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The Index of Manifolds with Toral Actions and Geometric Interpretations of the $\sigma(\infty, (S^1, M^n))$ Invariant of Atiyah and Singer

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In the theory of transformation groups the smooth theory is certainly the most tractable. However, not all operations that one would like to perform remain in the smooth category. For example, analysis of the orbit space usually must go outside the smooth category. Furthermore, one would also like to study symmetries of interesting geometric spaces which often fail to be locally Euclidean such as spaces having manifolds as ramified coverings and analytic spaces.

It has long been recognized that cohomology manifolds or generalized manifolds encompass all manifolds as well as many typical analytic and ramified spaces. The most important feature of generalized manifolds, from the point of view of transformation groups, is that one is often able to work with orbit spaces. In addition, one does have characteristic classes and with care one can often define workable invariant tubular neighborhoods of fixed point sets.

In this paper we develop and exploit some of these ideas to prove results which even when specialized to the smooth category seem to be unobtainable from standard smooth methods. For example, we establish by cohomological methods alone formulae for the Atiyah-Singer invariant $\sigma(\infty, (S^1, M^n))$, Theorems 3, 4, 5, and 6. This enables us to conclude that its value is an integer and also to give explicit geometric interpretations of it in terms of the orbit space and the mapping cylinder of the orbit map. This makes it much easier to compute than by using smooth techniques alone, e.g. § 4.

A very interesting geometric interpretation of $\sigma(S^1, M^{4k-1})=0$ is also given in terms of fibering M^{4k-1} over a circle, § 4.

There are several ingredients that enable us to obtain these results. First, we obtain by cohomological methods:

Theorem 1. *Let (T^s, M^n) be a smooth action on a closed oriented n-manifold. Then the fixed point set $F(T^s, M^n)$ can be naturally oriented and $I(M^n)=I(F(T^s, M^n))$, where I denotes the Thom-Hirzebruch index of the manifolds.*

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This theorem was initially proved in the fixed point free case by Conner and Floyd [5] and in the semi-free circle case by Kawakubo and Uchida [8]. We replace bundles by mapping cylinder neighborhoods and adapt the proof of [8] using many of the cohomological features of the index as well as the fact that the quotient space of a generalized manifold by a finite group of orientation preserving homeomorphisms is again a generalized manifold [13].

Secondly, we introduce the notion of *nicely embedded fixed points.* This enables us to find invariant closed tubular neighborhoods which are cohomology fiber spaces [3] over the fixed point sets. The mapping cylinder of the orbit map is an important construction. For circle actions, without fixed points on generalized manifolds, this mapping cylinder is a generalized manifold with boundary with a natural circle action having nicely embedded fixed points identical with the orbit space. We then obtain Theorem 2 which is the analogue of Theorem 1 for generalized manifolds.

We establish Theorems 1 and 2 first for a circle group. The general case $s \geqq 1$, follows immediately from the elementary observation that *one can always find a circle subgroup $S^1 \subset T^s$ so that $F(S^1, X) = F(T^s, X)$, whenever (T^s, X) has a finite number of distinct orbit types.*

It has been pointed out by a number of people that Theorems 1 and 3 also follow from the Atiyah-Singer G-Signature Theorem [1; 6.12]. We, on the other hand, use Theorem 2 to obtain our integrality conditions and formulae for $\sigma(\infty, (S^1, M^n))$.

§ 1. Proof of Theorem 1

One may proceed in several different ways. The quickest is to assume familiarity with [8] and to follow their proof adapting each step as needed. Even though the action is not free outside the fixed point set, F, the action there has only finite stability groups. Consequently, by [13], the orbit space outside of F, while not a manifold, is an orientable generalized (cohomology) manifold over the reals, $\mathbf{R}$. The bundle $\mathbf{CP}(\xi)$ of [8] is replaced by the orbit space $S(\xi)/S^1$ of the sphere bundle $S(\xi)$ of [8]. This bundle has fiber an oriented generalized manifold over $\mathbf{R}$ having the real cohomology type of a complex projective space. The argument then follows formally as in [8].

We shall give, however, a complete and somewhat different proof. The reason for this is to produce a variant of bundle methods which enables us to extend our arguments to certain cohomological settings which arise naturally from smooth problems. In fact, Theorem 1 is already known to be valid in the general cohomological case, provided that the fixed point set is empty, see [6; 7.3]. The proof that we offer is really a mixture of the techniques of [8] and [6].

As the reader will discern, smoothness is only used in obtaining a nice invariant neighborhood of the fixed point set. The point of § 2 is to properly formulate the conditions on S^1-actions on certain types of generalized manifolds that will yield nice enough invariant neighborhoods so that our arguments, to be given below, apply.

Let F_k be the $(n-2k)$-dimensional component of the fixed point set, $F=F(S^1, M^n)$, and $F_{k,j}$ be the j-th connected component of F_k. Each $F_{k,j}$ is a smooth orientable closed manifold. A Riemannian metric on M^n can be chosen so that it is invariant under the action of S^1. The normal bundle $N_{k,j}$ of the embedding $F_{k,j} \subseteq M^n$ is mapped via the exponential map and restriction onto a small closed invariant tubular neighborhood $T_{k,j}$ so that $T_{k,j} \cap T_{k',j'} = \emptyset$, if $(k,j) \neq (k',j')$.

Since $S^1 = U(1)$ operates on the invariant normal disk at each point of $F_{k,j}$ linearly, and since $F_{k,j}$ is connected, this representation is constant on $F_{k,j}$. Furthermore, the structure group $SO(2k)$ of the normal bundle over $F_{k,j}$ can be reduced to the unitary group $U(k)$ and $N_{k,j}$ thought of as a complex bundle. It has a natural orientation.

Let us assume that $T_{k,j}$ is oriented compatibly with M^n. Orient $F_{k,j}$ so that the orientation class of $F_{k,j}$ times the orientation class of the oriented normal disk is the orientation of M^n. Equivalently, the bundle map $\tau(F_{k,j}) \oplus N_{k,j} \to \tau(M^n)$ is orientation preserving where "$\tau(\ )$" denotes the tangent bundle.

Let us delete the interiors of $T_{k,j}$. We obtain an oriented n-manifold X with boundary, and $M^n = X \cup_{\partial X} T$, where $T = \bigcup_{k,j} T_{k,j}$.

In [13] it is shown that the orbit space of an orientation preserving finite group of homeomorphisms on an orientable **R**-generalized n-manifold is an orientable **R** generalized n-manifold. (By an **R**-generalized manifold we mean a cohomology manifold where the real numbers are used as coefficients [15] and [2].) Obviously, then, the quotient space of a fixed point free circle action on an orientable generalized n-manifold (with locally finite orbit structure) is an orientable generalized $(n-1)$-manifold.

The mapping cylinder of the orbit map, $\pi\colon M \to M/S^1$ will be denoted by $\mathrm{Map}(\pi)$. A point in $\mathrm{Map}(\pi)$ may be denoted by $((m,t))$, where $(m,t) \in M \times I$. There is a natural S^1 action on $\mathrm{Map}(\pi)$ with quotient space $M/S^1 \times I$, defined by $\tau \times ((m,t)) \to ((\tau m, t))$. Observe that on $D^2 \times M$ one can define an action of the 2-dimensional torus $S_1^1 \times S_2^1$ by $(\tau_1, \tau_2) \times (\rho e^{i\theta}, m) \to (\rho\, \tau_2 e^{i\theta}, \tau_2 \tau_1 m)$. The induced action of S_1^1 on the orbit space $(D^2 \times M)/S_2^1$ is equivalent to the S^1 action on $\mathrm{Map}(\pi)$ just described above. Thus,

Proposition 1 [6; 7.2]. *If (S^1, M^n) has no fixed points then* $\mathrm{Map}(\pi)$ *is an orientable* **R**-*generalized $(n+1)$-manifold with boundary M.*

We remark, although we shall not use this fact, that a result of C.T. Yang now implies that $\mathrm{Map}(\pi)$ is triangulable when (S^1, M^n) is smooth.

Now if we restrict the orbit map above to X, then $\mathrm{Map}(\pi|_X)$ is an oriented $\mathbf{R}$-generalized $(n+1)$-manifold with boundary $X \cup Y$, where Y is the mapping cylinder restricted to ∂X. The boundary, ∂X, is the disjoint union of $(2k-1)$-dimensional sphere bundles over the disjoint connected components $F_{k,j}$. Passing to the orbit space we obtain $\partial X/S^1$ as a fiber bundle over F. (The $U(1)$ representation is constant on each $F_{k,j}$.) A fiber over a point in $F_{k,j}$ is the orbit space S^{2k-1}/S^1. The action (S^1, S^{2k-1}) is fixed point free but may have non-trivial finite isotropy subgroups. The space S^{2k-1}/S^1 is a $(2k-2)$-real cohomology manifold having the real cohomology of complex projective space $\mathbf{CP}_{k-1}$. This is easily seen by considering the usual standard diagram (see [2; Chap. 4]),

$$\begin{tikzcd}
S^{2k-1} \arrow[d, "/S^1"] & S^{2k-1} \times S^{\infty} \arrow[l] \arrow[r] \arrow[d] & S^{\infty} \arrow[d, "/S^1"] \\
S^{2k-1}/S^1 & S^{2k-1} \underset{S^1}{\times} S^{\infty} \arrow[l, "\pi_2"'] \arrow[r, "S^{2k-1}"] & \mathbf{CP}_{\infty}.
\end{tikzcd}$$

The fibering π_1 induces a bijection of the integral cohomology ring up through dimension $2k-2$. The map π_2 induces a bijection of the real cohomology ring because the cohomology of the stalks of the Leray sheaf associated with the spectral sequence of the map π_2 vanishes.

It is known [4] that for a fibering of oriented manifolds with connected structure group the index of the total space is the product of the index of the base and the index of the fiber. The proof in [4] is purely cohomological via spectral sequences and Poincaré duality. It immediately extends to fiberings of Poincaré duality spaces with connected structure groups. Furthermore, if an orientable closed $\mathbf{R}$-generalized n-manifold is the boundary of a compact $\mathbf{R}$-generalized $(n+1)$-manifold with boundary then the index of the closed manifold is 0.

We have

(a) $$\partial(X/S^1) = \partial X/S^1,$$

hence

(i) $$0 = I(\partial(X/S^1)) = I(\partial X/S^1) = \sum_{k,j} I(\mathbf{CP}_{k-1})\, I(F_{k,j}).$$

We also have

(b) $$X \cup Y = \partial(X \times D^2)/S^1,$$

hence,

(ii) $$I(X+Y) = 0.$$

One may define the index of a $4t$ dimensional compact oriented **R**-generalized manifold M with boundary ∂M to be the signature of the quadratic form associated with the cup product pairing

$$H^{2t}(M, \partial M) \otimes H^{2t}(M, \partial M) \to H^{4t}(M, \partial M)$$

on the image

$$j^*: \; H^{2t}(M, \partial M) \to H^{2t}(M).$$

If dimension of M is not divisible by 4 then define $I(M)=0$. From the cohomological fact that this index is additive, see [1; Prop. 7.1], one obtains

(iii) $$I(X)+I(Y)=0,$$

(iv) $$I(X)+I(T)=I(M^n).$$

We shall now attach T along ∂X to $-Y$ to form an oriented closed generalized manifold Z which fibers over F. Each fiber is a $2k$-cell attached to the boundary of the mapping cylinder of the orbit map $S^{2k-1} \to S^{2k-1}/S^1$, where S^{2k-1} is the boundary of the normal disk in T of a point $x \in F_{k,j}$. Since S^{2k-1}/S^1 has the real cohomology type of $\mathbf{CP}_{k-1}$, the fiber has the real cohomology type of $\mathbf{CP}_k$. The bundle has connected structure group and consequently,

(v) $$I(Z) = I(-Y) + I(T) = \sum_{k,j} I(\mathbf{CP}_k)\, I(F_{k,j}).$$

Substituting in the above equations we have

(vi) $$I(M^n) = I(-Y) + I(T) = I(Z) = \sum_{k,j} I(\mathbf{CP}_k)\, I(F_{k,j}).$$

Since $I(\mathbf{CP}_l) = (1+(-1)^l)/2$, we have from (i) that,

$$0 = \sum_{k:\,\text{odd}} I(F_{k,j}).$$

Furthermore, from (vi) we have that

$$I(M^n) = \sum_{k:\,\text{even}} I(F_{k,j}).$$

Hence,

$$I(M^n) = \sum_{k,j} I(F_{k,j}), \qquad \left(= \sum_{k=0}^{[n/4]} I(F_{2k}) \right),$$

which is what we wanted to prove.

§ 2. Nicely Embedded Fixed Points

For the later sections, where we offer geometric interpretations of the Atiyah-Singer σ-invariant, it seems necessary to formulate the results of § 1 in a more general way. Suppose one takes a smooth S^1-action on an oriented closed smooth manifold M^n; let the finite cyclic group Z_p be a subgroup of S^1. For a typical non-trivial example of such a generalized setting one may consider the induced S^1/Z_p action on the oriented (triangulable) **R**-generalized n-manifold M^n/Z_p.

Let us assume that we have an action of the circle S^1 on an oriented closed **R**-generalized n-manifold M^n. In addition it will be necessary to postulate that the components of the fixed point set, which are orientable closed generalized manifolds over **R**, have very nice neighborhoods. Recall that a (closed) mapping cylinder neighborhood of a closed subset A of a space X is the closure of an open subset $U \supset A$ of X, a map f of the frontier of U onto the frontier of A, and a homeomorphism h of the closure of U minus the interior of A onto the mapping cylinder of f such that h restricted to the frontiers of U and A is the identity. Note that closure U minus A is homeomorphic to (frontier of U) $\times$ [0, 1), with (frontier of U) $\times$ (0, 1) being an open subset of X. (In [9] it is shown that any two such *open* mapping cylinder neighborhoods (MCN) are essentially unique.)

We shall say that an S^1-action on an orientable closed **R**-generalized n-manifold M^n has *nicely embedded fixed points* if each component $F_{k,j}$ of the fixed point set F has an invariant closed mapping cylinder neighborhood $T(F_{k,j})$ satisfying the conditions (i)–(iv) listed below.

Without loss of generality we may assume that S^1 is not acting trivially upon M and $F_{k,j}$ is an $(n-2k)$-dimensional orientable generalized manifold over **R** with $k>0$. If we let r_t: $T \to T$ denote the retraction along the "fibers," then $r_1|_{\partial T}=f$. We shall postulate that (i) $(r_1|_{\partial T})^{-1}(x)=S_x^{2k-1}$ *is a* $(2k-1)$*-generalized manifold over* **R** *having the real cohomology of the* $(2k-1)$*-sphere, for each* $x \in F_{k,j}$. This means that $r_1^{-1}(x)=D_x^{2k}$ is the cone over S_x^{2k-1} and is consequently a generalized $2k$-cell over **R**. We shall assume that (ii) $r_1^{-1}(x)$ *is invariant and the* S^1*-action on* $S_x^{2k-1} \times [0, t)$ *is independent of* t, $0 \leqq t < 1$. To handle how the "fibers" fit together homologically we assume that (iii) *the Leray sheaf of the map* f *is simple, coefficients in* **R**. Finally, to eliminate "exotic" pathology we assume that (iv) (S^1, M^n) *has only a finite number of distinct orbit types* (always holds if M^n is a Z-generalized manifold).

Finally, if (T^s, M^n) is an action of an s-dimensional torus on an orientable **R** generalized manifold, we shall say that it has *(very) nicely embedded fixed points* if *(every) some* circle subgroup $S^1 \subset T^s$ for which $F(S^1, M^n)=F(T^s, M^n)$ has nicely embedded fixed points.

Theorem 2. *If (T^s, M^n) denotes an action with nicely embedded fixed points on an oriented closed* **R**-*generalized manifold, then each component of the fixed point set can be oriented so that*

$$I(M^n) = I(F).$$

Proof. Let $s=1$. We shall discuss the orientability of the fixed point set. We shall regard the group S^1 as the unitary group $U(1)$ with its natural orientation. We shall use this orientation together with the orientation of M^n to define the orientation on F. The main point is really to orient $f^{-1}(x) = S_x^{2k-1}$. Now (S^1, S_x^{2k-1}) has an action so that S_x^{2k-1}/S^1 has the real cohomology ring of $\mathbf{CP}_{k-1}$. What we want is that using the orientation of S_x^{2k-1}/S^1 and S^1 we may orient S_x^{2k-1}. Having this orientation of S_x^{2k-1}, the simplicity of the Leray sheaf enables us to orient $F_{k,j}$ so that the product orientation is the orientation on $\partial T(F_{k,j})$ induced from M^n. It remains to orient S_x^{2k-1}/S^1. We take the Leray spectral sequence of the orbit map $S_x^{2k-1} \to S_x^{2k-1}/S^1$. We use the transgression of $1 \otimes g$, $g \in H^1(S^1; \mathbf{R})$ or the Gysin sequence to define the isomorphism $H^1(S^1; \mathbf{R}) \to H^2(S_x^{2k-1}/S^1; \mathbf{R})$. We let the image of the natural generator of $H^1(S^1; Z)$ define the generator of $H^2(S_x^{2k-1}/S^1; \mathbf{R})$. This orients S_x^{2k-1}/S^1 by the cohomology ring. Notice that this yields the orientation for $\mathbf{CP}_{k-1}$ for the standard sphere and the usual Hopf action.

The remaining part of the proof now follows formally as in § 1. We point out that when we attach $T(=N)$ along ∂X to $-Y$ to form an oriented closed **R**-generalized manifold Z the map onto F is a cohomology fiber space (in the sense of Bredon [3]). Each "fiber" becomes an oriented **R**-generalized manifold having the real cohomology type of $\mathbf{CP}_k$. The Leray sheaf is once again simple and we may use the Chern, Hirzebruch, Serre result to compute the index of Z in terms of F.

For $s>1$, we employ the remark made near the end of the introduction.

§ 3. An Invariant for Actions without Fixed Points

Let (T^s, M^n) be a smooth action without fixed points. Suppose (T^s, M^n) is a smooth equivariant boundary of (T^s, B^{n+1}) where B^{n+1} is a smooth oriented compact manifold with oriented boundary M^n. The action (T^s, B^{n+1}) may have fixed points. Define

$$'\sigma(T^s, M^n) = I(F(T^s, B^{n+1})) - I(B^{n+1}).$$

Theorem 3. *$'\sigma(T^s, M^n)$ depends only upon (T^s, M^n). When $s=1$, $'\sigma(S^1, M^n)$ is the invariant $'\sigma(\infty, (S^1, M^n))$ of* [1; 7.8].

Proof. Suppose (T^s, A) and (T^s, B) equivariantly bound (T^s, M^n). Form the oriented manifold $(T^s, A) \cup_{(T^s, M^n)} (T^s, -B) = (T^s, C)$.

Since,

$$F(T^s, C) = F(T^s, A) \cup F(T^s, -B),$$

Theorem 1 implies

$$I(F(T^s, A)) + I(F(T^s, -B)) = I(C) = I(A) + I(-B).$$

Hence,

$$I(F(T^s, A)) - I(A) = I(F(T^s, B)) - I(B).$$

When $s=1$, Atiyah and Singer arrive by application of their G-signature theorem to the same formula for their invariant $'\sigma(t, (S^1, M^n))$ when $t \to \infty$. Hence, they are the same.

It is known [12] that for every smooth action (S^1, M^n) without fixed points there is a multiple $r M^n$ (r is actually of the form 2^s), which is an equivariant smooth boundary. Thus one can define:

$$\sigma(\infty, (S^1, M^n)) = \frac{1}{r}\, '\sigma(\infty, (S^1, r M^n)).$$

Of course, from the Atiyah-Singer theorem it is not apparent that $\sigma(\infty, (S^1, M^n))$ is an integer. We shall now show using the technique developed here that this rational number is actually an integer. First, we extend Theorem 3 to actions on **R**-generalized manifolds.

We shall assume that all our actions considered on compact **R**-generalized manifolds have a finite number of distinct orbit types. This assumption can actually be avoided by a slightly different approach. Since our main interest is in smooth actions, we do not pursue this here.

Theorem 4. (i) *Let (S^1, M^n) be an action without fixed points on a closed oriented* **R***-generalized manifold. Then, the induced S^1 action on the* **R***-generalized $(n+1)$-manifold with boundary $(S^1, \mathrm{Map}(\pi))$, where $\pi: M \to M/S^1$ is the orbit map, has very nicely embedded fixed point sets.*

(ii) *Let (T^s, M^n) be an action without fixed points on an oriented* **R***-generalized manifold. Suppose (T^s, M^n) is an equivariant boundary of the oriented compact (T^s, B^{n+1}) with very nicely embedded fixed points. Then,*

$$'\sigma(T^s, M^n) = I(F(T^s, B^{n+1})) - I(B^{n+1})$$

depends only upon (T^s, M^n).

Proof. (i) As pointed out earlier (Prop. 1), $(S^1, \mathrm{Map}(\pi))$ is an action on an oriented **R**-generalized manifold with boundary $\partial(S^1, \mathrm{Map}(\pi)) = (S^1, M^n)$. The fixed point set $F(S^1, \mathrm{Map}(\pi)) = M^n/S^1$ is an oriented closed generalized manifold of codimension 2. The inverse image of each point x in M^n/S^1 under the equivariant retraction map $r_1: \mathrm{Map}(\pi) \to M^n/S^1$ is a 2-disk which "fills in" the orbit in M^n lying over x. The Leray sheaf associated to this map $r_1|_{\partial(\mathrm{Map}(\pi))}$ is simple. Thus $(S^1, \mathrm{Map}(\pi))$ has nicely embedded fixed point sets.

The proof of (ii) is exactly the same as the proof of Theorem 3 except that we replace Theorem 1 by Theorem 2. This completes the proof.

Suppose (S^1, M^n) is an action without fixed points on an oriented closed **R**-generalized n-manifold. Define:

$$\sigma(S^1, M^n) = \begin{cases} I(M^n/S^1), & \text{if } n=4k+1 \\ -I(\mathrm{Map}(\pi)), & \text{if } n=4k-1 \\ 0, & n \text{ even.} \end{cases}$$

Corollary 1. *$'\sigma(S^1, M^n) = \sigma(S^1, M^n)$. In particular, if (S^1, M^n) is smooth then the Atiyah-Singer invariant $\sigma(\infty, (S^1, M^n))$ is an integer and equals $\sigma(S^1, M^n)$.*

Proof. From (i) (S^1, M^n) is an equivariant boundary of $(S^1, \mathrm{Map}(\pi))$ with nicely embedded fixed point set M^n/S^1. By (ii) $'\sigma(S^1, M^n) = I(M^n/S^1) - I(\mathrm{Map}(\pi))$. But if n is even $'\sigma = 0$. If $n=4k+1$, then $I(\mathrm{Map}(\pi))=0$. If $n=4k-1$, $I(M^{4k-1}/S^1)=0$. Hence, $'\sigma(S^1, M^n) = \sigma(S^1, M^n)$. If (S^1, M^n) is smooth then

$$r\,\sigma(S^1, M^n) = \sigma(S^1, r\,M^n) = {}'\sigma(S^1, r\,M^n) = {}'\sigma(\infty, (S^1, r\,M^n)).$$

Hence, $\sigma(S^1, M^n) = \sigma(\infty, (S^1, M^n))$.

Remark. To obtain the analogue of Corollary 1 for toral actions we need to further strengthen the notion of very nicely embedded fixed points. Let (T^s, M^n) be an action without fixed points. For $S^1 \subset T^s$, such that $F(S^1, M^n) = F(T^s, M^n) = \emptyset$, we may write $T^s = S^1 \times T^{s-1}$. We form $(S^1 \times T^s, D^2 \times M^n)$, where S^1 acts diagonally and T^s on the second factor. We obtain a toral action, $(T^s, D^2 \times_{S^1} M^n) = (T^s, \mathrm{Map}(\pi))$, where $\pi\colon M^n \to M^n/S^1$. We require that $(T^s, D^2 \times_{S^1} M^n)$ has very nicely embedded fixed points. In particular, if (T^s, M^n) is a smooth action, then $(T^s, D^2 \times_{S^1} M^n)$ has very nicely embedded fixed points. We may then define

$$\sigma(T^s, M^n) = \begin{cases} I(F(T^{s-1}, M^n/S^1)), & \text{if } n=4k+1 \\ -I(\mathrm{Map}(\pi)), & \text{if } n=4k-1 \\ 0, & n \text{ even.} \end{cases}$$

Of course, $\partial(T^s, D^2 \times_{S^1} M^n) = (T^s, M^n)$ and we apply Theorem 4, (ii) and identify $'\sigma(T^s, M^n) = \sigma(T^s, M^n)$ for the purpose of computation. (Observe that $I(F(T^s, \mathrm{Map}(\pi))) = I(F(T^{s-1}, M^n/S^1)) = I(M^n/S^1)$.)

It is not known whether a smooth fixed point free (T^s, M^n) has a multiple $(T^s, r\,M^n)$ which is a smooth equivariant boundary. If this is true, then $'\sigma(T^s, rM^n)$ is defined as a smooth invariant and hence is clearly divisible by r and the quotient is $\sigma(T^s, M^n)$.

It should also be observed that the smooth $\sigma(T^k, M^n)$ invariant really is defined up to topological equivalence.

§ 4. Some Calculations of $\sigma(S^1, M^n)$ and Geometric Interpretation of $\sigma(S^1, M^n)=0$

(i) Consider (S^1, M^n) where M^n has the real cohomology of the n-sphere, and the action is fixed point free. Then let (S^1, CM^n) be the action defined on the cone over M^n. This is a generalized $(n+1)$-cell with the vertex of the cone being the fixed point set. It is nicely embedded. Hence, $\sigma(S^1, M^n)=\pm 1$.

(ii) The Dold manifold $\mathscr{D}_1^{2n+1}=\mathbf{P}(1, n)$, n even [7].

There is an involution $(Z_2, \mathbf{CP}_n)$ defined by

$$(z_0:z_1:\cdots:z_n)\to(\bar{z}_0:\bar{z}_1:\cdots:\bar{z}_n).$$

This involution preserves the orientation if n is even and reverses the orientation if n is odd.

An S^1 action without fixed points can be described by the equivariant diagram

$$\begin{tikzcd}
(S^1, S^1) \arrow[d] & (S^1\times Z_2, S^1\times \mathbf{CP}_n) \arrow[l] \arrow[r, "/S^1"] \arrow[d, "/Z_2"] & (Z_2, \mathbf{CP}_n) \arrow[d, "p", "/Z_2"'] \\
(S^1, S^1/Z_2) & (S^1, S^1\times_{Z_2}\mathbf{CP}_n) \arrow[l, "\mathbf{CP}_n"'] \arrow[r, "/S^1"] & \mathbf{CP}_n/Z_2
\end{tikzcd}$$

$(S^1, S^1\times_{Z_2}\mathbf{CP}_n)$ is the desired action on $(S^1, \mathscr{D}^{2n+1})$. $\mathscr{D}^{2n+1}$ is not a smooth boundary.

The space $\mathbf{CP}_n/Z_2=\mathscr{D}^{2n+1}/S^1$ and the monomorphism

$$p^*:\ H^j(\mathbf{CP}_n/Z_2;\mathbf{R})\to H^j(\mathbf{CP}_n;\mathbf{R})$$

goes onto the invariant cohomology. For $j\equiv 2(4)$ the map is trivial. For $n\equiv 2(4)$ then,

$$0=I(\mathbf{CP}_n/Z_2)=\sigma(S^1, \mathscr{D}^{8k+5}).$$

For n divisible by 4 then p^* is bijective, for $n/2$, and hence,

$$\sigma(S^1, \mathscr{D}^{8k+1})=\pm 1$$

(iii) (S^1, M^3).

Let (S^1, M^3) be closed connected oriented and without fixed points.

Theorem 5. *$\sigma(S^1, M^3)=0$ or ± 1. Furthermore, $\sigma(S^1, M^3)=0$, if and only if (S^1, M^3) fibers (equivariantly) over S^1 with finite structure group.*

Proof. From [14], M^3/S^1 is a closed oriented surface of genus $g \geqq 0$. Let $W = \mathrm{Map}(\pi)$ where $\pi\colon M^3 \to M^3/S^1$ is the orbit map. W is an oriented compact $\mathbf{R}$-generalized 4-manifold with boundary $\partial W = M^3$, which deforms onto the closed surface M^3/S^1. We shall now show that:

$$\begin{aligned} &\operatorname{rk} H_1(M^3;Z) \quad \text{odd implies} \quad \sigma(S^1, M^3) = 0 \\ &\operatorname{rk} H_1(M^3;Z) \quad \text{even implies} \quad \sigma(S^1, M^3) = \pm 1. \end{aligned} \tag{4.1}$$

Consider the exact sequence with coefficients in $\mathbf{R}$:

$$\begin{tikzcd} 0 \arrow[r] & H^1(W,\partial W) \arrow[r, "j^*"] \arrow[d, phantom, "\wr"] & H^1(W) \arrow[r, "i^*"] & H^1(\partial W) \arrow[r, "\delta^*"] & H^2(W,\partial W) \arrow[r, "j^*"] \arrow[d, phantom, "\wr"] & H^2(W) \arrow[dd, equal] \\ & H_3(W) \arrow[d, phantom, "\wr"] & & & H_2(W) \arrow[d, phantom, "\wr"] & \\ & 0 = H_3(M/S^1) \arrow[r, "j^*"] & H^1(M/S^1) \arrow[r, "\pi^*"] \arrow[uu, "r^\dagger", "\wr"'] \arrow[d, equal] & H^1(M) \arrow[r, "\delta^*"] \arrow[uu, "1^*", "\|"'] & \mathbf{R} \arrow[r, "j^*"] & \mathbf{R} \\ & & (\mathbf{R})^{2g} & & & \end{tikzcd}$$

Thus we have:

$$0 \to (\mathbf{R})^{2g} \xrightarrow{\pi} \begin{cases} (\mathbf{R})^{2g} \oplus \mathbf{R} & \text{or} \\ \mathbf{R}^{2g}. \end{cases}$$

If we let $A = \operatorname{im} j^*\colon H^2(W,\partial W) \to H^2(W)$, then $A = \mathbf{R}$ or 0 if $H^1(M) \cong (\mathbf{R})^{2g}$ or $(\mathbf{R})^{2g+1}$, respectively. The cup product pairing $H^2(W,\partial W) \otimes H^2(W,\partial W)$ restricted to A is non-singular. Therefore, $\sigma(S^1, M^3) = \pm 1$ or 0, respectively. This yields (4.1).

In dimension 3 all actions of S^1 are known and classified [14], [11] and [10]. For fixed point free circle actions on *orientable* manifolds all are Seifert fiber spaces. Consider Seifert's presentation of $H_1(M^3;Z)$:

$$\{a_1, b_1, \ldots, a_g, b_g, h, q_1, \ldots, q_n\colon \left. \begin{aligned} q_1 + q_2 \qquad & + \cdots + q_n - b\,h = 0 \\ \alpha_1 q_1 + 0 q_2 \qquad & + 0 q_n + \beta_1 h = 0 \\ \vdots \qquad & \\ 0 q_1 + \cdots + 0 q_{n-1} & + \alpha_n q_n + \beta_n h = 0 \end{aligned} \right\}.$$

Looking at the relations we have the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} 1 & 1 & 1 \ldots & 1 & -b \\ \alpha_1 & 0 & \ldots & 0 & \beta_1 \\ 0 & \alpha_2 & 0 & 0 & \beta_2 \\ \vdots & & & & \\ 0 & 0 & \ldots 0 & \alpha_n & \beta_n \end{pmatrix}$$

where $g \geqq 0$, $0 < \beta_i < \alpha_i$, $(\alpha_i, \beta_i) = 1$, $b \in Z$ and n may be any non-negative integer.

Thinking of $H_1(M^3; \mathbf{R})$ as $H_1(M^3; Z) \otimes \mathbf{R}$ we see that this relation matrix is of rank either n or $n+1$. Thus,

$$H_1(M^3; Z) = \begin{cases} (Z)^{2g} \oplus Z + \text{Torsion or} \\ (Z)^{2g} + \text{Torsion} \end{cases}$$

(where Torsion may be 0), which of course agrees with above.

Notice that h is of infinite order if and only if all the q_i are of infinite order. This occurs exactly where rk $H_1(M^3; Z)$ *is odd.*

By [6] (S^1, M^3) fibers equivariantly over S^1 if and only if the order of h is infinite (see also [11]). We may now conclude (S^1, M^3) fibers equivariantly over $(S^1, S^1/Z_p)$, some p.

We remark that fibering a space X over S^1 with finite structure group is equivalent to asking whether or not there exists an S^1 action on X and an equivariant map onto $(S^1, S^1/Z_p)$, for some p, [6]. That is, $(S^1, X) = (S^1, S^1 \times_{Z_p} Y)$. In dimension 3, all fiberings of an oriented closed 3 manifold which admit circle actions necessarily have finite structure groups with the exception of non-trivial principal circle bundles over the torus. None of these non-trivial principal bundles fiber with finite structure group. Thus, fibering a closed oriented 3 manifold (which admits a circle action) over the circle and with oriented fiber is equivalent to fibering it (equivariantly) and with finite structure group with the previously mentioned exception. All of these manifolds admit exactly one S^1-action. Caution: The fiberings may not be unique! We must also point out another exception: $(S^1, S^1 \times S^2)$. When it has no fixed points it fibers equivariantly over the circle; but when $(S^1, S^1 \times S^2)$ has fixed points, the fiberings cannot be equivariant.

We now can point out a generalization of part of Theorem 5. It seems to suggest a very interesting geometric meaning for the vanishing of $\sigma(S^1, M^{4k-1})$.

Theorem 6. *Let M^{4k-1} be an orientable closed* **R**-*generalized manifold. If M^{4k-1} fibers over the circle with finite structure group then there exists an S^1 action (S^1, M^{4k-1}) which fibers equivariantly over $(S^1, S^1/Z_p)$ and $\sigma(S^1, M^{4k-1}) = 0$. If M^{4k-1} and the fibering are smooth then (S^1, M^{4k-1}) is smooth.*

(We call attention to the theorem of Conner and Raymond [6]:
Whenever (S^1, X) is a given action where $H_1(X; Z)$ is finitely generated, then (S^1, X) fibers equivariantly over the circle for some finite structure group if and only if the homomorphism

$$f_{\#}^{x}: H_1(S^1, 1) \to H_1(X, x),$$

defined by $f^x(t) = t\,x$, $t \in S^1$, is a monomorphism.)

Proof. We need only show that $\sigma(S^1, M^{4k-1})$ is necessarily 0. We have $(S^1, M^{4k-1})=(S^1, S^1\times_{Z_p} Y)$ for some Z_p, where Y is an oriented closed $\mathbf{R}$-generalized $(n-1)$-manifold and (Z_p, Y) is orientation preserving (since $Y/Z_p=M^{4k-1}/S^1$ is oriented). We may describe $\mathrm{Map}(\pi)$ in this case in a very nice way as $(S^1, D^2\times_{Z_p} Y)$. To see this consider the commutative diagram of equivariant maps

$$\begin{array}{ccccc} (S^1, S^1) & \longleftarrow & (S^1\times Z_p, S^1\times Y) & \xrightarrow[\pi']{/S^1} & (Z_p, Y) \\ \Big|{/Z_p} & & \Big|{/Z_p} & & \Big\downarrow{/Z_p} \\ (S^1, S^1/Z_p) & \xleftarrow{Y} & (S^1, S^1\times_{Z_p} Y=M) & \xrightarrow[\pi']{/S^1} & Y/Z_p\,. \end{array}$$

There is associated to this diagram an associated diagram for the mapping cylinders (§ 1):

$$\begin{array}{ccccc} (S_1^1, D^2) & \longleftarrow & (S_1^1\times Z_p, D^2\times Y=D^2\times_{S_2^1}(S^1\times Y)) & \xrightarrow{/S_1^1} & (Z_p, I\times Y) \\ \Big\downarrow{/Z_p} & & q\Big\downarrow{/Z_p} & & \Big\downarrow{/Z_p} \\ (S_1^1, D^2/Z_p) & \xleftarrow{Y} & (S_1^1, D^2\times_{Z_p} Y)=(S_1^1, D^2\times_{S_2^1}(S^1\times_{Z_p} Y)) & \xrightarrow{/S_1^1} & (Z_p, I\times Y/Z_p). \end{array}$$

Now $(S_1^1, S^1\times_{Z_p} Y)=(S^1, M^{4k-1})$. Thus the quotient space of $(Z_p, \mathrm{Map}(\pi'))$ under the orbit map q is $\mathrm{Map}(\pi)$. The homomorphism

$$q^*\colon H^j(\mathrm{Map}(\pi); \mathbf{R})\to H^j(\mathrm{Map}(\pi'); \mathbf{R})$$

is a monomorphism onto the invariant cohomology. The same is true if we take relative cohomology modulo the boundary. The pairing

$$H^{2k}(\mathrm{Map}(\pi'), \partial(\mathrm{Map}(\pi')); \mathbf{R})\otimes H^{2k}(\mathrm{Map}(\pi'), \partial(\mathrm{Map}(\pi')); \mathbf{R})$$
$$\to H^{4k}(\mathrm{Map}(\pi'), \partial(\mathrm{Map}(\pi')); \mathbf{R})$$

is trivial since $\mathrm{Map}(\pi')=D^2\times Y^{4k-2}$. Since q^* is injective, the pairing on $\mathrm{Map}(\pi)$ is also trivial. Thus, $\sigma(S^1, M^{4k-1})=0$, which completes the proof.

Finally we point out that one could also interpret $\sigma(\infty, (S^1, M^n))$, when one knows that $\mathrm{Map}(\pi)$ or M^n/S^1 is triangulable (for example, if (S^1, M^n) is smooth) in terms of the rational Pontryagin classes of M^n/S^1 or $\mathrm{Map}(\pi)$. For $n=4k+1$, then $\sigma(\infty, (S^1, M^{4k+1}))$ is the value of Hirzebruch L-genus of the rational Pontryagin classes of M^{4k+1}/S^1.

References

1. Atiyah, M.F., Singer, I.M.: The index of elliptic operators: III. Annals of Math. **87**, 546–604 (1968).
2. Borel, A., with contributions by Bredon, G., Floyd, E.E., Montgomery, D., Palais, R.: Seminar on transformation groups. Annals of Mathematics Studies, Princeton **46** (1960).

3. Bredon, G. E.: Cohomology fiber spaces, the Smith-Gysin sequence, and orientation in generalized manifolds. Michigan Math. J. **10**, 321–333 (1963).
4. Chern, S. S., Hirzebruch, F., Serre, J. P.: On the index of a fibered manifold. Proc. Amer. Math. Soc. **8**, 587–596 (1957).
5. Conner, P. E., Floyd, E. E.: Differentiable periodic maps. Berlin-Göttingen-Heidelberg: Springer 1964.
6. — Raymond, F.: Injective operations of the toral groups. To appear in Topology.
7. Dold, A.: Erzeugende der Thomschen Algebra $\mathfrak{N}$. Math. Zeitschr. **65**, 25–35 (1956).
8. Kawakubo, K., Uchida, F.: On the index of a semi-free S^1-action. Proc. Japan Academy **46**, 620–622 (1970); and to appear in J. Math. Soc. Japan.
9. Kwun, K. W., Raymond, F.: Mapping cylinder neighborhoods. Michigan Math. J. **10**, 353–357 (1963).
10. Orlik, P., Raymond, F.: Actions of $SO(2)$ on 3-manifolds. Proceedings of the Conference on Transformation Groups. Berlin-Heidelberg-New York: Springer 1968.
11. — Vogt, E., Zieschang, H.: Zur Topologie gefaserter dreidimensionaler Mannigfaltigkeiten. Topology **6**, 49–64 (1967).
12. Ossa, E.: Fixpunktfreie S^1-Aktionen. Math. Ann. **186**, 45–52 (1970).
13. Raymond, F.: The orbit spaces of totally disconnected groups of transformations on manifolds. Proc. Amer. Math. Soc. **12**, 1–7 (1961).
14. — Classification of the actions of the circle on 3-manifolds. Trans. Amer. Math. Soc. **131**, 51–78 (1968).
15. Wilder, R. L.: Topology of manifolds. Amer. Math. Soc. Coll. Publ. **32** (1949).

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