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A COMBINATORIAL METHOD FOR PRODUCTS OF TWO MULTIPLE k -STATISTICS
WITH SOME GENERAL FORMULAE

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1. INTRODUCTION AND SUMMARY

Wishart has applied [1952a] and Kendall has justified [1952] the application of a combinatorial method to products of k -statistics. This combinatorial method is, with appropriate modification, that introduced by Fisher [1929] and formalized by Kendall [1940a], [1940b], [1940c]. But no one has shown how a combinatorial method can be applied to products of multiple k -statistics such as $k_{11} k_2$ and $k_{p_1 p_2 p_3} k_{111}$. Wishart obtained formulae for such products for all cases through the 6th order [1952a] by algebraic manipulation of the results for the products of the k -statistics. Tukey has suggested [1956] that the products can be obtained by direct calculation and has provided certain principles and tabular aids which are useful when the product consists of two factors. As illustrations, he has provided the details for the direct calculation of $k_{21} k_2$ and $k_{22} k_{22}$.

As opposed to the algebraic method and the direct method of calculation of formulae for products of multiple k -statistics, there is need for a combinatorial method. This paper not only provides a combinatorial method for products of two multiple k -statistics but also presents general formulae resulting from the application of the method.

An advantage of a combinatorial method is that it concentrates on the essential calculation without demanding a lot of superfluous writing as in the algebraic method and with the direct method. A combinatorial method might also be considered superior to the other methods (a) if it expresses the results in a systematic form so that there is reasonable assurance that no term will be missed, and (b) if it is adaptable to the expression of general formulae such as $k_{p_1 p_2 \dots p_r} k_2$, $k_{p_1 p_2 \dots p_r} k_{22}$, etc.

The purpose of this paper is to provide the fundamental basis for such a combinatorial method. The developmental plan of the paper is similar to that of Fisher [1929] in which the direct method is transformed, by a suitable notation and argument, to a combinatorial method.

2. NOTATION AND BASIC MATERIAL

We use a notation which, in general, is similar to the usage of Fisher [1929], Irwin and Kendall [1944], Kendall [1940a, b, c]; [1952], Tukey [1954], Wishart [1952a], [1952b], Abdel-Aty [1954], Barton, David, and Fix [1960], Barton and David [1961]. We denote the sample size by n and the finite population size by N . We also consider an infinite population of which the finite population may be considered to be a sample of size N . Then k_p represents the p^{th} k -statistic of the sample, K_p the corresponding value for the finite population and χ_p the p^{th} cumulant for the infinite population. The corresponding moment (about an arbitrary origin) of the infinite population is μ_p . We use the sample augmented symmetric functions [Kendall and Stuart, 1958]

$$\sum_{i=1}^n x_i^{p_1}, \quad \sum_{i \neq j} x_i^{p_1} x_j^{p_2}, \quad \sum_{i \neq j \neq k} x_i^{p_1} x_j^{p_2} x_k^{p_3}, \quad \dots$$

and their symmetric means [Tukey, 1956 p. 38]

$$\begin{aligned} \langle p_1 \rangle &= \frac{\sum_{i=1}^n x_i^{p_1}}{n}, \\ \langle p_1 p_2 \rangle &= \frac{\sum_{i \neq j} x_i^{p_1} x_j^{p_2}}{n(n-1)}, \\ \langle p_1 p_2 p_3 \rangle &= \frac{\sum_{i \neq j \neq k} x_i^{p_1} x_j^{p_2} x_k^{p_3}}{n(n-1)(n-2)}, \\ &\dots \end{aligned} \tag{2.1}$$

Following Tukey [1956, p. 38], these averages may be referred to as angle brackets or brackets. Now a basic theorem of both old and new finite sampling theory [Dwyer, 1938, p. 112], [Tukey, 1950, p. 504] is that

$$\begin{aligned} E_N \langle p_1 p_2 \dots p_r \rangle &= E_N \left(\frac{\sum_{i \neq j \neq \dots \neq r} x_i^{p_1} x_j^{p_2} \dots x_r^{p_r}}{n^{(r)}} \right) \\ &= \frac{\sum_{i \neq j \neq \dots \neq r} x_i^{p_1} x_j^{p_2} \dots x_r^{p_r}}{N^{(r)}} \end{aligned} \tag{2.2}$$

so that, in the new notation, the expected value of the sample bracket is the corresponding population bracket. If the sampling is from an infinite population, or from a finite population with replacement, we have at once

$$E\langle p_1 p_2 \dots p_r \rangle = E \left(\frac{\sum_{\substack{i_1 \neq i_2 \neq \dots \neq i_r \\ i_1, i_2, \dots, i_r}}^n x_{i_1}^{p_1} x_{i_2}^{p_2} \dots x_{i_r}^{p_r}}{n^{(r)}} \right) = \mu_{p_1} \mu_{p_2} \dots \mu_{p_r} \quad (2.3)$$

where the μ 's are moments about the origin. The sample bracket then is both the estimate of the corresponding finite population bracket and of the corresponding moment product for the infinite population.

The methods and formulae of this paper feature a combinatorial coefficient for a partition of p which is the number of ways that the distinct units of p may be collected into distinct parcels described by the specified partition of p . For example, the combinatorial coefficient of the partition 22 is 3. The general formula for the combinatorial coefficient of the r part partition $p_1^{\pi_1} \dots p_s^{\pi_s}$ is

$$\frac{p!}{(p_1!)^{\pi_1} \dots (p_s!)^{\pi_s}} \cdot \frac{1}{\pi_1! \dots \pi_s!}$$

which features the s different parts of the partition rather than the r parts. In the analysis below it is the number of parts rather than the number of different parts which is important, so with $\sum_{i=1}^r p_i = p$ we use $(p_1, p_2, \dots, p_r)^*$ or $(p_1 p_2 \dots p_r)$ to indicate the partition coefficient. Thus

$$(2, 2, 1, 1) \text{ or } (2211) = \frac{6!}{2! 2! 1! 1!} \cdot \frac{1}{2!} \cdot \frac{1}{2!} = 45.$$

In this notation, the formula for cumulants in terms of moments [Kendall and Stuart, 1958, p. 70] appears as

$$\chi_p = \sum_{r=1}^p \sum (-1)^{r-1} (r-1)! (p_1 p_2 \dots p_r) \mu_{p_1} \mu_{p_2} \dots \mu_{p_r} \quad (2.4)$$

where the second summation applies to every r part partition of p .

*The (...) is adapted from a common notation for the binomial coefficient.

For two-part partitions, the partition coefficients are the binomial coefficients except when $p_1 = p_2$.

Following Fisher [1929, p. 203] we define k_p to be sample functions such that $E(k_p) = \chi_p$. Then (2.3) and (2.4) give

$$k_p = \sum (-1)^{r-1} (r-1)! (p_1 \cdots p_r) \langle p_1 \cdots p_r \rangle \quad (2.5)$$

where \sum holds for all partitions. Thus k_p is defined in terms of partition brackets, partition coefficients and number of partition parts. It is well-known [Fisher, 1929], [Kendall and Stuart, 1958, p. 282] that the value of k_p , $p > 1$ is independent of the choice of origin.

The multiple k -statistics* are designed to have the basic property of being estimators of products of cumulants [Dressel, 1940]. We follow Tukey [1956, p. 52] in defining them by a symbolic multiplication of the k 's in which conventional products of brackets are replaced by brackets enclosing the product factors. Thus

$$\begin{aligned} k_{22} &= k_2 \circ k_2 = [\langle 2 \rangle - \langle 11 \rangle] \circ [\langle 2 \rangle - \langle 11 \rangle] \\ &= \langle 22 \rangle - 2\langle 211 \rangle + \langle 1111 \rangle \end{aligned}$$

and in general using (2.5) with r_i , the number of parts of a partition of p_i

$$\begin{aligned} k_{p_1 p_2 \cdots} &= k_{p_1} \circ k_{p_2} \circ \cdots \\ &= \sum (-1)^{\sum (r_i - 1)} \prod (r_i - 1)! [(p_{11} \cdots p_{1r_1}) (p_{21} \cdots p_{2r_2}) \cdots] \langle p_{11} \cdots p_{1r_1} p_{21} \cdots p_{2r_2} \cdots \rangle \quad (2.6) \end{aligned}$$

where the first summation extends over all combinations of partitions of p_1 , partitions of p_2 , etc. This expresses the multiple k -statistic in terms of combinatorial coefficients and brackets and is essentially the same formula as that given by Wishart [1952a, p. 1]. It follows at once that

$$E_N(k_{p_1 p_2 \cdots}) = K_{p_1 p_2 \cdots}$$

*Also called generalized k -statistics [Wishart, 1952a], polykays [Tukey, 1956], and l -statistics [Kendall and Stuart, 1958].

so $k_{p_1 p_2 \dots}$ is an unbiased estimate of $K_{p_1 p_2 \dots}$. Also,

$$E[k_{p_1 p_2 \dots}] = \sum (-1)^{\sum (x_i - 1)} \prod (x_i - 1) [(p_{11} \dots p_{1x_1}) (p_{21} \dots p_{2x_2}) \dots] \mu_{p_1} \dots \mu_{p_{x_1}} \mu_{p_2} \dots \mu_{p_{2x_2}} \dots \quad (2.7)$$

and this is just $\chi_{p_1} \chi_{p_2} \dots$ by virtue of (2.4). So $k_{p_1 p_2 \dots}$ is an unbiased estimate of $\chi_{p_1} \chi_{p_2} \dots$.

An important property of $k_{p_1 p_2 \dots}$ is that it is independent of the location of the origin when each subscript is greater than unity [Tukey, 1956, p. 42].

3. THE DIRECT METHOD USING ARRAYS

The objective is to provide formulae for products of multiple k -statistics in terms of linear functions of such statistics. Then expectation or estimation formulae, with N either finite or infinite, are immediately available. The direct method consists in (a) expressing each multiple k -statistic as a linear function of brackets using (2.6), (b) multiplying out the brackets and (c) converting the resultant brackets to multiple k -statistics. For example, the first step in obtaining $k_{21} k_2$ is

$$\begin{aligned} k_{21} k_2 &= [\langle 21 \rangle - \langle 111 \rangle] [\langle 2 \rangle - \langle 11 \rangle] \\ &= \langle 21 \rangle \langle 2 \rangle - \langle 21 \rangle \langle 11 \rangle - \langle 111 \rangle \langle 2 \rangle + \langle 111 \rangle \langle 11 \rangle \end{aligned} \quad (3.1)$$

The second step consists in multiplying the brackets. Thus

$$\langle 21 \rangle \langle 2 \rangle = \frac{1}{n} \langle 41 \rangle + \frac{1}{n} \langle 23 \rangle + \frac{n-2}{n} \langle 212 \rangle$$

since

$$\left(\sum_{i \neq j}^n x_i^2 x_j \right) \left(\sum_k^n x_k^2 \right) = \sum_{i \neq j} x_i^4 x_j + \sum_{i \neq j} x_i^2 x_j^3 + \sum_{i \neq j \neq k} x_i^2 x_j x_k$$

as shown in the bracket or μ -coefficient row of Table I.

The n -coefficient of the bracket is obtained by dividing the number of terms in the bracket by the product of the numbers of terms in the brackets whose product is being formed. Thus the n -coefficient of $\langle 212 \rangle$ is $n(n-1)(n-2) \div [n(n-1)][n] = \frac{n-2}{n}$. Tukey has provided a table [1956, p. 47] to assist in this.

We propose the use of arrays, similar to those introduced by Fisher [1929] and used by Wishart [1952a], for indicating the results of the second step.

The general formula for the n -coefficient of any ρ -rowed array in the expansion of $k_{p_1 p_2 \dots} k_{q_1 q_2 \dots}$ is $n^{(\rho)} / n^{(r)} n^{(s)}$ where r is the total number of parts of the partitions of the p_i and s is the total number of parts of the partitions of the q_j .

The brackets from 21 are placed in the first column, those from 2 in the second, and all possible pairings of subscripts, except for permutations by rows, are recorded. Thus the terms in the product $\langle 21 \rangle \langle 2 \rangle$ are given by

$\begin{array}{c c} 22 & 4 \\ \hline 10 & 1 \end{array}$	$\begin{array}{c c} 20 & 2 \\ \hline 12 & 3 \end{array}$	$\begin{array}{c c} 20 & 2 \\ \hline 10 & 1 \\ \hline 02 & 2 \end{array}$
--	--	---

with n-coefficients of $1/n$, $1/n$, $(n-2)/n$, respectively, and with the resulting brackets indicated by the marginal column. The arrays resulting from (3.1) are shown in Table I. In general the arrays appear in the order indicated by the expansion of (3.1) except that, for convenience in the later steps, they are grouped according to the number of rows. The coefficient, obtained from (3.1), is shown in the row directly below the arrays. To avoid extensive repetition, equivalent arrays resulting from the permutations of the second column have been grouped together and a compensatory combinatorial coefficient supplied in the row so labeled. The n-coefficient is obtained in the manner described above or by Tukey's table [1956, p. 47]. The product of these coefficients is a coefficient of the bracket indicated by the marginal column. There may be more than one array leading to the same bracket.

The result after the first two steps of the direct method is

$$k_{2_1} k_2 = \frac{1}{n} \langle 41 \rangle + \frac{1}{n} \langle 32 \rangle - \frac{2}{n(n-1)} \langle 32 \rangle - \frac{2(n-2)}{n(n-1)} \langle 311 \rangle + \dots$$

These results are equivalent to those obtained by Tukey [1956, p. 47] by the direct algebraic method though, since he does not use arrays, the coefficient of a specified array cannot be identified in his results. In the direct method using arrays, we do not collect the coefficients of arrays having the same marginal partition (bracket) at this stage, but continue with the development by examining the contribution of each individual array.

The third step in the derivation calls for the expansion of the various brackets in terms of multiple k -statistics. Tables are available [Tukey, 1956, p. 44], [Abdel-Aty, 1954], [David and Kendall, 1949] for assisting in this. Instead of using these, we use the device [Kendall, 1952, p. 15] of introducing the parent cumulants and obtaining the final formula by estimation.

We can take the expectation of the results in terms of brackets by (2.3) to obtain equivalent results in terms of μ 's. In our example, for instance, the results become

$$E[k_{2_1} k_2] = \frac{1}{n} \mu_4 \mu_1 + \frac{1}{n} \mu_3 \mu_2 - \frac{2}{n(n-1)} \mu_3 \mu_2 - \frac{2(n-2)}{n(n-1)} \mu_3 \mu_1^2 + \dots$$

TABLE I

AN ILLUSTRATION OF THE DIRECT METHOD USING ARRAYS

$$k_{21}k_2 = [< 21 > - < 111 >][< 2 > - < 11 >] = < 21 > < 2 > - < 21 > < 11 > + < 111 > < 2 > + < 111 > < 11 >$$

Arrays	22 4 20 2 21 3	21 3 12 3 20 2 11 2	20 2 10 1 11 2 10 1	10 1 12 3 11 2 01 1 02 2 01 1 01 1					
Formula Coeff.	1	1	1	-1	1	-1	1	1	1
Comb. Coeff.	1	1	1	2	1	2	6	1	1
μ -Coeff.	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{n-2}{n(n-1)}$	$\frac{n-2}{n}$	$\frac{n-2}{n(n-1)}$	$\frac{1}{n(n-1)}$	$\frac{n-3}{n(n-1)}$	$\frac{(n-3)(n-4)}{n(n-1)}$
Bracket or μ -Coeff.	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{-2(n-2)}{n(n-1)}$	$\frac{n-2}{n}$	$\frac{-2(n-2)}{n(n-1)}$	$\frac{6}{n(n-1)}$	$\frac{-n-3}{n}$	$\frac{(n-3)(n-4)}{n(n-1)}$
Transformation Coefficient	1	1	1	2	1	2	2	1	1
				1	1	2	1	1	1
				1	1	1	2	3	1
				1			2	2	1
K , or μ -Coeff.	$\frac{1}{n}$	$\frac{1}{n}$	0	0	1	0	0	0	0

Then we can expand the μ 's in terms of the κ 's and take estimates of each side to obtain the desired formula. The real task is the one of changing from μ 's to κ 's.

In the direct method the usual formulae [Kendall and Stuart, 1958, p. 68] can be used for changing from μ 's to κ 's. In general, except for the most trivial cases, the algebra is extensive though it can be reduced somewhat by use of modulo unit parts [Tukey, 1956]. In adapting the transformation from μ 's to κ 's using arrays, we need to distinguish the two components of the μ 's corresponding to the two columns of the arrays. This can be done by using a multipartite notation. Thus the μ_4 of the first row of the first array of Table I is treated as the bipartite μ_{22} . Then the expansion in terms of bipartite κ 's is

$$\begin{aligned} \mu_{22} = & \kappa_{22} + 2 \kappa_{21} \kappa_{01} + 2 \kappa_{12} \kappa_{10} + \kappa_{20} \kappa_{02} + 2 \kappa_{11} \kappa_{11} + \kappa_{20} \kappa_{01} \kappa_{01} + \kappa_{02} \kappa_{10} \kappa_{10} \\ & + 4 \kappa_{11} \kappa_{10} \kappa_{01} + \kappa_{10} \kappa_{10} \kappa_{01} \kappa_{01}^* \end{aligned} \quad (3.2)$$

and the transform of $\mu_{22} \mu_{10}$ becomes the right side of (3.2) multiplied by κ_{10} . The paired subscripts identify the rows of the different arrays of Table I. The coefficients of (3.2) are then written in the row under the bracket coefficients. Similar transformation coefficients for the μ 's indicated by the other arrays are written in the rows below. Coefficients from the unipartite expansions are available for checking.

The calculation of the κ -coefficient corresponding to each array is then direct. We simply multiply the transformation coefficient (as indicated in each column) by the μ -coefficient (as indicated by the column of the diagonal term) and form the sum.

From the results in Table I we see that

$$E(k_2, k_2) = \frac{1}{n} \kappa_4 \kappa_1 + \frac{1}{n} \kappa_3 \kappa_2 - \frac{2}{n(n-1)} \kappa_3 \kappa_2 + \kappa_2 \kappa_2 \kappa_1 + \frac{2}{n-1} \kappa_2 \kappa_2 \kappa_1$$

*This expansion may be obtained from the multipartite

$$\mu_{1111} = \kappa_{1111} + \kappa_{1110} \kappa_{0001} + \kappa_{1101} \kappa_{0010} + \kappa_{1011} \kappa_{0100} + \kappa_{0111} \kappa_{1000} + \kappa_{1100} \kappa_{0011} + \dots$$

by combining the first pair of subscripts to form a new first subscript and the second pair of subscripts to form a new second subscript.

so that, taking estimates, we have

$$k_{21} k_2 = \frac{1}{n} k_{41} + \frac{1}{n} k_{32} - \frac{2}{n(n-1)} k_{32} + k_{221} + \frac{2}{n-1} k_{221} \quad (3.3)$$

Wishart [1952a] and Tukey [1956] have written the result in the form

$$k_{21} k_2 = \frac{1}{n} k_{41} + \left(\frac{1}{n} - \frac{2}{n(n-1)} \right) k_{32} + \left(1 + \frac{2}{n-1} \right) k_{221}$$

Formula (3.3) is in good form for approximation with large n and it does show the contribution of each array.

The illustration in Table I presents the rather complete details of the direct method using arrays. The establishment of general principles which are applicable to the contributions of a given array makes possible considerable further condensation and the development of a true combinatorial method.

4. FURTHER CONDENSATION OF THE DIRECT METHOD

Examination of Table I shows that the k -coefficient of every array having a marginal partition with two or more unit parts is zero. This is in agreement with Tukey's result [1956] which states that the coefficient of every k ... having more unit subscripts than the set of original subscripts is zero. We now use Tukey's result to condense the results of Table I. The process is in a sense equivalent to Tukey's algebraic direct development in which he used modulo unit parts [1956, p. 43]. The amount of work is greatly shortened for this problem in that no entry in any column corresponding to an array with more than one marginal unit subscript need be computed. The condensed calculation is shown in Table II.

TABLE II

CALCULATION OF $k_{2,1}k_2$ USING ARRAYS

(Marginal Partitions With More Than One Unit Subscript Omitted)

$$k_{2,1}k_2 = [\langle 21 \rangle - \langle 111 \rangle] [\langle 2 \rangle - \langle 11 \rangle]$$

Array	22 4 10 1	20 2 12 3	21 3 11 2	20 2 10 1 02 2	20 2 11 2 01 1	11 2 11 2 10 1
Formula Coeff.	1	1	-1	1	-1	1
Comb. Coeff.	1	1	2	1	2	6
n-Coeff.	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n(n-1)}$	$\frac{n-2}{n}$	$\frac{n-2}{n(n-1)}$	$\frac{1}{n(n-1)}$
μ -Coeff.	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{-2}{n(n-1)}$	$\frac{n-2}{n}$	$\frac{-2(n-2)}{n(n-1)}$	$\frac{6}{n(n-1)}$
Separations Coefficient	1			1		2
		1		1	2	
			1		1	2
				1		
					1	1
k -Coeff.	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{-2}{n(n-1)}$	1	0	$\frac{2}{n-1}$

The results are those of Table I. It can be seen from Table I, and it is very noticeable in Table II, that the k -coefficient of the $\frac{20}{11}$ term is

also zero. This fact is not covered by Tukey's rule but it does follow from our rule of proper parts, established in Section 8, that the coefficient is zero for any array which has at least one row with a simple non-zero element which is a proper part* of a partition of some integer subscript. Since the 1 in the third row of the partition above is a proper part of the 2, the coefficient is zero and the array need not be considered.

Tukey's rule as applied to array coefficients follows as a corollary of this general rule since any additional unit subscript must come from a row with a single proper unit part. In more general problems, the condensation resulting from the application of rule of proper parts is extensive.

The array $\frac{11}{10}$, though having a combinatorial coefficient of 6, has a k -coefficient¹⁰ of only $2/(n-1)$ because only two of the six arrays, resulting from the matching of the unit parts of 2 in 2 with unit parts of 21, do not have rows with unit proper parts. Thus the six arrays obtained by permuting the unit elements of the second column are not really equivalent. A true combinatorial method in which each array makes a fixed contribution to the total result needs a more precise notation and appropriate modifications.

*A proper part of a partition of an integer is any positive integral value less than the integer.

5. BACKGROUND FOR A COMBINATORIAL METHOD

We first introduce a different terminology for the transformation coefficients. The various arrays which are obtained by separating the elements of the rows of a given array into more rows are called the separations [Fisher, 1929] of the given array. The transformation coefficient is then the number of ways the elements of the rows of a given array may be separated to form a new array. This number may be called a separations coefficient as indicated in Table II. Alternately the columns of Table I (and Table II) show all the arrays which have a given array as a separation together with the number of ways the separation can be made. The arrays represented by the columnar entries may be called amalgamations which are obtained by adding the rows of the given array in all possible ways. More specifically the arrays of Table I and Table II may be called conditional amalgamations since the rows may be amalgamated only to form arrays which are possible in accord with the conditions of the problem. In the expansion of $k_{2,1} k_2$, for example, no 3 may appear so it is possible to amalgamate the rows in all possible ways except that no 3 may appear in the first column. These are just the amalgamations revealed by Table I and Table II.

A significant part of the work deals with the calculation of the separations coefficient to be associated with each amalgamation. The calculation of this coefficient can be eliminated if the arrays are so defined that there is just one way in which one array can be separated to form another specific array. This can be achieved by considering elements of arrays composed of distinct units. Thus in $k_{2,1}$ we let e_1, e_2 be the unit elements of 2 and e_3 the unit element 1. In k_2 we use e_4, e_5 to be the unit elements of 2. Then the array $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$, with combinatorial coefficient 2, represents either of the arrays

$$\begin{array}{cc} e_1 + e_2 & e_4 \\ e_3 & e_5 \end{array} \quad \text{or} \quad \begin{array}{cc} e_1 + e_2 & e_5 \\ e_3 & e_4 \end{array}$$

Each of these arrays is unique and can be separated to form another unique array in only one way. Hence every non-zero separations coefficient is 1. The number of distinct arrays in Table II (the sum of the combinatorial coefficients) neglecting the penultimate array (which has a coefficient zero) is 11. Now four of the last six have a coefficient of zero since the proper part $e_1, e_2, e_4,$ or e_5 appears alone in a row. Hence there are but 7 arrays with distinct units having non-zero coefficients. These are shown in Table III. The algebra is simple since each conditional amalgamation has unity as coefficient. The results of equivalent arrays of distinct units are then collected to obtain the previous results.

TABLE III

k_1, k_2 WITH USE OF DISTINCT UNITS

	e_1e_2	e_4e_5	e_1e_2	e_4	e_1e_2	e_4	e_1e_2	e_5	e_1e_2	e_4	e_1e_2	e_5
Array	e_3	0	e_3	e_4e_5	e_3	e_5	e_3	e_4	e_3	0	e_2	e_4
	e_3	0	e_4e_5	e_3	e_5	e_3	e_4	e_3	0	e_4e_5	e_3	0
Formula Coeff.	1	-1	1	-1	-1					1	1	1
n-Coeff.	$\frac{1}{n}$	$\frac{1}{n(n-1)}$	$\frac{1}{n}$	$\frac{1}{n(n-1)}$	$\frac{1}{n(n-1)}$					$\frac{n-2}{n}$	$\frac{1}{n(n-1)}$	$\frac{1}{n(n-1)}$
k -Coeff.	$\frac{1}{n}$	$-\frac{1}{n(n-1)}$	$\frac{1}{n}$	$-\frac{1}{n(n-1)}$	$-\frac{1}{n(n-1)}$					$\frac{n-2}{n}$	$\frac{1}{n(n-1)}$	$\frac{1}{n(n-1)}$
Separations Coefficient	1		1	1	1					1	1	1
k -Coefficient (distinct units)	$\frac{1}{n}$	$-\frac{1}{n(n-1)}$	$\frac{1}{n}$	$-\frac{1}{n(n-1)}$	$-\frac{1}{n(n-1)}$					1	$\frac{1}{n-1}$	$\frac{1}{n-1}$

With permutations of the elements of the second column, the first two arrays, the next two arrays, and the last two arrays constitute three distinct array types with the same k -coefficients. Then the condensed direct method using arrays applied to the four array types gives

$$k_{21} k_2 = \frac{1}{n} k_{21+20} - \frac{1}{n(n-1)} k_{21+11} + k_{221} + \frac{1}{n-1} k_{11;1+11;0}$$

where the + in the subscript indicates the sum of the k 's with sums of permuted subscripts prior to the; . Thus

$$k_{21+20} = k_{41} + k_{23} , \quad k_{21+11} = 2k_{32} , \quad k_{11;1+11;0} = 2k_{221} .$$

With suitable notation it is not much more difficult to obtain a formula for $k_{p_1 p_2} k_2$ with this condensed direct method using arrays. We have

$$k_{p_1 p_2} k_2 = [\langle p_1 p_2 \rangle - \sum (p_{11} p_{12}) \langle p_{11} p_{12} p_2 \rangle - \sum (p_{21} p_{22}) \langle p_1 p_{21} p_{22} \rangle + \dots] [\langle 2 \rangle - \langle 11 \rangle] .$$

Using the rule of proper parts, the only bracket products producing array types with non-vanishing coefficients are given by

$$k_{p_1 p_2} k_2 = \langle p_1 p_2 \rangle \langle 2 \rangle - \langle p_1 p_2 \rangle \langle 11 \rangle + \sum (p_{11} p_{12}) \langle p_{11} p_{12} p_2 \rangle \langle 11 \rangle + \sum (p_{21} p_{22}) \langle p_1 p_{21} p_{22} \rangle \langle 11 \rangle .$$

This can be written

$$k_{p_1 p_2} k_2 = \langle p_1 p_2 \rangle \langle 2 \rangle - \langle p_1 p_2 \rangle \langle 11 \rangle + \sum T \langle p_{11} p_{12} p_2 \rangle \langle 11 \rangle + \sum T \langle p_1 p_{21} p_{22} \rangle \langle 11 \rangle$$

where $T \langle p_{11} p_{12} p_2 \rangle$ symbolizes $(p_{11} p_{12})$ equivalent brackets. With this method, the algebraic coefficient for each array type is determined and then multiplied by the type coefficient.

The separations (amalgamations) coefficient need not be recorded since it is unity. We now absorb the sign and the factorials in the n-coefficient and have the general formula

$$\text{n-coefficient} = \frac{\prod (-1)^{i-1} (r_i - 1)! \prod (-1)^{j-1} (\Delta_j - 1)!}{n^{(r)} n^{(\Delta)}} \cdot n^{(P)} \quad (5.1)$$

The algebraic coefficient of the array type is obtained by adding the n-coefficients of the array type and all its conditional amalgamations. Thus the algebraic coefficient of

$$\begin{matrix} p_{11} & 1 \\ p_{12} & 1 \\ p_2 & 0 \end{matrix}, \text{ with } \begin{matrix} p_1 & 2 \\ p_2 & 0 \end{matrix} \text{ the only conditional amalgamation,}$$

is $\frac{1}{n(n-1)} + \frac{1}{n} = \frac{1}{n-1}$.

TABLE IV

CONDENSED DIRECT METHOD FOR $k_{p_1 p_2} k_2$

Array Type	$p_1 \ 2$ $p_2 \ 0$	$p_1 \ 0$ $p_2 \ 0$ $0 \ 2$	$p_1 \ 1$ $p_2 \ 1$	$p_{11} \ 1$ $p_{12} \ 1$ $p_2 \ 0$	$p_1 \ 0$ $p_{21} \ 1$ $p_{22} \ 1$
Alg. Coeff.	$\frac{1}{n}$	1	$\frac{-1}{n(n-1)}$	$\frac{1}{n-1}$	$\frac{1}{n-1}$
Comb. Coeff.	1	1	1	$(p_{11} p_{12})$	$(p_{21} p_{22})$
k-Coeff.	$\frac{1}{n}$	1	$\frac{-1}{n(n-1)}$	$\frac{(p_{11} p_{12})}{n-1}$	$\frac{(p_{21} p_{22})}{n-1}$

The result in terms of the k 's associated with array types is then

$$\begin{aligned} k_{p_1 p_2} k_2 &= \frac{1}{n} k_{p_1 p_2 + 20} + k_{p_1 p_2 2} - \frac{1}{n(n-1)} k_{p_1 p_2 + 11} + \frac{1}{n-1} \sum (p_{11} p_{12}) k_{p_{11} p_{12} p_2 + 11; 0} \\ &+ \frac{1}{n-1} \sum (p_{21} p_{22}) k_{p_{21} p_{22}; p_1 + 11; 0} \end{aligned}$$

and the more explicit formula is

$$k_{p_1 p_2} k_2 = \frac{1}{n} k_{p_1+2, p_2} + \frac{1}{n} k_{p_1, p_2+2} + k_{p_1 p_2 2} - \frac{2}{n(n-1)} k_{p_1+1, p_2+1} \\ + \frac{2}{n-1} \sum (p_{11} p_{12}) k_{p_{11}+1, p_{12}+1, p_2} + \frac{2}{n-1} \sum (p_{21} p_{22}) k_{p_{21}+1, p_{22}+1, p_1} .$$

Values of p_1 and p_2 can be substituted. Some collection of the resultant terms is commonly possible. Thus with $p_1 = 4$ and $p_2 = 2$,

$$k_{42} k_2 = k_{422} + \frac{1}{n} k_{62} + \frac{1}{n} k_{44} - \frac{2}{n(n-1)} k_{53} + \frac{2}{n-1} [4k_{422} + 3k_{322}] \\ + \frac{2}{n-1} k_{422} \\ = \frac{1}{n} k_{62} + \frac{1}{n} k_{44} - \frac{2}{n(n-1)} k_{53} + \frac{6}{n-1} k_{322} + \left(1 + \frac{10}{n-1}\right) k_{422} .$$

No restriction has been placed on p_1 or p_2 (except, of course, that they are positive integers). The formula simplifies somewhat when one of them, say p_2 , is 1, since 1 has no 2-part partitions and the last term vanishes. The formula is even applicable to the case with $p_2 = 0$ if we drop the p_2 from all terms containing p_2 as a subscript, and drop all terms containing other functions of p_2 as subscripts. Thus

$$k_{p_1} k_2 = k_{p_1 2} + \frac{1}{n} k_{p_1+2} + \frac{2}{n-1} \sum (p_{11} p_{12}) k_{p_{11}+1, p_{12}+1} .$$

6. SOME GENERAL THEORY

Using bracket types we can write (2.6) with $p_1, p_2, \dots > 0$ as

$$k_{p_1, p_2, \dots} = \sum (-1)^{\sum (r_i - 1)} \prod (r_i - 1)! T \langle p_1, \dots, p_1 | \dots | p_2, \dots \rangle \\ = \sum f(r_i) T \langle p_{11}, \dots, p_{1r_1} | p_{21}, \dots, p_{2r_2}, \dots \rangle$$

Similarly,

$$k_{q_1, q_2, \dots} = \sum f(s_j) T \langle q_{11}, \dots, q_{1s_1} | q_{21}, \dots, q_{2s_2}, \dots \rangle$$

so,

$$k_{p_1, p_2, \dots} k_{q_1, q_2, \dots} = \sum f(r_i) f(s_j) T \langle p_{11}, \dots, p_{1r_1} | p_{21}, \dots, p_{2r_2}, \dots \rangle T \langle q_{11}, \dots, q_{1s_1} | q_{21}, \dots, q_{2s_2}, \dots \rangle \quad (6.1)$$

Array types based on (6.1) are used. The algebraic coefficient of each one of them must be multiplied by the product of the partition coefficients,

$$(p_{11}, \dots, p_{1r_1}) (p_{21}, \dots, p_{2r_2}) \dots (q_{11}, \dots, q_{1s_1}) (q_{21}, \dots, q_{2s_2}) \dots$$

to obtain the k -coefficient of the array type.

Formula (6.1) gives the product expansion in terms of every possible array type which results from the partitions of p_i and q_j . Moreover the formula coefficient, aside from $f(r_i) f(s_j)$, is unity for every array type in the complete expansion. The values of $f(r_i) f(s_j)$ are absorbed in the n -coefficient in calculating the algebraic coefficient.

Now consider any array type with $\sum r_i p_i$ in the first column and $\sum s_j q_j$ in the second. Suppose there are f rows in the array type. Then the n -coefficient for the array type, with the $f(r_i) f(s_j)$ absorbed, is (see (5.1))

$$\text{n-coefficient} = \frac{(-1)^{\sum(\tau_i - 1) + \sum(\Delta_j - 1)} \prod (\tau_i - 1)! \prod (\Delta_j - 1)! n^{(p)}}{n^{\sum \tau_i} n^{\sum \Delta_j}} \quad (6.2)$$

It is apparent at once that this modified n-coefficient is a function of the τ 's and Δ 's and not of the individual τ 's and Δ 's. This means that different numerical values can be substituted for the various τ 's and Δ 's without changing the n-coefficient as long as the pattern is not changed. Some important rules follow.

RULE I. RULE FOR ALGEBRAIC COEFFICIENT OF ARRAY TYPE

The algebraic coefficient of the array type is obtained by adding the n-coefficient of the array type and those of all its conditional amalgamations. This rule results from the facts that (a) every conditional amalgamation of each array type appears in (6.1) with coefficient of 1, aside from $f(\tau_i) f(\Delta_j)$, and (b) the expansion of each array type in terms of its separations has unit coefficients as illustrated by Table III using distinct units.

RULE II. PATTERN RULE

Array types with the same pattern have the same algebraic coefficient no matter what the values of the original τ 's and Δ 's. Array types have the same pattern when the various groups of partition parts correspond in location. Thus the array types

$$\begin{array}{ll} \tau_{11} = 3 & \Delta_{11} = 2 \\ \tau_{12} = 2 & \Delta_{12} = 2 \\ \tau_2 = 1 & \Delta_2 = 2 \end{array} \quad \text{and} \quad \begin{array}{ll} \tau_{11} = 4 & \Delta_{11} = 3 \\ \tau_{12} = 1 & \Delta_{12} = 1 \\ \tau_2 = 1 & \Delta_2 = 2 \end{array}$$

in the expansion of $k_{51} k_{42}$ and the array type

$$\begin{array}{ll} \tau_{11} = 5 & \Delta_{11} = 4 \\ \tau_{12} = 4 & \Delta_{12} = 3 \\ \tau_2 = 8 & \Delta_2 = 2 \end{array}$$

in the expansion of $k_{98} k_{72}$ have the same pattern. The algebraic coefficient is $1/n(n-2)$ in each case since the only conditional amalgamation results from adding the first two rows.

Rule II follows immediately from Rule I.

RULE III. RULE OF PROPER PARTS

The algebraic coefficient is zero for every array type in which there is at least one row in which a proper part appears alone.

By the pattern rule, the algebraic coefficient is the same no matter what the values of p_i and q_j and the partition elements in the pattern are. Consider the case in which p_i and q_j are all greater than 1 and take the proper part appearing alone in a row to be unity. Then each of the $k...$ terms arising from the array type has a unit subscript. Since the product expansion with $p_i > 1$, $q_j > 1$ does not have any $k...$ term with unit subscript, the k -coefficient must be zero.* Now the combinatorial coefficients are not zero, so it follows that the algebraic coefficient for the array type is zero.

These rules provide the basis of the combinatorial method for the products of two multiple k -statistics as developed in the sections below. The theory of this section is immediately extendable to multiple products of multiple k -statistics, and the rules stated above hold for general products.

As suggested above, the rule of proper parts, as applied to arrays, gives Tukey's rule as a special case since any unit subscript over and above the unit subscripts of the original set must result from the appearance of a unit proper part as a single element of a row. But the rule of proper parts is much more general, as it commonly eliminates many more array types.

The theory and rules above are in a sense generalizations of some of the Wishart (modified Fisher) rules for arrays of k_1, k_2, \dots in which partitions are replaced by x 's since only partitions of p can occur in the first column, partitions of q in the second, etc. Thus all γ_i vanish except γ_1 and all δ_j except δ_1 and the general rule for n -coefficient becomes

$$n\text{-coefficient} = \frac{(-1)^{(\gamma_1-1)+(\delta_1-1)} (\gamma_1-1)! (\delta_1-1)! \dots n^{(p)}}{n^{(\gamma_1)} n^{(\delta_1)} \dots} \quad (6.3)$$

All conditional amalgamations become amalgamations and the rule of proper parts becomes "The coefficient is zero for every array in which an x is the only entry in the row but not the only one in the column." This rule appears to be more general than the Wishart recommendation [1952a, p. 4] which is somewhat vague but explicitly applies to patterns with single unit proper parts.

*Since the algebraic coefficient is the same for a multiple infinity of values of p_i and q_j , it follows that the contribution of every array is in general unique so that the coefficient of every k with unit subscript must vanish.

7. THE STEPS OF THE GENERAL METHOD WITH ILLUSTRATION

The general method can now be stated.

1. Write each multiple k -statistic of the product in terms of bracket types as indicated by (6.1).

2. List all possible arrangements of the products of bracket types in which the bracket type components of the first factor are placed in the first column, those of the second factor in the second column, etc., to form the array types. In so doing ignore any array type which has a proper part as a single non-zero element of a row.

3. Compute the combinatorial coefficient for the array type by forming the product of all combinatorial coefficients associated with every partition appearing in the array type.

4. Compute the algebraic coefficient for each array type as indicated in the previous section.

5. Multiply the algebraic coefficient by the combinatorial coefficient to obtain the k -coefficient for each array type. The listing of the coefficient in the column for the k -term gives the result in combinatorial form. More explicitly,

6. Write the formula for the sums of the products of the k -coefficients and the k -terms.

If still more explicit form is desired,

7. Expand each of the k -terms to feature explicit k 's.

As an illustration we apply the method to $k_1 k_3$. The first five steps are shown in Table V.

TABLE V

COMBINATORIAL METHOD FOR $k_{p_1 p_2 k_3}$

$k_{p_1 p_2} = \langle p_1 p_2 \rangle - \sum T \langle p_{11} p_{12} p_2 \rangle - \sum T \langle p_1 p_2 p_{21} p_{22} \rangle + \sum T \langle p_{11} p_{12} p_{21} p_{22} \rangle + \dots$ $k_3 = \langle 3 \rangle - \sum T \langle 21 \rangle + 2T \langle 111 \rangle$										
(1) Array Type	p_1 3	p_1 0	p_1 2	p_{11} 2	p_1 0	p_{11} 1	p_1 1	p_{11} 1	p_{11} 1	p_1 0
	p_2 0	p_2 0	p_2 1	p_{12} 1	p_{21} 2	p_{12} 1	p_{21} 1	p_{12} 1	p_{21} 1	p_{21} 1
		0 3		p_2 0	p_{22} 1	p_2 1	p_{22} 1	p_{13} 1	p_{22} 1	p_{23} 1
(2) Comb. Coeff.	1	1	3	$3(p_{11} p_{12})$	$3(p_{21} p_{22})$	$(p_{11} p_{12})$	$(p_{21} p_{22})$	$(p_{11} p_{12} p_{13})$	$(p_{21} p_{22} p_{23})$	
(3) Alg. Coeff.	$\frac{1}{n}$	1	$\frac{-1}{n(n-1)}$	$\frac{1}{n-1}$	$\frac{1}{n-1}$	$\frac{-1}{(n-1)(n-2)}$	$\frac{-1}{(n-1)(n-2)}$	$\frac{n}{(n-1)(n-2)}$	$\frac{n}{(n-1)(n-2)}$	
(4) k -Coeff.	$\frac{1}{n}$	1	$\frac{-3}{n(n-1)}$	$\frac{3(p_{11} p_{12})}{n-1}$	$\frac{3(p_{21} p_{22})}{n-1}$	$\frac{-(p_{11} p_{12})}{(n-1)(n-2)}$	$\frac{-(p_{21} p_{22})}{(n-1)(n-2)}$	$\frac{n(p_{11} p_{12} p_{13})}{(n-1)(n-2)}$	$\frac{n(p_{21} p_{22} p_{23})}{(n-1)(n-2)}$	

Step 6 gives us

$$\begin{aligned}
k_{p_1 p_2} k_3 &= k_{p_1 p_2 3} + \frac{1}{n} k_{p_1 p_2 + 3 0} - \frac{3}{n(n-1)} k_{p_1 p_2 + 2 1} \\
&+ \frac{3}{n-1} \sum (p_{11} p_{12}) k_{p_{11} p_{12}; p_2 + 2 1; 0} + \frac{3}{n-1} \sum (p_{21} p_{22}) k_{p_{21} p_{22}; p_1 + 2 1; 0} \\
&- \frac{1}{(n-1)(n-2)} \sum (p_{11} p_{12}) k_{p_{11} p_{12} p_2 + 1 1 1} - \frac{1}{(n-1)(n-2)} \sum (p_{21} p_{22}) k_{p_1 p_{21} p_{22} + 1 1 1} \\
&+ \frac{n}{(n-1)(n-2)} \sum (p_{11} p_{12} p_{13}) k_{p_{11} p_{12} p_{13}; p_2 + 1 1 1; 0} \\
&+ \frac{n}{(n-1)(n-2)} \sum (p_{21} p_{22} p_{23}) k_{p_{21} p_{22} p_{23}; p_1 + 1 1 1; 0} \quad ,
\end{aligned}$$

and Step 7 gives

$$\begin{aligned}
k_{p_1 p_2} k_3 &= k_{p_1 p_2 3} + \frac{1}{n} k_{p_1 + 3, p_2} + \frac{1}{n} k_{p_1, p_2 + 3} \\
&- \frac{3}{n(n-1)} k_{p_1 + 2, p_2 + 1} - \frac{3}{n(n-1)} k_{p_1 + 1, p_2 + 2} \\
&+ \frac{3}{n-1} \sum (p_{11} p_{12}) k_{p_{11} + 2, p_{12} + 1, p_2} + \frac{3}{n-1} \sum (p_{11} p_{12}) k_{p_{11} + 1, p_{12} + 2, p_2} \\
&+ \frac{3}{n-1} \sum (p_{21} p_{22}) k_{p_1, p_{21} + 2, p_{22} + 1} + \frac{3}{n-1} \sum (p_{21} p_{22}) k_{p_1, p_{21} + 1, p_{22} + 2} \\
&- \frac{6}{(n-1)(n-2)} \sum (p_{11} p_{12}) k_{p_{11} + 1, p_{12} + 1, p_2 + 1} - \frac{6}{(n-1)(n-2)} \sum (p_{21} p_{22}) k_{p_1 + 1, p_{21} + 1, p_{22} + 1} \\
&+ \frac{6n}{(n-1)(n-2)} \sum (p_{11} p_{12} p_{13}) k_{p_{11} + 1, p_{12} + 1, p_{13} + 1, p_2} \\
&+ \frac{6n}{(n-1)(n-2)} \sum (p_{21} p_{22} p_{23}) k_{p_1, p_{21} + 1, p_{22} + 1, p_{23} + 1} \quad .
\end{aligned}$$

As a special case consider the value of $k_{22}k_3$ which is needed in the estimation of $M(32^2) = E_N \{ (k_3 - K_3)(k_2 - K_2)^2 \}$. We get at once

$$k_{22}k_3 = k_{322} + \frac{2}{n} k_{52} - \frac{6}{n(n-1)} k_{43} + \frac{12}{n-1} k_{322} - \frac{12}{(n-1)(n-2)} k_{322}.$$

A satisfactory combinatorial method needs rules which give the algebraic coefficient so that it need not be computed anew for each array type. The remainder of this paper is devoted, for the case of products of two factors, (a) to the determination of the algebraic coefficients for groups of array types, and (b) to the listing of general formulae resulting from the application of this combinatorial method.

8. ALGEBRAIC COEFFICIENTS

A useful rule is the rule of extended array types. We define an extended array type to be one which consists of an initial array type plus additional rows in which elements are ϕ 's (but not proper parts of ϕ 's) matched with zeros, or q 's matched with zeros. Then we have:

RULE IV. RULE OF EXTENDED ARRAY TYPES

The algebraic coefficient of an extended array type is equal to that of the initial array type.

This rule is very useful in appropriate cases in simplifying the calculation of the algebraic coefficient since one may cross out any row which has a ϕ term or a q term (which is not a proper part) in a row with a zero. For example, the algebraic coefficient of

$$\begin{array}{cc} \phi_1 & q_1 \\ \phi_2 & q_2 \\ \phi_2 & 0 \\ 0 & q_2 \end{array} \text{ is that of } \begin{array}{cc} \phi_1 & q_1 \\ \phi_2 & q_2 \end{array}$$

$1/n(n-1)+1/n = 1/(n-1)$. This rule should be applied after application of the rule of proper parts which eliminates all array types having a single entry in a row which is a proper part.

Suppose there are R rows of the initial array type with both ϕ and q entries, S with q entries only, and T with ϕ entries only. If the contribution to the n -coefficient of the signs and factorials is indicated by C , the value of the n -coefficient is

$$C \cdot \frac{1}{n^{(R+T)}} \cdot \frac{1}{n^{(R+S)}} \cdot n^{(R+S+T)}$$

Consider an extended array type which results from the addition of the row $\phi_i, 0$. Then the contribution of the conditional amalgamations resulting from adding the new row to the initial ones in all possible ways, but not as yet involving any amalgamations of the initial rows, is

$$C \cdot \frac{1}{n^{(R+T+1)}} \cdot \frac{1}{n^{(R+S)}} \cdot n^{(R+S+T+1)} + C S \cdot \frac{1}{n^{(R+T+1)}} \cdot \frac{1}{n^{(R+S)}} \cdot n^{(R+S+T)}$$

which reduces to the value of the n -coefficient of the initial array type indicated above. Since this equality describes the relationship between every conditional amalgamation of the initial array type and the corresponding contribution of the augmented array type, it follows that the two algebraic coefficients are the same. A similar argument holds for the addition of the row $0, q_j$. The result is immediately extendable to the case of more than one row.

As a result of the application of the rule of proper parts which eliminates many array types having one or more zeros, and of the rule of extended array types which makes possible the elimination of all rows having zero entries from the remaining array types, we need concern ourselves only with array types having no zero entry which we call reduced array types. Thus the only array types needed in applying the combinatorial method to

$k_{p_1 p_2} k_3$ (see Table V) are

$$p_1 \ 3, \quad p_1 \ 2, \quad p_{11} \ 2, \quad p_{21} \ 2,$$

$$p_2 \ 1, \quad p_{12} \ 1, \quad p_{22} \ 1$$

$$p_{11} \ 1, \quad p_1 \ 1, \quad p_{11} \ 1, \quad p_{21} \ 1$$

$$p_{12} \ 1, \quad p_{21} \ 1, \quad p_{12} \ 1, \quad p_{22} \ 1$$

$$p_2 \ 1, \quad p_{22} \ 1, \quad p_{13} \ 1, \quad p_{23} \ 1$$

(8.1)

whose algebraic coefficients are easily computed. The one possible exception is the array type which has a zero in every row. The algebraic coefficient of this array type is 1 as may be seen by eliminating all rows except the first. The algebraic coefficient is then the n-coefficient which is $(1/n)(n) = 1$. The term Case I is used to identify array types with a single non-zero entry in each row.

Additional notation is useful in expressing general formulae for the algebraic coefficients. We collect all the rows which cannot be added to any row, whether they contain proper parts of the p 's and q 's or not, at the lower part of the reduced array type and indicate this collection of rows by AA (or A). Thus the reduced array type

$$p_{11} \ q_{11}$$

$$p_{12} \ q_{12}$$

$$p_{13} \ q_2$$

$$p_{21} \ q_3$$

$$p_{22} \ q_4$$
(8.2)

can be written $p_{11} \ q_{11}$, where A consists of $a = 3$ rows.

$$p_{12} \ q_{12}$$

$$A \quad A$$

Now since no row of A can be amalgamated with any other row of the reduced array type, the signed factorial contribution to every conditional amalgamation (of the rows above A), and hence to the algebraic coefficient itself,

is the same. This is denoted by C_A and is the product of the signed factorials in A excluding all parts of partitions which also have parts in the rows above A. Thus in the illustration above, $C_A = -1$ since the sign of p_{13} is related to the other parts above and since the last two elements in the first column are parts of a two part partition.

The notation is general enough to include $\alpha = 0$ in which case $C_A = 1$. C_A is also 1 when no partitions of new p_i or q_j appear in A or when the partitions are two-part partitions and there is an even number of them.

Case II is used to denote any reduced array type in which no conditional amalgamation is possible. All the rows are included in A. The algebraic coefficient is the n-coefficient. Thus the algebraic coefficient of

$$\begin{array}{cc} p_{11} & q_{11} \\ p_{12} & q_{21} \\ p_{21} & q_{12} \\ p_{22} & q_{22} \end{array} \quad \text{is} \quad C_A \cdot \frac{1}{n^{(4)}} = \frac{1}{n^{(4)}} .$$

If A does not include any partitions of p_i or q_j , or if only even numbers of two-part partitions appear, as in the illustration, the algebraic coefficient is $1/n^{(a)}$.

In developing notation for more complex situations we let r_i be the number of parts of a partition of p_i appearing above A and R_i be the number of parts of p_i in the (reduced) array type. Similarly s_j is the number of parts of q_j appearing above A and S_j the number of parts of q_j appearing in the reduced array type. Thus in the illustration (8.2), $r_1 = 2$, $R_1 = 3$, $s_1 = 2$, $S_1 = 2$, $R_2 = 2$, $S_2 = 1$, $S_3 = 1$, $S_4 = 1$. Once the values of C_A and α are determined we need pay no more attention to the elements of A in determining the algebraic coefficient. Specification of these r 's and s 's enables us to identify extensive groups of reduced array types which are covered by a general formula for the algebraic coefficient. Thus $r_1 = R_1 = p$, $s_1 = S_1 = p$ and $\alpha = 0$ identify the p row, two column patterns of Fisher [1929].

Further β_1 and β_2 can be used to indicate groups of rows within which amalgamations can be made. Thus $r_1 = s_1 = \beta_1$, $r_2 = s_2 = \beta_2$, $\alpha > 0$ indicate that there is a group of β_1 rows which can be amalgamated with each other, a group of β_2 rows which can be similarly amalgamated, and a group of α rows which cannot be amalgamated. If also $R_1 = r_1$ and $S_1 = s_1$ then there are no parts of p_1 , nor parts of q_1 , in A. Of course if $\alpha = 0$, then $R_1 = r_1$, $S_1 = s_1$ and $C_A = 1$.

General algebraic coefficients are given for the cases indicated below. The derivation features the adding of the combinatorial coefficients for partitions having the same number of parts and transforming to (advancing) dif-

ferences of zero. Thus

$$(31) + (22) = 4+3 = 7 = \frac{1}{2!} \Delta^2(0^4).$$

Case III.

$$r_1 = s_1 = p \geq 2$$

$$\text{Algebraic coefficient} = C_A \sum_{u=1}^p \frac{(R_1 - p - 1 + u)! (S_1 - p - 1 + u)!}{n^{(a+u)}} \cdot \frac{\Delta^u(0^p)}{u!}. \quad (8.3)$$

This formula is very general and special cases of it are useful in practice. For example it covers the last six of the nine reduced array types of Table V (see (8.1)). The three others are taken care of by the rules for Case I and Case II. Of frequent occurrence is the case in which $r_1 = s_1 = p = 2$. The more explicit formula is then

$$\text{Alg. Coeff.} = C_A (R_1 - 2)! (S_1 - 2)! \left[\frac{(n-a) + R_1 S_1 - R_1 - S_1}{n^{(a+2)}} \right]. \quad (8.4)$$

This explicit formula covers four of the nine reduced array types of (8.1).

If in addition we have $R_1 = S_1 = 2$, then

$$\text{Alg. Coeff.} = \frac{C_A (n-a)}{n^{(a+2)}} \quad (8.5)$$

with obvious further simplification when $C_A = 1$ and when $a = 0$.

When $r_1 = s_1 = p = 3$, the more explicit formula is

$$\text{Alg. Coeff.} = C_A (R_1 - 3)! (S_1 - 3)! \left[\frac{(R_1 - 2)^{[2]} (S_1 - 2)^{[2]} + 3(R_1 - 2)(S_1 - 2)(n-a-2) + (n-a-2)^{[2]}}{n^{(a+3)}} \right]. \quad (8.6)$$

If in addition $R_1 = r_1 = S_1 = s_1 = 3$, we have

$$\text{Alg. Coeff.} = \frac{C_A (n-a)^2}{n^{(a+3)}} \quad (8.7)$$

with obvious further simplification when $C_A = 1$ and $a = 0$. For general P with $r_i = R_i$ and $s_i = S_i$, we have

$$\text{Alg. Coeff.} = C_A \sum_{u=1}^P \frac{(u-1)!}{u} \cdot \frac{\Delta^u(0^P)}{n^{(a+u)}} \quad (8.8)$$

which, when $a = 0$ simplifies to $\sum_{u=1}^P \frac{(u-1)!}{u} \cdot \frac{\Delta^u(0^P)}{n^{(u)}}$ as given by Fisher [1929, p. 226].

Case IV. $r_1 = s_1 = \rho_1$, $r_2 = s_2 = \rho_2$.

In this case the algebraic coefficient becomes

$$C_A \sum_{u_1=1}^{\rho_1} \sum_{u_2=1}^{\rho_2} \frac{(R_1 - \rho_1 - 1 + u_1)! (S_1 - \rho_1 - 1 + u_1)! (R_2 - \rho_2 - 1 + u_2)! (S_2 - \rho_2 - 1 + u_2)!}{n^{(a+u_1+u_2)}} \cdot \frac{\Delta^{(u_1)}_0(\rho_1)}{u_1!} \cdot \frac{\Delta^{(u_2)}_0(\rho_2)}{u_2!} \quad (8.9)$$

More explicit formulae can be written for special cases. Thus when

$r_1 = s_1 = r_2 = s_2 = \rho_1 = \rho_2 = 2$, we have

$$\begin{aligned} \text{Alg. Coeff.} = C_A \cdot \frac{(R_1-2)! (S_1-2)! (R_2-2)! (S_2-2)!}{n^{(a+4)}} & \left[(R_1-1)(S_1-1)(R_2-1)(S_2-1) + \right. \\ & \left. \{ (R_1-1)(S_1-1) + (R_2-1)(S_2-1) \} (n-a-3) + (n-a-2)(n-a-3) \right]. \end{aligned} \quad (8.10)$$

And if, in addition, $R_1 = S_1 = R_2 = S_2 = 2$, we have

$$\text{Alg. Coeff.} = C_A \cdot \frac{1}{n^{(a+4)}} \left[(n-a)^2 + 3(n-a) + 1 \right] \quad (8.11)$$

which simplifies to $\frac{n^2 + 3n + 1}{n^{(4)}}$ when $a = 0$. For general ρ_1, ρ_2 with

$$r_1 = s_1 = R_1 = S_1 = \rho_1, \quad r_2 = s_2 = R_2 = S_2 = \rho_2, \\ \text{Alg. Coeff.} = C_A \sum_{u_1=1}^{\rho_1} \sum_{u_2=1}^{\rho_2} \frac{(u_1-1)!}{u_1} \cdot \frac{(u_2-1)!}{u_2} \cdot \frac{\Delta^{u_1}(0^{\rho_1}) \Delta^{u_2}(0^{\rho_2})}{n^{(a+u_1+u_2)}}. \quad (8.12)$$

which simplifies further when $a = 0$.

Case V. $r_1 = \rho_1 + \rho_2$, $\rho_1 = \rho_1$, $\rho_2 = \rho_2$.

Case V is in a sense a modification of Case IV if we consider $r_i = r'_1 + r'_2$ where $r'_1 = \rho_1$ and $r'_2 = \rho_2$. The terms in the conditional amalgamations correspond exactly except for the factorial involving ρ_1 and ρ_2 which becomes $(R_1 - \rho_1 - \rho_2 - 1 + u_1 + u_2)!$ rather than $(R_1 - \rho_1 - 1 + u_1)!$ $(R_1 - \rho_2 - 1 + u_2)!$. Hence the formula is

$$\text{Alg. Coeff.} = C_A \sum_{u_1=1}^{\rho_1} \sum_{u_2=1}^{\rho_2} \frac{(R_1 - \rho_1 - \rho_2 - 1 + u_1 + u_2)! (S_1 - \rho_1 - 1 + u_1)! (S_2 - \rho_2 - 1 + u_2)!}{n^{(a+u_1+u_2)}} \frac{\Delta^{u_1}(0^{\rho_1}) \cdot \Delta^{u_2}(0^{\rho_2})}{u_1! u_2!} \quad (8.13)$$

Case VI. (Generalization of Case IV) $r_i = \rho_i = \rho_i$.

$$\text{Alg. Coeff.} = \prod \sum_{u_i=1}^{\rho_i} \frac{(R_i - \rho_i - 1 + u_i)! (S_i - \rho_i - 1 + u_i)!}{n^{(a+\sum u_i)}} \cdot \frac{\Delta^{u_i}(0^{\rho_i})}{u_i!} \quad (8.14)$$

where $\prod \sum$ indicates $\sum_{u_1} \sum_{u_2} \dots \sum_{u_i} \dots$.

Case VII. (Generalization of Case V) $r_i = \rho_{i1} + \dots + \rho_{ih_i} + \dots + \rho_{in_i}$
 $= \rho_{i1} + \dots + \rho_{ih_i} + \dots + \rho_{in_i}$.

$$\text{Alg. Coeff.} = C_A \prod \sum_{u_{ih_i}} \frac{(R_i - \sum_{h_i} \rho_{ih_i} - 1 + \sum u_{ih_i})! (S_i - \rho_{ih_i} - 1 + u_{ih_i})!}{n^{(a+\sum u_{ih_i})}} \frac{\Delta^{u_{ih_i}}(0^{\rho_{ih_i}})}{u_{ih_i}!} \quad (8.15)$$

These seven cases and adaptations of them take care of very general situations.

Formulae for algebraic coefficients which are more specific but which are general enough for broad classes of reduced array types can be obtained as special cases. These are indicated by the values of r_i , ρ_j , R_i , S_j and a . Unless otherwise specified it is understood that $R_i = r_i$ and $S_j = \rho_j$ for $i, j > 1$ and that all other $r_i = 0$ and $\rho_j = 0$.

<u>Case</u>	<u>Specification</u>	<u>Algebraic Coefficient</u>
I	All rows cancel.	1
II	The reduced array type is all A.	$C_A \cdot \frac{1}{n^{(p)}}$
III	* $r_1 = 2, s_1 = 2$	$\frac{1}{n-1}$
	* $r_1 = 3, s_1 = 3$	$\frac{n}{(n-1)(n-2)}$
	* $r_1 = 4, s_1 = 4$	$\frac{n(n+1)}{(n-1)(n-2)(n-3)}$
	* $r_1 = 5, s_1 = 5$	$\frac{n^2(n+5)}{(n-1)^{(4)}}$
	* $r_1 = p, s_1 = p$	$\sum_{u=1}^p \frac{(u-1)!}{u} \cdot \frac{\Delta^u(o^p)}{n^{(u)}}$
	$r_1 = 2, s_1 = 2, R_1 = 3, S_2 = 1$	$-\frac{1}{(n-1)(n-2)}$
	$r_1 = 2, s_1 = 2, R_2 = 1, S_1 = 3$	
	$r_1 = 2, s_1 = 2, R_2 = 1, S_2 = 1$	$\frac{1}{n(n-2)}$
	$r_1 = 2, s_1 = 2$ with other $R_i = 1, S_j = 1$	$\frac{n-a}{n^{(a+2)}}$
	$r_1 = 3, s_1 = 3, R_2 = 1, S_2 = 1$	$\frac{n-1}{n(n-2)(n-3)}$
$r_1 = 3, s_1 = 3$ with other $R_i = 1, S_j = 1$	$\frac{(n-a)^2}{n^{(a+3)}}$	
IV	$r_1 = s_1 = 2, r_2 = s_2 = 2$	$\frac{n^2 - 3n + 1}{n^{(4)}}$
	$r_1 = s_1 = 3, r_2 = s_2 = 2$	$\frac{n^2 - 5n + 5}{(n-1)^{(4)}}$
	$r_1 = s_1 = 4, r_2 = s_2 = 2$	$\frac{(n-1)(n^3 - 5n^2 + n + 4)}{n^{(6)}}$
	$r_1 = s_1 = 3, r_2 = s_2 = 3$	$\frac{n^4 - 8n^3 + 26n^2 - 91n + 196}{n^{(6)}}$

*Given essentially by Fisher [1929].

<u>Case</u>	<u>Specification</u>	<u>Algebraic Coefficient</u>
V	$\left. \begin{array}{l} r_1 = 4, \Delta_1 = 2, \Delta_2 = 2 \\ r_1 = 2, r_2 = 2, \Delta_1 = 4 \end{array} \right\}$	$-\frac{1}{(n-2)(n-3)}$
	$\left. \begin{array}{l} r_1 = 5, \Delta_1 = 3, \Delta_2 = 2 \\ r_1 = 3, r_2 = 2, \Delta_1 = 5 \end{array} \right\}$	$-\frac{n}{(n-2)(n-3)(n-4)}$
	$\left. \begin{array}{l} r_1 = 6, \Delta_1 = 3, \Delta_2 = 3 \\ r_1 = 3, r_2 = 3, \Delta_1 = 6 \end{array} \right\}$	$-\frac{n^2 - n + 4}{(n-2)^{(4)}}$
	$\left. \begin{array}{l} r_1 = 6, \Delta_1 = 4, \Delta_2 = 2 \\ r_1 = 4, r_2 = 2, \Delta_1 = 6 \end{array} \right\}$	$-\frac{(n-1)(n+4)}{(n-2)^{(4)}}$
VI	$r_1 = 2, r_2 = 2, r_3 = 2, \Delta_1 = 2, \Delta_2 = 2, \Delta_3 = 2$	$\frac{n^3 - 9n^2 + 23n - 14}{n^{(6)}}$
VII	$\left. \begin{array}{l} r_1 = 6, \Delta_1 = 2, \Delta_2 = 2, \Delta_3 = 2 \\ r_1 = 2, r_2 = 2, r_3 = 2, \Delta_1 = 6 \end{array} \right\}$	$\frac{2}{(n-3)(n-4)(n-5)}$
	$\left. \begin{array}{l} r_1 = 4, r_2 = 2, \Delta_1 = 2, \Delta_2 = 2, \Delta_3 = 2 \\ r_1 = 2, r_2 = 2, r_3 = 2, \Delta_1 = 4, \Delta_2 = 2 \end{array} \right\}$	$-\frac{(n-2)(n^2 - 5n + 2)}{n^{(6)}}$

9. SOME GENERAL FORMULAE

Some general formulae, obtained by the combinatorial method, are presented in this section. To condense the results, we use symbolic forms for the reduced array types. With $\{ \}$ representing the set p_1, p_2, \dots , we indicate the group of array types in which any q , say q_1 , may appear in a row with any element of $\{ \}$ by $\{q_1\}$, the group of array types in which q_1 and q_2 may appear by $\{q_1, q_2\}$ etc. The notation $\{ \} q_1, q_2$ is used for the extended array type in which q_1, q_2 appear alone in additional rows. The notation $\{q_1\} q_2$ symbolizes the array type with q_1 in a row with some element of $\{ \}$ with an additional row containing q_2 . Double subscripts indicate partitions. Thus $\{q_{11}, q_{12}\} q_2$ indicates an array type in which the two part partitions of q_1 are added concurrently to two elements of $\{ \}$ and the row for q_2 is added.

An adaptation makes possible the characterization of partitions of an element of $\{ \}$. The appearance of double subscripts on some p , say p_1 , indicates that partitions of p_1 , and not p_1 itself, appear in the array type. Then we use $\{p_{11}, p_{12} + q_{11}, q_{12}\}$ to indicate the array type in which the two part partitions of p_1 appear with the two part partitions of q_1 , the other p 's appearing alone. Similarly we use $\{p_{11}, p_{12} + q_{11}, q_{12} + q_2\}$ to indicate an array type like the previous one with, in addition, q_2 appearing in a row with some $p_i, i \neq 1$. And $\{p_{11}, p_{12} + q_{11}, q_{12} + q_2\} q_3, \dots$ indicates the extended array type with additional rows of q 's.

The results of the combinatorial method, in this condensed notation, for several general formulae are given below. Some collections of coefficients of array types have been made. Thus the coefficient of $k_{\{p_{i1}, p_{i2} + 11\}}$ in the expansion of $k_{\{ \} } k_{21}$ is $1/n(n-2) + 2/n(n-1)(n-2) = (n+1)/n(n-1)(n-2)$. The first formula although well-known, is given for completeness.

$$k_{\{ \} } k_1 = k_{\{ \} } k_{11} + \frac{1}{n} k_{\{1\}}$$

$$k_{\{ \} } k_2 = k_{\{ \} } k_{22} + \frac{1}{n} k_{\{2\}} - \frac{1}{n(n-1)} k_{\{11\}} + \frac{1}{n-1} \sum (p_{i1} p_{i2}) k_{\{p_{i1}, p_{i2} + 11\}}$$

The results of Table IV are a special case of this.

$$\begin{aligned}
k_{\{ \} } k_{11} &= k_{\{ \} 11} + \frac{2}{n} k_{\{1\} 1} + \frac{1}{n(n-1)} k_{\{11\}} - \frac{1}{n(n-1)} \sum (p_{i1} p_{i2}) k_{\{p_{i1} p_{i2} + 11\}} \quad * \\
k_{\{ \} } k_{33} &= k_{\{ \} 33} + \frac{1}{n} k_{\{3\} 3} - \frac{3}{n(n-1)} k_{\{21\}} + \frac{2}{n(n-1)(n-2)} k_{\{111\}} \\
&+ \frac{3}{n-1} \sum (p_{i1} p_{i2}) k_{\{p_{i1} p_{i2} + 21\}} - \frac{3}{(n-1)(n-2)} \sum (p_{i1} p_{i2}) k_{\{p_{i1} p_{i2} + 111\}} \\
&+ \frac{n}{(n-1)(n-2)} \sum (p_{i1} p_{i2} p_{i3}) k_{\{p_{i1} p_{i2} p_{i3} + 111\}}
\end{aligned}$$

The results of Table V are a special case of this with the fourth term vanishing.

$$\begin{aligned}
k_{\{ \} } k_{21} &= k_{\{ \} 21} + \frac{1}{n} k_{\{2\} 1} + \frac{1}{n} k_{\{1\} 2} + \frac{1}{n(n-1)} k_{\{21\}} - \frac{1}{n(n-1)} k_{\{11\} 1} \\
&- \frac{1}{n(n-1)(n-2)} k_{\{111\}} - \frac{1}{n(n-1)} \sum (p_{i1} p_{i2}) k_{\{p_{i1} p_{i2} + 21\}} \\
&+ \frac{n+1}{n(n-1)(n-2)} \sum (p_{i1} p_{i2}) k_{\{p_{i1} p_{i2} + 111\}} + \frac{1}{n-1} \sum (p_{i1} p_{i2}) k_{\{p_{i1} p_{i2} + 11\} 1} \\
&- \frac{1}{(n-1)(n-2)} \sum (p_{i1} p_{i2} p_{i3}) k_{\{p_{i1} p_{i2} p_{i3} + 111\}}
\end{aligned}$$

*This gives, as a very special case,

$$k_p k_{11} = k_{p11} + \frac{2}{n} k_{p+1,1} - \frac{2}{n(n-1)} \sum (p_1 p_2) k_{p_1+1, p_2+1}.$$

This result has been stated by Barton, David, and Fix [1960, p. 56] for the case of $p=r$ even and odd using binomial coefficients. The need for two formulae is avoided by using the combinatorial coefficient (p_1, p_2) . The subscripts of $K_{\frac{1}{2}r, \frac{1}{2}r}$ in the formula for $K_r K_{11}$, r even, should be $\frac{1}{2}r+1, \frac{1}{2}r+1$.

$$\begin{aligned}
k_{\{ \} } k_{\{ \} } k_{\{ \} } &= k_{\{ \} } k_{\{ \} } k_{\{ \} } k_{\{ \} } + \frac{3}{n} k_{\{ \} } k_{\{ \} } k_{\{ \} } + \frac{3}{n(n-1)} k_{\{ \} } k_{\{ \} } k_{\{ \} } + \frac{1}{n(n-1)(n-2)} k_{\{ \} } k_{\{ \} } k_{\{ \} } \\
&\quad - \frac{3}{n(n-1)(n-2)} \sum (p_{i_1} p_{i_2}) k_{\{ p_{i_1} p_{i_2} + \{ \} : 1 \} } \\
&\quad - \frac{3}{n(n-1)} \sum (p_{i_1} p_{i_2}) k_{\{ p_{i_1} p_{i_2} + \{ \} : 1 \} } \\
&\quad + \frac{2}{n(n-1)(n-2)} \sum (p_{i_1} p_{i_2} p_{i_3}) k_{\{ p_{i_1} p_{i_2} p_{i_3} + \{ \} : 1 \} }
\end{aligned}$$

$$\begin{aligned}
k_{\{ \} } k_{\{ \} } k_{\{ \} } k_{\{ \} } &= k_{\{ \} } k_{\{ \} } k_{\{ \} } k_{\{ \} } k_{\{ \} } + \frac{1}{n} k_{\{ \} } k_{\{ \} } k_{\{ \} } k_{\{ \} } - \frac{4}{n(n-1)} k_{\{ \} } k_{\{ \} } k_{\{ \} } k_{\{ \} } - \frac{3}{n(n-1)} k_{\{ \} } k_{\{ \} } k_{\{ \} } k_{\{ \} } \\
&\quad + \frac{12}{n(n-1)(n-2)} k_{\{ \} } k_{\{ \} } k_{\{ \} } k_{\{ \} } k_{\{ \} } - \frac{6}{n(n-1)(n-2)(n-3)} k_{\{ \} } k_{\{ \} } k_{\{ \} } k_{\{ \} } k_{\{ \} } \\
&\quad + \frac{4}{n-1} \sum (p_{i_1} p_{i_2}) k_{\{ p_{i_1} p_{i_2} + \{ \} : 1 \} } + \frac{3}{n-1} \sum (p_{i_1} p_{i_2}) k_{\{ p_{i_1} p_{i_2} + \{ \} : 2 \} } \\
&\quad - \frac{12}{(n-1)(n-2)} \sum (p_{i_1} p_{i_2}) k_{\{ p_{i_1} p_{i_2} + \{ \} : 1 \} } - \frac{6}{(n-1)(n-2)} \sum (p_{i_1} p_{i_2}) k_{\{ p_{i_1} p_{i_2} + \{ \} : 2 \} } \\
&\quad + \frac{12}{(n-1)(n-2)(n-3)} \sum (p_{i_1} p_{i_2}) k_{\{ p_{i_1} p_{i_2} + \{ \} : 1 \} } \\
&\quad - \frac{4(n+1)}{(n-1)(n-2)(n-3)} \sum (p_{i_1} p_{i_2} p_{i_3}) k_{\{ p_{i_1} p_{i_2} p_{i_3} + \{ \} : 1 \} } \\
&\quad + \frac{6n}{(n-1)(n-2)} \sum (p_{i_1} p_{i_2} p_{i_3}) k_{\{ p_{i_1} p_{i_2} p_{i_3} + \{ \} : 2 \} } \\
&\quad - \frac{1}{(n-2)(n-3)} \sum (p_{i_1} p_{i_2})(p_{j_1} p_{j_2}) k_{\{ p_{i_1} p_{i_2} p_{j_1} p_{j_2} + \{ \} : 1 \} } \\
&\quad + \frac{n(n+1)}{(n-1)(n-2)(n-3)} \sum (p_{i_1} p_{i_2} p_{i_3} p_{i_4}) k_{\{ p_{i_1} p_{i_2} p_{i_3} p_{i_4} + \{ \} : 1 \} }
\end{aligned}$$

$$\begin{aligned}
& k_{\{31\}} = k_{\{31\}} + \frac{1}{n} k_{\{31\}} + \frac{1}{n(n-1)} k_{\{31\}} - \frac{3}{n(n-1)} k_{\{211\}} - \frac{3}{n(n-2)} k_{\{211\}} \\
& + \frac{2}{n(n-2)} k_{\{111\}} + \frac{2}{n(n-2)(n-3)} k_{\{111\}} - \frac{1}{n(n-2)} \sum (p_{11} p_{12}) k_{\{p_{11} p_{12} + 31\}} \\
& + \frac{3}{n-1} \sum (p_{11} p_{12}) k_{\{p_{11} p_{12} + 21\}} - \frac{3}{(n-1)(n-2)} \sum (p_{11} p_{12}) k_{\{p_{11} p_{12} + 211\}} \\
& + \frac{3}{n(n-2)} \sum (p_{11} p_{12}) k_{\{p_{11} p_{12} + 111\}} - \frac{3}{(n-1)(n-2)} \sum (p_{11} p_{12}) k_{\{p_{11} p_{12} + 111\}} \\
& - \frac{3(n+1)}{n(n-1)(n-2)(n-3)} \sum (p_{11} p_{12}) k_{\{p_{11} p_{12} + 111\}} + \frac{n^2 + n + 4}{n(n-1)(n-2)(n-3)} \sum (p_{11} p_{12} p_{13}) k_{\{p_{11} p_{12} p_{13} + 111\}} \\
& + \frac{n}{(n-1)(n-2)} \sum (p_{11} p_{12} p_{13}) k_{\{p_{11} p_{12} p_{13} + 111\}} - \frac{3}{(n-1)(n-2)} \sum (p_{11} p_{12} p_{13}) k_{\{p_{11} p_{12} p_{13} + 211\}} \\
& - \frac{1}{n(n-2)(n-3)} \sum (p_{11} p_{12})(p_{11} p_{12}) k_{\{p_{11} p_{12} p_{11} p_{12} + 1111\}} \\
& - \frac{n+1}{(n-1)(n-2)} \sum (p_{11} p_{12} p_{13}) k_{\{p_{11} p_{12} p_{13} p_{14} + 1111\}}
\end{aligned}$$

$$\begin{aligned}
R_{\{22\}} &= R_{\{22\}} + \frac{2}{n} R_{\{22\}2} + \frac{1}{n(n-1)} R_{\{22\}} - \frac{2}{n(n-1)} R_{\{11\}2} - \frac{2}{n(n-1)(n-2)} R_{\{112\}} \\
&+ \frac{1}{n(n-1)(n-2)(n-3)} R_{\{1111\}} - \frac{1}{n(n-1)} \sum (p_{i1} p_{i2}) R_{\{p_{i1} p_{i2} + 22\}} \\
&+ \frac{2}{n(n-2)} \sum (p_{i1} p_{i2}) R_{\{p_{i1} p_{i2} + 112\}} - \frac{2}{(n-1)(n-2)(n-3)} \sum (p_{i1} p_{i2}) R_{\{p_{i1} p_{i2} + 111\}} \\
&+ \frac{2}{n-1} \sum (p_{i1} p_{i2}) R_{\{p_{i1} p_{i2} + 11\}2} + \frac{4}{n(n-1)(n-2)} \sum (p_{i1} p_{i2}) R_{\{p_{i1} p_{i2} + 211\}} \\
&- \frac{2}{(n-1)(n-2)} \sum (p_{i1} p_{i2} p_{i3}) R_{\{p_{i1} p_{i2} p_{i3} + 211\}} \\
&+ \frac{4}{n(n-2)(n-3)} \sum (p_{i1} p_{i2} p_{i3}) R_{\{p_{i1} p_{i2} p_{i3} + 1111\}} \\
&+ \frac{n^2 - 3n + 3}{3n(n-1)(n-2)(n-3)} \sum (p_{i1} p_{i2}) (p_{j1} p_{j2}) R_{\{p_{i1} p_{i2} p_{j1} p_{j2} + 1111\}} \\
&- \frac{1}{(n-2)(n-3)} \sum (p_{i1} p_{i2} p_{i3} p_{i4}) R_{\{p_{i1} p_{i2} p_{i3} p_{i4} + 1111\}}
\end{aligned}$$

$$\begin{aligned}
R_{\{2\}} &= R_{\{2\}} + \frac{1}{n} R_{\{2\}} + \frac{2}{n} R_{\{2\}} + \frac{2}{n(n-1)} R_{\{2\}} + \frac{1}{n(n-1)} R_{\{2\}} \\
&+ \frac{1}{n(n-1)(n-2)} R_{\{2\}} - \frac{1}{n(n-1)} R_{\{2\}} - \frac{2}{n(n-1)(n-2)} R_{\{2\}} - \frac{1}{n(n-1)(n-2)(n-3)} R_{\{2\}} \\
&- \frac{2}{n(n-1)} \sum (p_i p_j) R_{\{p_i p_j + 2\}} - \frac{1}{n(n-1)} \sum (p_i p_j) R_{\{p_i p_j + 1\}} \\
&- \frac{2}{n(n-1)(n-2)} \sum (p_i p_j) R_{\{p_i p_j + 2\}} - \frac{1}{n(n-1)(n-2)} \sum (p_i p_j) R_{\{p_i p_j + 1\}} \\
&+ \frac{n+3}{n(n-1)(n-2)(n-3)} \sum (p_i p_j) R_{\{p_i p_j + 1\}} + \frac{1}{n-1} \sum (p_i p_j) R_{\{p_i p_j + 1\}} \\
&+ \frac{2(n+1)}{n(n-1)(n-2)} \sum (p_i p_j) R_{\{p_i p_j + 1\}} \\
&- \frac{2}{(n-1)(n-2)} \sum (p_i p_j p_k) R_{\{p_i p_j p_k + 1\}} \\
&- \frac{2(n+1)}{n(n-1)(n-2)(n-3)} \sum (p_i p_j p_k) R_{\{p_i p_j p_k + 1\}} \\
&- \frac{1}{3(n-1)(n-2)(n-3)} \sum (p_i p_j) (p_k p_l) R_{\{p_i p_j p_k p_l + 1\}} \\
&+ \frac{2}{(n-1)(n-2)(n-3)} \sum (p_i p_j p_k p_l) R_{\{p_i p_j p_k p_l + 1\}}
\end{aligned}$$

$$\begin{aligned}
& k_{\{1111\}} = k_{\{1111\}} + \frac{4}{n} k_{\{1111\}} + \frac{6}{n(n-1)} k_{\{1111\}} + \frac{4}{n(n-1)(n-2)} k_{\{1111\}} \\
& + \frac{1}{n(n-1)(n-2)(n-3)} k_{\{1111\}} - \frac{6}{n(n-1)} \sum (p_{i1} p_{i2}) k_{\{p_{i1} p_{i2} + 1111\}} \\
& - \frac{12}{n(n-1)(n-2)} \sum (p_{i1} p_{i2}) k_{\{p_{i1} p_{i2} + 1111\}} \\
& - \frac{6}{n(n-1)(n-2)(n-3)} \sum (p_{i1} p_{i2}) k_{\{p_{i1} p_{i2} + 1111\}} \\
& + \frac{8}{n(n-1)(n-2)} \sum (p_{i1} p_{i2} p_{i3}) k_{\{p_{i1} p_{i2} p_{i3} + 1111\}} \\
& + \frac{8}{n(n-1)(n-2)(n-3)} \sum (p_{i1} p_{i2} p_{i3}) k_{\{p_{i1} p_{i2} p_{i3} + 1111\}} \\
& + \frac{1}{n(n-1)(n-2)(n-3)} \sum (p_{i1} p_{i2} p_{j1} p_{j2}) k_{\{p_{i1} p_{i2} p_{j1} p_{j2} + 1111\}} \\
& - \frac{6}{n(n-1)(n-2)(n-3)} \sum (p_{i1} p_{i2} p_{i3} p_{i4}) k_{\{p_{i1} p_{i2} p_{i3} p_{i4} + 1111\}}
\end{aligned}$$

If $\{ \} = p^r$, we get

$$k_{p^r} k_1 = k_{p^r, 1} + \frac{r}{n} k_{p^{r-1}, p+1}$$

$$k_{p^r} k_2 = k_{p^r, 2} + \frac{r}{n} k_{p^{r-1}, p+2} - \frac{r(r-1)}{n(n-1)} k_{p^{r-2}, (p+1)^2} \\ + \frac{2r}{n-1} \sum (p_1, p_2) k_{p^{r-1}, p_1+1, p_2+1}^*$$

Special cases are obtained by putting $p = 2, 3, \dots$:

$$k_{2^r} k_2 = \frac{r}{n} k_{42^{r-1}} - \frac{r(r-1)}{n(n-1)} k_{3^2 2^{r-2}} + \frac{n+2r-1}{n-1} k_{2^{r+1}}$$

$$k_{3^r} k_2 = \frac{r}{n} k_{53^{r-1}} - \frac{r(r-1)}{n(n-1)} k_{4^2 3^{r-2}} + \frac{n+6r-1}{n-1} k_{3^r 2}$$

as indicated by Barton, David, and Fix [1960].

$$k_{p^r} k_{11} = k_{p^r, 11} + \frac{2r}{n} k_{p^{r-1}, p+1, 1} + \frac{r(r-1)}{n(n-1)} k_{p^{r-2}, (p+1)^2} \\ - \frac{2r}{n(n-1)} \sum (p_1, p_2) k_{p^{r-1}, p_1+1, p_2+1}$$

$$k_{p^r} k_3 = k_{p^r, 3} + \frac{r}{n} k_{p^{r-1}, p+3} - \frac{3r(r-1)}{n(n-1)} k_{p^{r-2}, p+2, p+1} \\ + \frac{2r(r-1)(r-2)}{n(n-1)(n-2)} k_{p^{r-3}, (p+1)^3} \\ + \frac{3r}{n-1} \sum (p_1, p_2) \left[k_{p^{r-1}, p_1+2, p_2+1} + k_{p^{r-1}, p_1+1, p_2+2} \right] \\ - \frac{6r(r-1)}{(n-1)(n-2)} \sum (p_1, p_2) k_{p^{r-2}, p_1+1, p_2+1, p+1} \\ + \frac{6nr}{(n-1)(n-2)} \sum (p_1, p_2, p_3) k_{p^{r-1}, p_1+1, p_2+1, p_3+1}$$

*The single subscript of p in these formulae indicate proper parts of p .

$$\begin{aligned}
R_{\{22\}} &= R_{\{22\}} + \frac{2}{n} R_{\{22\}2} + \frac{1}{n(n-1)} R_{\{22\}} - \frac{2}{n(n-1)} R_{\{11\}2} - \frac{2}{n(n-1)(n-2)} R_{\{112\}} \\
&+ \frac{1}{n(n-1)(n-2)(n-3)} R_{\{1111\}} - \frac{1}{n(n-1)} \sum (p_{i1} p_{i2}) R_{\{p_{i1} p_{i2} + 22\}} \\
&+ \frac{2}{n(n-2)} \sum (p_{i1} p_{i2}) R_{\{p_{i1} p_{i2} + 1112\}} - \frac{2}{(n-1)(n-2)(n-3)} \sum (p_{i1} p_{i2}) R_{\{p_{i1} p_{i2} + 1111\}} \\
&+ \frac{2}{n-1} \sum (p_{i1} p_{i2}) R_{\{p_{i1} p_{i2} + 11\}2} + \frac{4}{n(n-1)(n-2)} \sum (p_{i1} p_{i2}) R_{\{p_{i1} p_{i2} + 211\}} \\
&- \frac{2}{(n-1)(n-2)} \sum (p_{i1} p_{i2} p_{i3}) R_{\{p_{i1} p_{i2} p_{i3} + 2111\}} \\
&+ \frac{4}{n(n-2)(n-3)} \sum (p_{i1} p_{i2} p_{i3}) R_{\{p_{i1} p_{i2} p_{i3} + 1111\}} \\
&+ \frac{n^2 - 3n + 3}{3n(n-1)(n-2)(n-3)} \sum (p_{i1} p_{i2}) (p_{j1} p_{j2}) R_{\{p_{i1} p_{i2} p_{j1} p_{j2} + 1111\}} \\
&- \frac{1}{(n-2)(n-3)} \sum (p_{i1} p_{i2} p_{i3} p_{i4}) R_{\{p_{i1} p_{i2} p_{i3} p_{i4} + 1111\}}
\end{aligned}$$

$$\begin{aligned}
k_{p^r k_4} &= k_{p^4} + \frac{r}{n} k_{p^{r-1}, p+4} - \frac{4r(r-1)}{n(n-1)} k_{p^{r-2}, p+3, p+1} - \frac{3r(r-1)}{n(n-1)} k_{p^{r-2}, (p+2)^2} \\
&+ \frac{12r(r-1)(r-2)}{n(n-1)(n-2)} k_{p^{r-3}, p+2, (p+1)^2} - \frac{6r(r-1)(r-2)(r-3)}{n(n-1)(n-2)(n-3)} k_{p^{r-4}, (p+1)^4} \\
&+ \frac{4r}{n-1} \sum (p_1, p_2) [k_{p^{r-1}, p+3, p_2+1} + k_{p^{r-1}, p_1+1, p_2+3}] + \frac{6r}{n-1} \sum (p_1, p_2) k_{p^{r-1}, p_1+2, p_2+2} \\
&- \frac{12r(r-1)}{(n-1)(n-2)} \sum (p_1, p_2) [k_{p^{r-2}, p_1+2, p_2+1, p+1} + k_{p^{r-2}, p_1+1, p_2+2, p+1}] \\
&- \frac{12r(r-1)}{(n-1)(n-2)} \sum (p_1, p_2) k_{p^{r-2}, p_1+1, p_2+1, p+2} + \frac{2+4r(r-1)(r-2)}{(n-1)(n-2)(n-3)} \sum (p_1, p_2) k_{p^{r-3}, p_1+1, p_2+1, (p+1)^2} \\
&- \frac{24(n+1)r(r-1)}{(n-1)(n-2)(n-3)} \sum (p_1, p_2, p_3) k_{p^{r-2}, p_1+1, p_2+1, p_3+1, p+1} \\
&+ \frac{12nr}{(n-1)(n-2)} \sum (p_1, p_2, p_3) [k_{p^{r-1}, p_1+2, p_2+1, p_3+1} + k_{p^{r-1}, p_1+1, p_2+2, p_3+1} + k_{p^{r-1}, p_1+1, p_3+1, p_2+2}] \\
&- \frac{12r(r-1)}{(n-2)(n-3)} \sum (p_1, p_2) (p'_1, p'_2) k_{p^{r-2}, p_1+1, p_2+1, p'_1+1, p'_2+1} \\
&+ \frac{24n(n+1)r}{(n-1)(n-2)(n-3)} \sum (p_1, p_2, p_3, p_4) k_{p^{r-1}, p_1+1, p_2+1, p_3+1, p_4+1}.
\end{aligned}$$

$$\begin{aligned}
k_{p^r k_{31}} &= k_{p^{r+3}} + \frac{r}{n} k_{p^{r+1}, p+3, 1} + \frac{r}{n} k_{p^{r+1}, p+1, 3} + \frac{r(r-1)}{n(n-1)} k_{p^{r-2}, p+3, p+1} \\
&- \frac{3r(r-1)}{n(n-1)} k_{p^{r-2}, p+2, p+1, 1} - \frac{3r(r-1)(r-2)}{n(n-1)(n-2)} k_{p^{r-3}, p+2, (p+1)^2} \\
&+ \frac{2r(r-1)(r-2)}{n(n-1)(n-2)} k_{p^{r-3}, (p+1)^3, 1} + \frac{2r(r-1)(r-2)(r-3)}{n(n-1)(n-2)(n-3)} k_{p^{r-4}, (p+1)^4} \\
&- \frac{r}{n(n-1)} \sum (p_1 p_2) [k_{p^{r-1}, p_1+3, p_2+1} + k_{p^{r-1}, p_1+1, p_2+3}] \\
&+ \frac{3r}{n-1} \sum (p_1 p_2) [k_{p^{r-1}, p_1+2, p_2+1, 1} + k_{p^{r-1}, p_1+1, p_2+2, 1}] \\
&+ \frac{3r(r-1)}{(n-1)(n-2)} \sum (p_1 p_2) [k_{p^{r-2}, p_1+2, p_2+1, p+1} + k_{p^{r-2}, p_1+1, p_2+2, p+1}] \\
&+ \frac{6r(r-1)}{n(n-1)(n-2)} \sum (p_1 p_2) k_{p^{r-2}, p_1+1, p_2+1, p+2} - \frac{6r(r-1)}{(n-1)(n-2)} \sum (p_1 p_2) k_{p^{r-2}, p_1+1, p_2+1, p+1, 1} \\
&- \frac{6(n+1)r(r-1)(r-2)}{n(n-1)(n-2)(n-3)} \sum (p_1 p_2) k_{p^{r-3}, p_1+1, p_2+1, (p+1)^2} \\
&+ \frac{6(n^2+n+4)r(r-1)}{n(n-1)(n-2)(n-3)} \sum (p_1 p_2 p_3) k_{p^{r-2}, p_1+1, p_2+1, p_3+1, 1} + \frac{6nr}{(n-1)(n-2)} \sum (p_1 p_2 p_3) k_{p^{r-1}, p_1+1, p_2+1, p_3+1} \\
&- \frac{6r}{(n-1)(n-2)} \sum (p_1 p_2 p_3) [k_{p^{r-1}, p_1+2, p_2+1, p_3+1} + k_{p^{r-1}, p_1+1, p_2+2, p_3+1} + k_{p^{r-1}, p_1+1, p_2+1, p_3+2}] \\
&+ \frac{12r(r-1)}{n(n-2)(n-3)} \sum (p_1 p_2) (p_1' p_2') k_{p^{r-2}, p_1'+1, p_2'+1, p_1'+1, p_2'+1} \\
&- \frac{24(n+1)r}{(n-1)(n-2)(n-3)} \sum (p_1 p_2 p_3 p_4) k_{p^{r-1}, p_1+1, p_2+1, p_3+1, p_4+1}
\end{aligned}$$

$$\begin{aligned}
k_{p^r} k_{22} &= k_{p^{r+2} 22} + \frac{2r}{n} k_{p^{r+1}, p+2, 2} + \frac{r(r-1)}{n(n-1)} k_{p^{r-2}, (p+2)^2} - \frac{2r(r-1)}{n(n-1)} k_{p^{r-2}, (p+1)^2, 2} \\
&- \frac{2r(r-1)(r-2)}{n(n-1)(n-2)} k_{p^{r-3}, (p+1)^2, p+2} + \frac{r(r-1)(r-2)(r-3)}{n(n-1)(n-2)(n-3)} k_{p^{r-4}, (p+1)^4} \\
&- \frac{2r}{n(n-1)} \sum (p_1, p_2) k_{p^{r-1}, p_1+2, p_2+2} + \frac{4r(r-1)}{n(n-2)} \sum (p_1, p_2) k_{p^{r-2}, p_1+1, p_2+1, p+2} \\
&- \frac{4r(r-1)(r-2)}{(n-1)(n-2)(n-3)} \sum (p_1, p_2) k_{p^{r-3}, p_1+1, p_2+1, (p+1)^2} + \frac{4r}{n-1} \sum (p_1, p_2) k_{p^{r-1}, p_1+1, p_2+1, 2} \\
&+ \frac{4r(r-1)}{n(n-1)(n-2)} \sum (p_1, p_2) [k_{p^{r-2}, p_1+2, p_2+1, p+1} + k_{p^{r-2}, p_1+1, p_2+2, p+1}] \\
&- \frac{4r}{(n-1)(n-2)} \sum (p_1, p_2, p_3) [k_{p^{r-1}, p_1+2, p_2+1, p_3+1} + k_{p^{r-1}, p_1+1, p_2+2, p_3+1} + k_{p^{r-1}, p_1+1, p_2+1, p_3+2}] \\
&+ \frac{24r(r-1)}{n(n-2)(n-3)} \sum (p_1, p_2, p_3) k_{p^{r-2}, p_1+1, p_2+1, p_3+1, p+1} \\
&+ \frac{4}{n(n-1)(n-2)(n-3)} \sum (p_1, p_2) (p_1', p_2') k_{p^{r-2}, p_1+1, p_2+1, p_1'+1, p_2'+1} \\
&- \frac{24r}{(n-2)(n-3)} \sum (p_1, p_2, p_3, p_4) k_{p^{r-1}, p_1+1, p_2+1, p_3+1, p_4+1}
\end{aligned}$$

$$\begin{aligned}
k_{p^+} k_{211} &= k_{p^+ 211} + \frac{r}{n} k_{p^+ 1, 1} + \frac{2r}{n} k_{p^+ 1, 1} + \frac{2r(x-1)}{n(n-1)} k_{p^+ 2, 1} + \frac{2r}{n(n-1)} k_{p^+ 2, 1} + \frac{2r(x-1)}{n(n-1)} k_{p^+ 2, 1} \\
&+ \frac{r(x-1)}{n(n-1)} k_{p^+ 2, 2} + (p+1)^2 \cdot 2 + \frac{r(x-1)(x-2)}{n(n-1)(n-2)} k_{p^+ 3, p+2, (p+1)^2} - \frac{r(x-1)}{n(n-1)} k_{p^+ 2, (p+1)^2, 1, 1} \\
&- \frac{2r(x-1)(x-2)}{n(n-1)(n-2)} k_{p^+ 3, (p+1)^3, 1} - \frac{r(x-1)(x-2)(x-3)}{n(n-1)(n-2)(n-3)} k_{p^+ 4, (p+1)^4} \\
&- \frac{2r}{n(n-1)} \sum (p_1, p_2) [k_{p^+ 1, p_1+2, p_2+1, 1} + k_{p^+ 1, p_1+1, p_2+2, 1}] - \frac{2r}{n(n-1)} \sum (p_1, p_2) k_{p^+ 1, p_1+1, p_2+1, 2} \\
&- \frac{2r(x-1)}{n(n-1)(n-2)} \sum (p_1, p_2) [k_{p^+ 2, p_1+2, p_2+1, p+1} + k_{p^+ 2, p_1+1, p_2+2, 1} + k_{p^+ 2, p_1+1, p_2+2, p+1}] \\
&- \frac{2r(x-1)}{n(n-1)(n-2)} \sum (p_1, p_2) k_{p^+ 2, p_1+1, p_2+1, p+2} + \frac{2(n+3)r(x-1)(x-2)}{n(n-1)(n-2)(n-3)} \sum (p_1, p_2) k_{p^+ 3, p_1+1, p_2+1, (p+1)^2} \\
&+ \frac{2r}{n-1} \sum (p_1, p_2) k_{p^+ 1, p_1+1, p_2+1, 1, 1} + \frac{4(n+1)r(x-1)}{n(n-1)(n-2)} \sum (p_1, p_2) k_{p^+ 2, p_1+1, p_2+1, p+1, 1} \\
&- \frac{12r}{(n-1)(n-2)} \sum (p_1, p_2, p_3) k_{p^+ 1, p_1+1, p_2+1, p_3+1, 1} \\
&- \frac{12(n+1)r(x-1)}{n(n-1)(n-2)(n-3)} \sum (p_1, p_2, p_3) k_{p^+ 2, p_1+1, p_2+1, p_3+1, p+1} \\
&- \frac{4r(x-1)}{(n-1)(n-2)(n-3)} \sum (p_1, p_2) (p_1', p_2') k_{p^+ 2, p_1+1, p_2+1, p_1'+1, p_2'+1} \\
&+ \frac{48r}{(n-1)(n-2)(n-3)} \sum (p_1, p_2, p_3, p_4) k_{p^+ 1, p_1+1, p_2+1, p_3+1, p_4+1}
\end{aligned}$$

$$\begin{aligned}
k_{p^+} k_{1111} &= k_{p^+ 1111} + \frac{4r}{n} k_{p^{r-1}, p+1, 1, 1, 1} + \frac{6r(r-1)}{n(n-1)} k_{p^{r-2}, (p+1)^2, 1, 1} \\
&+ \frac{4r(r-1)(r-2)}{n(n-1)(n-2)} k_{p^{r-3}, (p+1)^3, 1} + \frac{r(r-1)(r-2)(r-3)}{n(n-1)(n-2)(n-3)} k_{p^{r-4}, (p+1)^4} \\
&- \frac{12r}{n(n-1)} \sum (\beta_1 \beta_2) k_{p^{r-1}, \beta_1+1, \beta_2+1, 1, 1} - \frac{24r(r-1)}{n(n-1)(n-2)} \sum (\beta_1 \beta_2) k_{p^{r-2}, \beta_1+1, \beta_2+1, \beta_2+1, 1} \\
&- \frac{12r(r-1)(r-2)}{n(n-1)(n-2)(n-3)} \sum (\beta_1 \beta_2) k_{p^{r-3}, \beta_1+1, \beta_2+1, (p+1)^2} \\
&+ \frac{48r}{n(n-1)(n-2)} \sum (\beta_1 \beta_2 \beta_3) k_{p^{r-1}, \beta_1+1, \beta_2+1, \beta_3+1, 1} \\
&+ \frac{48r(r-1)}{n(n-1)(n-2)(n-3)} \sum (\beta_1 \beta_2 \beta_3) k_{p^{r-2}, \beta_1+1, \beta_2+1, \beta_3+1, p+1} \\
&+ \frac{12r(r-1)}{n(n-1)(n-2)(n-3)} \sum (\beta_1 \beta_2) (\beta'_1 \beta'_2) k_{p^{r-2}, \beta_1+1, \beta_2+1, \beta'_1+1, \beta'_2+1} \\
&- \frac{144r}{n(n-1)(n-2)(n-3)} \sum (\beta_1 \beta_2 \beta_3 \beta_4) k_{p^{r-1}, \beta_1+1, \beta_2+1, \beta_3+1, \beta_4+1} .
\end{aligned}$$

If $p = 1$, all the partition terms vanish, so,

$$\begin{aligned}
k_{1^+} k_{1111} &= k_{1^{r+4}} + \frac{4r}{n} k_{21^{r+2}} + \frac{6r(r-1)}{n(n-1)} k_{2^2 1^r} + \frac{4r(r-1)(r-2)}{n(n-1)(n-2)} k_{2^3 1^{r-2}} \\
&+ \frac{r(r-1)(r-2)(r-3)}{n(n-1)(n-2)(n-3)} k_{2^4 1^{r-4}} .
\end{aligned}$$

Similarly, compact expressions are easily obtained for $k_{1^+} k_n$ and $k_{1^+} k_{111}$.
The general formula for $k_{1^+} k_{1^{\Delta}}$ ($r \geq \Delta$) is

$$\begin{aligned}
k_{1^+} k_{1^{\Delta}} &= k_{1^{r+\Delta}} + \frac{r\Delta}{n} k_{21^{r+\Delta-2}} + \frac{r(r-1)\Delta(\Delta-1)}{2n(n-1)} k_{2^2 1^{r+\Delta-4}} \\
&+ \frac{r(r-1)(r-2)\Delta(\Delta-1)(\Delta-2)}{3! n(n-1)(n-2)} k_{2^3 1^{r+\Delta-6}} + \dots + \frac{r(r-1)\dots(r-\Delta+1)}{n(n-1)\dots(n-\Delta+1)} k_{2^{\Delta} 1^{r-\Delta}} \\
&= \sum_{t=0}^{\Delta} \frac{r \binom{r}{t} \binom{\Delta}{t}}{t! n \binom{r}{t}} k_{2^t 1^{r+\Delta-2t}}
\end{aligned}$$

This section provides formulae for $k_{p_1 \dots p_r} k_{q_1 \dots q_s}$ for r finite and $\sum q_j \leq 4$. Additional formulae for products of two multiple k -statistics can be found by the combinatorial method developed above.

While this paper is primarily concerned with products of two multiple k -statistics, some of the results (such as the rule of proper parts) hold for multiple products. The establishment of rules for algebraic coefficients for array types with more than two columns, the specification of the resultant combinatorial method, and the presentation of general and specific formulae for multiple products are objectives of our continuing study.

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