QUARTERLY PROGRESS REPORT NO. 1

May 31, 1954

Contract No.:	AF18(600)-1050
Budget Project No.:	670-193
Contract Title:	Development of Generalized Mathematical Procedures for Optimum Assembly of Potentially Effective Combat Crews.
Issuing Office:	The Air Research and Development Command.
Contractor:	The Regents of the University of Michigan.
Monitoring Agency:	Director, Detachment 4 (Crew Research Laboratory), Air Force Personnel and Training Research Center, Randolph Field, Texas.
Principal Investigator:	Dr. Paul S. Dwyer
Period:	March 1, 1954 to May 31, 1954.

PERSONNEL

Name	Title	Portion of Time Devoted to Contract Work
Dwyer, Paul S.	Professor of Mathematics Consultant in Statistical Laboratory	*
Hubbell, Charles H.	Assistant in Research	Full time **
Lott, Fred W.	Assistant in Research	1/2 time ***

* During the three summer months, June 13 to Sept. 13, Dr. Dwyer will be working full time on this project. During the three month interval of time covered by this report, Dr. Dwyer, on full time work at the University, was limited to a maximum of 40 hours per month on the project.

** Mr. Hubbell will start working full time on this project on June 7. He has not worked on this project during the time covered by this report.

*** Mr. Lott will start working 1/2 time on this project on June 7. He has not worked on this project during the time covered by this report.

RESEARCH PROGRESS

1. General

The work of this first quarter has been introductory in nature, though some important results, reported below, are available. Although the contract went into effect as of March 1, official notification was not received till late in March. Since then two capable graduate students, each nearing the doctorate, have been secured as assistants in research. One will work full time and the other half time during the three summer months. Arrangements have not been completed as yet for assistants during the next academic year though prospects are being considered. The results in the sections of this report below are thus the results of the principal investigator who is limited by college rules to 40 hours per month during the academic year, of which this first quarter is a part.

2. Orientation

A first phase of the research, according to the specifications of the contract, is "Familiarization, under the guidance and with the assistance of the monitoring agency, with the general problem and the conditions of group assembly of bomber crews." The office of the monitor advised the principal investigator in March that an immediate trip to the Combat Crew Training Research Laboratory was probably not necessary since the principal investigator had received considerable pre-contract orientation. The investigator had available Research Bulletin 52-[5]* which, with the knowledge he has gained in working on the personnel classification problem, has given him an excellent orientation to the problems covered by phase 3 of the contract. He feels that he needs more orientation with reference to some of the specific methods covered by phase 2, and hopes that this can be attained, along with other objectives,

* Numbers in square brackets refer to items in bibliography, page

during a trip to the Combat Crew Training Research Laboratory in mid-June. As a result of these considerations and after carefully studying Research Bulletin 52-, he has concentrated for the most part on phase 3 during this first quarter. The nature of the results found thus far are indicated in the sections below. (It is planned, during the next quarter when the project will be manned with the equivalent of 2 1/2 full time workers, to make available to the monitor specific reports, which may well be combined later to make chapters of a more extensive phase report, on topics dealing with certain aspects of the project.)

3. Relation of the Group Assembly Problem to the Personnel Classification Problem

In a formal sense, phase 3 of the group assembly problem is a generalization of the personnel classification problem [1]. We use N as the number of men for each of k classes (specialties). The number of possible distinct groups is N^k, and this becomes N^2, the number of C_{ij} values when the general form [1, p. 13] of the personnel classification problem is used. Other (condensed) forms of the group assembly problem result from grouping equivalent or approximately equivalent group scores just as other (condensed) forms of the personnel classification problem result from grouping equivalent or approximately equivalent c_{ij} values. In this first quarterly report the discussion is limited for the most part to the basic form which features the N^k possible group scores. Condensed forms resulting from grouping will be discussed in later reports.

Again, the number of possible assemblies, $(N!)^{k-1}$, is a generalization of the number of possible allocation sums, $N!$, of the personnel classification problem.

It appears that the group assembly maximization problem is a direct generalization of the personnel classification problem when each is expressed in the non-grouped general form. The question naturally arises as to the extension of methods used in solving the personnel classification problem to the group assembly maximization problem. This question is one of the important features of phase 3.

4. The Subtraction of Constants

There is an important possible generalization of a property of the personnel classification problem which deserves consideration before discussion of the specific methods. This deals with the subtraction of a constant from any row, column, layer,... of the $G_{ijl...}$. In the personnel classification problem it is known that any constant, positive or negative, may be subtracted from any row (column) without changing the assignments (though the allocation sum will be decreased by the amount subtracted) if the number of men available equals the number of jobs, N. A corresponding property holds for the more general group assembly maximization problem as long as there are exactly N^k group scores (or appropriate groupings of them). Since one and only one selection is to be made from each row, column, layer...., it follows that the general conditions for maximum sums are the same after some constant has been subtracted from the row (column, layer,...) and that the maximum sum is decreased by the amount subtracted. Subtractions of this sort may be compounded so that transformations involving subtractions of constants from rows, columns, layers,... may be made simultaneously. It is only necessary that the constant subtracted from the row i be the fixed c_i, the constant subtracted from the column j be the fixed c_j, the constant subtracted from the fixed layer l be the fixed c_l,... .

5. Methods used in the Personnel Classification Problem

Methods seriously proposed for the solution of the personnel classification problem include the simplex method [8], the method of bounding sets [6], the method of optimal regions [1], the method of interchange [2], and methods using transformations [3]. These methods and others are derived, described, and illustrated in reports prepared for the Department of the Army [2,3].

6. The Conditions of Solution

In general the simplex method, the method of bounding sets, and the method of optimal regions are based, directly or indirectly, on certain conditions of solution. The allocation sum is known to be a maximum when values of u_i for each row i and v_j for each column j, which always exist, are found [1, p. 20] such that the condition of solution

$$\text{6.1} \qquad \begin{array}{ll} c_{ij} = u_i + v_j & \text{for assigned values,} \\ c_{ij} \leq u_i + v_j & \text{for unassigned values} \end{array}$$

hold when the assigned values include one and only one selection from each row i and each column j. (A slight modification enables us to apply these conditions to problems resulting from grouped c_{ij} .) With the use of the results of section 4 the condition of solution reduce to

$$\text{6.2} \qquad \begin{array}{ll} c_{ij} - u_i - v_j = c_{ij}^{(t)} = 0 & \text{for assigned values,} \\ c_{ij} - u_i - v_j = c_{ij}^{(t)} \leq 0 & \text{for unassigned values,} \end{array}$$

where the $c_{ij}^{(t)}$ result from a series of t subtraction from the original c_{ij} values. The conditions 6.2 may be called the reduced conditions of solution. The personnel classification problem is essentially solved when the values $c_{ij}^{(t)}$ can be found such that it is possible to select zero values of $c_{ij}^{(t)}$ so as to satisfy row and column specifications. This can always be done when k = 2.

7. The Method of Marginal Zeros

One method based on these reduced conditions of solution rather than on the conditions of solution is here called the method of marginal zeros. This terminology results from the fact that in several methods using the conditions of solution, the values of u_i and v_j are used as marginal values for the c_{ij} matrix, while the objective of the method of marginal zeros is to perform transformations which have the effect of reducing to zero all the values of u_i and v_j associated with the $c_{ij}^{(t)}$ matrix.

The method is illustrated in Appendix A where a problem used by Votaw and Dailey [7] to exemplify the simplex method is solved with the method of marginal zeros. The frequencies of the different personnel categories, f_i , and the quotas, q_j , are indicated, as are the values c_{ij} . A first subtraction of the largest element in the row gives all $c_{ij}^{(1)} \leq 0$. A second subtraction of the largest element in each column results in the $c_{ij}^{(2)}$ matrix with at least one zero in each row and in each column. However, it is impossible to complete the quotas with the zeros available, since row 2 and row 3 do not have enough zeros to satisfy the quotas. The additional subtraction of -1 from row 2, and -1 from row 3, with the compensatory subtraction of 1 from column 3 and 1 from column 4 to keep the previous zeros in row 2 and row 3 intact, result in the $c_{ij}^{(3)}$ matrix having marginal zeros and having more than half of its elements zero. It is possible to show from the $c_{ij}^{(3)}$ matrix that the frequencies and quotas may be assigned in many ways. One of these ways is shown by the superscripts associated with the $c_{ij}^{(3)}$ values in Appendix A. These assignments and alternative assignments, when applied to the original c_{ij} matrix, give the maximum sum of 671 units.

The method of marginal zeros can always be used to solve the group assembly problem for the special case in which k = 2 (the personnel classification problem) since there are always solutions of the reduced conditions of solution. In general, these solutions may be found by continued subtractions of constants from rows and columns until there is at least one zero (maximum) in each row and each column (the marginal zero property) followed by additional subtractions from rows and columns which do not change the property of marginal zeros but do permit an allocation, involving only zero terms, which satisfies the frequencies and quotas. One convenient scheme for reducing to marginal zeros involves the subtraction of the largest element in the row followed by the subtraction of the largest element in the column as in Appendix A. An alternative is the subtraction of the largest element in the column followed by the subtraction of the largest element in the row. These do not necessarily result in the same matrices and it may be easier to make the final reduction from one of these matrices than from the other.

8. Inapplicability of the Method of Marginal Zeros to the General Group Assembly Problem

The question confronting us is whether the conditions of solution can be extended directly to apply to the k- dimensional case. For example are the conditions

8.1
$$G_{ijl} = u_i + v_j + w_l \quad \text{for assigned values}$$
$$G_{ijl} \leq u_i + v_j + w_l \quad \text{for unassigned values}$$

conditions of solution for all problems when k = 3? They are not, as is shown by a counterexample. We attempt to use the method of marginal zeros with the reduced (proposed) conditions of solution

$$G^{(t)}_{ijl} = 0 \quad \text{for assigned values,}$$
$$G^{(t)}_{ijl} \leq 0 \quad \text{for unassigned values}$$

and show that problems exist for which the maximal sums do not satisfy these conditions. For example with k = 3, N = 2, and

8.3
$$G_{111} = 2 \qquad G_{121} = 3$$
$$G_{211} = 2 \qquad G_{221} = 0$$
$$G_{112} = 3 \qquad G_{122} = 1$$
$$G_{212} = 0 \qquad G_{222} = 2$$

we note that $(N!)^{k-1} = 2^2 = 4$ possible group assembly scores. They are $G_{111} + G_{222} = 4$, $G_{121} + G_{212} = 3$, $G_{211} + G_{122} = 3$, $G_{221} + G_{112} = 3$. We subtract the largest value, 3, from each column and then -1 from the second rows to arrive at the matrix

8.4
$$G^{(2)}_{111} = -1 \qquad G^{(2)}_{121} = 0$$
$$G^{(2)}_{211} = 0 \qquad G^{(2)}_{221} = -2$$
$$G^{(2)}_{112} = 0 \qquad G^{(2)}_{122} = -2$$
$$G^{(2)}_{212} = -2 \qquad G^{(2)}_{222} = 0$$

which has marginal zeros for rows, columns, and layers, but the maximum sum $G^{(2)}_{111} + G^{(2)}_{222} = -1$ does not consist entirely of zeros. Furthermore, it is impossible to perform subtractions from rows, columns, or layers which maintain the marginal zeros and which transform the $G_{111}^{(2)} = -1$ to 0. Now if the reduced conditions of solution do not hold for all problems, the conditions of solution do not hold for all problems since, in the light of section 4, they are formally equivalent.

The counterexample above can be extended to higher values of k so that we can state that the conditions of solution good for $k = 2$ are not universal conditions of solution when $k > 2$. However there are some problems in which the conditions hold and the method of marginal zeros is directly applicable. Such a problem, $k = 3$ and $N = 3$, is shown in Appendix B1. Subtraction is made by columns, then by rows. The resulting matrix has marginal zeros and it is possible to find a group assembly composed of zeros. The maximum assembly sum is $G_{331} + G_{122} + G_{213} = 6 + 8 + 5 = 19$ units. The treatment of a somewhat similar problem is shown in Appendix B2. The subtraction of constants in turn from column, row, and layer result in the $G_{ijl}^{(3)}$ matrix with marginal zeros. However it is not possible to select an assembly using only zero values, and it appears to be impossible to subtract additional constants, maintaining the marginal zeros, so that an assembly results which consists only of zeros.

9. Inapplicability of Several Methods of Solving the Personnel Classification Problem

The results of the section above have a definite import for the question of the applicability of such methods as the simplex method, the method of optimal regions, etc. to the group assembly problem. Certainly methods which are based on the conditions of solution are not generally applicable. Now, the simplex method and the method of bounding sets are methods which make constant use of the conditions of solution. It follows at once that generalizations of these methods which include w_l as well as u_i and v_j are not applicable to the general group assembly problem. Of course there are certain types of group assembly problems (and there are some prospects that study will enable us to work out methods for identifying these types) where these generalized conditions of solution hold. But present indications are that the conditions of solution of the general group assembly problem are so much more general than the conditions of solution for the case with k = 2 that the specialized techniques applicable to the personnel classification problem are not applicable to the general problem.

Similar remarks apply to the method of optimal regions. For this method the conditions of solution are transformed to the generalized Brogden condition [1, p. 20]

9.1 $$c_{1J} - v_{1J} \geq c_{1j} - v_j ,$$

where J indicates the assigned value. But since the conditions of solution are not applicable, the transformed generalized Brodgen condition (really its k-dimensional generalization) is not applicable.

To sum up, we may say that no immediate generalization of any method which is based on the conditions of solution is applicable. This does not necessarily hold for the method of interchange, unless the conditions of solution are used, but the process of interchange becomes very much more complicated as k increases. We are led to the conclusion that these basic methods which are useful when k = 2 are not, except under special conditions, applicable to the case of larger k.

10. Approximate Results with the Method of Marginal Zeros

The difficulties are with the conditions of solution and not with the subtraction of constants. We can use the method of marginal zeros to obtain at least one zero in each row, column, layer,... . If additional transformations can be made, keeping marginal zeros, so that groups can be assembled with the use of zero elements as in Appendix B1, well and good; but if it cannot, as in the problem of section 8 and the problem of Appendix B2, the transformed assembly sum can usually be made small with the use of zeros for the most part.

The group assembly sum thus obtained may in fact be a maximum, though the procedures indicated thus far may not show it; but the sum is in principle an approximation to the maximum sum, since it cannot differ from the maximum sum by more than its absolute value. Thus in the illustration of section 8, we know at once that the sum $G^{(t)}_{111} + G^{(t)}_{222}$ (= -1) is either optimal or is within one unit of being optimal. Also in Appendix B2, the sum indicated by $G_{211} + G_{332} + G_{123}$ (= 6+4+11=21) is either a maximum or is within one unit of being a maximum. The method of marginal zeros is thus useful in finding an approximation to the maximum (which may in fact be the maximum) and at the same time a bound for the error of the approximation.

11. The Use of Deviate Transformations

A device which is very useful in the two-dimensional case is the use of deviate transformations. A discussion of the use of large deviates, double deviates, and approximate deviates k = 2 is presented in the report [3], to which the reader of this report is referred. In general, using the results of section 4, we may make such transformations as

11.1
$$\begin{aligned} k = 2, \quad g_{ij} &= G_{ij} - \bar{G}_{i.} - \bar{G}_{.j} + \bar{G}_{..} \\ k = 3, \quad g_{ijl} &= G_{ijl} - \bar{G}_{i..} - \bar{G}_{.j.} - \bar{G}_{..l} + 2\,\bar{G}_{...}\,, \\ k = 4, \quad g_{ijlm} &= G_{ijlm} - \bar{G}_{i...} - \bar{G}_{.j..} - \bar{G}_{..l.} - \bar{G}_{...m} + 3\bar{G}_{....}\,, \\ &\text{etc.,} \end{aligned}$$

where the G's are the means of the respective rows, columns, layers, etc., without changing the assembly (though the assembly sum is changed by the amount subtracted).

The purpose of the transformation is to make all rows, columns, layers, etc. comparable in the sense that all means are zero. Approximate maximum solutions may be obtained by selecting the largest g's. Approximate means, $\tilde{G}$, are recommended for most problems to avoid the decimal places of the $\bar{G}$. The values of $\tilde{G}$ are rounded off to the same units as the G's so that g's are computed in the same units. Of course the approximate deviates do not ordinarily sum exactly to 0.

The computation is illustrated in Appendix C where the method is applied to a problem with $k = 3$ and $N = 4$. The values of G_{ijl} were taken from a table of random numbers [4, p. 108]. Various approximations to the maximum sum may be obtained by selecting the larger values.

12. The Group Assembly Problem and the Analysis of Variance

The transformations of section 11 might be called analysis-of-variance transformations since they remove the main effects in a k-way classification in which each classification has N categories. In the illustration of Appendix C, for example, the analysis of the $N^k = 4^3 = 64$ elements of the 3-way classification with four categories each gives the mean square for the main effects and for the interactions of different orders. The analysis of the approximate deviates results in equivalent interaction terms with the main effects approximately zero. Several problems have been worked out illustrating this point. In connection with these results we are led to the conclusion that the main effects of analysis of variance may be eliminated without changing the assembly maximization. The application of analysis of variance in analyzing the relative importance of various interaction terms gives considerable promise of usefulness in directing our efforts toward suitable mathematical models for specific problems. Alternatively, the known mathematical model, when group scores are computed on the basis of certain interactions (if the model is consistent with the mathematical model for the analysis of variance), should be useful in designing techniques appropriate to different types of special problems.

13. The Method of Marginal Zeros after the Deviate Transformation

We next combine the two methods, each involving subtraction of constants, which are useful in reducing the group assembly problem. We first apply the deviate transformation whose objective is to place each row, column, layer, etc. on a somewhat equivalent basis by reducing the mean of each to zero. Then we apply the method of marginal zeros to the results of the deviate transformation. From these transformed results one can usually select elements which are all zero or are all small.

The method is illustrated in Appendix C where the values of $g_{ijl}^{(2)}$ are found by subtracting the maximum elements in the rows, followed by the subtraction of the maximum elements in the layers, these results satisfying the condition of marginal zeros. The elements $\leq$ - 16 are 0, 0, 0, 0, 0, 0, -4, -10, -12, -14, -16, -16, -16. Study shows that $g_{311}^{(2)} = 0$, $g_{122}^{(2)} - -4$, $g_{433}^{(2)} = 0$, $g_{244}^{(2)} = -12$ gives a solution which is either maximum or is within 16 units of being maximum. The value of the group assembly sum is $G_{311} + G_{122} + G_{433} + G_{244} = 95 + 91 + 96 + 78 =$ 360 units. This actually is a maximum as is shown below.

14. A General Method for the Group Assembly Problem

The values of $g_{ij\ldots}^{(t)}$ of the section above may be used as the basis of a non-approximation method. An approximate solution which is either maximum, or within a units of being maximum, is first obtained. Then all values of $g_{ij\ldots}^{(t)} \leq a$ are recorded. In the illustration above, the recorded values are

$g_{311}^{(2)} = 0$	$g_{122}^{(2)} = -4$
$g_{441}^{(2)} = 0$	$g_{421}^{(2)} = -10$
$g_{142}^{(2)} = 0$	$g_{244}^{(2)} = -12$
$g_{422}^{(2)} = 0$	$g_{444}^{(2)} = -14$
$g_{433}^{(2)} = 0$	$g_{121}^{(2)} = -16$
$g_{214}^{(2)} = 0$	$g_{313}^{(2)} = -16$
	$g_{241}^{(2)} = -16$.

The zero values are first tried to see if a possible solution exists consisting of zeros only. Such a solution must consist of one and only one value of 1,2,3,4,... for each set i, j, l,... . In the problem above it is not possible to form such a set. Then the next smaller value of $g^{(t)}_{1j1\ldots}$ is introduced and tried. The process continues until a solution with sum $\leq$ a is reached. In the illustration, this first solution is possible with the introduction of $g^{(2)}_{244} = -12$. The solution is indicated by 311,122,433,244. This is the approximate solution of section 11. It is now established that that solution is a maximum. This general method has been applied to several problems expressed in the individual form as summarized below:

Type of problem	Number of problems solved
k = 3, N = 2	2
k = 3, N = 3	3
k = 3, N = 4	3
k = 3, N = 5	1
k = 4, N = 2	2
k = 4, N = 3	1
k = 4, N = 4	1
k = 4, N = 5	1

Some of these problems involved considerable hand calculation but the method did lead to a solution. The operations could in general be performed with punch-card equipment, as there is every indication that standard punched card equipment will be ideal for performing the transformation.

15. Conclusions

The work of the first quarter has led to several significant conclusions among which are:

A. Specific methods, such as the simplex method and the method of optimal regions, which are very useful in handling the case when k = 2, are not applicable to the more general problem of maximizing the group assembly sum. Other methods must be developed.

B. The general device of subtracting constants from the rows, columns, layers, etc. is applicable to the general problem and is useful.

C. Deviate transformations appear to be very useful in the general problem as well as in the case with $k = 2$.

D. There is considerable evidence that the theory of solution for types of problems can be closely linked with analysis of variance particularly when specific types of interaction are known to exist.

E. The method of marginal zeros, though not necessarily leading directly to the maximum solution as in the two-dimensional case, leads at least to approximations to the maximum assembly sum.

F. A general method based on the application of the method of marginal zeros to the results of the approximate deviate transformation and followed by an examination of the combinations of the largest elements in the resulting matrix, has led to solutions in all the problems attempted thus far. Some of these problems are not trivial. The use of punched-card equipment and the use of grouping devices give promise that this general method can be modified to produce solutions (or at least good approximations to solutions) encountered in the optimum assembly of potentially effective medium bomber combat crews.

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APPENDIX A

Solution of the Votaw-Dailey Problem with the Method of Marginal Zeros

	c_{ij}					
q_j \ f_i	15	20	25	28	12	$c_i^{(0)}$
12	7	8	6	7	8	8
30	6	5	8	6	7	8
35	5	4	5	7	6	7
23	8	9	8	7	9	9
$c_j^{(0)}$	0	0	0	0	0	

	$c_{ij}^{(1)}$					
q_j \ f_i	15	20	25	28	12	$c_i^{(1)}$
12	-1	0	-2	-1	0	0
30	-2	-3	0	-2	-1	0
35	-2	-3	-2	0	-1	0
23	-1	0	-1	-2	0	0
$c_j^{(1)}$	-1	0	0	0	0	

	$c_{ij}^{(2)}$						
q_j \ f_i	15	20	25	28	12	u_i	$c_i^{(2)}$
12	0	0	-2	-1	0	0	0
30	-1	-3	0	-2	-1	0	-1
35	-1	-3	-2	0	-1	0	-1
23	0	0	-1	-2	0	0	0
v_j	0	0	0	0	0		
$c_j^{(2)}$	0	0	1	1	0		

	$c_{ij}^{(3)}$					
q_j \ f_i	15	20	25	28	12	u_i
12	0^{10}	0^{2}	-3	-2	0	0
30	0^{5}	-2	0^{25}	-2	0	0
35	0	-2	-2	0^{28}	0^{7}	0
23	0	0^{18}	-2	-3	0^{5}	0
v_j	0	0	0	0	0	

APPENDIX B1

Solution of a Problem, k = 3, N = 3, with the Method of Marginal Zeros

	G_{ijl}						$c_i^{(1)}$				$\bar{u}_i$	$\bar{w}_i$
	2	4	3	-4	-4	-4	0	-4	-4	-4	0	0
	1	3	5	-5	-5	-2	-1	-4	-4	-1	0	0
	2	1	6*	-4	-7	-1	-1	-3	-6	0*	0	0
	4	8*	5	-2	0	-2	0	-2	0*	-2	0	0
	2	1	6	-4	-7	-1	-1	-3	-6	0	0	0
	5	3	2	-1	-5	-5	-1	0	-4	-4	0	0
	6	6	7	0	-2	0	0	0	-2	0	0	0
	5*	4	3	-1	-4	-4	-1	0*	-4	-4	0	0
	1	5	1	-5	-3	-6	-1	-4	-2	-5	0	0
c_j	6	8	7	0	0	0		0	0	0	←v_j	

*indicates solution. Sum is 19 units.

APPENDIX B2

Reduction of a Problem, k = 3, N = 3, to Marginal Zeros.

	G_{ijl}			$G_{ijl}^{(1)}$			$c_i^{(1)}$	$G_{ijl}^{(2)}$			$c_i^{(2)}$	$G_{ijl}^{(3)}$		
	2	4	6	-5	-7	-2	0	-5	-7	-2	0	-5	-7	-2
	6	4	8	-1	-7	0	0	-1	-7	0	0	-1	-7	0
	5	7	2	-2	-4	-6	-2	0	-2	-4	0	0	-2	-4
	3	7	2	-4	-4	-6	0	-4	-4	-6	-2	-2	-2	-4
	4	7	5	-3	-4	-3	0	-3	-4	-3	-2	-1	-2	-1
	2	2	4	-5	-9	-4	-2	-3	-7	-2	-2	-1	-5	0
	7	11	7	0	0	-1	0	0	0	-1	0	0	0	-1
	2	5	6	-5	-6	-2	0	-5	-6	-2	0	-5	-6	-2
	3	1	4	-4	-10	-4	-2	-2	-8	-2	0	-2	-8	-2
c_j	7	11	8	0	0	0		0	0	0				

APPENDIX C

Group Assembly Problem with k = 3, N = 4.

G_{ijl} taken from Fisher and Yates [4, p. 108].

	G_{ijl}				$T_{i.l}$	$T_{i..}$	$T_{..l}$	$\tilde{G}_{i..}$	$\tilde{G}_{..l}$	g_{ijl}			
	28	89	65	87	269	1023	963	64	60	-32	6	-2	9
	30	29	43	65	167	622	963	39	60	-5	-29	1	12
	95	74	62	60	291	827	963	52	60	47	3	7	-6
	01	85	54	96	236	865	963	54	60	-49	12	-3	28
	10	91	46	96	243	1023	559	64	35	-25	33	4	43
	05	33	18	08	64	622	559	39	35	-5	0	1	-20
	04	43	13	37	97	827	559	52	35	-19	-3	-17	-4
	05	85	40	25	155	865	559	54	35	-20	37	8	-18
	84	90	90	65	329	1023	949	64	59	25	8	24	-12
	28	55	53	09	145	622	949	39	59	-6	-2	12	-43
	89	83	40	69	281	827	949	52	59	42	13	-14	4
	73	20	96	05	194	865	949	54	59	24	-52	40	-62
	10	89	07	76	182	1023	866	64	54	-44	12	-54	4
	91	50	27	78	246	622	866	39	54	62	-2	-9	31
	03	45	44	66	158	827	866	52	54	-39	-20	-5	6
	89	41	59	91	280	865	866	54	54	45	-26	8	29
$T_{.j.}$	645	1002	757	933	3337	3337	3337			(-5)	(-10)	(1)	(1)
$\tilde{G}_{.j.}$	40	63	47	58				52	52	62	39	40	43

	g_{ijl}				$c_i^{(1)}$	$g_{ijl}^{(2)}$				u_i	w_l
	-94	-31	-42	-34	-15	-79	-16	-27	-19	0	0
	-67	-66	-39	-31	-15	-52	-51	-24	-16	0	0
	-15	-34	-33	-49	-15	0	-19	-18	-34	0	0
	-111	-25	-43	-15	-15	-96	-10	-28	0	0	0
	-87	-4	-36	0	0	-87	-4	-36	0	0	0
	-67	-37	-39	-63	0	-67	-37	-39	-63	0	0
	-81	-40	-57	-47	0	-81	-40	-57	-47	0	0
	-82	0	-32	-61	0	-82	0	-32	-61	0	0
	-37	-29	-16	-55	0	-37	-29	-16	-55	0	0
	-68	-39	-28	-86	0	-68	-39	-28	-86	0	0
	-20	-24	-54	-39	0	-20	-24	-54	-39	0	0
	-38	-89	0	-105	0	-38	-89	0	-105	0	0
	-106	-25	-94	-39	0	-106	-25	-94	-39	0	0
	0	-39	-49	-12	0	0	-39	-49	-12	0	0
	-101	-57	-45	-37	0	-101	-57	-45	-37	0	0
	-17	-63	-32	-14	0	-17	-63	-32	-14	0	0
v_j	0	0	0	0		0	0	0	0		