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Unitary structure in representations of infinite-dimensional groups and a convexity theorem*

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In this paper, we show that a Kac-Moody algebra $\mathfrak{g}(A)$ associated to a symmetrizable generalized Cartan matrix A carries a contravariant Hermitian form which is positive-definite on all root spaces. We deduce that every integrable highest weight $\mathfrak{g}(A)$-module $L(\Lambda)$ carries a contravariant positive-definite Hermitian form. This allows us to define the moment map and prove a generalization of the Schur-Horn-Kostant-Heckman-Atiyah-Pressley convexity theorem. The proofs are based on an identity which also gives estimates for the action of $\mathfrak{g}(A)$ on $\mathfrak{g}(A)$ and $L(\Lambda)$.

We hope that the main idea behind the paper is apparent: it is to use the interplay between the coadjoint and the highest weight representations.

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§ 1. Basic definitions (see [6, 8, 9] for details)

1.1. Let $A=(a_{ij})_{i,j=1}^n$ be a symmetrizable generalized Cartan matrix, i.e., $a_{ii}=2$, a_{ij} are non-positive integers for $i \neq j$ $(i,j=1,\dots,n)$, and there exists an invertible diagonal matrix $D=\operatorname{diag}(d_1,\dots,d_n)$ such that $D^{-1}A$ is symmetric. Then we can (and will) choose the d_i to be positive rational. Choose a triple $(\mathfrak{h}_{\mathbb{R}}, \Pi, \Pi^\vee)$, unique up to isomorphism, where $\mathfrak{h}_{\mathbb{R}}$ is a vector space over $\mathbb{R}$ of dimension $2n-\operatorname{rank} A$, and $\Pi=\{\alpha_1,\dots,\alpha_n\}\subset\mathfrak{h}_{\mathbb{R}}^*$, $\Pi^\vee=\{h_1,\dots,h_n\}\subset\mathfrak{h}_{\mathbb{R}}$ are linearly independent sets satisfying $\alpha_j(h_i)=a_{ij}$. We put $\mathfrak{h}=\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{h}_{\mathbb{R}}$.

The *Kac-Moody algebra* $\mathfrak{g}=\mathfrak{g}(A)$ is the Lie algebra over $\mathbb{C}$ generated by the vector space $\mathfrak{h}$ and symbols e_i and f_i $(i=1,\dots,n)$, with defining relations: $[\mathfrak{h},\mathfrak{h}]=(0)$; $[e_i,f_j]=\delta_{ij}h_i$; $[h,e_i]=\alpha_i(h)e_i$, $[h,f_i]=-\alpha_i(h)f_i (h\in\mathfrak{h})$; $(\operatorname{ad} e_i)^{1-a_{ij}}(e_j)=0=(\operatorname{ad} f_i)^{1-a_{ij}}(f_j)$ $(i\neq j)$.

We have the canonical embedding $\mathfrak{h}\subset\mathfrak{g}$. Let $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) be the subalgebra of $\mathfrak{g}$ generated by the e_i (resp. f_i), $i=1,\dots,n$. We have the *triangular decomposition*: $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$. Every ideal of $\mathfrak{g}$ which intersects $\mathfrak{h}$ trivially is zero [3].

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We have the *root space decomposition* $\mathfrak{g}=\bigoplus_{\alpha\in\mathfrak{h}^*}\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha=\{x\in\mathfrak{g}\,|\,[h,x]=\alpha(h)x$ for all $h\in\mathfrak{h}\}$, so that $\mathfrak{g}_{\alpha_i}=\mathbb{C}e_i$, $\mathfrak{g}_{-\alpha_i}=\mathbb{C}f_i$, $\mathfrak{g}_0=\mathfrak{h}$. A *root* is an element of $\Delta:=\{\alpha\in\mathfrak{h}^*\,|\,\alpha\neq 0,\ \mathfrak{g}_\alpha\neq(0)\}$. Put $Q=\sum_i \mathbb{Z}\alpha_i$ and $Q_+=\sum_i\mathbb{Z}_+\alpha_i$, where $\mathbb{Z}_+=\{0,1,\ldots\}$, and put $ht(\alpha)=\sum_i k_i$ for $\alpha=\sum_i k_i\alpha_i\in Q$. Introduce an ordering on $\mathfrak{h}^*$ by: $\lambda\geqq\mu$ if $\lambda-\mu\in Q_+$. Put $\Delta_+=\Delta\cap Q_+$. We have: $\mathfrak{n}_\pm=\bigoplus_{\alpha\in\Delta_+}\mathfrak{g}_{\pm\alpha}$.

The root space decomposition of $\mathfrak{g}$ gives us a Q-gradation of the universal enveloping algebra: $U(\mathfrak{g})=\bigoplus_\beta U(\mathfrak{g})_\beta$.

We choose a nondegenerate symmetric $\mathbb{C}$-bilinear form $(.|.)$ on $\mathfrak{h}$ such that $(h_i|h)=d_i\alpha_i(h)$ for $i=1,\ldots,n$ and $h\in\mathfrak{h}$. This form extends uniquely to a nondegenerate $\mathfrak{g}$-invariant symmetric $\mathbb{C}$-bilinear form $(.|.)$ on $\mathfrak{g}$ (see [6], Proposition 7 and Lemma 2). We have:

$$(e_i|f_i)=d_i. \tag{1.1}$$

The form $(.|.)$ induces an isomorphism $\nu\colon\mathfrak{h}\to\mathfrak{h}^*$ and a form $(.|.)$ on $\mathfrak{h}^*$. Then $\nu(h_i)=d_i\alpha_i$. Furthermore, $(\mathfrak{g}_\alpha|\mathfrak{g}_\beta)=(0)$ if $\alpha\neq-\beta$, and $\mathfrak{g}_\alpha$ and $g_{-\alpha}$ are nondegenerately paired; we have:

$$[x,y]=(x|y)\,\nu^{-1}(\alpha)\quad \text{if } x\in\mathfrak{g}_\alpha \text{ and } y\in\mathfrak{g}_{-\alpha}. \tag{1.2}$$

Define a conjugate-linear involution ω_0 of $\mathfrak{g}$ by requiring $\omega_0(e_i)=-f_i$, $\omega_0(f_i)=-e_i$ $(i=1,\ldots,n)$, $\omega_0(h)=-h$ for $h\in\mathfrak{h}_{\mathbb{R}}$, and define the following nondegenerate Hermitian form on $\mathfrak{g}$:

$$(x|y)_0=-(x|\omega_0(y)).$$

Then the root space decomposition is orthogonal with respect to $(.|.)_0$.

Choose $\rho\in h^*_{\mathbb{R}}$ satisfying $(\rho|\alpha_i)=\frac{1}{2}(\alpha_i|\alpha_i)$ (or, equivalently, $\rho(h_i)=1$) for $i=1,\ldots,n$. For $\Lambda,\alpha\in\mathfrak{h}^*$, put

$$T_\Lambda(\alpha)=(\Lambda+\rho|\alpha)-\tfrac{1}{2}(\alpha|\alpha).$$

In the sequel we will need

$$T_0(\alpha)>0 \quad \text{if } \alpha\in\Delta_+\setminus\Pi. \tag{1.3}$$

Indeed, (1.3) is clear when $(\alpha|\alpha)\leqq 0$; otherwise, using [6, Lemma 14 and formula (23)], $2\nu^{-1}(\alpha)/(\alpha|\alpha)\in\sum_i\mathbb{Z}_+h_i\setminus\Pi^\vee$, proving (1.3) in this case also.

1.2. Given $\Lambda\in\mathfrak{h}^*$, a $\mathfrak{g}$-module V is called a *module with highest weight* Λ if there exists a non-zero cyclic vector $v_\Lambda\in V$ such that $\mathfrak{n}_+(v_\Lambda)=(0)$ and $h(v_\Lambda)=\Lambda(h)v_\Lambda$ for all $h\in\mathfrak{h}$. Such a module is $\mathfrak{h}$-diagonalizable.

Given an $\mathfrak{h}$-diagonalizable module V, we have the *weight space decomposition* $V=\bigoplus_{\lambda\in\mathfrak{h}^*}V_\lambda$, where $V_\lambda=\{v\in V\,|\,h(v)=\lambda(h)v$ for all $h\in\mathfrak{h}\}$. Elements of $P(V):=\{\lambda\in\mathfrak{h}^*\,|\,V_\lambda\neq(0)\}$ are called *weights* of V. For a $\mathfrak{g}$-module V with highest weight Λ, we have: $V_\Lambda=\mathbb{C}v_\Lambda$ and $P(V)\subset\Lambda-Q_+$.

Let V be a $\mathfrak{g}$-module such that for every $v \in V$, the set $\{\alpha \in \Delta_+ | \mathfrak{g}_\alpha(v) \neq (0)\}$ is finite. Such a module is called *restricted.* Note that every highest weight module is a restricted module. Following [7], we define the *partial Casimir operator* Ω_0 on a restricted module V as follows. For each $\alpha \in \Delta$, choose bases $\{x_\alpha^{(k)}\}$ of $\mathfrak{g}_\alpha$ and $\{y_\alpha^{(k)}\}$ of $\mathfrak{g}_{-\alpha}$ such that $(x_\alpha^{(k)} | y_\alpha^{(l)}) = \delta_{kl}$, and put

$$\Omega_0(v) = \sum_{\alpha \in \Delta_+} \sum_k y_\alpha^{(k)}(x_\alpha^{(k)}(v)).$$

Lemma 1.1. a) *If* $\alpha, \beta \in \Delta$ *and* $z \in \mathfrak{g}_{\alpha-\beta}$, *then, in* $\mathfrak{g} \otimes \mathfrak{g}$, *we have:*

$$\sum_k x_\alpha^{(k)} \otimes [z, y_\alpha^{(k)}] = \sum_k [x_\beta^{(k)}, z] \otimes y_\beta^{(k)}.$$

b) *If* V *is a restricted* $\mathfrak{g}$-*module and* $u \in U(\mathfrak{g})_\beta$, *then we have on* V:

$$\Omega_0 u - u\Omega_0 = u(T_0(-\beta) I_V - v^{-1}(\beta)). \tag{1.4}$$

Proof. a) is checked by pairing with an element $e \otimes f$, where $e \in \mathfrak{g}_{-\alpha}$, $f \in \mathfrak{g}_\beta$:

$$\begin{aligned}\sum_k (x_\alpha^{(k)}|e)([z, y_\alpha^{(k)}]|f) &= \sum_k (x_\alpha^{(k)}|e)(y_\alpha^{(k)}|[f, z]) = (e|[f, z]) = ([z, e]|f) \\ &= \sum_k (x_\beta^{(k)}|[z, e])(y_\beta^{(k)}|f) = \sum_k ([x_\beta^{(k)}, z]|e)(y_\beta^{(k)}|f).\end{aligned}$$

Since $\mathfrak{g}_\gamma$ and $\mathfrak{g}_{-\gamma}$ are nondegenerately paired under $(.|.)$, this verifies a).

If b) holds for $u \in U(\mathfrak{g})_\beta$ and $u' \in U(\mathfrak{g})_{\beta'}$, then it holds for $uu' \in U(\mathfrak{g})_{\beta+\beta'}$. Hence, it suffices to check b) for $u = x_{\alpha_i}$ or y_{α_i} (for $u \in U(\mathfrak{h})$, b) is obvious). Using a) and

$$(\gamma + \mathbb{Z}\alpha_i) \cap \Delta \subset \Delta_+ \quad \text{for } \gamma \in \Delta_+ \setminus \{\alpha_i\},$$

we have, on V:

$$\begin{aligned}[\Omega_0, x_{\alpha_i}] &= [y_{\alpha_i} x_{\alpha_i}, x_{\alpha_i}] = -v^{-1}(\alpha_i) x_{\alpha_i} \\ &= -(\alpha_i|\alpha_i) x_{\alpha_i} - x_{\alpha_i} v^{-1}(\alpha_i) = x_{\alpha_i}(T_0(-\alpha_i) I_V - v^{-1}(\alpha_i)); \\ [\Omega_0, y_{\alpha_i}] &= [y_{\alpha_i} x_{\alpha_i}, y_{\alpha_i}] = y_{\alpha_i} v^{-1}(\alpha_i) \\ &= y_{\alpha_i}(T_0(\alpha_i) I_V - v^{-1}(-\alpha_i)), \quad \text{proving b).} \quad \square\end{aligned}$$

Among $\mathfrak{g}$-modules with highest weight Λ, there is a module $M(\Lambda)$ which is free of rank 1 as a $U(\mathfrak{n}_-)$-module, and an irreducible module $L(\Lambda)$. $M(\Lambda)$ and $L(\Lambda)$ are unique up to isomorphism.

A Hermitian form $F(.,.)$ on a $\mathfrak{g}$-module V is called *contravariant* if $F(g(u), v) = -F(u, \omega_0(g)v)$ for all $u, v \in V$ and $g \in \mathfrak{g}$. For example, the form $(.|.)_0$ on $\mathfrak{g}$ is contravariant. It is standard that for $\Lambda \in \mathfrak{h}_{\mathbb{R}}^*$, $L(\Lambda)$ carries a unique contravariant Hermitian form, denoted by $H(.,.)$, such that $H(v_\Lambda, v_\Lambda) = 1$; this form is nondegenerate and the weight space decomposition is orthogonal with respect to it.

Fix *fundamental weights* $\Lambda_i \in \mathfrak{h}_{\mathbb{R}}^*$, $1 \leq i \leq n$, satisfying $\Lambda_i(h_j) = \delta_{ij}$, $1 \leq j \leq n$, and put $P_+ = \sum_i \mathbb{Z}_+ \Lambda_i$. A $\mathfrak{g}$-module $L(\Lambda)$, $\Lambda \in P_+$, is called an *integrable* highest weight module. We have [9, Proposition 2.4d]:

$$T_\Lambda(\beta) > 0 \quad \text{if } \Lambda \in P_+ \text{ and } \Lambda - \beta \in P(L(\Lambda)) \setminus \{\Lambda\}. \tag{1.5}$$

§ 2. The crucial lemma

By analogy with the partial Casimir operator, we define an operator Ω_1 on $\mathfrak{n}_-$ by:

$$\Omega_1(z)=\sum_{\alpha\in\Delta_+}\sum_k [y_\alpha^{(k)},[x_\alpha^{(k)},z]_-],$$

where the subscript "minus" denotes projection on $\mathfrak{n}_-$ with respect to the triangular decomposition.

Lemma 2.1. *If $\alpha\in\Delta_+$ and $z\in\mathfrak{g}_{-\alpha}$, then*

$$\Omega_1(z)=2T_0(\alpha)\,z.$$

Proof. Put $R=\Delta_+\cap(\alpha-\Delta_+)$ and calculate in $M(0)$:

$$\begin{aligned}2T_0(\alpha)\,z(v_0)&=2\Omega_0(z(v_0))=2\sum_{\beta\in\Delta_+}\sum_k y_\beta^{(k)}x_\beta^{(k)}z(v_0)=2\sum_{\beta\in\Delta_+}\sum_k y_\beta^{(k)}[x_\beta^{(k)},z](v_0)\\&=2\sum_{\beta\in R}\sum_k y_\beta^{(k)}[x_\beta^{(k)},z](v_0)=\sum_{\beta\in R}\sum_k [y_\beta^{(k)},[x_\beta^{(k)},z]](v_0)\\&\quad+\sum_{\beta\in R}\sum_k (y_\beta^{(k)}[x_\beta^{(k)},z]+[x_\beta^{(k)},z]\,y_\beta^{(k)})(v_0)=(\Omega_1(z))(v_0).\end{aligned}$$

The first equality follows from (1.4) and the last one from Lemma 1.1a.
As $M(0)$ is a free $U(\mathfrak{n}_-)$-module, the lemma follows. □

Remark. Lemmas 1.1 and 2.1 hold (by the same proof) for the Lie algebra $\mathfrak{g}(A)$ associated to an arbitrary symmetrizable matrix A over a field.

§ 3. Unitary structure on $L(\Lambda)$ and $\mathfrak{g}$

3.1. **Theorem 1.** *Let $\mathfrak{g}(A)$ be the Kac-Moody algebra associated to a symmetrizable generalized Cartan matrix A. Then:*

a) *The Hermitian form $(.|.)_0$ is positive-definite on $\mathfrak{n}_-\oplus\mathfrak{n}_+$.*

b) *Every integrable highest weight $\mathfrak{g}(A)$-module $L(\Lambda)$ carries a positive-definite contravariant Hermitian form.*

Proof. We first prove a). Using ω_0, it suffices to show that $(.|.)_0$ is positive-definite on $\mathfrak{g}_{-\alpha}$ for all $\alpha\in\Delta_+$. We do this by induction on $ht(\alpha)$. The case $ht(\alpha)=1$ is clear by (1.1). Otherwise, put $R=\Delta_+\cap(\alpha-\Delta_+)$ and use the inductive assumption to choose, for every $\beta\in R$, an orthonormal basis $\{y_\beta^{(k)}\}$ of $\mathfrak{g}_{-\beta}$ with respect to $(.|.)_0$. Then, setting $x_\beta^{(k)}=-\omega_0(y_\beta^{(k)})$, we have $(x_\beta^{(k)}|y_\beta^{(l)})=\delta_{kl}$. Now we apply Lemma 2.1 with this choice of $x_\beta^{(k)}$ and $y_\beta^{(k)}$ (the choice for the $\beta\in\Delta_+\setminus R$ is unimportant) and $z\in\mathfrak{g}_{-\alpha}$:

$$\begin{aligned}2T_0(\alpha)(z|z)_0=(\Omega_1(z)|z)_0&=\sum_{\beta\in R}\sum_k([y_\beta^{(k)},[x_\beta^{(k)},z]]|z)_0\\&=\sum_{\beta\in R}\sum_k([x_\beta^{(k)},z]|[x_\beta^{(k)},z])_0.\end{aligned}$$

By the inductive assumption, the last sum is non-negative; using (1.3), we get $(z|z)_0 \geqq 0$. Since $(.|.)_0$ is nondegenerate on $\mathfrak{g}_{-\alpha}$, we deduce that it is positive-definite, proving a).

By remarks in Sect. 1.2, the contravariant Hermitian form $H(.,.)$ on $L(\Lambda)$ satisfies: $H(v_\Lambda, v_\Lambda)=1$, and the weight spaces are pairwise orthogonal. We prove by induction on $ht(\Lambda-\lambda)$ that the restriction of $H(.,.)$ to $L(\Lambda)_\lambda$ is positive-definite. Let $\lambda \in P(L(\Lambda))$ and $v \in L(\Lambda)_\lambda$. Thanks to a), we can choose bases $\{x_\alpha^{(k)}\}$ of $\mathfrak{g}_\alpha$, $\alpha \in \Delta_+$, such that $(x_\alpha^{(k)}|x_\alpha^{(l)})_0=\delta_{kl}$. Note that $v=u(v_\Lambda)$ for some $u \in U_{\lambda-\Lambda}$. Hence, by (1.4), we have:

$$\Omega_0(v)=T_\Lambda(\Lambda-\lambda)\,v. \tag{3.1}$$

Therefore, we have: $T_\Lambda(\Lambda-\lambda)H(v,v)=H(\Omega_0(v),v)=\sum_{\alpha\in\Delta_+}\sum_k H(x_\alpha^{(k)}(v), x_\alpha^{(k)}(v))$. In the same way as in the proof of a), we conclude, using (1.5), that $H(.,.)$ is positive-definite on $L(\Lambda)_\lambda$. This proves b). □

Remark. Positivity of $H(.,.)$ in the affine case is due to Garland [4]; our argument in the proof of b) is similar to his.

3.2. We now derive estimates for the action of $\mathfrak{g}$ on $\mathfrak{g}$ and on $L(\Lambda)$. For this we need:

Lemma 3.1. *Let $\{x_k\}$ be a basis of $\mathfrak{n}_+$ such that $(x_k|x_l)_0=\delta_{kl}$. Let $y \in \mathfrak{n}_-$, and let $v \in L(\Lambda)$, $\Lambda \in P_+$. Then:*

a) $H(\Omega_0(v),v)=\sum_k H(x_k(v),x_k(v))$.

b) $(\Omega_1(y)|y)_0=\sum_k([x_k,y]_-|[x_k,y]_-)_0$.

Proof. Putting $x_k^*=-\omega_0(x_k)$, $\{x_k\}$ and $\{x_k^*\}$ are bases of $\mathfrak{n}_+$ and $\mathfrak{n}_-$, dual under $(.|.)$. Since the operators Ω_0 and Ω_1 can be expressed using arbitrary dual bases of $\mathfrak{n}_-$ and $\mathfrak{n}_+$, we have:

$$\Omega_0(v)=\sum_k x_k^*(x_k(v)) \quad \text{and} \quad \Omega_1(y)=\sum_k [x_k^*,[x_k,y]_-].$$

The lemma follows. □

To state our estimates, we need some notation. Choose an inner product $(\ ,\ )$ on $\mathfrak{h}$. The induced norm on $\mathfrak{h}^*$ satisfies: $|\Lambda(h)|\leqq|\Lambda|\,|h|$ for all $\Lambda\in\mathfrak{h}^*$, $h\in\mathfrak{h}$. For $z\in\mathfrak{g}$, write $z=z_-+z_0+z_+$, where $z_\pm\in\mathfrak{n}_\pm$, $z_0\in\mathfrak{h}$, and define an inner product $(\ ,\ )$ on $\mathfrak{g}$ by: $(z,z')=(z_-|z'_-)_0+(z_0,z'_0)+(z_+|z'_+)_0$. We will write $|z|$ for $(z,z)^{\frac{1}{2}}$, and also $|v|$ for $H(v,v)^{\frac{1}{2}}$, where $v\in L(\Lambda)$, $\Lambda\in P_+$. Define $d\in\mathrm{Der}(\mathfrak{g})$ by $d(x)=ht(\alpha)x$ for $x\in\mathfrak{g}_\alpha$, and $d\in\mathrm{End}\,L(\Lambda)$ by $d(v)=-ht(\Lambda-\lambda)v$ for $v\in L(\Lambda)_\lambda$.

Put $C_1=0$ if $n=1$, $C_1=(\max_{1\leqq i,j\leqq n}-(\alpha_i|\alpha_j))^{\frac{1}{2}}$ otherwise; $C_2=\max_{1\leqq i\leqq n}|\alpha_i|$; $C_3=\max_{1\leqq i\leqq n}|\nu^{-1}(\alpha_i)|$. Then we have, for all $\alpha\in Q_+$:

$$T_0(\alpha)\leqq\tfrac{1}{2}C_1^2\,ht(\alpha)^2; \qquad |\alpha|\leqq C_2\,ht(\alpha); \tag{3.2}$$
$$|\nu^{-1}(\alpha)|\leqq C_3\,ht(\alpha)$$

Indeed, for $\alpha=\sum k_i \alpha_i \in Q_+$, we have:

$$C_1^2\, ht(\alpha)^2 - 2T_0(\alpha) = C_1^2 \sum_i k_i^2 + \sum_i (k_i^2 - k_i)(\alpha_i|\alpha_i) + \sum_{i \neq j} (C_1^2 + (\alpha_i|\alpha_j))\, k_i k_j \geqq 0.$$

The rest of (3.2) is obvious.

Put $C_4 = 4C_1 + 2C_2 + 2C_3$.

Below, we shall use the Schwarz inequality, etc., without comment.

Proposition 3.1. *If* $x \in \mathfrak{n}_+$, $z, z' \in \mathfrak{g}$, $\Lambda \in P_+$ *and* $v \in L(\Lambda)$, *then:*

a) $|[x,z]_-| \leqq C_1 |x|\, |d(z)|$.

b) $|[z,z']| \leqq C_4(|d(z)|\, |z'| + |z|\, |d(z')|)$.

c) $|x(v)| \leqq |\Lambda|\, |x|\, |v| + C_4 |x|\, |d(v)|$.

d) $|z(v)| \leqq 3|\Lambda|\, |z|\, |v| + C_4(|d(z)|\, |v| + |z|\, |d(v)|)$.

Proof. Let $x \in \mathfrak{n}_+$; $y, y' \in \mathfrak{n}_-$; $z, z' \in \mathfrak{g}$; $h \in \mathfrak{h}$. We claim:

$$|[x,y]_-|^2 \leqq |x|^2 (\Omega_1(y), y).$$

Indeed, we may assume that $|x|=1$, complete $\{x\}$ to an orthonormal basis of $\mathfrak{n}_+$, and apply Lemma 3.1b. Furthermore, $(\Omega_1(y), y) \leqq C_1^2 |d(y)|^2$ by (3.2) and Lemma 2.1, yielding:

$$|[x,y]_-| \leqq C_1 |x|\, |d(y)|. \tag{3.3}$$

Let $y'' \in \mathfrak{n}_-$ satisfy $d(y'') = [y, y']$. Then:

$$\begin{aligned} |[y,y']|^2 &= |([y,y'], d(y''))| = |(d([y,y']), y'')| = |([d(y), y'] + [y, d(y')], y'')| \\ &= |(d(y), [\omega_0(y'), y'']_-) - (d(y'), [\omega_0(y), y'']_-)| \\ &\leqq |d(y)|\, |[\omega_0(y'), y'']_-| + |d(y')|\, |[\omega_0(y), y'']_-|. \end{aligned}$$

Applying (3.3) to estimate the right-hand side, we obtain, using $d(y'') = [y, y']$:

$$|[y,y']| \leqq C_1(|d(y)|\, |y'| + |y|\, |d(y')|). \tag{3.4}$$

Write $z = \Sigma z_\alpha$, $z' = \Sigma z'_\alpha$, where $z_\alpha, z'_\alpha \in \mathfrak{g}_\alpha$. Then

$$|[h,z]|^2 = \Sigma |\alpha(h)|^2 |z_\alpha|^2 \leqq C_2^2 |h|^2\, \Sigma\, ht(\alpha)^2 |z_\alpha|^2 = C_2^2 |h|^2 |d(z)|^2,$$

so:

$$|[h,z]| \leqq C_2 |h|\, |d(z)|. \tag{3.5}$$

Using (1.2), we have:

$$\begin{aligned} |[z,z']_0| &\leqq \Sigma |(z_\alpha|z'_{-\alpha})|\, |\nu^{-1}(\alpha)| \leqq \Sigma |z_\alpha|\, |z'_{-\alpha}|\, |\nu^{-1}(\alpha)| \leqq C_3\, \Sigma |z_\alpha|\, |z'_{-\alpha}|\, |ht(\alpha)| \\ &= C_3\, \Sigma |d(z_\alpha)|\, |z'_{-\alpha}| \leqq C_3 (\Sigma |d(z_\alpha)|^2)^{\frac{1}{2}} (\Sigma |z'_{-\alpha}|^2)^{\frac{1}{2}} \\ &= C_3 |d(z)|\, |z'|, \end{aligned}$$

so:

$$|[z,z']_0| \leqq C_3 |d(z)|\, |z'|. \tag{3.6}$$

Applying (3.3–5) and the triangle inequality to

$$[z,z']_- = [z_-,z'_-] + [z_+,z'_-]_- + [z_-,z'_+]_- + [z_0,z'_-] + [z_-,z'_0],$$

we obtain:

$$|[z,z']_-| \leqq (2C_1 + C_2)(|d(z)|\,|z'| + |z|\,|d(z')|). \tag{3.7}$$

Using (3.6), (3.7) and an analogous estimate for $|[z,z']_+|$, the triangle inequality applied to $[z,z'] = [z,z']_- + [z,z']_0 + [z,z']_+$ gives:

$$|[z,z']| \leqq (4C_1 + 2C_2 + C_3)(|d(z)|\,|z'| + |z|\,|d(z')|). \tag{3.8}$$

Now, let $\Lambda \in P_+$. Using (3.2) and $T_\Lambda(\Lambda-\lambda) = T_0(\Lambda-\lambda) + (\Lambda|\Lambda-\lambda)$, we have, for all $\lambda \in P(L(\Lambda)) \subset \Lambda - Q_+$:

$$T_\Lambda(\Lambda-\lambda) \leqq \tfrac{1}{2} C_1^2 (ht(\Lambda-\lambda))^2 + C_3 |\Lambda|\, ht(\Lambda-\lambda).$$

Hence, by (3.1) and Lemma 3.1a, we have

$$\begin{aligned}|x(v)|^2 &\leqq |x|^2 H(\Omega_0(v), v) \leqq |x|^2 (\tfrac{1}{2} C_1^2 |d(v)|^2 + C_3 |\Lambda|\, H(-d(v), v))\\ &\leqq |x|^2 (|\Lambda|\,|v| + (C_1 + C_3)\,|d(v)|)^2,\end{aligned}$$

so:

$$|x(v)| \leqq |\Lambda|\,|x|\,|v| + (C_1 + C_3)\,|x|\,|d(v)|. \tag{3.9}$$

We also have, for $v = \sum v_\lambda,\ v_\lambda \in L(\Lambda)_\lambda$,

$$\begin{aligned}|h(v)|^2 &= \Sigma\,|\lambda(h)|^2\,|v_\lambda|^2 \leqq |h|^2\,\Sigma\,|\lambda|^2\,|v_\lambda|^2 \leqq |h|^2\,\Sigma\,(|\Lambda| + C_2\, ht(\Lambda-\lambda))^2\,|v_\lambda|^2\\ &= |h|^2\,\big||\Lambda|\,v - C_2\, d(v)\big|^2,\end{aligned}$$

so that:

$$|h(v)| \leqq |\Lambda|\,|h|\,|v| + C_2\,|h|\,|d(v)|. \tag{3.10}$$

We now take $y \in \mathfrak{n}_-$, and put $x' = \omega_0(y)$ and $s = [y, x']$, so that $\omega_0(s_-) = -s_+$. Using the contravariance of H, we obtain:

$$|y(v)|^2 = |x'(v)|^2 + 2\,\mathrm{Re}\, H(s_+(v), v) + H(s_0(v), v),$$

so:

$$|y(v)|^2 \leqq |x'(v)|^2 + 2\,|s_+(v)|\,|v| + |s_0(v)|\,|v|.$$

We estimate $|x'(v)|$ and $|s_+(v)|$ using (3.3 and 9), and $|s_0(v)|$ using (3.6 and 10). From this, we obtain:

$$|y(v)| \leqq |\Lambda|\,|y|\,|v| + (C_1 + C_3)\,|d(y)|\,|v| + (C_1 + C_2 + C_3)\,|y|\,|d(v)|. \tag{3.11}$$

Finally, (3.9–11) combine to show:

$$|z(v)| \leqq 3\,|\Lambda|\,|z|\,|v| + (C_1 + C_3)\,|d(z)|\,|v| + 2(C_1 + C_2 + C_3)\,|z|\,|d(v)|. \tag{3.12}$$

(3.3, 8, 9 and 12) prove the proposition. □

Remark. It is not difficult to sharpen these inequalities. Also, using $|H(v,d(v))|\leqq|d(v)|^2$, we have an alternative version of (3.9):

$$|x(v)|\leqq(|\Lambda|+C_1+C_3)\,|x|\,|d(v)|. \tag{3.13}$$

§ 4. A convexity theorem

4.1. We first recall the construction of the group G associated to $\mathfrak{g}$ and its *unitary form* K, and related results from [11]. Put $V=\bigoplus_{\Lambda\in P_+} L(\Lambda)$ and $V^0=\sum \mathbb{C}\, v_\Lambda \subset V$. We endow V and $\mathfrak{g}$ with the finest topology which induces the metric topology on finite-dimensional subspaces. Since the elements e_i and f_i are locally nilpotent on V, we have the one-parameter groups $\exp t e_i$ and $\exp t f_i$ ($t\in\mathbb{C}$) for all i; they generate a subgroup G of $GL(V)$. G acts on each $L(\Lambda)$, $\Lambda\in P_+$, say by π_Λ, and also on $\mathfrak{g}$ via the adjoint action Ad. We have: $\pi_\Lambda(\mathrm{Ad}(g)\,x)=\pi_\Lambda(g)\,\pi_\Lambda(x)\,\pi_\Lambda(g)^{-1}$ for $g\in G$ and $x\in\mathfrak{g}$.

The involution ω_0 lifts to G; let $\mathfrak{k}$ and K be its fixed point sets in $\mathfrak{g}$ and G. Note that K preserves the Hermitian forms $(.|.)_0$ on $\mathfrak{g}$ and H on $L(\Lambda)$.

Let $B=\{g\in G\,|\,g(V^0)\subset V^0\}$, $H=B\cap\omega_0(B)$, and let N be the normalizer in G of H. These definitions are equivalent to the ones in [11]. (B is denoted B_+ in [11].)

$\mathfrak{h}\subset\mathfrak{g}$ is Ad(N)-invariant and Ad(H)-fixed. Hence, we have an action of the *Weyl group* $W:=N/H$ on $\mathfrak{h}$; moreover, this action is faithful. W is generated by the set $S=\{r_i\}_{i=1}^n$, where $r_i(h)=h-\alpha_i(h)\,h_i$ ($h\in\mathfrak{h}$), and (W,S) is a Coxeter system (cf. [8] or [9]). $C:=\{h\in\mathfrak{h}_{\mathbb{R}}\,|\,\alpha_i(h)\geqq 0$ for $i=1,\dots,n\}$ is called the *fundamental chamber* for W. The set $X=\bigcup_{w\in W} w(C)$ is a convex cone in $\mathfrak{h}_{\mathbb{R}}$, called the *Tits cone*. Note that $X=\mathfrak{h}_{\mathbb{R}}$ if and only if $\dim\mathfrak{g}<\infty$ (cf. [9]). Let $\leqq$ be the Bruhat order on W (see e.g. [10]).

For $w\in W$, put $K_w=K\cap BwB$. Then [11]:

$$K=\coprod_{w\in W} K_w. \tag{4.1}$$

Fix $\Lambda\in P_+$. Since $G=KB$ [11], and hence $K_w B=BwB$, we deduce:

$$\mathbb{C}^*\,K_w(v_\Lambda)=\mathbb{C}^*\,B\,w\,B(v_\Lambda). \tag{4.2}$$

For $v\in L(\Lambda)$, denote by $\mathrm{supp}\,v$ the set of all $\lambda\in\mathfrak{h}^*$ such that v has a non-zero component in $L(\Lambda)_\lambda$. We have by [11, Theorem 1]:

$$\text{If}\quad v\in K_w(v_\Lambda),\quad\text{then}\ \ \mathrm{supp}\,v\subset[\{w'(\Lambda)\,|\,w'\leqq w\}]. \tag{4.3}$$

(Here and further on, the convex hull of a subset M of a real vector space is denoted by $[M]$.)

Put $\tilde{H}=\mathrm{Hom}(Q,\mathbb{C}^*)$; this is a group isomorphic to $(\mathbb{C}^*)^n$. Define a homomorphism Ad: $\tilde{H}\to\mathrm{Aut}(\mathfrak{g})$ by $\mathrm{Ad}(h)\,x=h(\alpha)\,x$ if $x\in\mathfrak{g}_\alpha$, and an action of $\tilde{H}$ on $L(\Lambda)$ by: $h(v)=h(\beta)\,v$ if $v\in L(\Lambda)_{\Lambda+\beta}$. $\tilde{H}$ normalizes G and B under these actions and commutes with H; since the centralizer of H in G is H [11], we have, using (4.2):

$$\mathbb{C}^* \tilde{H} K_w(v_\Lambda) = \mathbb{C}^* K_w(v_\Lambda). \tag{4.4}$$

There exists a finest topology on G such that (cf. [10]):

a) G is a topological group;

b) the maps $t \mapsto \exp t e_i$ $(i=1, \ldots, n)$ are continuous on $\mathbb{C}$ with the usual topology.

We fix this topology on G. Then G is Hausdorff and the action of G on V and $\mathfrak{g}$ is continuous. ω_0 is continuous and hence K is a closed subgroup of G. Furthermore, each $\overline{K}_{r_i}$ is a compact subgroup of K, and the commutator subgroup $(\overline{K}_{r_i}, \overline{K}_{r_i})$ is isomorphic to SU_2 as a topological group. (Here and further on, $\overline{M}$ denotes the closure of M.) Let $w \in W$, and write $w = r_{i_1} \ldots r_{i_s}$, where s is minimal. Then

$$K_w = K_{r_{i_1}} \ldots K_{r_{i_s}}.$$

This is shown by the same argument as in [13, Sect. 8]. In particular, $\overline{K_w}$ is compact.

Fix $w \in W$. Put $L(\Lambda; w) = \bigoplus_{\lambda \geqq w(\Lambda)} L(\Lambda)_\lambda$; clearly, $\dim L(\Lambda; w) < \infty$. Then $BwB(v_\Lambda) \subset L(\Lambda; w)$ is irreducible in the Zariski topology (see [11, Theorem 1]). Since the closure of $BwB(\mathbb{C} v_\Lambda)$ in the Zariski topology is $\bigcup_{w' \leqq w} Bw'B(\mathbb{C} v_\Lambda)$ [11, Theorem 1c], we obtain, using (4.2):

There exists $v \in K_w(v_\Lambda)$ such that $\operatorname{supp} v \supset \{w'(\Lambda) | w' \leqq w\}$. (4.5)

4.2. Denote by p the projection of $\mathfrak{g}$ on $\mathfrak{h}$ with respect to the root space decomposition. Now we can prove the following convexity theorem.

Theorem 2. a) *If* $h \in C$ *and* $w \in W$, *then*

$$p(\mathrm{Ad}(\overline{K_w})\, h) = [\{w'(h) | w' \leqq w\}]. \tag{4.6}$$

b) *If* $h \in X$, *then*

$$p(\mathrm{Ad}(K)\, h) = [W(h)]. \tag{4.7}$$

Proof. Formula (4.7) follows from (4.6) and the following

Lemma 4.1. *If* $w_1, w_2 \in W$, *then there exists* $w \in W$ *such that* $w \geqq w_1$ *and* $w \geqq w_2$.

Proof of Lemma 4.1. Induction on $l(w_1) + l(w_2)$ using the following two facts: $l(r_i w) > l(w)$ implies $r_i w > w$; $r_i w \geqq w, w'$ implies $r_i w \geqq r_i w'$. □

To prove (4.6), we employ the *moment map*. Let $\mathfrak{a}$ be $i\mathfrak{k}$ or $\mathfrak{h}_{\mathbb{R}}$. Fix $\Lambda \in P_+$. We define the moment map $M_{\mathfrak{a}}$ from the projective space $\mathbb{P}(L(\Lambda))$ to $\mathfrak{a}$ by:

$$(x | M_{\mathfrak{a}}(v))_0 = H(x(v), v)/H(v, v) \quad \text{for } x \in \mathfrak{a}$$

(we can make this definition thanks to Theorem 1). Notice that $M_{i\mathfrak{k}}$ is K-equivariant and that $M_{i\mathfrak{k}}(v_\Lambda) = v^{-1}(\Lambda)$. We also have:

$$M_{\mathfrak{h}_{\mathbb{R}}}(\sum_\lambda v_\lambda) = \sum_\lambda H(v_\lambda, v_\lambda)\, v^{-1}(\lambda) / \sum_\lambda H(v_\lambda, v_\lambda), \quad \text{where } v_\lambda \in L(\Lambda)_\lambda. \tag{4.8}$$

Let $w \in W$. The crucial observation is:

$$M_{\mathfrak{h}_{\mathbb{R}}}(\mathbb{C}^* \overline{K}_w(v_\Lambda)) = [\{w'(v^{-1}(\Lambda)) | w' \leqq w\}]. \tag{4.9}$$

The inclusion $\subset$ in (4.9) follows from (4.8) and (4.3). On the other hand, by Theorem 2 from [1] applied to the action of the complex torus $\tilde{H}$ on the finite-dimensional projective space $\mathbb{P}(L(\Lambda; w))$, $M_{\mathfrak{h}_{\mathbb{R}}}(\tilde{H}(v))$ is convex for every non zero $v \in L(\Lambda; w)$. The reverse inclusion now follows from (4.4), (4.5) and (4.8).

The following properties of the moment map are clear:

$$M_{\mathfrak{h}_{\mathbb{R}}} = p \circ M_{i\mathfrak{k}}.$$
$$M_{i\mathfrak{k}}(k(v_\Lambda)) = \mathrm{Ad}(k)\, v^{-1}(\Lambda) \quad \text{for } k \in K.$$

Using this, (4.9) implies (4.6) for all $h \in C$ such that $\alpha_i(h) \in \mathbb{Q}$ $(i = 1, \ldots, n)$. Using the compactness of $\overline{K}_w$, we deduce that (4.6) holds for all $h \in C$. □

Remarks. a) If $\dim \mathfrak{g} < \infty$, then K is a compact group and $X = h_{\mathbb{R}}$; in this case, (4.7) is due to Schur-Horn-Kostant and (4.6) to Heckman (references may be found in [1]).

b) We have $p(\mathrm{Ad}(K)X) = X$ by Theorem 2. What are $[\mathrm{Ad}(K)X]$ and $\{x \in i\mathfrak{k} | p(\mathrm{Ad}(K)x) \subset X\}$?

c) Let $h \in C$ have finite stabilizer in W and let $h' \in h + \sum_s \mathbb{R} h_s$. Put

$$|h'|_s = \sup_{k \in K} \Lambda_s(p(\mathrm{Ad}(k)h')) \quad \text{for } s = 1, \ldots, n.$$

Then the following are equivalent:

(i) $h' \in [\mathrm{Ad}(K)h]$.

(ii) $h' \in [W(h)]$.

(iii) $|h'|_s \leqq \Lambda_s(h)$ for $s = 1, \ldots, n$.

This is immediate from Theorem 2, $W(h') \subset \mathrm{Ad}(K)h'$, and:

$$[W(h)] = \bigcap_{w \in W} w(h - \sum_s \mathbb{R}_+ h_s), \quad \text{where } \mathbb{R}_+ = \{t \in \mathbb{R} | t \geqq 0\}.$$

The latter formula follows from [14, Proposition 2.4] and the fact that if M is a bounded subset of $[W(h)]$, then $\rho(h - w(h')) \to \infty$ as $l(w) \to \infty$, uniformly for $h' \in M$.

d) The previous remark implies that, in the case $\dim \mathfrak{g} < \infty$, one has for $h \in \mathfrak{h}_{\mathbb{R}}$ (using that all K-orbits in $i\mathfrak{k}$ intersect $\mathfrak{h}_{\mathbb{R}}$): $[\mathrm{Ad}(K)h] = \{x \in i\mathfrak{k} | (p(\mathrm{Ad}(k)x)) \leqq \Lambda_s(h)$ for $s = 1, \ldots, n$ and all $k \in K\}$.

e) Using the proposition below and the fact that $M_{\mathfrak{h}_{\mathbb{R}}}(e^h(v)) = \frac{1}{2} \operatorname{grad} \log H(e^h(v), e^h(v))$ for $h \in \mathfrak{h}_{\mathbb{R}}$, one can avoid the reference to [1].

Proposition 4.1. *Let V be a finite-dimensional real vector space and let S be a finite subset of V^* such that $[S]$ has non-empty interior. Let c_λ $(\lambda \in S)$ be positive real numbers. Put $G(v) = \sum_{\lambda \in S} c_\lambda e^{\lambda(v)}$ and $F(v) = \log G(v)$. Then the image of* $(\operatorname{grad} F)$: $V \to V^*$ *is the interior of* $[S]$.

Proof. First, let $f\colon V\to\mathbb{R}$ be an arbitrary convex function. For $l\in V^*$, put $\tilde{f}(l)=\inf\{f(v)-l(v)\mid v\in V\}\in\mathbb{R}\cup\{-\infty\}$; put $T_f=\{l\in V^*\mid \tilde{f}(l)>-\infty\}$, $T'_f=\{l\in T_f\mid f(v)-l(v)=\tilde{f}(l)$ for some $v\in V\}$. Then:

$$\operatorname{Interior}(T_f)\subset T'_f\subset T_f. \tag{4.10}$$

Indeed, if $l\in T_f\setminus T'_f$, choose $v_1, v_2, \ldots\in V$ such that $f(v_n)-l(v_n)\to\tilde{f}(l)$. The continuity of f forces $|v_n|\to\infty$, so that by choosing a subsequence if necessary, there exists $l'\in V^*$ such that $l'(v_n)\to+\infty$. Hence, $l+\varepsilon l'\notin T_f$ for all $\varepsilon>0$, and so $l\notin\operatorname{Interior}(T_f)$. This proves (4.10).

Let G and F satisfy the hypothesis of the proposition. Then it is easy to check that $T_F=[S]$ and that F is of class C^∞. Moreover, one calculates that

$$2G^2(D^2_\beta F)=\sum_{\lambda,\mu\in S} c_\lambda c_\mu(\lambda(\beta)-\mu(\beta))^2 e^{\lambda+\mu},$$

so that $(D^2_\beta F)(v)>0$ for all $\beta, v\in V$ such that $\beta\neq 0$. In particular, F is convex and of class C^1, and therefore $T'_F=(\operatorname{grad} F)(V)$. Moreover, $D(\operatorname{grad} F)$ is surjective at each $v\in V$, so that $\operatorname{grad} F$ is an open map and hence $(\operatorname{grad} F)(V)$ is open. Applying (4.10), the proposition follows. □

4.3. *Example.* Let $K\subset GL_r(\mathbb{C})$ be a connected simply-connected compact simple Lie group, with Lie algebra $\mathfrak{k}\subset\mathfrak{gl}_r(\mathbb{C})$, and let T be a maximal torus of K, with Lie algebra $\mathfrak{t}\subset\mathfrak{k}$. For a subset A of a finite-dimensional vector space U over $\mathbb{C}$, we denote by $\tilde{A}$ the set of all polynomial loops on A, i.e., the set of all maps $f\colon S^1\to A$ such that $f(e^{i\theta})=p(e^{i\theta}, e^{-i\theta})$ for some polynomial map $p\colon \mathbb{C}^2\to U$. If $f\in\tilde{U}$, then we write f' for $\frac{d}{d\theta}f(e^{i\theta})$ and $\int f$ for $(2\pi)^{-1}\int_0^{2\pi} f(e^{i\theta})\,d\theta$. We regard $GL_r(\mathbb{C})$ and $\mathfrak{gl}_r(\mathbb{C})$ as subsets of $\operatorname{Mat}_r(\mathbb{C})$. Using pointwise multiplication and addition in $\widetilde{\operatorname{Mat}_r(\mathbb{C})}$, $\widetilde{GL_r(\mathbb{C})}$ becomes a group and $\widetilde{\mathfrak{gl}_r(\mathbb{C})}$ a Lie algebra.

The unitary form of the Kac-Moody algebra $\hat{\mathfrak{g}}$ associated to the extended Cartan matrix of K (a "non-twisted affine Lie algebra") is $\hat{\mathfrak{k}}:=\mathbb{R}d\oplus\tilde{\mathfrak{k}}\oplus\mathbb{R}c$, with bracket (for $x, y\in\tilde{\mathfrak{k}}$):

$$[x,y]=(xy-yx)+\left(\int tr(x'y)\right)c;\qquad [d,x]=x';\qquad [c,\hat{\mathfrak{k}}]=(0).$$

We put $\hat{\mathfrak{t}}:=\mathbb{R}d\oplus\mathfrak{t}\oplus\mathbb{R}c$ (its complexification is the Cartan subalgebra of $\hat{\mathfrak{g}}$). We define a $\hat{\mathfrak{k}}$-invariant symmetric $\mathbb{R}$-bilinear form $(.|.)$ on $\hat{\mathfrak{k}}$ by (for $x, y\in\tilde{\mathfrak{k}}$):

$$(x|y)=\int tr(xy);\qquad (c|d)=1;\qquad (x|c)=(x|d)=(c|c)=(d|d)=0. \tag{4.11}$$

As we shall see in a moment, the unitary form $\hat{K}$ of the group $\hat{G}$ associated to $\hat{\mathfrak{g}}$ is a central extension $\sigma\colon\hat{K}\to\tilde{K}$ of the loop group $\tilde{K}$. (One can show that $\operatorname{Ker}\sigma\simeq S^1$.) We proceed to compute the adjoint representation Ad of $\hat{K}$ on $\hat{\mathfrak{k}}$.

We extend the obvious action of $\tilde{\mathfrak{k}}$ on $\widetilde{\mathbb{C}^r}$ to a representation $\hat{\pi}$ of $\hat{\mathfrak{k}}$ by putting $\hat{\pi}(c)=0$, $\hat{\pi}(d)f=f'$. Then $\hat{\pi}$ is an integrable representation (cf. [11]), and hence induces a representation $\hat{\pi}$ of $\hat{K}$ on $\widetilde{\mathbb{C}^r}$ satisfying:

$$\hat{\pi}(\operatorname{Ad}(k)z)=\hat{\pi}(k)\hat{\pi}(z)\hat{\pi}(k)^{-1}\qquad\text{for } k\in\hat{K},\ \alpha\in\hat{\mathfrak{k}}.$$

On the other hand, it is easy to check that the obvious representation $\tilde{\pi}$ of the group $\tilde{K}$ on $\widetilde{\mathbb{C}^r}$ is faithful, and that $\hat{\pi}(\hat{K}) \subset \tilde{\pi}(\tilde{K})$. Hence, there exists a homomorphism σ: $\hat{K} \to \tilde{K}$ such that $\hat{\pi} = \tilde{\pi} \circ \sigma$; we write $\tilde{k}$ for $\sigma(k)$.

For $a \in \tilde{K}$ and $x \in \tilde{\mathfrak{k}}$, one easily calculates:

$$\tilde{\pi}(a)\,\hat{\pi}(x)\,\tilde{\pi}(a)^{-1} = \hat{\pi}(a\,x\,a^{-1}); \qquad \tilde{\pi}(a)\,\hat{\pi}(d)\,\tilde{\pi}(a)^{-1} = \hat{\pi}(d - a'\,a^{-1}).$$

For $k \in \hat{K}$ and $x \in \tilde{\mathfrak{k}}$, we find:

$$\hat{\pi}(\mathrm{Ad}(k)\,x) = \hat{\pi}(k)\,\hat{\pi}(x)\,\hat{x}(k)^{-1} = \tilde{\pi}(\tilde{k})\,\hat{\pi}(x)\,\tilde{\pi}(\tilde{k})^{-1} = \hat{\pi}(\tilde{k}\,x\,\tilde{k}^{-1})$$

and, similarly, $\hat{\pi}(\mathrm{Ad}(k)\,d) = \hat{\pi}(d - \tilde{k}'\,\tilde{k}^{-1})$. Now, $\mathrm{Ker}(\hat{\pi}) = \mathbb{R}\,c$, and $(.|.)$ is $\mathrm{Ad}(\hat{K})$-invariant. Hence, using (4.11), we obtain for $k \in \hat{K}$ (cf. [2]):

$$\begin{aligned} &\mathrm{Ad}(k)\,d = d - \tilde{k}'\,\tilde{k}^{-1} - \tfrac{1}{2}\left(\textstyle\int tr(\tilde{k}'\,\tilde{k}^{-1})^2\right) c; \\ &\mathrm{Ad}(k)\,x = \tilde{k}\,x\,\tilde{k}^{-1} + \left(\textstyle\int tr(\tilde{k}'\,x\,\tilde{k}^{-1})\right) c \qquad (x \in \tilde{\mathfrak{k}}); \\ &\mathrm{Ad}(k)\,c = c. \end{aligned} \tag{4.12}$$

We proceed to write (4.7) more explicitly. Put $Q^\vee = \{\gamma \in \mathfrak{t} \,|\, \exp(2\pi\gamma) = 1 \in K\}$, and define an injective homomorphism ψ: $Q^\vee \to \tilde{K}$ by: $(\psi(\gamma))(e^{i\theta}) = \exp(\theta\,\gamma)$. Now, regard K as the group of constant loops in $\tilde{K}$. Letting N (resp. $\tilde{N}$) be the normalizer in K (resp. $\tilde{K}$) of T, it is easy to see that $\tilde{T} = T \times \psi(Q^\vee)$ and $\tilde{N} = N \ltimes \psi(Q^\vee)$. Put

$$\hat{T} = \{k \in \hat{K} \,|\, \mathrm{Ad}(k)\,x = x \text{ for all } x \in \hat{\mathfrak{t}}\}, \qquad \hat{N} = \{k \in \hat{K} \,|\, \mathrm{Ad}(k)\,\hat{\mathfrak{t}} = \hat{\mathfrak{t}}\}.$$

Using (4.12), we have:

$$\hat{T} = \sigma^{-1}(T), \qquad \hat{N} = \sigma^{-1}(\tilde{N}).$$

Since K is connected and simply-connected, the standard construction of $\hat{N}$ using the Chevalley generators of $\hat{\mathfrak{g}}$ shows that $\tilde{N} \subset \sigma(\hat{N})$. Putting $W = N/T$ and $\hat{W} = \hat{N}/\hat{T}$, we therefore have:

$$\begin{aligned} &\sigma \text{ induces an isomorphism } \hat{W} \to \tilde{N}/T; \\ &\tilde{N}/T = (N \ltimes \psi(Q^\vee))/T \cong W \ltimes Q^\vee. \end{aligned} \tag{4.13}$$

Moreover, $\hat{W}$ is the Weyl group of $\hat{K}$, and its natural action on $\hat{\mathfrak{t}}$ is that described in Sect. 4.1.

Theorem 2c in [11] implies that $\tilde{K} = \tilde{N}\,\sigma(\hat{K})$. But, as we have seen above, $\tilde{N} = \sigma(\hat{N}) \subset \sigma(\hat{K})$. Hence, we have:

$$\tilde{K} = \sigma(\hat{K}). \tag{4.14}$$

Let p be the orthogonal projection of $\mathfrak{k}$ onto $\mathfrak{t}$ (i.e., the projection along $[\mathfrak{t}, \mathfrak{k}]$). Then the projection of Sect. 4.2, denoted here by $\hat{p}$, is given by:

$$\hat{p}(\lambda\,d + x + \mu\,c) = \lambda\,d + p\left(\textstyle\int x\right) + \mu\,c.$$

Now, let $x \in t$; then $d+x \in \pm iX$, where X is the Tits cone (cf. [9, Proposition 1.9]). Applying (4.7), we obtain:

$$\hat{p}(\mathrm{Ad}(\hat{K})(d+x)) = [\hat{W}(d+x)] \qquad \text{for all } x \in t. \tag{4.15}$$

Finally, taking $x=0$ and transforming (4.15) using (4.12–14), we obtain:

$$\{p(\int a' a^{-1}) + \tfrac{1}{2}(\int tr(a' a^{-1})^2)\, c \,|\, a \in \tilde{K}\} = [\{\gamma + \tfrac{1}{2}(\gamma|\gamma)\, c \,|\, \gamma \in Q^\vee\}], \tag{4.16}$$

which is due to Atiyah-Pressley (according to Guillemin).[1]

§ 5. On a KAK-decomposition

The following results are due to the second author [12]. Proof will appear elsewhere.

We regard G and $\tilde{H}$ as subgroups of $GL(V)$, $V = \bigoplus_{\Lambda \in P_+} L(\Lambda)$, so that $\tilde{H} \cap G = (I)$ and $\tilde{H}$ normalizes G, defining $\tilde{G} := \tilde{H} \ltimes G \subset GL(V)$. We denote the action of $\tilde{G}$ on $L(\Lambda)$ by π_Λ.

If $\mathscr{H}$ is a Hilbert space, let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded operators on $\mathscr{H}$. For $\Lambda \in P_+$, let $\mathscr{H}_\Lambda$ be the completion of the pre-Hilbert space $L(\Lambda)$ with inner product $H(.,.)$. Put $\tilde{G}^{\mathrm{cont}} = \{g \in \tilde{G} \,|\, \pi_\Lambda(g)$ is a bounded operator on $L(\Lambda)$ for all $\Lambda \in P_+\}$, and define π_Λ: $\tilde{G}^{\mathrm{cont}} \to \mathscr{B}(\mathscr{H}_\Lambda)$, $\Lambda \in P_+$, in the obvious way. Then $K \subset \tilde{G}^{\mathrm{cont}}$ acts unitarily on $\mathscr{H}_\Lambda$.

Let $\bar{K}$ be the closure, in the strong operator topology, of

$$\{(\pi_\Lambda(k), \pi_\Lambda(k)^*)_{\Lambda \in P_+} \,|\, k \in K\} \subset \mathscr{B}(\bigoplus_{\Lambda \in P_+} (\mathscr{H}_\Lambda \oplus \mathscr{H}_\Lambda)).$$

We identify K with a subset of $\bar{K}$, and extend the π_Λ to $\bar{K}$ in the obvious way. We have the following strong "rigidity" statement.

Proposition 5.1. *A sequence $k_1, k_2, \ldots$ of elements of $\bar{K}$ converges in the strong operator topology if and only if the sequences*

$$\pi_{\Lambda_i}(k_1)\, v_{\Lambda_i}, \pi_{\Lambda_i}(k_2)\, v_{\Lambda_i}, \ldots \qquad \textit{and} \qquad \pi_{\Lambda_i}(k_1)^*\, v_{\Lambda_i}, \pi_{\Lambda_i}(k_2)^*\, v_{\Lambda_i}, \ldots$$

converge, in norm, for all i, $1 \leqq i \leqq n$.

Put $\tilde{G}^{cpt} = \{g \in \tilde{G}^{\mathrm{cont}} \,|\, \pi_\Lambda(g)$ is a compact operator for all $\Lambda \in P_+\}$, and let $\bar{\tilde{G}}^{cpt}$ be the norm-closure of $\tilde{G}^{cpt}$ in $\mathscr{B}\left(\bigoplus_{i=1}^{n} \mathscr{H}_{\Lambda_i}\right)$, so that $\bar{\tilde{G}}^{cpt}$ is a semigroup of compact operators. Then $\bar{\tilde{G}}^{cpt}$ acts on $\mathscr{H}_\Lambda$, $\Lambda \in P_+$, in a natural way, denoted by π_Λ. Let A_c be the norm closure of $(\tilde{H} \times H) \cap \tilde{G}^{cpt}$ in $\mathscr{B}\left(\bigoplus_{i=1}^{n} \mathscr{H}_{\Lambda_i}\right)$. Then $\bar{\tilde{G}}^{cpt} = \bar{K} A_c \bar{K}$ in the following sense.

[1] Note added in proof. This result has already appeared in [15]

Theorem 3. *If* $g \in \bar{G}^{cpt}$, *then there exist* $k_1, k_2 \in \bar{K}$ *and* $a \in A_c$ *such that for all* $\Lambda \in P_+$:

$$\pi_\Lambda(g) = \pi_\Lambda(k_1)\, \pi_\Lambda(a)\, \pi_\Lambda(k_2)^*,$$

and

$$\pi_\Lambda(a)\, v_\Lambda = \|\pi_\Lambda(g)\|\, v_\Lambda.$$

Moreover, a is uniquely determined by these conditions, and is, in the norm-topology on $\mathscr{B}\left(\bigoplus_{i=1}^{n} \mathscr{H}_{\Lambda_i}\right)$, *a continuous function of g.*

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Note added in proof

(a) We take this opportunity to correct a misprint in [11]: in line 5 of the proof of Theorem 1, replace $\mathbb{Q}(\lambda - S(v))$ by $\mathbb{Q}_+(-\lambda + S(v))$.

(b) We have recently computed the cohomology ring of the topological space K and of the Lie algebra $\mathfrak{g}'$. In particular, it turned out that $H^*(\mathfrak{g}', \mathbb{C}) \simeq H^*(K, \mathbb{C})$, and that in the case when A is indecomposable and not of finite or affine type, the algebra $H^*(K, \mathbb{C})$ is a free graded commutative algebra on ε generators of degree 3 and a_j generators of degree $2j$, $j = 2, 3, \ldots$, where $\varepsilon = 1$ or 0 according as A is symmetrisable or not and $a_2, a_3, \ldots$ are determined from the formula:

$$\sum_{w \in W} t^{l(w)} = (1-t)^{-n}(1-t^2)^{\varepsilon - a_2}(1-t^3)^{-a_3}(1-t^4)^{-a_4} \ldots$$