarrow A$; equivalently, this hypothesis states that the sections $f_\gamma, \gamma \in \pi_1(X)$, obtained by applying deck transformations to the graph of f are uniformly bounded on compacts with respect to any trivialization. The hull condition is interpreted by noting that $f(p^{-1}(z))$ is a subset of the affine fiber A_z .

Theorem 5 *Let X be a Riemann surface with the property that for all affine bundles $A \rightarrow X$ and for all holomorphic sections f of $p^*A \rightarrow \tilde{X}$ which are bounded on $p^{-1}(K)$ for compact $K \subset X$ there exists a holomorphic section \hat{f} of $A \rightarrow X$ with $\hat{f}(z)$ contained in the closed convex hull of $f(p^{-1}(z))$ for all $z \in X$. Then $\pi_1(X)$ is abelian.*

Remark 1 The abelian case.

If X is simply-connected then X admits a holomorphic averaging sequence by default.

If X is biholomorphic to an annulus, punctured disk, or punctured plane then \tilde{X} may be represented as a (finite, semi-infinite, or infinite) horizontal strip with deck group consisting of translations $w \mapsto w + n$. The functions $g_j(w) = \frac{e^{-(w/j)^2}}{j\sqrt{\pi}}$ define a holomorphic averaging sequence for X . (Interpret δ_j and v_j as Riemann sums!)

The remaining case is that of an elliptic curve $X = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. In this case Liouville's theorem shows that any holomorphic averaging sequence must have a non-empty exceptional set; nevertheless, the meromorphic functions

$$g_j(w) = \frac{1}{(2j+1)^2} \frac{\sum_{|n|, |m| \leq j} (z - n - m\tau)^{-3}}{\sum_{n, m \in \mathbb{Z}} (z - n - m\tau)^{-3}}$$

do define a holomorphic averaging sequence with exceptional set $E = \{[0], [\frac{1}{2}], [\frac{3}{2}], [\frac{1}{2} + \frac{3}{2}]\}$. (This can be proved along the lines of [M, Theorem 9.1]; one shows, for example, by separate arguments that $\sum_{\max(|m|, |n|) \leq j - \sqrt{j}} |g_j| \rightarrow 1$, $\sum_{\max(|m|, |n|) \geq j + \sqrt{j}} |g_j| \rightarrow 0$, and $\sum_{j - \sqrt{j} < \max(|m|, |n|) < j + \sqrt{j}} |g_j| \rightarrow 0$.) \square

Proof of Theorem 4 Let g_j be a holomorphic averaging sequence for X . Since $A \rightarrow X$ is affine and $\sum_{w \in p^{-1}(z)} \frac{g_j(w)}{v_j(z)} = \sum_{w \in p^{-1}(z)} \frac{|g_j(w)|}{\delta_j(z)} = 1$, the formulae

$$\hat{f}_j(z) = \sum_{w \in p^{-1}(z)} \frac{g_j(w)}{v_j(z)} f(w)$$

and

$$\check{f}_j(z) = \sum_{w \in p^{-1}(z)} \frac{|g_j(w)|}{\delta_j(z)} f(w)$$

define sections of A over any given compact subset of $X \setminus E$ when j is large enough; \hat{f}_j is holomorphic while \check{f}_j satisfies $\check{f}_j(z) \in$ closed convex hull of $f(p^{-1}(z))$.

Now

$$\begin{aligned} \sum_{w \in p^{-1}(z)} \left| \frac{|g_j(w)|}{\delta_j(z)} - \frac{g_j(w)}{v_j(z)} \right| &= \sqrt{2} \sum_{w \in p^{-1}(z)} \sqrt{|g_j(w)| (|g_j(w)| - \operatorname{Re} g_j(w))} \\ &\leq \sqrt{2 \left(\sum_{w \in p^{-1}(z)} |g_j(w)| \right) \left(\sum_{w \in p^{-1}(z)} |g_j(w)| - \operatorname{Re} g_j(w) \right)} \\ &= \sqrt{2 \delta_j(z) (\delta_j(z) - \operatorname{Re} v_j(z))}, \end{aligned}$$

so

$$\begin{aligned} \sum_{w \in p^{-1}(z)} \left| \frac{|g_j(w)|}{\delta_j(z)} - \frac{g_j(w)}{v_j(z)} \right| &\leq \sum_{w \in p^{-1}(z)} \left| \frac{|g_j(w)|}{\delta_j(z)} - \frac{g_j(w)}{\delta_j(z)} \right| + \sum_{w \in p^{-1}(z)} \left| \frac{g_j(w)}{\delta_j(z)} - \frac{g_j(w)}{v_j(z)} \right| \\ &\leq \sqrt{\frac{2(\delta_j(z) - \operatorname{Re} v_j(z))}{\delta_j(z)}} + \delta_j(z) \left| \frac{1}{\delta_j(z)} - \frac{1}{v_j(z)} \right|. \end{aligned}$$

It follows that

$$\operatorname{dist}(\hat{f}_j(z), \text{closed convex hull of } f(p^{-1}(z))) \leq \operatorname{dist}(\hat{f}_j(z), \check{f}_j(z)) \rightarrow 0$$

uniformly on compact subsets of $X \setminus E$ as $j \rightarrow \infty$. Thus some subsequence of the \hat{f}_j converges to a holomorphic section \hat{f} over $X \setminus E$ with $\hat{f}(z) \in$ closed convex hull of

$f(p^{-1}(z))$ for all $z \in X \setminus E$. Now \hat{f} is bounded near each point in E , so the singularities of \hat{f} at points of E are removable, and the extended section satisfies the required hull condition over all of X . \square

Proof of Theorem 5 (Compare [B, Theorem 2].) We may assume that \tilde{X} is biholomorphic to the upper half-plane H (since all exceptional Riemann surfaces have abelian fundamental group). Fix a biholomorphic identification of \tilde{X} with H , and let $\psi: H \rightarrow X$ denote the corresponding projection. For each loop $\gamma \in \pi_1(X)$ let $\lambda_\gamma \in \text{Aut}(H)$ denote the corresponding deck transformation.

Define a properly discontinuous fixed-point free action of $\pi_1(X)$ on $H \times \mathbb{C}$ by the formula

$$\mu_{\gamma^{-1}}: (\lambda_\gamma(w), \zeta) \mapsto \left(w, \zeta \cdot \lambda'_\gamma(w) + \frac{\lambda''_\gamma(w)}{\lambda'_\gamma(w)} \right).$$

Since λ_γ is fractional linear, a routine computation shows that this action may also be defined by the formula

$$\mu_\gamma: \left(w, \frac{2}{\xi - w} \right) \mapsto \left(\lambda_\gamma(w), \frac{2}{\lambda_\gamma(\xi) - \lambda_\gamma(w)} \right).$$

Let A denote the quotient of $H \times \mathbb{C}$ by this action. The map $A \rightarrow X$, $[(w, \zeta)] \mapsto \psi(w)$ gives A the structure of a one-dimensional affine bundle over X .

The motivation for defining A is that a non-constant meromorphic function η on X induces a meromorphic section $\zeta = f(w) := \frac{(\eta \circ \psi)''(w)}{(\eta \circ \psi)'(w)}$ of $H \times \mathbb{C} \rightarrow H$ passing down to a well-defined section $\mathcal{U}\eta$ of $A \rightarrow X$. Similarly, multivalued functions η on X coming from single-valued functions $\tilde{\eta}$ on $\tilde{X} = H$ give rise to sections $\zeta = f(w) := \frac{\tilde{\eta}''(w)}{\tilde{\eta}'(w)}$ on H which may be viewed as multivalued sections of $A \rightarrow X$.

If we apply these observations to the multivalued function $\eta = \psi^{-1}$ we have $\tilde{\eta} = \text{Id}$ yielding $f(w) \equiv 0$. Choosing an alternate branch $\tilde{\eta} = \lambda_\gamma$ we have instead

$$f_\gamma(w) = \frac{\lambda''_\gamma(w)}{\lambda'_\gamma(w)} = \frac{2}{\lambda_\gamma^{-1}(\infty) - w}.$$

Since $\lambda_\gamma^{-1}(\infty) \in \mathbb{R} \cup \{\infty\}$ it follows from standard properties of fractional linear transformations that the closed convex hull of $f(\psi^{-1}(z))$ is contained in the closed disk $D_z \subset A_z$ defined by $D_{\psi(w)} = \left\{ \left[\left(w, \frac{2}{\tau - w} \right) \right] : \text{Im } \tau \leq 0 \text{ or } \tau = \infty \right\}$. The center and radius of D_z with respect to a local trivialization of $A \rightarrow X$ vary smoothly with z ; it follows that f is bounded on $\psi^{-1}(K)$ for compact $K \subset X$.

Now suppose that \hat{f} is a section of $A \rightarrow X$ such that $\hat{f}(z) \in D_z$ for all $z \in X$. Such a section \hat{f} would pull back to a section $\zeta = F(w)$ of $H \times \mathbb{C} \rightarrow H$. Let $h(w) = w + \frac{2}{F(w)}$ so that $F(w) = \frac{2}{h(w) - w}$. Since $\hat{f}(z) \in D_z$ we have $\text{Im } h(w) \leq 0$ or $h(w) = \infty$. Also, the invariance of the graph of F under the deck transformations μ_γ yields $h(\lambda_\gamma(w)) = \lambda_\gamma(h(w))$.

Suppose that $h(w_0) \in \mathbb{R} \cup \{\infty\}$ for some $w_0 \in H$. Then the open mapping theorem yields $h(w) \equiv k$. The constant k must be a fixed point of each λ_γ ; but the existence of a such a constant fixed point implies that $\pi_1(X)$ is abelian [FK, IV.9.4, IV.9.9].

Hence we may assume that $\text{Im } h(w) < 0$. Thus \bar{h} is an anti-holomorphic mapping from H to H with $\bar{h}(\lambda_\gamma(w)) = \lambda_\gamma(\bar{h}(w))$. This implies that h induces an anti-holomorphic map $[h]: X \rightarrow X$ with $[h]_* = \text{Id}: \pi_1(X) \rightarrow \pi_1(X)$. But by [B, Theorem 6] the existence of such a map implies that $\pi_1(X)$ is abelian. \square

Remark 2 If χ is a holomorphic section of $A \rightarrow X$ then local solutions of $\mathbf{U}\eta = \chi$ satisfy transition relations of the form $\eta_2 = B\eta_1 + C$ and hence induce an affine structure on X . If X is compact of genus $\neq 1$ then X admits no affine structure [G, §9(a), Corollary 3], so $A \rightarrow X$ admits no holomorphic section.

Remark 3 Curt McMullen has pointed out to the authors that A admits the more elegant representation

$$A = \{(w, \zeta) \in H \times (\mathbb{C} \cup \{\infty\}) : w \neq \zeta\} / (w, \zeta) \sim (\lambda_\gamma(w), \lambda_\gamma(\zeta))$$

with the obvious projection map. For f we may choose any constant section $f(w) = k, k \in \bar{H} \cup \{\infty\}$.

Proof of Theorem 2 Let $\varphi_j \in B(\Delta)$ with $\Theta_\psi \varphi_j \rightarrow \Phi \in S(X)$, and let E be the zero set of Φ .

Recall that if γ is a holomorphic function on an open subset U of $X \setminus E$ then the C^1 -norm of γ on any compact subset K of U is bounded by a constant $C_{K,U}$ times $\int_U |\gamma \Phi|$ (see [H, Theorem 1.2.4]).

Write $\varphi_j = g_j \psi^* \Phi$ so that g_j is holomorphic on $\Delta \setminus \psi^{-1}(E)$. Since $\Theta_\psi \varphi_j = v_j \Phi \rightarrow \Phi$ in $Q(X)$ it follows that $v_j \rightarrow 1$ uniformly on compact subsets of $X \setminus E$.

To handle the δ_j note that $\psi_* |\varphi_j| = \delta_j |\Phi|$, where ψ_* denotes push-forward of measures. The C^1 estimates cited above show that the δ_j are uniformly Lipschitz and hence equicontinuous on compact subsets of $X \setminus E$; hence, passing to a subsequence, we may assume that the δ_j converge to some function δ_∞ uniformly on compact subsets of $X \setminus E$. Now $\delta_\infty \geq 1$ since $\delta_j \geq |v_j|$; moreover

$$\begin{aligned} \int_X (\delta_\infty - 1) |\Phi| &\leq \lim_{j \rightarrow \infty} \int_X \psi_* |\varphi_j| - 1 \\ &= \lim_{j \rightarrow \infty} \|\varphi_j\|_{Q(\Delta)} - 1 \\ &\leq 0, \end{aligned}$$

so $\delta_\infty \equiv 1$. \square

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Note. Since writing this paper the authors have been informed by Curt McMullen that he has found a short proof of the full Theorem 3.