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Legendrian distributions with applications to relative Poincaré series

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1 Introduction

Let X be a compact Kähler manifold of complex dimension n and Kähler form Ω, equipped with a holomorphic Hermitian line bundle $L \to X$ such that the curvature of its natural connection is Ω. Such an L is called a quantizing line bundle. For each positive integer k, let

$$\mathfrak{S}_k = H^0(X, L^{\otimes k}) \tag{1}$$

be the complex inner-product space of holomorphic sections of the k-th tensor power of L. Philosophically, $\mathfrak{S}_k$ is the quantum phase space of X where k is the inverse of Planck's constant. In this paper we do the following:

1. We associate, to certain immersed Lagrangian submanifolds $\Lambda \to X$, sequences of sections $u_k \in \mathfrak{S}_k$, $k = 1, 2, \ldots$. These sections represent quantum-mechanical states that are associated semi-classically with Λ. The Λ's in question (defined below) will be called *Bohr-Sommerfeld Lagrangians*.
2. To each such sequence, we associate a symbol which is a half-form on Λ.

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3. We compute the large k asymptotics of matrix coefficients $\langle Tu_k, u_k\rangle$ where T is a Toeplitz operator. The dependence on T of the leading order term is an integral of the symbol of T over Λ, proving that these sections concentrate on Λ. By taking $T = I$ we obtain estimates on the L^2 norms of the u_k. In particular we show that these sections are not zero for k large. We also estimate $\langle Tu_k, v_k\rangle$ where $\{v_k\}$ is a sequence associated with a second immersed Lagrangian intersecting Λ cleanly.
4. For X a Riemann surface, the elements of $\mathfrak{S}_k$ are holomorphic cusp forms of weight $2k$. We show that the relative Poincaré series associated to geodesics [12] are particular cases of our construction. As a corollary of the asymptotic expansion, we find that the Poincaré series associated to a fixed periodic geodesic is non-vanishing for large weight. We extend this result to hypercycles and circles. The classical Poincaré series associated with cusps are also a particular case of our construction, but the proof of the estimates needs the fact that X is a manifold.

Our main results are Theorem 3.2, which gives the asymptotics of the matrix elements, and Theorem 3.12, in which we establish the local realization:

$$u_k(x) = k^{n/2}(Gaussian) + O(k^{(n-1)/2})\,. \tag{2}$$

We thus establish a precise correspondence between Bohr-Sommerfeld Lagrangian submanifolds of X (equipped with half-forms, see below) and certain sequences of states depending on $k = 1/\hbar$. Motivation for this comes from general quantization/semi-classical ideas in the context of Kähler phase spaces. As with many others working in this area, we were very influenced by the pioneering work of F.A. Berezin, [2]. He was one of the first to study the semi-classical (i.e. large k) limit of Toeplitz operators with multipliers given by functions of X. For further developments of Berezin's ideas, see [7] and references therein, and also [13], [4], [3]. To our knowledge, no systematic method of quantization of Bohr-Sommerfeld Lagrangians has been developed. In addition to the applications to Poincaré series presented in Sect. 4 of this paper, our construction can be applied, e.g., to the quantization and semi-classical limit of symplectomorphisms $X \to X$, and to the construction of quasi-modes for Toeplitz operators (both in progress).

Our methods use heavily the machinery of Fourier integral operators of Hermite type, developed by Louis Boutet de Monvel and Victor Guillemin in [5]. In fact, we associate to closed Legendrian submanifolds of a strictly pseudoconvex domain, P, distributions in the generalized Hardy space of P (see Sect. 2). The Szegö projector is an Hermite FIO, and we show that our Legendrian distributions posses a symbol calculus inherited from that of Hermite distributions (symplectic spinors).

The sections u_k are defined as follows. Let $P \subset L^*$ the unit circle bundle in the dual of L. We denote by α the connection form on P; then the pair (P, α) is a contact manifold and so it has a natural volume form,

$$dp = \frac{\alpha}{2\pi} \wedge (d\alpha)^n\,. \tag{3}$$

The disk bundle in L^* is a strictly pseudoconvex domain; we will consider the Hardy space of $P, \mathfrak{S} \subset L^2(P)$, and the Szegö projector

$$\Pi : L^2(P) \to \mathfrak{S} \tag{4}$$

given by orthogonal projection onto $\mathfrak{S}$. The natural action of S^1 on P commutes with Π, and hence $\mathfrak{S}$ decomposes as a Hilbert space direct sum of isotypes. Only positive frequencies arise in the decomposition, and in fact the k-th summand is naturally identified with $\mathfrak{S}_k$. Therefore we identify

$$\mathfrak{S} = \bigoplus_{k=0}^{\infty} \mathfrak{S}_k . \tag{5}$$

Since we will use the calculus of Hermite Fourier integral operators, we will actually need a metalinear structure on P. This is a way of keeping track of the Maslov factors.

Let $\Lambda \subset P$ be a compact Legendrian submanifold, and ν a half-form on Λ. It turns out that Π extends to a class of distributions including the delta function defined by (Λ, ν). We will suppress ν from the notation, and denote the latter by δ_Λ.

Definition 1.1 *For each k, we denote by u_k the k-th component of $u := \Pi(\delta_\Lambda)$ in the decomposition* (5).

Remarks. 1. Instead of a delta function along Λ one can just as well take a conormal distribution to Λ, but to leading order asymptotics the resulting states are not more general.

2. We regard the sequence $\{u_k\}$ as being associated with the immersed Lagrangian $\Lambda_0 := \pi(\Lambda)$, where $\pi : P \to X$ is the projection. Not all immersed Lagrangians in X are of this form; those that are labeled *Bohr-Sommerfeld Lagrangians.*

3. In case the restriction $\pi|_\Lambda : \Lambda \to \Lambda_0$ is a covering map with deck transformation group the group of k_0 roots of unity, and the density ν is chosen invariant under it, then the Fourier coefficients u_k will be zero unless k_0 divides k.

The matrix element estimate, Theorem 3.2, gives in particular the asymptotics of the L^2 norms $\|u_k\|$. The resulting estimate can be explained rather simply as follows. For every $p \in P$, let

$$\varphi_p^{(k)} := \Pi_k(\delta_p) \tag{6}$$

be the orthogonal projection of the delta function at p into $\mathfrak{S}_k$. (In case X is a coadjoint orbit of a Lie group these are the "coherent states" of the physicists.) That is, if $\mathcal{K}_k(q, p)$ is the Schwartz kernel of the orthogonal projection Π_k,

$$\varphi_p^{(k)}(q) := \mathcal{K}_k(q, p) , \tag{7}$$

and the reproducing property follows:

$$\forall f \in \mathfrak{S}_k \quad f(p) = \langle f, \varphi_p^{(k)} \rangle . \tag{8}$$

Applying this to $f = \varphi_p^{(k)}$ itself gives

$$\mathscr{K}_k(p, p) = \|\varphi_p^{(k)}\|^2 , \tag{9}$$

and so

$$\dim \mathfrak{S}_k = \int_P \mathscr{K}_k(p, p)dp = \int_P \|\varphi_p^{(k)}\|^2 dp . \tag{10}$$

By Riemann-Roch, we know this is a polynomial in k of degree n and leading term $(2\pi)^{-n}\mathrm{Vol}(X)k^n$. Thus we get that on average $\|\varphi_p^{(k)}\|^2$ is of size $(2\pi)^{-n}k^n$. If we assume that $\|\varphi_p^{(k)}\|^2$ is independent of p (true for example if there is a transitive symmetry group present), then we actually get

$$\|\varphi_p^{(k)}\|^2 = (2\pi)^{-n} k^n + \mathrm{l.o.t} . \tag{11}$$

On the other hand, by definition

$$u_k = \int_\Lambda \varphi_p^{(k)} \nu_p , \tag{12}$$

and so the square of the norm is

$$\langle u_k, u_k \rangle = \int\int_{\Lambda\times\Lambda} \langle \varphi_p^{(k)}, \varphi_q^{(k)} \rangle \nu_p \bar{\nu}_q . \tag{13}$$

It turns out that this integral can be estimated for large k by the method of stationary phase. For this it is crucial that Λ be a Legendrian submanifold. The relevant critical points are on the diagonal, $p = q \in \Lambda$, which is a non-degenerate manifold of critical points. Since the dimension of Λ is n, we see from (11) that we should have

$$\langle u_k, u_k \rangle \sim \left(\frac{k}{\pi}\right)^{n/2} \int_\Lambda |\nu|^2 . \tag{14}$$

We will prove that this is indeed the case.

2 Legendrian distributions and their symbols

In [5] Boutet de Monvel and Guillemin associate spaces of distributions $I^m(\mathscr{M}, \Sigma)$ on $\mathscr{M}$ (called Hermite distributions) to a conic closed isotropic submanifold $\Sigma \subset T^*\mathscr{M}\backslash\{0\}$. In case Σ is Lagrangian, these distributions are precisely the classical Lagrangian distributions of Hörmander except that amplitudes of elements in $I^m(\mathscr{M}, \Sigma)$ have asymptotic expansions decreasing by half-integer powers of the fibre variables. (There is also a discrepancy in the definition of order; we will follow the conventions in [5].) Elements in $I^m(\mathscr{M}, \Sigma)$ have symbols, which are symplectic spinors on Σ. Boutet de Monvel and Guillemin prove a series of composition theorems regarding Hermite distributions. We will review this material as needed.

2.1 The definition

For all of Sect. 2 the setting is the following. Let P be a strictly pseudoconvex domain in a Stein manifold, and let α be the pull-back to P of $\Im\bar{\partial}\rho$, where ρ is a defining function for P. Then (P, α) is a contact manifold, and at every point p the null space of α is the maximal complex subspace of T_pP. The null space has a symplectic structure, namely the one induced from $d\alpha$; therefore it has an associated Hermitian structure. Thus P has a so-called pseudo-Hermitian structure. Denote by $\mathscr{H} \subset L^2(P)$ the Hardy space and let $\Pi : L^2(P) \to \mathscr{H}$ be the Szegö projector. Let $\Lambda \subset P$ be a closed Legendrian submanifold.

Definition 2.1 *The space of Legendrian distributions of order m associated with Λ is defined to be*

$$J^m(P, \Lambda) = \Pi(I^{m+n/2}(P, N^*\Lambda)) .$$

Here $I^*(P, N^*\Lambda)$ denotes the spaces of conormal distributions to Λ. We must justify this definition; that is, we must show that Π extends to distributions conormal to Λ.

Define a submanifold $\mathscr{X}^\sharp \subset T^*P$ by

$$\mathscr{X}^\sharp := \{(p, r\alpha_p); p \in P, r > 0\} , \qquad (15)$$

where α is the connection form on P. $\mathscr{X}^\sharp$ is in fact a symplectic submanifold of T^*P.

Theorem 2.2 ([5], *Thm.* 11.1) *Let*

$$\mathscr{X} = \{(p, r\alpha_p; p, -r\alpha_p); r > 0 \text{ and } p \in P\} , \qquad (16)$$

where α is the connection form on P. Then $\mathscr{X}$ is an isotropic submanifold of $T^*(P \times P)$, and the Schwartz kernel of Π is an Hermite distribution in the space $I^{1/2}(P \times P, \mathscr{X})$.

We will use various composition theorems of [5]. For completeness, we make a definition that encapsulates the hypotheses of all of these theorems ([5], conditions (7.4)).

Definition 2.3 *Let P and Q be manifolds, and $\Gamma \subset T^*(Q \times P)\backslash\{0\}$ and $\Sigma \subset T^*P\backslash\{0\}$ be two closed homogeneous submanifolds. We will say that Γ and Σ are composible iff the following hold:*

1. *Γ should not contain vectors of the form $(q, \mu; p, 0)$.*
2. *$\Gamma \circ \Sigma$ should not contain zero vectors.*
3. *If Γ_0 is the projection of Γ onto $Q \times P$, then the projection $\Gamma_0 \to Q$ is proper.*
4. *The fiber product*

$$\begin{array}{ccc} \mathscr{F} & \to & \Gamma \\ \downarrow & & \downarrow\rho \\ \Sigma & \hookrightarrow & T^*P \end{array} \qquad (17)$$

where the right vertical arrow ρ is the obvious projection, is clean.

5. *The map* $\tau : \mathcal{F} \to T^*Q$ *defined as the composition of the top arrow in* (17) *and the projection* $\Gamma \to T^*Q$ *is of constant rank.*

We now recall the two conditions which define the cleanness of a fiber product such as (17). The first requirement is that $\mathcal{F}$, which by definition is

$$\mathcal{F} = \{(\gamma, \sigma) \in \Gamma \times \Sigma; \rho(\gamma) = \sigma\}, \tag{18}$$

is a submanifold of $\Gamma \times \Sigma$. In addition, we require that for all $(\gamma, \sigma) \in \mathcal{F}$,

$$T_{\gamma,\sigma}\mathcal{F} = d\rho_\gamma^{-1}(T_\sigma \Sigma). \tag{19}$$

The following integer plays an important role in the calculations:

Definition 2.4 *The excess of the diagram* (17) *is*

$$e = \dim(\mathcal{F}) + \dim(T^*P) - \dim(\Gamma) - \dim(\Sigma).$$

The geometrical meaning of the clean intersection is this: that locally near every point in $\mathcal{F}$ there is a submanifold of T^*M of codimension e, containing neighborhoods of the point in the intersecting manifolds, which intersect transversely in the submanifold.

Now consider

$$\Lambda^\sharp := \mathcal{Z}^\sharp \cap N^*\Lambda = \{(p, r\alpha_p); r > 0 \text{ and } p \in \Lambda\}, \tag{20}$$

where the second equality follows from Λ being Legendrian. $\Lambda^\sharp$ is a submanifold of the conormal bundle of Λ and hence is an isotropic submanifold of T^*P. It is a Lagrangian submanifold of $\mathcal{Z}^\sharp$.

Proposition 2.5 *The Szegö projector extends by continuity of the space of distributions on* P *conormal to* Λ. *The extension maps* $I^m(P, N^*\Lambda)$ *to the space* $I^{m-n/2}(P, \Lambda^\sharp)$.

Proof. We apply Theorem 9.4 in [5], which in the present case says that if $\mathcal{Z}$ and $N^*\Lambda$ are composible (in the sense of Definition 2.3), then an Hermite FIO associated with $\mathcal{Z}$ can be applied to a Lagrange distribution associated with $N^*\Lambda$, and the result is an Hermite distribution associated with $\Lambda^\sharp$. Therefore all we need to do is to check that $\mathcal{Z}$ and $N^*\Lambda$ are composible, which is straightforward. (The excess of the composition diagram turns out to be equal to n.) □

By Proposition (2.5) not only the spaces J are well-defined, but in fact one has the inclusion

$$J^m(P, \Lambda) \subset I^m(P, \Lambda^\sharp). \tag{21}$$

Although we won't need it here, we mention that these distributions can also be described as "marked Lagrangian distributions" in the sense of Melrose, [14], associated to the conormal bundle of Λ marked by the submanifold $\Lambda^\sharp$.

2.2 The symbols of Legendrian distributions

Our next task is to show that one can identify the symbol of an element in $J^m(P, \Lambda)$ with a half-form on Λ. The symbol of a Lagrangian distribution is a half form on the Lagrangian submanifold. The symbol of an Hermite distribution is more complicated object, a symplectic spinor. For the sake of completeness, we review briefly the construction of symplectic spinors.

To any symplectic vector space V there is naturally associated a Heisenberg Lie algebra, denoted by heis(V), which as a vector space is just $V \oplus \mathbb{R}$. The Stone-von Neumann theorem gives us a unitary representation ρ of the associated Heisenberg group on a Hilbert space $H(V) \cong L^2(\mathbb{R}^{\dim V/2})$. If V carries a metaplectic structure, then we can use the action of Sp(V) on heis(V) to construct a unitary representation of Mp(V) on $H(V)$, the Segal-Shale-Weyl representation. Denote by $H_\infty(V)$ the space of smooth vectors for this representation, which is identified with the Schwartz functions $\mathcal{S}(\mathbb{R}^{\dim V/2})$. Now let $Y \subset V$ be an isotropic subspace of dimension k. Then $Y^\perp/Y$ inherits a symplectic structure from V. Moreover, the metaplectic structure on V gives us the product of a metalinear structure on Y and a metaplectic structure on $Y^\perp/Y$. The space Spin(Y) is defined by

$$\mathrm{Spin}(Y) := H_\infty(Y^\perp/Y) \otimes \textstyle\bigwedge^{1/2} Y \; . \tag{22}$$

Let $H_\infty(V)'$ denote the topological dual to $H_\infty(V)$, which is identified with the space of tempered distributions $\mathcal{S}'(\mathbb{R}^{\dim V/2})$. The representation $d\rho$ of heis(V) on $H(V)$ extends to a representation on $H_\infty(V)'$. Identifying a subspace $Y \subset V$ as a subalgebra of heis(V), we define

$$\ker d\rho(Y) = \{f \in H_\infty(V)' : d\rho(u)f = 0 \;\forall u \in Y\} \; . \tag{23}$$

Theorem 2.6 (*Kostant*) *For a Lagrangian subspace $Y \subset V$, the space* ker *$d\rho(Y)$ is one-dimensional and isomorphic to $\bigwedge^{1/2} Y$.*

The bundle of symplectic spinors is defined as follows. Let M be a metalinear manifold (a manifold possesses a metalinear structure whenever the square of the first Stiefel-Whitney class vanishes). The choice of a metalinear structure on M gives a canonical metaplectic structure on T^*M. Now let $\Sigma \subset T^*M$ be an isotropic subspace. Let Σ_x denote the tangent space to Σ at the point x. The symplectic normal bundle, $\Sigma^\perp/\Sigma$, is the bundle whose fiber at x is the space $\Sigma_x^\perp/\Sigma_x$, where $\Sigma_x^\perp$ is the perpendicular space to Σ_x in $T_x(T^*M)$. As above the metaplectic structure on T^*M gives us the product of a metalinear structure on Σ with a metaplectic structure on $\Sigma^\perp/\Sigma$. We can now define Spin(Σ) to be the vector bundle on Σ whose fiber at a point x is given by

$$\mathrm{Spin}\,(\Sigma)_x = H_\infty(\Sigma_x^\perp/\Sigma_x) \otimes \textstyle\bigwedge^{1/2} \Sigma_x \; . \tag{24}$$

Note that the fiber of Spin(Σ) is an infinite dimensional vector space.

A symplectic spinor on Σ is a smooth section of Spin(Σ). There is an action of $\mathbb{R}^+$ on Spin(Σ) coming from the action of $\mathbb{R}^+$ on $T^*M\backslash\{0\}$ given by $r \cdot (x, \xi) \mapsto (x, r\xi)$. Denote by SS^m the space of symplectic spinors which are homogeneous of degree m under this action.

Proposition 2.7 ([5], *Proposition* 7.4) *There is a canonical symbol map,*

$$I^m(M,\Sigma) \to SS^m(\Sigma)\,, \tag{25}$$

whose kernel is $I^{m-1/2}(M,\Sigma)$.

In the present case, all of the relevant metalinear and metaplectic structures derive from the choice of a metalinear structure on P. In particular, T^*P inherits a metaplectic structure, as do the horizontal subspaces of TP. These metaplectic structures on the horizontal subspaces of TP in turn give rise to metalinear structures on Legendrian submanifolds of P.

Lemma 2.8 *A pseudo-hermitian manifold P always possesses a metalinear structure.*

Proof. Let w_1 be the first Stiefel-Whitney class of TP, an element of $H^1(P, \mathbb{Z}/2)$. The obstruction to the existence of a metalinear structure is w_1^2. Now TP is orientable if and only if $w_1 = 0$. Note that this would take care of the case where P is a circle bundle over X as in Sect. 1. In general, we have

$$TP = H \oplus E \tag{26}$$

(a Whitney direct sum of bundles), where H is the horizontal distribution (the kernel of α), and E is the trivial rank-one bundle spanned by ∂_θ. By the axioms of Stiefel-Whitney classes, $w_1 = w_1(H)$. H is a complex vector bundle, so in fact $w_1(H) = 0$ since a complex bundle is always orientable. □

A metalinear structure of P is not necessarily unique. The set of all metalinear structures on P has the same cardinality as $H^1(P, \mathbb{Z}/2)$.

We will next describe the symbol of the Szegö projector, Π. Define $\mathscr{X}^\sharp \subset T^*P$ as in Sect. 2.1, and $\mathscr{X}^\flat$ by

$$\mathscr{X}^\flat = \{(p, -r\alpha_p) : p \in P, r > 0\}\,, \tag{27}$$

and note that the space $\mathscr{X}$ is the diagonal subspace $\mathscr{X}^\sharp_+ \diamond \mathscr{X}^\flat_-$. Note that $\mathscr{X}^\sharp$ and $\mathscr{X}^\flat$ are symplectic submanifolds, whereas $\mathscr{X}$ is an isotropic submanifold. From Theorem 2.2, the Schwarz kernel of Π is an Hermite distribution in $I^{1/2}(P \times P, \mathscr{X})$, so $\sigma(\Pi)$ is an element of $SS^{1/2}(\mathscr{X})$.

Since it is sufficient to describe $\sigma(\Pi)$ locally, we begin by linearizing the problem. Choose a point $(p, r\alpha_p) \in \mathscr{X}^\sharp$ and let $V = T_{(p,r\alpha_p)}(T^*P)$. Define

$$Z = T_{(p,r\alpha_p)}\mathscr{X}^\sharp\,, \tag{28}$$

which is a symplectic subspace of V. As vector spaces we can identify V with $T_{(p,-r\alpha_p)}(T^*P)$ and Z with $T_{(p,-r\alpha_p)}\mathscr{X}^\flat$, but then V and Z carry the opposite symplectic structures. To avoid notational complications, we will use V and

Z to denote both of the respective vector spaces and point out the differing symplectic structures where necessary. We therefore write

$$T_{(p, r\alpha_p, p, -r\alpha_p)}\mathscr{Y} = Z \diamondtimes Z . \tag{29}$$

Note that $Z \diamondtimes Z$ is isomorphic to Z but is an isotropic subspace of $V \times V$. At the point $(p, r\alpha_p,\ p, -r\alpha_p)$, the fiber of Spin($\mathscr{Y}$) is

$$\mathrm{Spin}\,(Z \diamondtimes Z) = H_\infty((Z \diamondtimes Z)^\perp/(Z \diamondtimes Z)) \otimes \bigwedge^{1/2}(Z \diamondtimes Z) . \tag{30}$$

Lemma 2.9 *We have the canonical identification*

$$\mathrm{Spin}(Z \diamondtimes Z) = H_\infty(Z^\perp) \otimes H_\infty(Z^\perp) \otimes \bigwedge^{1/2} Z . \tag{31}$$

Proof. The spaces of half forms are identified through the isomorphism between Z and $Z \diamondtimes Z$. By definition $(Z \diamondtimes Z)^\perp$ is the space of all $(v, w) \in V \times V$ such that $w - v \in Z^\perp$. Since $V = Z \oplus Z^\perp$, it is clear that $(Z \diamondtimes Z)^\perp = (Z \diamondtimes Z) \oplus (Z^\perp \times Z^\perp)$. Thus we have

$$(Z \diamondtimes Z)^\perp/(Z \diamondtimes Z) \cong Z^\perp \times Z^\perp , \tag{32}$$

which is a symplectic isomorphism. □

The pseudo-Hermitian structure of P enters into the description of $\sigma(\Pi)$ in the following.

Proposition 2.10 *Associated to the pseudo-Hermitian structure of P is a positive definite Lagrangian subspace of* $Z^\perp \otimes \mathbb{C}$.

Proof. A Kähler structure on a vector space V is equivalent to the combination of a symplectic structure on V and the choice of a positive definite Lagrangian subspace of $V \otimes \mathbb{C}$ (the type (1,0) subspace). Thus we need to show only that $Z^\perp$ inherits a Kähler structure. This follows from:

Claim. *Under the projection* $T_{(p, r\alpha_p)}(T^*P) \to T_pP$, *the image of* $Z^\perp$ *is the null space of* α *in* T_pP.

Indeed, let $s : P \times \mathbb{R}^+ \to T^*P$ be the map $(p, r) \mapsto (p, r\alpha_p)$, whose image is $\mathscr{Y}^\sharp$. Then Z is the image of the differential map ds at the point (p, r) singled out above. Explicitly,

$$ds_{(p,r)}(v_r, v_p) = (v_p, v_r\alpha_p + r\,d\alpha_p(v_p)) . \tag{33}$$

We quickly see that the perpendicular space to Z is given by

$$Z^\perp = \{(w, -r(\nabla_p\alpha, w)) : (\alpha_p, w) = 0\} , \tag{34}$$

from which the Claim follows. □

Proposition 2.11 ([15], *Prop.* 4.2) *If* W *is a positive definite Lagrangian subspace of* $Z^\perp \otimes \mathbb{C}$, *then* ker $d\rho(W) \subset H_\infty(Z^\perp)'$ *is one-dimensional and contained in* $H_\infty(Z^\perp)$.

Combining Propositions 2.10 and 2.11, the pseudo-Hermitian structure on P determines a one-dimensional subspace of $H_\infty(Z^\perp)$. To write the symbol of Π, we need to choose an element e of norm one in this space (the symbol is of course independent of the choice). For our purposes it is convenient to fix a particular choice of e. We can do this because $Z^\perp$ is a symplectic vector space with a Kähler structure. Therefore there is a canonical realization of the metaplectic representation $H(Z^\perp)$ on Bargmann space. We require that e be real and positive under this representation.

Theorem 2.12 ([5], *Thm.* 11.2) *The symbol of the Szego projector is*

$$\sigma(\Pi) = e \otimes \bar{e} \otimes \sqrt{\mathrm{vol}_Z} \in H_\infty(Z^\perp) \otimes H_\infty(Z^\perp) \otimes \bigwedge{}^{1/2} Z\,, \tag{35}$$

where vol_Z *is the canonical volume form on* $\mathbb{Z}$ *given by the symplectic structure.*

We now will describe the symbol of a Legendrian distribution, as a symplectic spinor. Let $\mathscr{K}$ denote the Schwartz kernel of Π, and define the maps:

$$\begin{array}{ccc} P \times P & \xrightarrow{\Delta} & P \times P \times P \\ \pi \downarrow & & \\ P & & \end{array} \tag{36}$$

where $\Delta : (p_1, p_2) \mapsto (p_1, p_2, p_2)$ and $\pi : (p_1, p_2) \mapsto p_1$. Let ζ be a distribution on P conormal to Λ. Then $u = \Pi(\zeta)$ is given by

$$u = \pi_* \Delta^*(\mathscr{K} \boxtimes \zeta)\,. \tag{37}$$

$\mathscr{K} \boxtimes \zeta$ is an Hermite distribution on $P \times P \times P$ whose isotropic relation is $\mathscr{Z} \times N^*\Lambda \subset T^*(P \times P \times P)$. The operator $\pi_*\Delta^*$ is an ordinary FIO with associated relation

$$\mathscr{B} = \{(p_1, \xi_1), (p_1, p_2, p_2, -\xi_1, \xi_2, -\xi_2)\} \subset T^*P \times T^*(P \times P \times P). \tag{38}$$

Once again we proceed by linearizing the problem. Fix a point $(p, \xi) \in \Lambda^\sharp$, and define V and Z as in Sect. 4.1. Define

$$B = T_{((p,\xi),(p,p,p,-\xi,\xi,-\xi))}\mathscr{B} \tag{39}$$

We identify B as a subspace of $V \times W$, where $W = V \times V \times V$, keeping track of the signs of the symplectic forms as needed. We further define

$$Y = T_{(p,\xi)}(N^*\Lambda)\,, \tag{40}$$

and

$$A = (Z \mathbin{\hat{\times}} Z) \times Y\,. \tag{41}$$

Our starting point for the calculation of $\sigma(u)$ is $\sigma(\mathscr{K} \boxtimes \zeta) \in \mathrm{Spin}(A)$. Because Y is Lagrangian,

$$A^\perp / A \cong (Z \mathbin{\hat{\times}} Z)^\perp / (Z \mathbin{\hat{\times}} Z)\,. \tag{42}$$

Using Lemma 2.9, we can thus canonically identify

$$\mathrm{Spin}(\Lambda) = H_\infty(Z^\perp) \otimes H_\infty(Z^\perp) \otimes \bigwedge^{1/2} Z \otimes \bigwedge^{1/2} Y . \tag{43}$$

Denote the symbol of ζ by $\mu \in \bigwedge^{1/2} N^*\Lambda$. This corresponds to an element of $\bigwedge^{1/2} Y$ which we also denote by μ. According to Theorem 2.12, we therefore have

$$\sigma(\mathscr{K} \boxtimes \zeta) = e \otimes \bar{e} \otimes \sqrt{\mathrm{vol}_Z} \otimes \mu . \tag{44}$$

The symbol of u will be an element of $\mathrm{Spin}(\Lambda^\sharp)$. The linearization of $\Lambda^\sharp$ is $T_{(p,\xi)}\Lambda^\sharp = Y \cap Z$. As remarked in Sect. 2.1, $\Lambda^\sharp$ is a Lagrangian submanifold of $\mathscr{Z}^\sharp$, and thus $Y \cap Z$ is a Lagrangian subspace of Z.

Lemma 2.13 *We have*

$$(Y \cap Z)^\perp/(Y \cap Z) \cong Z^\perp , \tag{45}$$

so that we can identify,

$$\mathrm{Spin}(Y \cap Z) = H_\infty(Z^\perp) \otimes \bigwedge^{1/2}(Y \cap Z) . \tag{46}$$

Proof. Since $Y \cap Z$ is a Lagrangian subspace of Z and $V = Z \oplus Z^\perp$, we have $(Y \cap Z)^\perp = Z^\perp \oplus (Y \cap Z)$. □

Lemma 2.14 *For a fixed choice of $e \in H_\infty(Z^\perp)$ there is a natural isomorphism*

$$\varphi_e : \bigwedge^{1/2} Y \to \bigwedge^{1/2}(Y \cap Z) . \tag{47}$$

Proof. We begin by noting that the direct sum decomposition $V = Z \oplus Z^\perp$ induces the direct sum decomposition

$$Y = (Y \cap Z) \oplus (Y \cap Z^\perp) . \tag{48}$$

Since Y and $Y \cap Z$ are Lagrangian subspaces of V and Z respectively, $Y \cap Z^\perp$ must be a Lagrangian subspace of $Z^\perp$.

We thus have

$$\bigwedge^{1/2} Y \cong \bigwedge^{1/2}(Y \cap Z) \otimes \bigwedge^{1/2}(Y \cap Z^\perp) . \tag{49}$$

Theorem 2.6 gives the identification

$$\bigwedge^{1/2}(Y \cap Z^\perp) \cong \mathrm{Ker}\, d\rho(Y \cap Z^\perp) \subset H_\infty(Z^\perp)' . \tag{50}$$

Thus the Hilbert space inner product extends to a pairing

$$H_\infty(Z^\perp) \otimes \bigwedge^{1/2}(Y \cap Z^\perp) \to \mathbb{C} , \tag{51}$$

which, combined with (49), yields the map φ_e.

To see that φ_e is an isomorphism, note that the unitary group $U(Z^\perp) \subset \mathrm{Sp}(Z^\perp)$ acts transitively on the set of Lagrangian subspaces of $Z^\perp$, while preserving the Kähler structure. Thus we can choose an identification $Z^\perp \cong \mathbb{R}^{2n}$

such that $e \in \delta(\mathbb{R}^n)$ is a Gaussian centered at the origin, and $\operatorname{Ker} d\rho(Y \cap Z^\perp)$ consists of constant multiples of the delta function at the origin. Therefore (47) is an isomorphism. □

Proposition 2.15 *As a symplectic spinor, the symbol of* $u = \Pi(\zeta)$ *is*

$$\sigma(u) = e \otimes \varphi_e(\mu) \in H_\infty(Z^\perp) \otimes \bigwedge^{1/2}(Y \cap Z) \tag{52}$$

(*which is independent of the choice of* e).

Proof. Consider $B \subset V \times W$ as a canonical relation from W to V:

$$\begin{array}{ccc} & B & \\ {}^{\alpha}\swarrow & & \searrow{}^{\beta} \\ W & & V \end{array} \tag{53}$$

where α and β are the obvious projections. The result of the composition is $B \circ A := \beta(\alpha^{-1}(A)) = Y \cap Z$. Proposition 6.5 of [5] gives, in the present case, the symbol map

$$\operatorname{Spin}(A) \otimes \bigwedge^{1/2} B \to \operatorname{Spin}(B \circ A)\,. \tag{54}$$

The construction of the map (54) has two essential components. The first is an exact sequence

$$0 \to \operatorname{Ker} \rho \to B \oplus A \xrightarrow{\tau} U_1^\perp \to 0\,, \tag{55}$$

where $\tau : B \oplus A \to W$ is defined by $((a,b),c) \mapsto b - c$, and $U_1 = \alpha(B)^\perp \cap A^\perp \subset W$. A simple computation reveals $U_1 \cong Y \cap Z^\perp$. In our case, $\operatorname{Ker} \rho \cong B \circ A = Y \cap Z$, so that this exact sequence gives an isomorphism

$$\bigwedge^{1/2} B \otimes \bigwedge^{1/2} Z \otimes \bigwedge^{1/2} Y \cong \bigwedge^{1/2}(U_1^\perp) \otimes \bigwedge^{1/2}(Y \cap Z) \tag{56}$$

Let U be the image of U_1 in the quotient $A^\perp/A$ (note that $U \cong U_1$). Under the identification of $A^\perp/A$ with $Z^\perp \times Z^\perp$, U is just given by to $\{0\} \times (Y \cap Z^\perp)$.

The other component of the map (54) is the isomorphism ([5], 4.15)

$$\operatorname{Ker} d\rho(U) \cong \bigwedge^{1/2} U \otimes H_\infty(U^\perp/U)'\,. \tag{57}$$

It is clear from the remarks above that $U^\perp/U = Z^\perp \times \{0\}$. Since $\operatorname{Ker} d\rho(U) \subset H_\infty(A^\perp/A)'$, taking the dual of the isomorphism (57) gives a map

$$H_\infty(Z^\perp) \otimes H_\infty(Z^\perp) \otimes \bigwedge^{1/2}(Y \cap Z^\perp) \mapsto H_\infty(Z^\perp)\,. \tag{58}$$

Note that this is not an isomorphism.

These components fit together as follows. We begin with

$$\operatorname{Spin}(A) \otimes \bigwedge^{1/2} B = H_\infty(Z^\perp) \otimes H_\infty(Z^\perp) \otimes \bigwedge^{1/2} A \otimes \bigwedge^{1/2} B\,. \tag{59}$$

The isomorphism (56) takes us to

$$H_\infty(Z^\perp) \otimes H_\infty(Z^\perp) \otimes \bigwedge^{1/2}(U_1^\perp) \otimes \bigwedge^{1/2}(Y \cap Z). \tag{60}$$

Now because of the natural isomorphism $U_1^\perp \cong (W/U_1)^*$, we have

$$\bigwedge^{1/2} U_1^\perp \cong \bigwedge^{-1/2} W \otimes \bigwedge^{1/2} U. \tag{61}$$

W possesses a canonical half-form, which gives us a map $\bigwedge^{-1/2} W \to \mathbb{C}$, so that we can naturally identify

$$\bigwedge^{1/2} U_1^\perp \cong \bigwedge^{1/2}(Y \cap Z^\perp). \tag{62}$$

Thus (60) is isomorphic to

$$H_\infty(Z^\perp) \otimes H_\infty(Z^\perp) \otimes \bigwedge^{1/2}(Y \cap Z^\perp) \otimes \bigwedge^{1/2}(Y \cap Z). \tag{63}$$

The map (58), the only stage which is not an isomorphism, completes construction of the map (54).

We now simply trace what happens to the combination of $\sigma(\mathscr{K} \boxtimes \zeta) \in \mathrm{Spin}(A)$ and $\sigma(\pi_* \Delta^*) \in \bigwedge^{1/2} B$ under this map. First of all, we note that $\pi_* \Delta^*$ is a naturally defined operator, and $\sigma(\pi_* \Delta^*)$ is just the canonical element of $\bigwedge^{1/2} B$ determined by the symplectic structure on $B \cong V \times V$.

Consider the point (60) in the construction of the map. In the present case,

$$U_1^\perp \cong V \times V \times [Z \oplus (Y \cap Z^\perp)] \subset W, \tag{64}$$

so that the isomorphism

$$\bigwedge^{1/2} B \otimes \bigwedge^{1/2} Z \otimes \bigwedge^{1/2} Y \cong \bigwedge^{1/2}(U_1^\perp) \otimes \bigwedge^{1/2}(Y \cap Z) \tag{65}$$

simply arises from the decomposition $Y = (Y \cap Z) \oplus (Y \cap Z^\perp)$. In view of (64), the isomorphism (62) consists simply of dividing out by the canonical half-forms on B and Z. Since the symbol of $\pi_* \Delta^*$ and the half-form part of the symbol of Π are in fact just the canonical half-forms, these cancel out. At the stage (63) we end up with

$$\sigma(\mathscr{K} \boxtimes \zeta) \otimes \sigma(\pi_* \Delta^*) = e \otimes \bar{e} \otimes \mu, \tag{66}$$

where μ is thought of as an element of $\bigwedge^{1/2}(Y \cap Z^\perp) \otimes \bigwedge^{1/2}(Y \cap Z)$. The final stage is to apply the map (58), which takes $e \otimes \bar{e} \otimes \mu$ to $e \otimes \varphi_e(\mu)$. □

Observe that $\varphi_e(\mu) \in \bigwedge^{1/2}(T_{(p, r\alpha_p)} \Lambda^\sharp)$. Letting p and r vary, $\varphi_e(\mu)$ defines a half-form on $\Lambda^\sharp$ which is homogeneous. In view of the previous results, this half-form is the non-trivial part of the symbol of u, as an Hermite distribution. Since $\varphi_e(\mu)$ is homogeneous, it is determined by the restriction to the image of the section

$$s_\alpha : \Lambda \to \Lambda^\sharp$$
$$p \mapsto \alpha_p \,. \qquad (67)$$

Upon division by the natural radial half-form, this restriction becomes a half-form on Λ. We will refer to this half-form as the pull-back of $\varphi_e(\mu)$ to Λ via s_α and denote it by $s_\alpha^* \varphi_e(\mu)$.

Definition 2.16 *For a Legendrian distribution u, we will identify the symbol of u with the half-form on Λ, $s_\alpha^*\varphi_e(\mu)$. Precisely, we call the (well-defined) map*

$$\sigma^{(m)} : J^m(P,\Lambda) \to \bigwedge^{1/2}\Lambda$$
$$u = \Pi(\zeta) \mapsto s_\alpha^*\varphi_e(\mu)$$

the symbol map of order m.

2.3 Exactness of the symbol sequence

Our goal here is to prove the following:

Theorem 2.17 *The following is an exact sequence:*

$$0 \to J^{m-1/2}(P,\Lambda) \to J^m(P,\Lambda) \to \bigwedge^{1/2}\Lambda \to 0 \,. \qquad (68)$$

Moreover this sequence has a natural splitting, namely

$$\bigwedge^{1/2}\Lambda \to J^m(P,\Lambda)$$
$$v \mapsto T^{m-\frac{1}{2}}\Pi(\delta_v) \qquad (69)$$

where $T = \Pi\partial_T\Pi$ and ∂_T is the contact vector field (defined by the conditions $\iota_{\partial_T}\alpha = 1$ and $\iota_{\partial_T}d\alpha = 0$).

Remark. The operator T is non-negative. This follows from the fact that the symbol of ∂_T restricted to $\mathscr{X}$ is positive. Then, as in Proposition 2.14 of [5], there exists a *non-negative, elliptic* self-adjoint pseudodifferential operator, A, on P such that

$$[A,\Pi] = 0 \quad \text{and} \quad \Pi A \Pi = \Pi\partial_T\Pi = T \,. \qquad (70)$$

Therefore the powers $T^s = \Pi A^s \Pi$ (A^s defined to be zero in the kernel of A) are Toeplitz operators of order s, $\forall s \in \mathbb{R}$.

We first check that (69) is a right inverse of the symbol map. We need the following result on the behavior of the spaces J under Toeplitz operators:

Lemma 2.18 *If $u \in J^m(P,\Lambda)$ and $S = \Pi B\Pi$ is a Toeplitz operator of order p, then $S(u) \in J^{m+p}(P,\Lambda)$ and its symbol is $s_\alpha^*(\sigma_B)\sigma(u)$. In particular, $\forall s \in$*

$\mathbb{R}$, *T^s maps $J^m(P,\Lambda)$ into $J^{m+s}(P,\Lambda)$, and this map is the identity at the symbol level.*

Proof. By [5], without loss of generality $[B,\Pi]=0$. Therefore $S(u)=\Pi B(u)$, and since $B(u)$ is another conormal distribution the proof is complete. □

Corollary 2.19 *Indeed* (69) *is a right inverse of the symbol map.*

Next we prove that the kernel of the symbol map is precisely $J^{m-1/2}(P,\Lambda)$. The non-trivial part is to show that if $u=\Pi(\zeta)\in J^m(P,\Lambda)$ has zero symbol of order m, then it is the projection of a conormal distribution of order $\mathrm{ord}(\zeta)-1/2$. This is a consequence of the following:

Theorem 2.20

$$\{u\in I^m(P,\Lambda^\sharp);\Pi(u)=u\}=J^m(P,\Lambda)\ \textit{modulo smooth functions}\,.$$

Proof. Suppose that $u\in I^m(P,\Lambda^\sharp)$ is invariant under Π. Then the symbol σ_u of u (as an Hermite distribution) is a symplectic spinor which equals its own composition with the symbol of Π. From the discussion of the symbol of Π one can see that this implies that σ_u is of the form

$$\sigma_u=e\otimes\mu\,, \tag{71}$$

where $\mu\in\bigwedge^{1/2}\Lambda^\sharp$. By Corollary 2.19 it is possible to construct a conormal distribution $\zeta_1\in I^{m+n/2}(P,N^*\Lambda)$ such that $\Pi(\zeta_1)$ and u are Hermite distributions with the same symbol. Therefore, by the general symbol calculus of [10],

$$u_1:=u-\Pi(\zeta_1)\in I^{m-1/2}(P,\Lambda^\sharp)\,. \tag{72}$$

Observe furthermore that $\Pi(u_1)=u_1$; therefore we can repeat the same argument with u_1. Continuing by induction, we see that there is a sequence of conormal distributions, $\{\zeta_j\}$ whose orders are monotonically decreasing such that $\forall k\in\mathbb{Z}^+$

$$u-\Pi\left(\sum_{j=1}^{k}\zeta_j\right)\in I^{m-k/2}(P,\Lambda^\sharp)\,. \tag{73}$$

Now let ζ be a conormal distribution such that $\zeta\sim\sum_{j=1}^{\infty}\zeta_j$. Then $u-\Pi(\zeta)$ is a smooth function. The converse inclusion is trivial. □

3 Matrix element estimates

For this section we return to the case described in Sect. 1, where P is a unit circle bundle over a compact Kahler manifold X. In Sect. 2, we saw that at the symbolic level all Legendrian distributions look like delta functions or their derivatives. In view of this fact, we will restrict ourselves to the delta function case for the sake of simplicity. Let Λ_1 and Λ_2 be compact Legendrian submanifolds of P, and define Legendrian distributions $u=\Pi(\delta_{\Lambda_1})\in J^{1/2}(P,\Lambda_1)$ with symbol $v_1 \in \bigwedge^{1/2} \Lambda_1$ and $v = \Pi(\delta_{\Lambda_2}) \in J^{1/2}(P, \Lambda_2)$ with symbol $v_2 \in \bigwedge^{1/2} \Lambda_2$. Let A be a zeroth order classical pseudodifferential operator on P and let $T_A = \Pi A \Pi$ be the corresponding Toeplitz operator. In this section we estimate the matrix elements $\langle T_A u_k, v_k \rangle = \langle A u_k, v_k \rangle$.

3.1 The main statements

Let $F : P \times S^1 \to P$ be the action map, and define

$$\Theta_2 := F^{-1}(\Lambda_2) = \{(p, \omega); p \cdot \omega \in \Lambda_2\}, \tag{74}$$

where $\omega \in S^1$ and the action is denoted by a dot. Θ_2 is a submanifold since F is a submersion; in fact $\Theta_2 \cong \Lambda_2 \times S^1$ by the map $(p, \omega) \mapsto (p \cdot \omega, \omega)$.

Assumption. *We will assume that $\Lambda_1 \times S^1$ and Θ_2 intersect cleanly, meaning:*

1. *The intersection*

$$\mathscr{P} := (\Lambda_1 \times S^1) \cap \Theta_2 = \{(p, \omega); p \in \Lambda_1 \text{ and } p \cdot \omega \in \Lambda_2\} \tag{75}$$

is a submanifold of $P \times S^1$. Different connected components of $\mathscr{P}$ are allowed to have different dimensions.

2. *At every $(p, \omega) \in \mathscr{P}$,*

$$T_{(p,\omega)} \mathscr{P} = T_{(p,\omega)} \Theta_2 \cap T_{(p,\omega)}(\Lambda_1 \times S^1).$$

Equivalently, we may assume that the image of the map $\Phi : \mathscr{P} \to S^1$ induced by the natural projection is finite, and that for $\omega \in \Phi(\mathscr{P})$, the intersection $(\Lambda_1 \cdot \omega) \cap \Lambda_2$ is clean. That this is equivalent to the above assumption follows from the fact that Λ_1 and Λ_2 are Legendrian.

We label the points in the image of Φ by

$$\Phi(\mathscr{P}) = \{\omega_1, \ldots, \omega_N\}, \tag{76}$$

For each $l \in \{1, \ldots, \}$ let d_l be the dimension of the fiber $\Phi^{-1}(\omega_l)$, i.e.,

$$d_l = \dim(\Lambda_1 \cdot \omega_l) \cap \Lambda_2. \tag{77}$$

Lemma 3.1 *If Λ_1 and Λ_2 are two cleanly intersecting Legendrian submanifolds of P and μ_1 and μ_2 are half-forms on the respective submanifolds, then the intersection $\Lambda_1 \cap \Lambda_2$ inherits a top degree form $\mu_1 \# \mu_2$.*

Proof. Let Z by a symplectic vector space, with Lagrangian subspaces L_1 and L_2. The exact sequence,

$$0 \to L_1 \cap L_2 \to L_1 \oplus L_2 \to L_1 + L_2 \to 0, \tag{78}$$

where the third arrow takes $(v, w) \mapsto v - w$, leads to an isomorphism

$$\bigwedge^{1/2} L_1 \otimes \bigwedge^{1/2} L_2 \cong \bigwedge^{1/2} (L_1 \cap L_2) \otimes \bigwedge^{1/2} (L_1 + L_2). \tag{79}$$

Now, since $(L_1 \cap L_2)^{\perp} = L_1 + L_2$, we have

$$L_1 + L_2 \cong [Z/(L_1 \cap L_2)]^* . \tag{80}$$

This in turn allows us to identify

$$\bigwedge^{1/2}(L_1 + L_2) \cong \bigwedge^{1/2}(L_1 \cap L_2), \tag{81}$$

by using the canonical half-form on Z to map $\bigwedge^{-1/2} Z \to \mathbb{C}$. Finally, there is a natural map

$$\bigwedge^{1/2}(L_1 \cap L_2) \otimes \bigwedge^{1/2}(L_1 \cap L_2) \to \wedge^d (L_1 \cap L_2), \tag{82}$$

where $d = \dim(L_1 \cap L_2)$.

This construction thus associates to a pair of half-forms on the Lagrangian subspaces a top degree form on their intersection. Clearly the procedure generalizes to the case of two cleanly intersecting Lagrangian submanifolds of a symplectic manifold.

Define $\Lambda_1^\sharp, \Lambda_2^\sharp$, and $\mathscr{Y}^\sharp$ as in Sect. 2.1. $\mathscr{Y}^\sharp$ is a symplectic manifold, and $\Lambda_1^\sharp$ and $\Lambda_2^\sharp$ are Lagrangian submanifolds. To a half-form v_j on Λ_j we naturally associate a half-form on $\Lambda_j^\sharp$ by

$$v_j \mapsto v_j \otimes \sqrt{dr} . \tag{83}$$

The result proven above gives us a top degree form $(v_1 \otimes \sqrt{dr}) \# (v_2 \otimes \sqrt{dr})$ on $\Lambda_1^\sharp \cap \Lambda_2^\sharp$. We define $v_1 \# v_2$ as this form divided by α. □

Proceeding by analogy with the Fourier integral operator calculus, one might guess that the leading coefficient in the estimates for the matrix elements would involve only universal constants and the natural pairing described in Lemma 3.1. In fact, this coefficient involves an additional term, which we now describe.

Consider the tangent space T_xP at a point $x \in P$. In Sect. 2 we noted that the null space of α in T_xP (the horizontal space) is a symplectic vector space. Thus we can define an action of the symplectic group $\mathrm{Sp}(n)$ on T_xP by its action on the null space of α, and acting trivially on vertical vectors. In a symplectic vector space the unitary group, $U(n)$, regarded as a subgroup of $\mathrm{Sp}(n)$, acts transitively on the set of Lagrangian subspaces. In our case $U(n)$ acts transitively on the set of tangent spaces (at x) to oriented Legendrian submanifolds, with isotropy subgroup $\mathrm{SO}(n)$. Thus, given two oriented Legendrian subspaces Λ_1 and Λ_2 of P, we have a well-defined function on $\Lambda_1 \cap \Lambda_2$ which is the determinant of the unitary matrix mapping $T_x\Lambda_1$ to $T_x\Lambda_2$. We denote this function by $\det\{\Lambda_1, \Lambda_2\}$. Alternatively, if positive orthonormal bases $\{e_j\}$ and $\{f_j\}$ are chosen for the respective tangent spaces at x, we may define

$$\det\{\Lambda_1, \Lambda_2\}(x) = \det\{h(e_j, f_j)\} , \tag{84}$$

where h is the hermitian form (on the null space of α) at x.

In what follows, we will need to make sense of the square root of this function. This is precisely the role of the metalinear and metaplectic structures. As we remarked in Sect. 2, the metalinear structure on P gives rise to a metaplectic structure on the horizontal subspace of TP and to metalinear structures on the Legendrian submanifolds. To a unitary transformation as described above, we can associate a well-defined element in the double cover of $U(n)$ by taking the unique element of $Mp(n)$ which lies over the given transformation and which is a metalinear map from the tangent space of one Legendrian to the other. This association allows us to define the square root of $\det\{\Lambda_1, \Lambda_2\}$: the function $\sqrt{\det}$ is well-defined on the double cover of $U(n)$.

With these assumption and notation, we are now prepared to state our main result.

Theorem 3.2 *As $k \to \infty$ there is an asymptotic expansion*

$$\langle T_A u_k, v_k \rangle \sim \sum_{l=1}^{N} \omega_l^k \sum_{j=0}^{\infty} c_{j,l}\, k^{(d_l - j)/2}\,, \tag{85}$$

with

$$c_{0,l} = 2^{(n-d_l)/2} \pi^{-d/2} \int_{(\Lambda_1 \cdot \omega_l) \cap \Lambda_2} \det\{\Lambda_1 \cdot \omega, \Lambda_2\}^{-1/2} a v_1 \# \overline{v_2}\,, \tag{86}$$

where a is the pull-back to Λ_1 of the symbol of A by the connection one-form, α. Furthermore, if $\pi(\Lambda_1)$ and $\pi(\Lambda_2)$ do not intersect, then $\langle T_A u_k, v_k \rangle$ decreases rapidly in k.

Consider now the case of a single $u \in J^m(P, \Lambda)$, with symbol v. Observe that $\Lambda \times \{1\} \subset \mathscr{P}$ and is always a component of maximal dimension, n. Furthermore, if $\pi : \Lambda \to \pi(\Lambda)$ is a covering map with covering group the group of k_0-th roots of unit, then $\Lambda \times \{e^{2\pi i j/k_0}\}$, $j = 0, \ldots, k_0 - 1$, are n-dimensional components as well. This leads us to the following corollary.

Corollary 3.3 *Let $\pi|_\Lambda$ be a covering map with covering group the group of k_0-th roots of unity, and suppose the half-form v on Λ is invariant under the action of the covering group. Then $\langle u_k, u_k \rangle$ has the asymptotic behavior:*

$$\langle u_k, u_k \rangle \sim k_0 \left(\frac{k}{\pi}\right)^{n/2} \int_\Lambda |v|^2 \tag{87}$$

if k_0 divides k ($u_k = 0$ otherwise). In particular, for k a sufficiently large multiple of k_0, $\langle u_k, u_k \rangle$ is non-zero.

The remainder of Sect. 3 is devoted to the proof of the asymptotic expansion of Theorem 3.2. We begin by dealing with the case where the immersed Lagrangians corresponding to the Legendrian submanifolds do not intersect. The strategy is to study the singularities of the periodic distribution

$$\Upsilon(\theta) := \sum_{k=0}^{\infty} \langle A u_k, v_k \rangle e^{ik\theta}\,, \tag{88}$$

as in [10]. Knowledge of the singularities of Υ translates into the asymptotic expansion of its Fourier coefficients. In particular, we will show that the wave-front set of Υ is empty when $\mathscr{P} = \emptyset$. In this case, Υ is smooth and the matrix elements must decrease rapidly in k.

To construct Υ we proceed as follows. Choose $\zeta \in I^{(n+1)/2}(P, N^*\Lambda_2)$, such that $v = \Pi(\zeta)$. We will demonstrate that

$$\mathscr{V} := F^*(\overline{\zeta}) \tag{89}$$

(well-defined because F is a submersion), is a Lagrangian distribution on $P \times S^1$. Here $\overline{\zeta}$ is the complex conjugate of the distribution ζ as defined by the identity

$$(\overline{\zeta}, \varphi) = \overline{(\zeta, \overline{\varphi})}\,.$$

Furthermore,

$$\mathscr{V}(p, e^{i\theta}) = \overline{\zeta(p \cdot e^{i\theta})}\,. \tag{90}$$

We can regard $\mathscr{V}$ as the Schwartz kernel of an operator, V, from P to the circle. Since Π is an orthogonal projection which has already been applied to obtain u, it is clear that

$$\Upsilon = V \circ T_A(u)\,, \tag{91}$$

independently of the choice of ζ.

We begin by describing the canonical relation of the standard FIO F^*. If $\eta \in T_p^*P$, denote by η° the horizontal part of η. Thus we decompose

$$\eta = \eta(\partial_\theta)\alpha_p + \eta^\circ\,.$$

For every $\omega \in S^1$, we denote by $R_\omega : P \to P$ the map induced by the action of ω on the right. We define an operator $\tilde{R}_\omega : T^*_{p\,\cdot\,\omega}P \to T_p^*P$ by

$$\tilde{R}_\omega : \eta \mapsto \eta(\partial_\theta)\alpha_p + R_\omega^*\eta^\circ \tag{92}$$

The canonical relation of the Schwartz kernel of F^* is

$$\mathscr{C} = \{(p, \omega; \tilde{R}_\omega(\eta), \eta(\partial_\theta)), (p \cdot \omega; \eta)\} \subset T^*(P \times S^1) \times T^*P\,. \tag{93}$$

The following result is well-known.

Proposition 3.4 *The pull-back operator, F^*, extends to a map*

$$F^* : I^m(P, N^*\Lambda_2) \to I^m(P \times S^1, \Gamma)\,, \tag{94}$$

where

$$\Gamma = \{(p, \omega; \tilde{R}_\omega(\eta), \eta(\partial_\theta)); (p \cdot \omega, \eta) \in N^*\Lambda_2\} \subset T^*(P \times S^1)\,. \tag{95}$$

In particular, $\mathscr{V} \in I^{(n+1)/2}(P \times S^1, \Gamma)$.

As noted above, we can now consider $\mathcal{V}$ as the Schwartz kernel of an operator, V, from P to the circle. Specifically,

$$V(f)(\omega) = \int_P f(p)\overline{\zeta(p \cdot \omega)}\, dp\,. \tag{96}$$

We apply V to $T_A(u)$, to obtain the following

Proposition 3.5 *$\Upsilon = V(T_A(u))$, and $WF(\Upsilon) = \bigcup_{j=1}^{N}\{\omega_j\} \times \mathbb{R}^+$. In particular, if $\mathscr{P} = \emptyset$ then Υ is smooth.*

Proof. By Proposition 2.13 of [5], we can assume that $[A, \Pi] = 0$, and so $T_A : I^m(P, \Lambda_1^\sharp) \to I^m(P, \Lambda_1^\sharp)$. Therefore we need only to apply e.g. Theorem 8.2.13 in [11]. We omit the details. □

Corollary 3.6 *If $\pi(\Lambda_1) \cap \pi(\Lambda_2) = \emptyset$ then the matrix elements $\langle T_A u_k, v_k\rangle$ decrease rapidly in k.*

Remark. If $\mathscr{P}$ is non-empty one can ask whether the above construction shows that Υ is a Lagrangian distribution, i.e. whether the composition fiber product diagram

$$\begin{array}{ccc} \mathscr{G} & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \Lambda_1^\sharp & \longrightarrow & T^*P \end{array} \tag{97}$$

(where

$$\mathscr{G} = \{[(q, \omega, r\alpha_q, r), (p, r)];\ p = q \cdot \omega,\ p \in \mathscr{P}\} \tag{98}$$

and the arrows are the obvious ones) is clean. This is not so because condition (19) is not satisfied. To proceed, we will estimate the Fourier coefficients of Υ directly using the stationary phase lemma. This is the content of Sects. 3.2 and 3.3.

3.2 Oscillatory integrals

The strategy for computing the asymptotic estimates of Sect. 3.1 is to write the Hermite distributions as oscillatory integrals and approximate the matrix elements by stationary phase. By definition, the space $I^m(P, \Lambda^\sharp)$ consists of distributions that are locally expressible as oscillatory integrals of a certain type. For the remainder of this section, we will be working with open neighborhoods in P and Λ. References to P and Λ below are to be interpreted as statements concerning local neighborhoods in these spaces (else the notation becomes excessively complex).

In order to write the oscillatory integrals, we must first find a phase function parametrizing $\Lambda^\sharp$, in a sense to be described below. We need only consider the case of a non-degenerate phase function. The set up is as follows. Let B be an open conic subset of $(\mathbb{R} \times \mathbb{R}^n)\backslash\{0\}$. We give $\mathbb{R} \times \mathbb{R}^n$ the coordinates (τ, η).

Definition 3.7 *A non-degenerate phase function is a function $\phi \in C^{\infty}(P \times B, \mathbb{R})$ which satisfies:*

1. *ϕ is homogeneous in (τ, η).*
2. *$d\phi$ is nowhere zero.*
3. *The critical set of ϕ,*

$$C_{\phi} = \{(x, \tau, \eta); (d_{\tau}\phi)_{(x,\tau,\eta)} = (d_{\eta}\phi)_{(x,\tau,\eta)} = 0\} , \tag{99}$$

intersects the the space $\eta_1 = \ldots = \eta_n = 0$ transversally.

4. *The map $(x, \tau, \eta) \mapsto (\frac{\partial\phi}{\partial\tau}, \frac{\partial\phi}{\partial\eta_1} \ldots, \frac{\partial\phi}{\partial\eta_n})$ has rank $n+1$ at every point of C_{ϕ}, i.e. ϕ is non-degenerate.*

Define the map $F : C_{\phi} \to T^*P\backslash\{0\}$ by $(x, \tau, \eta) \mapsto (x, (d_x\phi)_{(x,\tau,\eta)})$. We quote the following result.

Proposition 3.8 *The image under F of the subspace of C_{ϕ} given by $\eta_1 = \cdots = \eta_n = 0$ is a homogeneous isotropic submanifold of $T^*P\backslash\{0\}$ of dimension $n+1$.*

Definition 3.9 *A phase function ϕ is said to parametrize $\Lambda^{\sharp}$ provided $\Lambda^{\sharp}$ is the image under F of $C_{\phi} \cap \{\eta_1 = \cdots = \eta_n = 0\}$.*

We are now prepared to describe the oscillatory integrals. A distribution (generalized half-form) in $I^m(P, \Lambda^{\sharp})$ can be written as a finite sum of locally defined oscillatory integrals. Specifically, given a non-degenerate phase function ϕ parametrizing $\Lambda^{\sharp}$ locally, we can write the distribution locally as

$$\int e^{i\phi(x,\tau,\eta)} a\left(x, \tau, \frac{\eta}{\sqrt{\tau}}\right) d\tau\, d\eta , \tag{100}$$

where the amplitude $a(x, \tau, u)$ has the following properties (see Sect. 3 of [5] for the precise formulation of the estimates):

1. $a(x, \tau, u)$ is rapidly decreasing as a function of u.
2. $a(x, \tau, u)$ is cutoff to be zero near $\tau = 0$.
3. For sufficiently large $\tau, a(x, \tau, u)$ admits an expansion of the form

$$a(x, \tau, u) \sim \sum_{i=0}^{\infty} \tau^{m_i} a_i(x, u) , \tag{101}$$

where each m_i is either integer or half-integer, with $m_0 = m - 1/2$ and $m_i \to -\infty$.

A change in the cutoff function used to enforce item 2 results only in a smooth correction to (100). Because of this, the cutoff function is generally suppressed from the notation.

Our next task is to actually construct a phase function that is linear in τ and τ. Suppose that $\phi(x, \tau, \eta) = \tau f(x) + \sum_{j=1}^{n} \eta_j g_j(x)$. We will hereafter adopt a vector notation

$$\eta \cdot g := \sum_{j=1}^{n} \eta_j g_j(x), \quad \eta \cdot dg := \sum_{j=1}^{n} \eta_j dg_j . \tag{102}$$

The critical set is $C_\phi = \{(x,\tau,\eta); f(x) = g_1(x) = \cdots = g_n(x) = 0\}$, and the map F is given by $F(x,\tau,\eta) = (x; \tau\, df(x) + \eta \cdot dg(x))$. In order for ϕ to parametrize $\Lambda^\sharp$, we take the conic subset B to be $(\mathbb{R}_+ \times \mathbb{R}^n)\backslash\{0\}$, and choose functions f and g_j satisfying two conditions. We require that the zero locus $\{f(x) = g_1(x) = \cdots = g_n(x) = 0\}$ define Λ in our local patch, and also that $df_x = \alpha_x$ for $x \in \Lambda$ (locally).

By the Darboux theorem for contact manifolds, we can introduce local coordinates $\{q_i, p_i, \theta\}$ on P such that

$$\alpha = \theta - p \cdot dq\,. \tag{103}$$

Because Λ is Legendrian, by taking a small enough neighborhood we can assume there exists a local generating function which gives the relationship between p and q on Λ. At least one of the following two cases will occur:

Case 1. There exists a function $h(q)$ such that $p_j = \frac{\partial h}{\partial q_j}$ on Λ. In this case we take

$$f = \theta - h, \quad g_j = p_j - \frac{\partial h}{\partial q_j}\,. \tag{104}$$

Case 2. There exists a function $h(p)$ such that $q_j = \frac{\partial h}{\partial p_j}$ on Λ. We take

$$f = \theta + h - p \cdot q, \quad g_j = q_j - \frac{\partial h}{\partial p_j}\,. \tag{105}$$

Note that in either case $df = \alpha$ on Λ.

If we write $u \in J^m(P,\Lambda)$ as an integral of the form (100), the highest order term in the expansion (101) is determined by the symbol of u computed in Sect. 2. The symbol map associating amplitudes with symplectic spinors is given as follows. Let π be the projection $P \times B \to P$. The pull-back π^* extends to a morphism on half-forms once we fix the convention that $\pi^*\sqrt{dx} = \sqrt{dx\, d\tau\, d\eta}$. This map is an FIO with canonical relation Γ given by the conormal bundle of the graph π in $T^*(P \times P \times B)$. We can parametrize

$$\Gamma = \{(x,x,\tau,\eta;\xi,\xi,0,0)\}\,. \tag{106}$$

The symbol of π^* is just the canonical half-form on Γ, which in terms of these coordinates is just $\sqrt{dx\, d\xi\, d\tau\, d\eta}$.

Consider $d\phi$ as a map $P \times B \to T^*(P \times B)$. We have

$$d\phi : (x,\tau,\eta) \mapsto (x,\tau,\eta; \tau df + \eta \cdot dg, f d\tau, g \cdot d\eta)\,. \tag{107}$$

Let Σ_ϕ denote the image under $d\phi$ of the subspace $\eta_1 = \cdots = \eta_n = 0$,

$$\Sigma_\phi = \left\{\left(x,\tau,0; \tau df, f d\tau, \textstyle\sum g_j d\eta_j\right)\right\}\,. \tag{108}$$

Σ_ϕ is an isotropic submanifold of $T^*(P \times B)$. Furthermore, Γ intersects Σ_ϕ transversally and $\Gamma \circ \Sigma_\phi = \Lambda^\sharp$.

We define a symplectic spinor on Σ_ϕ by

$$\kappa = \sqrt{dx\,d\tau} \otimes \tau^{m-1/2} a_0(x,\eta) \sqrt{d\eta}\,, \tag{109}$$

where $a_0(x,\eta)$ is the leading term of the expansion (101).

Definition 3.10 *The symbol $\sigma(u) \in SS^m(\Lambda^\sharp)$ is the image of κ under the canonically defined map (see [8])*

$$SS^m(\Sigma_\phi) \to SS^m(\Gamma \circ \Sigma_\phi)\,. \tag{110}$$

The symbol map in this definition comes from the composition formula used in Proposition 2.15. We note that in [5], there is an apparent typo in the degree of homogeneity in τ of the amplitude a_0.

For the following Proposition, let f and g be chosen as above in accordance with either Case 1 or Case 2. For Case 1 let $H_{jk} = \frac{\partial^2 h}{\partial q_j \partial q_k}$, and for Case 2, $H_{jk} = \frac{\partial^2 h}{\partial p_j \partial p_k}$. In terms of the Darboux coordinates, we write the metric as a matrix

$$g = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix}\,, \tag{111}$$

with A and D symmetric. The matrix of the symplectic form is

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\,, \tag{112}$$

and the complex structure is given by $J = \Omega^t g$. The requirement that $J^2 = -I$ implies the following conditions:

$$\begin{aligned} AD - B^2 &= I\,, \\ B^t D &= DB\,, \\ AB^t &= BA\,. \end{aligned} \tag{113}$$

Proposition 3.11 *Let $u \in J^m(P,\Lambda)$ with symbol $e \otimes v \in H_\infty((\Lambda^\sharp)^\perp/\Lambda^\sharp) \otimes \bigwedge^{1/2}\Lambda^\sharp$. We can write u locally as an oscillatory integral of the form* (100) *with*

$$a_0(x,\eta) = C_n \tilde{v}(x) \det M^{-1/2} e^{-\frac{1}{2}\eta^t M^{-1}\eta}\,, \tag{114}$$

where C_n depends only on the dimension, $\tilde{v}$ is an extension of v to be defined below, and

$$M = \begin{cases} (I + iB^t + iDH)^{-1} D & \text{for Case } 1\,, \\ (I - iB - iAH)^{-1} A & \text{for Case } 2 \end{cases} \tag{115}$$

(*note that M is symmetric in either case*).

Proof. We need to compute the preimage of $e \otimes v$ under the symbol map (110) associated to π^*. The details of the map are given in Proposition 6.5 of [5]. This map can be broken into two parts: the map of half-forms which takes $\tilde{v}$ to

v and the map of Schwarz functions which takes the Gaussian above to e. Fix a point $x = (p, q, \theta) \in \Lambda$ and $\tau \in \mathbb{R}_+$. Let $W = T_{(x,\tau,0;\tau\alpha_x,0,0)}T^*(P \times G)$. Recall that $G = \mathbb{R}_+ \times \mathbb{R}^n$ and $\Sigma = T_{(x,\tau)}$. Similarly let Γ now denote the tangent space to the Γ defined above.

The half-form part of the map is particularly trivial in this case. We have an exact sequence

$$0 \to \Gamma \circ \Sigma \to \Gamma \oplus \Sigma \to W \to 0\,, \tag{116}$$

which, together with the natural half-forms on Γ and W, furnishes an isomorphism

$$\bigwedge^{1/2} \Sigma \cong \bigwedge^{1/2}(\Gamma \circ \Sigma)\,. \tag{117}$$

The map of half-forms reduces simply to this isomorphism, so that $\tilde{v}$ can be any smooth function on P such that the isomorphism (117) takes $\tilde{v}(x)\sqrt{dx\,d\tau}$ to v at points of $\Lambda^\#$.

The non-trivial portion of the symbol map is really the isomorphism of the Schwarz spaces $H_\infty(\Sigma^\perp/\Sigma) \cong H_\infty((\Gamma \circ \Sigma)^\perp/(\Gamma \circ \Sigma))$, which arises from a canonical symplectic isomorphism $\Sigma^\perp/\Sigma \cong (\Gamma\circ\Sigma)^\perp/(\Gamma\circ\Sigma)$ (Proposition 6.4 of [5]). This map is given as follows. Given $a \in \Sigma^\perp/\Sigma$, we choose $(b, c) \in \Gamma$ such that $c \in \Sigma^\perp$ and the image of c in $\Sigma^\perp/\Sigma$ is a. Then $b \in \Gamma\circ\Sigma^\perp = (\Gamma\circ\Sigma)^\perp$ and the association $a \to b$ descends to an isomorphism when we mod out by Σ. Because e was defined through the identification of $(\Gamma \circ \Sigma)^\perp/(\Gamma \circ \Sigma)$ with the horizontal subspace of T_xP, we will construct the map directly to this horizontal subspace.

We break the problem up into the two cases described above. Assume first that we are in Case 1, where $p = \frac{\partial h}{\partial q}$ on Λ and $f = \theta - h$. Define $H_{jk} = \frac{\partial^2 h}{\partial q_j \partial q_k}$. In terms of the basis $\{\frac{\partial}{\partial q_j}, \frac{\partial}{\partial p_j}, \frac{\partial}{\partial \theta}, \dots\}$, a straightforward computation gives

$$\Sigma = \{(v, w, t, r, 0; -rp - Hv, 0, r, t - p \cdot v, w - Hv)\} \tag{118}$$

(where v and w are n-vectors and t and r are real numbers). From this we compute that

$$\Sigma^\perp = \{(v, w, t, r, \beta; v - rp - w - H\beta, \beta, r, t - p \cdot v, \gamma)\}\,. \tag{119}$$

Define $\psi : \Sigma \to \mathbb{R}^n \times \mathbb{R}^n$ by

$$\psi : (v, w, t, r, \beta; v - rp - w - H\beta, \beta, r, t, -p \cdot v, \gamma) \mapsto (\beta, \gamma - w + Hv)\,. \tag{120}$$

The kernel of this map is Σ, so it descends to an isomorphism $\Sigma^\perp/\Sigma \to \mathbb{R}^n \oplus \mathbb{R}^n$ (with the natural symplectic structure on the latter). We will henceforth identify these spaces through this isomorphism, giving $\Sigma^\perp/\Sigma$ the coordinates (β, σ).

To $a = (\beta, \sigma) \in \Sigma^\perp/\Sigma$ we associate the vector $c = (0, -\sigma, 0, 0, \beta; \sigma - H\beta, \beta, 0, 0, 0) \in \Sigma^\perp$, so that $\psi(c) = a$. In view of (106), for (b, c) to be in Γ we must take $b = (0, -\sigma, 0; \sigma - H\beta, \beta, 0) \in \Gamma \circ \Sigma^\perp$. In the notation of Sect. 2, we have a decomposition $V = T_{(x,\tau\alpha_x)}(T^*P) = Z \oplus Z^\perp$, and terms of the Darboux coordinates:

$$Z = \{(v, w, t; -rp - w, 0, r)\}\,, \tag{121}$$

and

$$Z^{\perp} = \{(v, w, p \cdot v; -rp - w, 0, r)\} . \tag{122}$$

Recall that the linearization of the symplectic normal to $\Lambda^{\sharp}$ (here written $(\Gamma \circ \Sigma)^{\perp}/(\Gamma \circ \Sigma)$) was isomorphic to the symplectic vector space $Z^{\perp}$. Thus to complete the map we need only project $b \in \Gamma \circ \Sigma^{\perp}$ into $Z^{\perp}$. In order to pull-back the Gaussian e we then identify $Z^{\perp}$ with the horizontal subspace of T_xP by $(v, w, p \cdot v; -rp - w, 0, r) \to (v, w)$. The result is that the symplectic isomorphism from $\Sigma^{\perp}/\Sigma$ to $Z^{\perp}$ can be written

$$(\beta, \sigma) \mapsto (-\beta, -\sigma - H\beta) , \tag{123}$$

where the symplectic structure on both sides is the natural structure on $\mathbb{R}^n \oplus \mathbb{R}^n$.

Consider how e was constructed (Proposition 4.2 of [5]). If the set of complex vectors $\{x_j + iy_j\}$ gives a basis for the (1,0)-subspace of the complexified horizontal tangent space to P at x, then e is defined as a solution (of norm one) to the equations $(x_j + iy_j)e = 0$, where y_j acts as $-i\frac{\partial}{\partial x_j}$. In the present case, given that

$$J = \begin{pmatrix} -B^t & -D \\ A & B \end{pmatrix} , \tag{124}$$

the (1,0)-subspace is spanned by vectors of the form $\{v + iB^t v + iDw\}$. Composing with the inverse of the symplectic map (123), we see that $e(\beta)$ should satisfy the differential equation

$$\left[\beta + iB^t\beta + iD\left(-i\frac{\partial}{\partial\beta} + H\beta\right)\right] e(\beta) = 0 . \tag{125}$$

The solution (up to a constant depending only on the dimension) is

$$e(\beta) = \det M^{-1/2} e^{-\frac{1}{2}\beta^t M^{-1}\beta} \sqrt{d\beta} , \tag{126}$$

where $M = (I + iB^t + iDH)^{-1}D$. The factor $\det M^{-1/2}$ appears because e transforms as a half-form under symplectic transformations. This completes the proof for Case 1.

The proof is quite similar for Case 2. Here we have $q_j = \frac{\partial h}{\partial p_j}$ as the defining relation of Λ. Taking $H_{jk} = \frac{\partial^2 h}{\partial p_j \partial p_k}$, the symplectic map (123) turns out to be

$$(\beta, \sigma) \mapsto (-\sigma + H\beta, \beta) . \tag{127}$$

This leads to the Gaussian given above. □

Theorem 3.12 *Let $u \in J^m(P, \Lambda)$ with symbol v. Choose f and g locally as above, and define $f_0(p, q) := \theta - f(p, q, 0)$. For sufficiently large k the k-th isotype of u under the S^1 action has the local representation*

$$u_k = C_{m,n} k^{m+(n-1)/2} \tilde{v}(p, q, f_0) e^{ikf - \frac{k}{2} g^t M g} + O(k^{(m+n/2-1)}) \tag{128}$$

(in the sup norm topology), where $C_{m,n}$ is a constant depending only on m and n. (In fact, there is full asymptotic expansion in decreasing half-integer powers of k.)

Proof. We start with the local oscillatory integral representation

$$u(p,q,\theta) = \int e^{i\tau f + i\eta \cdot g} a\left(p,q,\theta,\tau,\frac{\eta}{\sqrt{\tau}}\right) d\tau\, d\eta \tag{129}$$

with

$$a(p,q,\theta,\tau,u) \sim \sum_{j=0}^{\infty} \tau^{m_j} a_j(p,q,\theta,u)\,. \tag{130}$$

To pick off the k-th isotype, we project onto the $e^{ik\theta}$ component by integrating the above expression against $e^{-ik\theta}$ for $0 \leqq \theta \leqq 2\pi$. The expression (129) is cutoff in θ so we may in fact extend the integration limits to infinity.

We will consider one term in the expansion at a time. Let

$$W_k(p,q,\theta) = e^{ik\theta} \int e^{-ik\theta'} e^{i\tau f(p,q,\theta')} \tau^l a\left(p,q,\theta',\frac{\eta}{\sqrt{\tau}}\right) d\tau\, d\theta'\,. \tag{131}$$

Rescaling $\tau \to k\tau$ yields

$$W_k(p,q,\theta) = k^{l+1} e^{ik\theta} \int e^{-ik\theta'} e^{ik(-\theta'+\tau\theta'-\tau f_0)} \tau^l a\left(p,q,\theta',\frac{\eta}{\sqrt{k\tau}}\right) d\tau\, d\theta' \tag{132}$$

This expression can be estimated for large k by stationary phase. The only stationary point occurs at $\tau = 1$, $\theta' = f_0(p,q)$, so we obtain the estimate

$$\left| W_k(p,q,\theta) - 2\pi k^l e^{ikf}\left[1 + k^{-1} L_{\tau,\theta'} \tau^l a\left(p,q,\theta',\frac{\eta}{\sqrt{k\tau}}\right)\right]\right|$$

$$\leqq C k^{l-1} \sum_{|\alpha| \leqq 4} \sup \left| D^\alpha \tau^l a\left(p,q,\theta',\frac{\eta}{\sqrt{k\tau}}\right)\right|\,, \tag{133}$$

where $L_{\tau,\theta'}$ is a second order differential operator in τ and θ', evaluated at the critical point, and D represents only derivaties with respect to τ and θ'. Note that the sup is finite if and only if $l \leqq 0$. By applying successive integrations by parts in the original expression, we may assume that this is the case.

Derivatives of a with respect to τ bring out a factor of $k^{-1/2}$, and derivatives with respect to θ' have no effect in terms of k. The first correction and the error term are thus both well-behaved in terms of k. So in terms of the sup norm we have

$$W_k(p,q,\theta) = 2\pi k^l e^{ikf} + O(k^{l-1})\,. \tag{134}$$

Applying this result to u_k, we obtain

$$u_k(p,q,\theta) = 2\pi k^{m-1/2} e^{ik(\theta - f_0)} \int e^{i\eta\cdot g} a_0\left(p,q,f_0,\frac{\eta}{\sqrt{k}}\right) d\eta + O(k^{m-1})\,. \tag{135}$$

From Proposition 3.11, we see that the remaining η integration is just the Fourier transform of a Gaussian:

$$\int e^{i\eta\cdot g} e^{-\frac{1}{2k}\eta^t M^{-1}\eta} d\eta = Ck^{n/2} \det M^{1/2} e^{-\frac{k}{2}g^t Mg} . \qquad \square \tag{136}$$

3.3 Proof of the estimates

For this subsection, we assume that $u = \Pi(\delta_{v_1}) \in J^{1/2}(P, \Lambda_1)$ and $v = \Pi(\delta_{v_2}) \in J^{1/2}(P, \Lambda_2)$. Consider the inner product $\langle u_k, v_k \rangle$. Because of the reproducing property of the kernel of Π and the fact that the states come from delta functions, $\langle u_k, v_k \rangle$ can be written as the integral of u_k (represented as a function on P) over Λ_2 (with a measure determined by v_2).

Because of this fact, the following proposition (a restatement of Corollary 3.3) follows from Theorem 3.12.

Proposition 3.13 *Let $u = \Pi(\delta_v) \in J^{1/2}(P, \Lambda)$, and suppose that Λ is a k_0-fold covering of $\pi(\Lambda)$, with v invariant under the action of the covering group. Then as $k \to \infty$*

$$\langle u_k, u_k \rangle \sim k_0 \left(\frac{k}{\pi} \right)^{n/2} \int_\Lambda |v|^2 , \tag{137}$$

for k a sufficiently large multiple of k_0.

Proof. From Theorem 3.12 and the reproducing property we find

$$\langle u_k, u_k \rangle \sim C \sum_{j=1}^{k_0} e^{2\pi i \frac{kj}{k_0}} k^{n/2} \int_\Lambda |v|^2 . \tag{138}$$

The sum is zero unless k_0 divides k, in which case it yields the factor k_0.

Since the constant out front is universal, it may be computed in a particular example. This is easily done for the Bargmann space $\mathbb{C}^n$, where the kernel of Π can be written explicitly. $\square$

Let Λ_1 and Λ_2 be two distinct intersecting Legendrian submanifolds. The following lemmas are a prelude to taking the stationary phase approximation of the inner product of states defined on Λ_1 and Λ_2. Let f and g_j be chosen to parametrize Λ_1 as in the last subsection (according to whether Λ_1 satisfies Case 1 or 2). Recall that the highest order term in u_k involved the phase function $\psi = f + \frac{i}{2} g^t Mg$, with M determined by f and g. Choose a set of local parameters $\{t_1, \ldots, t_n\}$ to describe Λ_2, so that $\{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n}\}$ gives locally an orthonormal frame for $T\Lambda_2$. We will use primes to denote derivatives with respect to these parameters.

Lemma 3.14 *The stationary points of ψ, with respect to the parametrization described above, correspond precisely with points of $\mathcal{P}$.*

Proof. The proof is similar for either case, so we assume Case 1. Then

$$\psi(p,q,\theta) = \theta - h(q) + \frac{i}{2}\left(p - \frac{\partial h}{\partial q}\right) M \left(p - \frac{\partial h}{\partial q}\right), \tag{139}$$

with derivative

$$\psi'(p,q,\theta) = \theta' - \frac{\partial h}{\partial q}q' + i\left(p - \frac{\partial h}{\partial q}\right) M(p' - Hq') + \frac{i}{2}\left(p - \frac{\partial h}{\partial q}\right) M' \left(p - \frac{\partial h}{\partial q}\right). \tag{140}$$

Since the horizontality of Λ_2 implies that $\theta' = pq'$, we see immediately that $p = \frac{\partial h}{\partial q}$ implies $\psi' = 0$. This is the case whenever there exists an $\omega \in S^1$ and $x \in \Lambda_1$ such that $x \cdot \omega \in \Lambda_2$.

It follows directly from Proposition 3.5 that there are no other possible stationary points. □

Lemma 3.15 *The Hessian of $\psi = f + \frac{i}{2}g^t Mg$ at a stationary point x*

$$\psi'' = \xi_2^t \Omega^t \xi_1 (\xi_1^t g \xi_1)^{-1} \xi_1^t (g + i\Omega)\xi_2 , \tag{141}$$

where $\xi_2 = \begin{pmatrix} q' \\ p' \end{pmatrix}$ *and* $\xi_1 = \begin{pmatrix} I \\ H \end{pmatrix}$.

Proof. The Hessian of ψ is

$$\psi'' = p''q' - q''Hq' + i(p'' - q''H)M(p' - Hq'), \tag{142}$$

where we have used the fact that $\theta' = pq'$ because Λ_2 is horizontal. Recalling the definition of M, we can write this as

$$\begin{aligned}\psi'' &= (p' - Hq')^t[q' + i(I + iB^t + iDH)^{-1}D(p' - Hq')] \\ &= (p' - Hq')^t(I + iB^t + iDH)^{-1}[q' + iB^t q' + iDp'] .\end{aligned} \tag{143}$$

We insert the matrix $(I - iB^t - iDH)$ and its inverse and use the identities (113) to obtain

$$\begin{aligned}\psi'' &= (p' - Hq')^t(A + BH + HB^t + HDH)^{-1} \\ &\quad \times [(A + HB^t)q' + (B + HD)p' + i(p' - Hq')] .\end{aligned} \tag{144}$$

Note that this is exactly the expression given above. □

Proposition 3.16 *The inner product $\langle u_k, v_k\rangle$ has an asymptotic expansion whose terms correspond to elements of $\Phi(\mathscr{P})$. The leading contribution from a particular $\omega \in \Phi(\mathscr{P})$ is*

$$(2i)^{(n-d)/2}\left(\frac{k}{\pi}\right)^{d/2} \omega^k \int_{\Lambda_1 \cdot \omega \cap \Lambda_2} \det\{\Lambda_1 \cdot \omega, \Lambda_2\}^{-1/2} v_1 \# \overline{v_2} , \tag{145}$$

where d is the dimension of $\Phi^{-1}(\omega)$.

Proof. We start with the representation of Theorem 3.12. Choose the parametrization of Λ_2 so that the first d variables parametrize $\Lambda_1 \cdot \omega \cap \Lambda_2$. The method of proof is to apply stationary phase to the integral over the $n-d$ transverse parameters of Λ_2. The integral over the remaining d variables survives in the final expression. For notational clarity, consider only the case where $\Lambda_1 \cdot \omega$ and Λ_2 intersect transversally ($d = 0$). From Theorem 3.12, the highest order term contribution to the inner product is

$$\langle u_k, v_k \rangle = \left(\frac{k}{\pi}\right)^{n/2} \int \overline{v_2}(p,q,\theta)\tilde{v}_1(p,q,f_0)e^{ikf - \frac{k}{2}g^t M g} d^n t + O(k^{(n-1)/2}), \quad (146)$$

where we have filled in the constant based on the computation in Proposition 3.13. According to Lemma 3.14, when we apply stationary phase to this integral, we obtain a term for each component of $\mathscr{P}$, i.e. for each point in $\Phi(\mathscr{P})$.

In the transverse case, at the point $x \in \Lambda_1 \cdot \omega \cap \Lambda_2$, the stationary phase lemma yields the term

$$\left(\frac{k}{\pi}\right)^{n/2} v_1(x \cdot \omega)\overline{v_2}(x)\omega^k \left(\frac{2\pi i}{k}\right)^{n/2} (\det \psi'')^{-1/2} . \quad (147)$$

We can reinterpret Lemma 3.15 in the following way. Given an orthonormal basis $\{e_i\}$ for $T_x\Lambda_1$ and $\{f_i\}$ for Λ_2,

$$\det \psi'' = \det\{\omega(e_i, f_j)\} \det\{h(e_i, f_j)\} , \quad (148)$$

where ω is the symplectic form and h the hermitian form. The first term on the right-hand side is (when raised to the $-1/2$ power) the factor which appears in the construction of $v_1 \# \overline{v_2}$ when we divide out by the square root of the Liouville form. The second term is the function $\det\{\Lambda_1 \cdot \omega, \Lambda_2\}$, as defined in Sect. 3.1.

In general, the stationary phase approximation is done over $n-d$ variables, so the last factors in (147) are $(2\pi i/k)^{(n-d)/2}(\det \psi'')^{-1/2}$, with the determinant taken over the transverse variables. It is straightforward to see the this determinant yields again the intersection of the half-forms with the same unitary factor. □

To conclude this section, we note that Theorem 3.2 follows directly from Propositions 3.13 and 3.16. The insertion of the Toeplitz operator T_A in the inner product is an essentially trivial modification.

4 Bohr-Sommerfeld curves and Poincaré series

In this Section we examine in the case where X is the quotient of the upper half plane by a Fuchsian group of the first kind. The natural quantizing line bundle L is simply the holomorphic tangent bundle. We will perform the general constructions outlined in Section I quite explicitly for this case. In particular, we compute the states associated to Bohr-Sommerfeld curves given by hypercycles, horocycles, or circles in H and show that these correspond to well-known Poincaré series.

4.1 Bohr-Sommerfeld curves

Let H be the upper-half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, and let SH denote the unit circle bundle of the holomorphic cotangent bundle of H,

$$SH = \{(z,\zeta) \in H \times \mathbb{C} : |\zeta| = \operatorname{Im} z\} . \tag{149}$$

The group $G = \mathrm{SL}(2,\mathbb{R})$ acts on SH by fractional linear transformations. In fact, there is a homeomorphism $SH \cong G/\{\pm id\}$, such that the action of G on SH corresponds to the left action of G on $G/\{\pm id\}$. Explicitly, for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G , \tag{150}$$

we have

$$g \cdot (z,\zeta) = (g \cdot z, j(g,z)^{-2}\zeta) , \tag{151}$$

where

$$g \cdot z = \frac{az+b}{cz+d} \tag{152}$$

and $j(g,z) = cz + d$. G is represented on the space of functions on SH by

$$(g \cdot F)(z,\zeta) = F(g^{-1} \cdot (z,\zeta)) . \tag{153}$$

The contact form α is given by

$$\alpha = d\phi - \frac{dx}{y} , \tag{154}$$

where $z = x + iy$, and $\zeta = ye^{i\phi}$. The volume form $dV = (2\pi)^{-1}\alpha \wedge d\alpha$ is the G-invariant volume form on SH:

$$dV = \frac{dx\,dy}{y^2}\frac{d\phi}{2\pi} . \tag{155}$$

The connection on SH corresponding to α is naturally defined as follows. Letting I denote the point $(i,1) \in SH$, we identify $G/\{\pm id\}$ with SH by the map $g \mapsto g \cdot I$. We thus have $T_I(SH) \cong \mathrm{sl}(2,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$. Since we also know that $T_iH \cong \mathfrak{p}$, we can define a left-invariant connection simply by declaring $\mathfrak{p}$ to be the horizontal space at $I \in SH$. Using the identification $T_{g \cdot I}SH \cong T_gG$, we see that the horizontal tangent space of $T_{g \cdot I}SH$ is $g \cdot \mathfrak{p}$. Therefore, if $g(t) : \mathbb{R} \to G$ is a smooth curve, $g(t) \cdot I$ will be horizontal iff

$$g(t)^{-1} \cdot \dot{g}\,(t) \in \mathfrak{p} , \tag{156}$$

i.e., $g(t)^{-1} \cdot \dot{g}\,(t)$ must be traceless and symmetric.

Recall that SH is the boundary of a strictly pseudoconvex domain (the unit disk bundle over H). Let $\mathfrak{S}(H) \subset L^2(SH)$ denote the Hardy space of boundary values of holomorphic functions on the unit disk bundle. The k-th isotype of $\mathfrak{S}(H)$ under the action of S^1 is the space of holomorphic k-differentials on H, which we denote by $\mathfrak{S}_k(H)$. This consists of functions $F : SH \to \mathbb{C}$ of the form $F(z,\zeta) = \zeta^k f(z)$, where $f : H \to \mathbb{C}$ is holomorphic. In other words,

$$\mathfrak{S}_k(H) = \left\{ \zeta^k f(z) : \int_H |f(z)|^2 y^{2k} \frac{dx\,dy}{y^2} < \infty \right\} . \tag{157}$$

Let Γ be a discrete subgroup of G such that $X = \Gamma \backslash H$ has finite volume, i.e., a Fuchsian group of the first kind. The unit circle bundle of the holomorphic cotangent bundle of X, denoted by SX, is again the boundary of a strictly pseudoconvex domain, and we let $\mathfrak{S}(X)$ denote the Hardy space for this domain. As above, $\mathfrak{S}_k(X)$ denotes the space of holomorphic k-differentials on X. Define the orthogonal projections

$$\Pi : L^2(SX) \to \mathfrak{S}(X) \quad \text{and} \quad \Pi_k : L^2(SX) \to \mathfrak{S}_k(X) . \tag{158}$$

If F is a function on SH which is invariant under the action of G, then F corresponds to a function on SX. Thus we can identify $\mathfrak{S}_k(X)$ with the space of cusp forms $S_{2k}(\Gamma)$:

$$\begin{aligned} \mathfrak{S}_k(X) &= S_{2k}(\Gamma) \\ &= \left\{ f(z) : f(g \cdot z) = f(z) j(g,z)^{2k}, \int_{\mathscr{F}} |f(z)|^2 y^{2k} \frac{dx\,dy}{y^2} < \infty \right\} , \end{aligned} \tag{159}$$

where $\mathscr{F}$ is a fundamental domain for Γ.

In what follows, by a smooth closed curve with domain $[0, T]$ we mean the restriction to $[0, T]$ of a smooth T-periodic map with domain $\mathbb{R}$. Generally, we will think of a closed curve γ on X as the projection to X of a curve $\gamma : \mathbb{R} \to H$ such that the points $\gamma(t)$ and $\gamma(t + T)$ are related by an element of Γ.

Definition 4.1 *Let k be a positive integer. A parametrized smooth closed curve, $\gamma : [0, T] \to X$ is said to satisfy the Bohr-Sommerfeld condition of order k, or BS_k for short, iff its holonomy in SX is an k-th root of unit.*

Note that the BS_k property is invariant under reparametrizations. Also note that a curve which satisfies BS_k satisfies the BS condition for any integer multiple of k.

To any curve satisfying BS_{k_0} we now describe how to associate a vector in $\mathfrak{S}_k(X)$, where k is a multiple of k_0.

Definition 4.2 *Assume γ satisfies BS_{k_0}, and let $\tilde{\gamma}$ be its horizontal lift as in definition* (4.1)*. If $\delta_{\tilde{\gamma}}$ denotes the delta function integrated along $\tilde{\gamma}$ using the parametrization, for k a multiple of k_0 we define*

$$|\gamma, k\rangle = \Pi_k(\delta_{\tilde{\gamma}}) . \tag{160}$$

Remark. If k is not a multiple of k_0 then the projection in (160) clearly zero gives zero. The definition of $|\gamma, k\rangle$ depends on the choice of the horizontal lift, $\tilde{\gamma}$, but it's easy to see that changing the horizontal lift amounts to multiplying the state by a complex number of modulus one.

Lemma 4.3 *Two BS_k curves defined as above are immersed Lagrangian submanifolds of X satisfying the cleanness assumption of Sect.* 3.1 *provided they have no common tangents.*

Note that in particular any pair of geodesic BS_k curves satisfy the assumption (including a geodesic with itself). We will see below that all geodesic curves are BS_1.

In order to apply the theory developed in Sect. 3, X must be a compact manifold, i.e. a Riemann surface. In addition, SX must be given a metalinear structure. To do this, note that SX is naturally identified with G/Γ, so that $T(SX) \cong P \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Thus SX inherits a metalinear structure from the metalinear structure on the vector space $\mathfrak{g}$. From this, we obtain metalinear structures on the BS_k curves. For our purposes here these structures will be invisible, since by Definition 4.2 we will deal only with half-forms defined through parametrizations.

From Theorem 3.2 and Corollary 3.3 we obtain the following result.

Theorem 4.4 *Let X be a Riemann surface and γ a BS_{k_0} curve with no self-tangents, parametrized by arclength. For k a sufficiently large multiple of k_0 we have*

$$\langle \gamma, k | \gamma, k \rangle = \left(\frac{k}{\pi}\right)^{1/2} k_0^2 T + O(1)\,. \tag{161}$$

Furthermore, if γ_1 and γ_2 are two distinct intersecting BS_{k_0} curves with no common tangents, then for k a sufficiently large multiple of k_0,

$$\langle \gamma_1, k | \gamma_2, k \rangle = 2^{1/2} k_0^2 \sum_{p \in \gamma_1 \cap \gamma_2} \frac{\omega_p^k e^{-i(\vartheta_p/2 - \pi/4)}}{\sqrt{\sin \vartheta_p}} + O(k^{-1/2})\,, \tag{162}$$

where ϑ_p is the angle from γ_1 to γ_2 at p, and $\omega_p \in S^1$ is determined by the condition that $\tilde{\gamma}_1 \cdot \omega_p$ intersects $\tilde{\gamma}_2$ over the point p.

4.2 Relative Poincaré series

We seek to write out the state $|\gamma, k\rangle$ explicitly as a function on SH that is invariant under Γ. Let $\psi_{(w,\eta)}$ denote the coherent state in $\mathfrak{S}_k(H)$ associated to the point $(w, \eta) \in SH$, i.e., the function on SH which is the orthogonal projection of the delta function at (w, η) into $\mathfrak{S}_k(H)$. By definition, the coherent states are equivariant under the action of G,

$$g \cdot \psi_{(w,\eta)} = \psi_{g \cdot (w,\eta)}\,. \tag{163}$$

To obtain coherent states in $\mathfrak{S}_k(X)$, we average over the action of G. It follows from a theorem of Katok [12] that for any function $F \in \mathfrak{S}_k(H)$,

$$\sum_{g \in \Gamma} g \cdot F \in \mathfrak{S}_k(X) , \tag{164}$$

where the convergence is absolute and uniform on compact sets. The coherent state in $\mathfrak{S}_k(X)$ associated to an equivalence class $[(w, \eta)] \in SX \cong \Gamma \backslash SH$ is thus

$$\Psi_{[(w,\eta)]} = \sum_{g \in \Gamma} g \cdot \psi_{(w,\eta)} , \tag{165}$$

Because of the equivariance (163), the sum on the right depends only on the class $[(w, \eta)]$.

The following proposition realizes our states $|\gamma, k\rangle$ as relative Poincaré series for functions given by integrals over coherent states. The proof is clear from the absolute convergence of (164).

Proposition 4.5 *Let γ be a BS_k curve on X and let $\tilde{\gamma}$ be a horizontal lift of γ to SX. Then the state $|\gamma, k\rangle \in \mathfrak{S}_k(X)$ corresponds to the function*

$$\Phi_\gamma(z, \zeta) = \sum_{g \in \Gamma} g \cdot F(z, \zeta) , \tag{166}$$

where $F(z, \zeta)$ is given by

$$F(z, \zeta) = \int_0^T \psi_{\tilde{\gamma}(t)}(z, \zeta) dt . \tag{167}$$

We can improve upon the realization given above if γ is not closed as a curve on H, using the Rankin-Selberg technique.

Proposition 4.6 *Suppose that $\gamma_0 \in \Gamma$ is not elliptic, and let γ be a BS_k curve defined as a map $\gamma : \mathbb{R} \to H$ such that $\gamma_0 \cdot \gamma(t) = \gamma(t + T)$. Then the state $|\gamma, k\rangle$ corresponds to the function*

$$\Phi_\gamma(z, \zeta) = \sum_{g \in \Gamma_0 \backslash \Gamma} \int_{-\infty}^{\infty} \psi_{\tilde{\gamma}(t)}(g \cdot (z, \zeta)) \, dt , \tag{168}$$

where Γ_0 is the cyclic group $\langle \gamma_0 \rangle$ and $\tilde{\gamma}$ is a horizontal lift of γ to SH.

Proof. First of all, note that since the connection is left-invariant, the horizontal lift $\tilde{\gamma}$ also satisfies $\gamma_0 \cdot \tilde{\gamma}(t) = \tilde{\gamma}(t + T)$. We break the sum over Γ in (166) up into a sum over cosets of Γ_0:

$$\Phi_\gamma(z, \zeta) = \sum_{g \in \Gamma / \Gamma_0} \sum_{n=-\infty}^{\infty} \int_0^T (g \gamma_0^n) \cdot \psi_{\tilde{\gamma}(t)}(z, \zeta) \, dt . \tag{169}$$

Now, since

$$\gamma_0^n \cdot \psi_{\tilde{\gamma}(t)} = \psi_{\tilde{\gamma}(t+nT)} , \tag{170}$$

we can reduce the sum in the above integral to

$$\Phi_\gamma(z,\zeta) = \sum_{g\in\Gamma/\Gamma_0} \int_{-\infty}^{\infty} g\cdot\psi_{\tilde{\gamma}(t)}(z,\zeta)\,dt\,. \tag{171}$$

To complete the proof, we note that $g\in\Gamma/\Gamma_0$ implies $g^{-1}\in\Gamma_0\backslash\Gamma$. □

In order to realize the relative Poincaré series given in these propositions more concretely we need to compute explicitly the coherent state $\psi_{(w,\eta)}$.

Lemma 4.7 *The orthogonal projection of the delta function at* $(w,\eta)\in SH$ *into* $\mathfrak{S}_k(H)$ *is the function*

$$\psi_{(w,\eta)}(z,\zeta) = A_k\frac{\zeta^k\bar{\eta}^k}{(z-\bar{w})^{2k}}\,, \tag{172}$$

where

$$A_k = (-1)^k 2^{2k-2}\frac{2k-1}{\pi}\,. \tag{173}$$

Proof. The fact that $\psi_{(w,\eta)} = \Pi_k(\delta_{(w,\eta)})$ is equivalent to the reproducing property:

$$F(w,\eta) = \int_{SH} \overline{\psi_{(w,\eta)}(z,\zeta)}F(z,\zeta)\,dV\,, \tag{174}$$

for all $F\in\mathfrak{S}_k(H)$. Given any orthonormal basis $\{F_{l,k}\}$ for $\mathfrak{S}_k(H)$, we can write the reproducing kernel as the series

$$\psi_{(w,\eta)}(z,\zeta) = \sum_l \overline{F_{l,k}(w,\eta)}F_{l,k}(z,\zeta)\,, \tag{175}$$

which converges absolutely and uniformly on compact sets. Using the well-known orthonormal basis

$$F_{l,k}(z,\zeta) = 2^{2k-1}\left[\frac{1}{\pi}\frac{(2k+l-1)!}{(2k-2)!\,l!}\right]^{1/2}\zeta^k\frac{(z-i)^l}{(z+i)^{l+2k}}\,, \tag{176}$$

we obtain the result given above. □

4.3 Hypercycles and geodesics

We consider now the application of Proposition 4.6 to the special case when γ_0 is hyperbolic and the associated curve γ is a hypercycle in H. We first consider the question of when a hypercycle corresponds to a BS_k curve on X.

Proposition 4.8 *Let* $\gamma_0\in\Gamma$ *be hyperbolic, and suppose that* γ *is a hypercycle* $\mathbb{R}\to H$ *such that* $\gamma_0\cdot\gamma(t)=\gamma(t+T)$. *Let* τ *denote the cotangent of the angle from the real axis to* γ *at the* $-\infty$ *limit point of* γ. *Then* γ *is* BS_k *as a curve on* X *if and only if*

$$\tau = \frac{2\pi j}{kT} \tag{177}$$

for some $j \in \mathbb{Z}$. In particular, all geodesics (for which $\tau = 0$) are BS_k curves for any value of k.

Proof. It suffices to consider the case where $\xi(t) = e^t(\tau + i)$, since the BS_k property is equivariant. The horizontal lifting of ξ to a curve on G is

$$g(t) = \begin{pmatrix} e^{t/2} & \tau e^{t/2} \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} \cos t\tau/2 & -\sin t\tau/2 \\ \sin t\tau/2 & \cos t\tau/2 \end{pmatrix} . \tag{178}$$

On SH this corresponds to

$$\tilde{\xi}(t) = (e^t(i+\tau), e^{t(1-i\tau)}) \tag{179}$$

The BS_k condition requires that $(e^{-iT\tau})^k = 1$, which implies that $kT\tau = 2\pi j$ for some $j \in \mathbb{Z}$. □

For the remainder of this subsection we will assume that γ is a BS_{k_0} curve such that

$$\tau = \frac{2\pi j}{k_0 T}, \tag{180}$$

for some $j \in \mathbb{Z}$. To compute the state associated to a general hypercycle, we first consider the hypercycle which connects the origin to the point at infinity, and then use the equivariance of the coherent states.

Lemma 4.9. *Let ξ be the hypercycle $\xi(t) = e^t(\tau + i)$ in H, with the lifting $\tilde{\xi}$ defined as in (179). Then*

$$\int_{-\infty}^{\infty} \psi_{\tilde{\xi}(t)}(z,\zeta)dt = B_{k,\tau}\zeta^k z^{ik\tau - k}, \tag{181}$$

where

$$B_{k,0} = (-i)^k \frac{(k-1)!^2}{(2k-1)!}, \tag{182}$$

and for $\tau \neq 0$,

$$B_{k,\tau} = \frac{2\pi i}{1 - e^{-2\pi k\tau}} (\tau - i)^{-k-i\tau} \frac{\Gamma(ik\tau + k)}{(2k-1)!\Gamma(ik\tau - k + 1)} . \tag{183}$$

Proof. We integrate the coherent states along the curve $\tilde{\xi}$:

$$\int_{-\infty}^{\infty} \psi_{\tilde{\xi}(t)}(z,\zeta)dt = \int_{-\infty}^{\infty} \frac{\zeta^k e^{kt(1+i\tau)}}{(z - e^t(-i+\tau))^{2k}} dt \tag{184}$$

The results are obtained by substituting $u = e^t$ and performing a contour integration. □

Given a hypercycle γ whose limit point both lie on the real axis, define

$$w_1 = \lim_{t\to-\infty} \gamma(t) \quad \text{and} \quad w_2 = \lim_{t\to\infty} \gamma(t) . \tag{185}$$

If $w_1 < w_2$, then we can set define $h \in G$ by

$$h = \frac{1}{\sqrt{w_2 - w_1}} \begin{pmatrix} 1 & -w_1 \\ -1 & w_2 \end{pmatrix}, \tag{186}$$

such that $h \cdot \gamma(t) = \xi(t)$. We make the obvious modifications to h if $w_2 < w_1$. In what follows we will define the lifting $\tilde{\gamma}$ by taking $\tilde{\gamma}(t) = h^{-1} \cdot \tilde{\xi}(t)$, where $\tilde{\xi}$ is given in (179). From Proposition 4.6 we immediately obtain the following.

Proposition 4.10 *Suppose γ_0 is hyperbolic and that γ is a corresponding hypercyclic BS_{k_0} curve as in Proposition (4.8). Suppose further the limit points w_1 and w_2 for γ lie on the real axis, and that the lifting $\tilde{\gamma}$ is defined as discussed above. Then $|\gamma, k\rangle$ is given by the function*

$$\Phi_\gamma(z,\zeta) = A_k B_{k,\tau} \zeta^k \sum_{g \in \Gamma_0 \backslash \Gamma} \left(\frac{g \cdot z - w_1}{w_2 - g \cdot z} \right)^{ik\tau} \frac{1}{j(g,z)^{2k}}$$
$$\times \left[\frac{w_2 - w_1}{(w_2 - g \cdot z)(g \cdot z - w_1)} \right]^k . \tag{187}$$

Consider the quadratic polynomial

$$\frac{(w_2 - z)(z - w_1)}{w_2 - w_1}, \tag{188}$$

which appears on the right side in the preceding proposition. In [12], Katok associates to a hyperbolic transformation $\gamma_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ the quadratic polynomial

$$Q_{\gamma_0}(z) = cz^2 + (d - a)z - b . \tag{189}$$

Since the roots of this polynomial are the fixed points of γ_0, w_1 and w_2, it differs from (188) by a constant factor. In fact

$$\frac{(w_2 - z)(z - w_1)}{w_2 - w_1} = -\frac{\operatorname{sgn} \operatorname{Tr}(\gamma_0)}{D_{\gamma_0}^{1/2}} Q_{\gamma_0}(z), \tag{190}$$

where D_{γ_0} is the discriminant of γ_0 as a matrix, i.e. $D_{\gamma_0} = (d - a)^2 + 4bc = (\operatorname{Tr} \gamma_0)^2 - 4$. From Theorem 4.4 we obtain the following.

Theorem 4.11 *For X a Riemann surface, the relative Poincaré series*

$$\sum_{g \in \Gamma_0 \backslash \Gamma} \left(\frac{g \cdot z - w_1}{w_2 - g \cdot z} \right)^{ik\tau} \frac{1}{j(g,z)^{2k} Q_{\gamma_0}(g \cdot z)^k} \tag{191}$$

associated to a BS_{k_0} hypercycle γ, is non-vanishing for sufficiently large values of the weight k (such that k_0 divides k).

For the remainder of this subsection we focus on the case of geodesics on Riemann surfaces. We begin by illustrating the implications of Theorem 3.12 for Poincaré series.

Theorem 4.12 *Suppose X is a Riemann surface and that $\gamma_0 \in \Gamma$ is a dilation. In a sufficiently narrow band surrounding the imaginary axis we can estimate*

$$\sum_{g\in\Gamma_0\backslash\Gamma} \frac{y^k}{j(g,z)^{2k}} \frac{1}{(g \cdot z)^k} \sim e^{ik(\frac{x}{y}-\frac{\pi}{2})-\frac{k}{2}\frac{x^2}{y^2}} \tag{192}$$

for k sufficiently large.

Proof. Consider the geodesic $\xi(t) = ie^t$. The associated state is

$$\Phi_\xi(z,\zeta) = A_k B_{k,0} \sum_{g\in\Gamma_0\backslash\Gamma} \frac{\zeta^k}{j(g,z)^{2k}} \frac{1}{(g \cdot z)^k}, \tag{193}$$

where

$$A_k B_{k,0} = i^k \frac{2^{2k-2}}{\pi} \binom{2k-2}{k-1}^{-1}, \tag{194}$$

A direct application of Theorem 3.12 yields the following: near the imaginary axis and for k sufficiently large

$$\Phi_\xi(z,\zeta) \sim \sqrt{\frac{k}{\pi}} e^{ik(0+\frac{x}{y})-\frac{k}{2}\frac{x^2}{y^2}}. \tag{195}$$

The above result follows because, by Stirling's formula,

$$2^{2k-2}\binom{2k-2}{k-1}^{-1} \sim \sqrt{\pi k} \tag{196}$$

for large k. □

Lemma 4.13 *Let $r = k^\alpha x$. If $\alpha > 1/2$ then for fixed r and y,*

$$\left(\frac{y}{z}\right)^k \sim e^{ik(\frac{x}{y}-\frac{\pi}{2})-\frac{k}{2}\frac{x^2}{y^2}} \tag{197}$$

as $k \to \infty$.

This lemma is a simple calculus exercise, which allows us to conclude the following.

Corollary 4.14 *We can find a band surrounding the imaginary axis whose width decreases as $k^{-\alpha}$ for $\alpha > 1/2$, in which the relative Poincaré series appearing in* (192) *is dominated for large k by the $g = id$ term.*

Similar results can of course established for the other Poincaré series defined above.

We turn next to applications of our results to the relative Poincaré series associated to geodesics by Katok [12]. Given any hyperbolic element $\gamma_0 \in \Gamma$, we have a parametrized geodesic $\gamma(t) = h^{-1} \cdot \xi(t)$ with a distinguished lifting $\tilde{\gamma}(t) = h^{-1} \cdot \tilde{\xi}(t)$, where $\tilde{\xi}$ is given by (179). The resulting states $|\gamma, k\rangle$ can be related to Katok's relative Poincaré series, $\Theta_{k,\gamma_0}(z)$. Katok's definition is

$$\Theta_{k,\gamma_0}(z) := D_{\gamma_0}^{k-1/2}(-\operatorname{sgn}\operatorname{Tr}\gamma_0)\frac{2^{2k-2}}{\pi}\binom{2k-2}{k-1}^{-1} \sum_{g\in\Gamma_0\backslash\Gamma} \frac{1}{j(g,z)^{2k} Q_{\gamma_0}(g\cdot z)^k} \,. \tag{198}$$

Denote the function associated to $|\gamma, k\rangle$ by $\Phi_\gamma(z,\zeta)$. By Proposition 4.10 and (194) we have

$$\Phi_\gamma(z,\zeta) = i^k(-\operatorname{sgn}\operatorname{Tr}\gamma_0)^{k-1} D_{\gamma_0}^{-(k-1)/2}\zeta^k\,\Theta_{k,\gamma_0}(z)\,. \tag{199}$$

For geodesics there is a nice relation between the intersection angles φ_p and the phases ω_p. Assume that the fixed points of γ_0 are given by real numbers $c_0 \pm r_0$. By the prescription above, we have

$$\tilde{\gamma}(t) = \left(c_0 + r_0 \tanh \varepsilon t + \frac{ir_0}{\cosh t}, -r_0\frac{\sinh \varepsilon t + i}{\cosh^2 t}\right), \tag{200}$$

where $\varepsilon = \pm 1$ depending on the orientation. We can parametrize the curve by angle instead or arclength by taking $\theta = \cos^{-1}(\tanh \varepsilon t)$, which gives

$$\tilde{\gamma}(\theta) = (c_0 + r_0\, e^{i\theta}, -r_0\, e^{i\theta}\sin\theta)\,, \tag{201}$$

where $0 < \theta < \pi$. Now consider the case of two intersecting geodesics, parametrized by angles θ_1 and θ_2. Since at the point where they intersect we have $r_1 \sin\theta_1 = r_2 \sin\theta_2$, it is easy to see from (201) that the relative phase at such a point is

$$\omega_p = \frac{\zeta_2}{\zeta_1} = e^{i(\theta_2-\theta_1)}\,, \tag{202}$$

A simple geometric exercise shows that $\theta_2 - \theta_1 = \vartheta_p$, the intersection angle.

Theorem 4.15 *Let X be a Riemann surface and $\gamma_0 \in \Gamma$. Then $\Theta_{k,\gamma_0}(z)$ is non-vanishing for sufficiently large k. Moreover,*

$$\|\Theta_{k,\gamma_0}\|_2^2 = D_{\gamma_0}^{k-1}\left[\left(\frac{k}{\pi}\right)^{1/2} T + O(1)\right], \tag{203}$$

where $T = 2\cosh^{-1}(\frac{1}{2}\operatorname{Tr}\gamma_0)$. For $\gamma_1, \gamma_2 \in \Gamma$ not conjugate to each other, we have

$$\langle\Theta_{k,\gamma_1}|\Theta_{k,\gamma_2}\rangle = 2^{1/2}(D_{\gamma_1}D_{\gamma_2})^{(k-1)/2}(\operatorname{sgn}\operatorname{Tr}\gamma_0\operatorname{Tr}\gamma_1)^{k-1} \times\left[\sum_{p\in[\gamma_1]\cap[\gamma_2]} \frac{e^{i(k-1/2)\vartheta_p + i\pi/4}}{\sqrt{\sin\vartheta_p}} + O(k^{-1/2})\right],$$

where $[\gamma_j]$ denotes the geodesic on X corresponding to γ_j.

In [12], Katok gives a period formula for the case in which Γ is symmetric (with respect to the imaginary axis), and γ_1 and γ_2 are primitive. This yields the following exact result for the imaginary part of the inner product:

$$\operatorname{Im}\langle\Theta_{k,\gamma_1}|\Theta_{k,\gamma_2}\rangle = 2^{2k-2}(D_{\gamma_1}D_{\gamma_2})^{(k-1)/2}(\operatorname{sgn}\operatorname{Tr}\gamma_0\operatorname{Tr}\gamma_1)^{k-1}\binom{2k-2}{k-1}^{-1}$$
$$\times \sum_{p\in[\gamma_1]\cap[\gamma_2]} \operatorname{sgn}(\sin\vartheta_p)P_{k-1}(\cos\vartheta_p),$$

where P_{k-1} is the Legendre polynomial of order $k-1$. This can be compared to our formula using (196) and the theorem of Darboux on the large n asymptotics of $P_n(\cos\theta)$:

$$P_{k-1}(\cos\theta) = \frac{\sin((k-1/2)\theta+\pi/4)}{\sqrt{\frac{\pi}{2}(k-1)\sin\theta}} + O(k^{-3/2}) \tag{204}$$

(Theorem 8.21.2 in [15]). The asymptotic estimate of Theorem 4.15 is seen to agree precisely with Katok's result.

4.4 Horocycles

We turn now to the horocycles, curves in H which correspond to parabolic elements of Γ. Note that these do not exist when X is a Riemann surface, since Γ must be hyperbolic in this case. Horocyclic curves are given either by circles tangent to the real axis or straight lines parallel to the real axis. Given a horocycle γ and parabolic $\gamma_0 \in \Gamma$ such that $\gamma_0 \cdot \gamma(t) = \gamma(t+T)$, there exists a unique $g \in G$ such $g\gamma_0 g^{-1} : z \mapsto z+T$ and $g\cdot\gamma(0)$ lies on the imaginary axis. The real number $\lambda = -ig\cdot\gamma(0)$ depends only on γ and T, Alternatively, we may use the definition

$$\lambda = \frac{T}{2\sinh[\frac{1}{2}\rho(\gamma(0),\gamma(T))]}, \tag{205}$$

where ρ is the hyperbolic distance.

Proposition 4.16 *Let $\gamma_0 \in \Gamma$ be parabolic, and suppose that γ is a curve $\mathbb{R} \to H$ such that $\gamma_0 \cdot \gamma(t) = \gamma(t+T)$. Then γ satisfies BS_k if and only if*

$$\lambda = \frac{kT}{2\pi m} \tag{206}$$

for some $m \in \mathbb{Z}$, where λ is defined as above.

Proof. By equivariance, we assume that $\gamma(t) = i\lambda + z$ and $\gamma_0 : z \mapsto z+T$. The horizontal lift of γ to G is

$$g(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}\begin{pmatrix} \cos\frac{t}{2\lambda} & -\sin\frac{t}{2\lambda} \\ \sin\frac{t}{2\lambda} & \cos\frac{t}{2\lambda} \end{pmatrix}, \tag{207}$$

which corresponds to

$$\tilde{\gamma}(t) = (i\lambda + t, \lambda e^{-it/\lambda}). \tag{208}$$

The BS_k condition then reduces to the requirement $kT/\lambda = 2\pi m$ for some $m \in \mathbb{Z}$. □

For the remainder of the subsection we assume that γ in a BS_{k_0} curve and that λ and T satisfy

$$\lambda = \frac{k_0 T}{2\pi m}, \tag{209}$$

for some $m \in \mathbb{Z}$.

Lemma 4.17 *Let γ be the horocycle $\gamma(t) = i\lambda + t$. Then*

$$\int_{-\infty}^{\infty} \psi_{\tilde{\gamma}(t)}(z,\zeta)dt = C_{k,\lambda}\zeta^k e^{\frac{ikz}{\lambda}}, \tag{210}$$

where

$$C_{k,\lambda} = \frac{2\pi(-1)^k}{(2k-1)!}\frac{k^{2k-1}}{\lambda^{k-1}}e^{-k} \tag{211}$$

Proof. The curve $\tilde{\gamma}$ is given by (208), so that

$$\int_{-\infty}^{\infty} \psi_{\tilde{\gamma}(t)}(z,\zeta)dt = \int_{-\infty}^{\infty} \frac{\zeta^k \lambda^k e^{\frac{ikt}{\lambda}}}{(z + i\lambda - t)^{2k}}dt. \tag{212}$$

The result follows from a contour integration. □

Proposition 4.18 *Suppose $\gamma_0 \in \Gamma$ is parabolic element that fixes ∞ and that γ is a corresponding BS_{k_0} curve. The state $|\gamma, k\rangle$ corresponds to the function*

$$\Phi_\gamma(z,\zeta) = A_k C_{k,\tau} \sum_{g\in\Gamma/\Gamma_0} \frac{e^{\frac{ik}{\lambda}g\cdot z}}{j(g,z)^{2k}}. \tag{213}$$

Corollary 4.19 *Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, and let $\gamma(t) = \frac{ik}{2\pi m} + t$, which is a BS_k curve. Then the state $|\gamma, k\rangle$ is represented by $\zeta^k P_{m,k}(z)$ (up to a constant depending on m and k), where $P_{m,k}$ is the classical Poincaré series:*

$$P_{m,k}(z) = \sum_{g\in\Gamma_\infty\backslash\Gamma} \frac{e^{2\pi i m(g\cdot z)}}{j(g,z)^{2k}}, v \tag{214}$$

with Γ_∞ the subgroup fixing ∞.

4.≦ Circles

We complete our discussion of specific BS_k curves on X by considering the circles on H.

Proposition 4.20 *Let γ be a circle in H with (hyperbolic) radius μ, with $\gamma_0 \in \Gamma$ such that $\gamma_0 \cdot \gamma(t) = \gamma(t+T)$ (γ_0 is either elliptic of finite order or the identity). Let n denote the minimal integer such that γ_0^n is the identity transformation. Then γ satisfies BS_k if and only if*

$$\cosh \mu = \frac{nl}{k}, \tag{215}$$

for some $l \in \mathbb{Z}$.

Proof. By equivariance, we assume that the center of the circle is i, and that $\gamma(0) = ie^{-\mu}$. The location of the center implies that

$$\gamma_0 = \begin{pmatrix} \cos \pi/n & \sin \pi/n \\ -\sin \pi/n & \cos \pi/n \end{pmatrix}. \tag{216}$$

Let $a = \frac{\pi}{nT}$. The curve,

$$g(t) = \begin{pmatrix} \cos at & \sin at \\ -\sin at & \cos at \end{pmatrix} \begin{pmatrix} e^{-\mu/2} & 0 \\ 0 & e^{\mu/2} \end{pmatrix} \begin{pmatrix} \cos bt/2 & \sin bt/2 \\ \sin bt/2 & \cos bt/2 \end{pmatrix}, \tag{217}$$

lies over γ and will be horizontal provided

$$b = 2a \cosh \mu. \tag{218}$$

The corresponding curve in SH is

$$\tilde{\gamma}(t) = \left(\frac{e^{\mu}\sin at + i\cos at}{e^{\mu}\cos at - i\sin at}, \frac{e^{\mu}e^{-ibt}}{(e^{\mu}\cos at - i\sin at)^2} \right), \tag{219}$$

so that γ is BS_k iff $kbT = 2\pi l$ for some $l \in \mathbb{Z}$, i.e., $b = \frac{2nl}{k}a$. The claim follows.

□

For the remainder of this section, we assume that γ is a BS_{k_0} curve and that μ and l satisfy

$$\cosh \mu = \frac{nl}{k_0}, \tag{220}$$

for some $l \in \mathbb{Z}$. Note that the BS_{k_0} condition requires $nl \geqq k_0$.

Proposition 4.21 *Let γ be a circle in $\mathbb{H}$ of radius μ which is a BS_k curve on X as above. Then*

$$\int_0^{nT} \psi_{\tilde{\gamma}(t)}(z,\zeta) = D_{l,k,\mu} F_{nl-k,k}(z,\zeta), \tag{221}$$

where $F_{nl-k,k}$ is the element of the orthonormal basis of $\mathfrak{S}_k(H)$ given by (176), and

$$D_{k,\mu} = 2^{1-2k} nT \left[\frac{\pi}{2k-1} \frac{(nlk/k_0 + k - 1)!}{(2k-1)!(nlk/k_0 + k)!} \right]^{1/2} \frac{(\sinh \mu/2)^{nlk/k_0 - k}}{(\cosh \mu/2)^{nlk/k_0 + k}} \tag{222}$$

Remark. In case $n = 1$, the states coming from BS_k circles on H give an orthonormal basis of $\mathfrak{S}_k$. This is an example of a more general phenomenon: *If we have a* $\mathbb{H}$*amiltonian action of the n torus of the Kahler manifold preserving all structures, our construction applied to the BS_k level sets of the moment map yield an orthonormal basis for $\mathfrak{S}_k$.* This actually follows from the quantum reduction theorem of Guillemin and Sternberg, [9]. In the case envisioned the reduced spaces are points, and so their quantization is one-dimensional. Other examples of this situation include the Bargmann metric on $\mathbb{C}$, where the BS_k circles correspond to eigenstates of the 1-dimensional harmonic oscillator problem, and the sphere, where the BS_k circles give spherical harmonics.

Proof. The curve $\tilde{\gamma}$ is given by (219), and we seek to compute

$$\int_0^{nT} \psi_{\tilde{\gamma}(t)}(z,\zeta) = \int_0^{nT} \frac{\zeta^k e^{k\mu} e^{ibkt}}{[(e^\mu \cos at + i \sin at)z - (e^\mu \sin at - i \cos at)]^{2k}} dt\,. \tag{223}$$

Using the fact that $b = \frac{2nl}{k_0} a$ we can rewrite this as

$$\int_0^{nT} \psi_{\tilde{\gamma}(t)}(z,\zeta) = 4^k e^{k\mu} \zeta^k \int_0^{nT} \frac{e^{2iatk(nl/k_0+1)}}{[(z+i)(e^\mu+1)e^{2iat} + (z-i)(e^\mu - 1)]^{2k}} dt\,. \tag{224}$$

Changing variables to $u = e^{2iat}$, we have

$$\int_0^{nT} \psi_{\tilde{\gamma}(t)}(z,\zeta) = 4^k e^{k\mu} \zeta^k \oint \frac{u^{nlk/k_0+k}}{[(z+i)(e^\mu+1)u + (z-i)(e^\mu - 1)]^{2k}} dt\,, \tag{225}$$

where the contour is the unit circle. Noting that the pole is always inside the contour, we find

$$\int_0^{nT} \psi_{\tilde{\gamma}(t)}(z,\zeta) = nT \binom{nlk/k_0 + k - l}{2k-1} \frac{(\sinh \mu/2)^{nlk/k_0 - k}}{(\cosh \mu/2)^{nlk/k_0+k}} \zeta^k \frac{(i-z)^{nlk/k_0-k}}{(i+z)^{nlk/k_0+k}}\,. \tag{226}$$

□

Proposition 4.22 *Suppose $\gamma_0 \in \Gamma$ is an elliptic element which fixes i, and that γ is a corresponding BS_{k_0} circle on X. The state $|\gamma, k\rangle$ is given by the function*

$$\Phi_\gamma(z,\zeta) = A_k D_{k,\mu} \sum_{g \in \Gamma_0 \backslash \Gamma_0} F_{nlk/k_0 - k,k}(g \cdot (z,\zeta))\,. \tag{227}$$

If X is a Riemann surface, then the only possibility for γ_0 is the identity, so the curves must close on H (thus $n = 1$).

Theorem 4.23 *Let X be a Riemann surface. The relative Poincaré series*

$$\sum_{g\in\Gamma} \frac{1}{j(g,z)^{2k}} \frac{(i - g \cdot z)^{lk/k_0 - k}}{(i + g \cdot z)^{lk/k_0+k}}\,, \tag{228}$$

is non-vanishing for sufficiently large k.

4.6 Towards a geometric construction of a basis

By Riemann-Roch,

$$\dim \mathfrak{S}_k = k(2g-2) - (g-1), \tag{229}$$

where g is the genus of X. We now indicate a strategy for choosing, $\forall k$, the same number of non-intersecting BS_k curves on X. We conjecture that the associated states form a basis of $\mathfrak{S}_k$.

Divide X into $2g-2$ pairs of pants, each bounded by three simple closed geodesics (therefore there are $3g-3$ different geodesics on X involved as boundaries). Consider a pair of pants, Y. By Gauss-Bonnet, it has an area of 2π. By the Collar Theorem, there are collar neighborhoods of the boundary geodesics of Y which are hyperbolic cylinders. Let A_c denote their total area. Their complement is the union of two identical hexagons, let A_h the area of one hexagon so that $A_c + 2A_h = 2\pi$.

We choose BS_k curves according to the following principle: •

- On each hyperbolic cylinder, the BS_k hypercycles parallel to the boundary geodesic.
- In each of the hexagons, the BS_k curves of a function with a single critical point in the interior. Now count BS_k curves in each region:

Proposition 4.24 *The above scheme produces exactly* $\dim \mathfrak{S}_k$ *BS_k curves,* $\forall k$.

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