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Alternative equations of motion for the radiating electron

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Abstract. We have previously written down a simple second-order equation for the radiating electron and pointed out that its solutions are well behaved. A key feature of this equation is the presence of a term involving the time derivative of the external field $f(t)$. Here, we show that a completely equivalent, but less elegant, equation may be written down which contains no derivatives of $f(t)$ but, instead, an infinite number of derivatives of the coordinate. However, it has the merit of displaying explicitly how our result differs from that of Abraham-Lorentz.

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The equation of motion of a radiating electron has a long history, too long to discuss it here, except to reference some current textbook [1] and review [2,3] expositions on the subject. One highlight of these discussions is the emphasis given to the problem of "runaway solutions" associated with the celebrated Abraham-Lorentz (AL) equation. Thus, we were motivated to bring a new approach to the problem [4–7] with the goal of eliminating the problem of "runaway solutions". Our approach has four key features:

(*i*) The use of techniques from the realm of stochastic physics to treat the electron–radiation-field system. In particular, account was taken of the time dependence of both the electron and the radiation field.
(*ii*) The incorporation of electron structure via an electron form factor [8] $\Omega^2/(\Omega^2+\omega^2)$, where $\Omega \to \infty$ corresponds to the limiting case of a point electron, and ω is the photon frequency.

This enabled us to write down the equation of motion of a quantum oscillator with charge e and bare mass m, dipole interacting with the electromagnetic field and moving to a potential $V(r)$, in the form of a generalized quantum Langevin equation [4]:

$$m\ddot{x}(t) + \int_{-\infty}^{t} \mathrm{d}t'\,\mu(t-t')\dot{x}(t') + V'(x) = F(t) + f(t) \qquad (1)$$

where $x(t)$ is the coordinate operator, $F(t)$ is the operator-valued random (fluctuating) force, $f(t)$ is the external force, $\mu(t)$ is the memory function, and where the dot and prime denote, respectively, the derivative with respect to t and x.

This is an exact result and explicit values are known for $\mu(t)$ and $F(t)$. However, as is usual, mass renormalization is required.
(*iii*) The equation of motion is quantum mechanical.
(*iv*) The Fourier transform of the equation of motion may be written in the form of a response equation involving a generalized susceptibility with well-known analyticity properties. This enables us to use the vast mathematical machinery associated with such functions; in particular, it enables us to rule out a point-electron model (which is the basis of the AL equation). We then showed that choosing the cut-off frequency Ω to have its maximum value τ_e^{-1} (where $\tau_e = 2e^2/3\,Mc^3 = 6\times10^{-24}\,\mathrm{s}$, M being the renormalized mass of the electron) compatible with causuality considerations led to a considerable simplification. In this large cut-off limit, which corresponds to taking the bare-mass of the electron $m = 0$, (1) becomes [4]:

$$M\tau_e^{-1}\dot{x}(t) - M\tau_e^{-2}\int_{-\infty}^{t} \mathrm{d}t' \exp[-(t-t')\tau_e^{-1}]\dot{x}(t') + V'(x) = F(t) + f(t). \qquad (2)$$

Next, if we multiply this equation across by $\exp(-\tau_e^{-1}t)(\mathrm{d}l/\mathrm{d}t)\exp(\tau_e^{-1}t)$, we obtain the operator equation

$$M\ddot{x}(t) + V'_{\mathrm{eff}}(x) = F_{\mathrm{eff}}(t) + f_{\mathrm{eff}}(t), \qquad (3a)$$

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where

$$f_{\text{eff}}(t) \equiv f(t) + \tau_e \dot{f}(t), \tag{3b}$$

and similarly for the other "effective" quantities. It will be noted that (3) above [which is the same as (5) of [4] in the large cut-off limit] is a simple second-order equation for the radiating electron and, in contrast to, say, the Abraham-Lorentz equation, its solutions are well behaved.

Let us now return to (2) with the aim of writing it as a differential equation involving only derivatives of $x(t)$. First of all, we rewrite (2) in the form

$$\frac{M}{\tau_e}\dot{x}(t) - \frac{M}{\tau_e^2} I(t) + V'(x) = F(t) + f(t), \tag{4}$$

where

$$I = \int_{-\infty}^{t} \mathrm{d}t'\, G(t-t')\dot{x}(t'), \tag{5}$$

with

$$G(t-t') = \exp[-(t-t')/\tau_e]. \tag{6}$$

The following properties of $G(t-t')$ are found to be useful:

$$\frac{\mathrm{d}G(t-t')}{\mathrm{d}t'} = \frac{1}{\tau_e} G(t-t'), \tag{7}$$

$$G(0) = 1, \tag{8}$$

$$G(\infty) = 0. \tag{9}$$

Thus, from (5) and (7), we have

$$I(t) = \tau_e \int_{-\infty}^{t} \mathrm{d}t'\, G^{(1)}(t-t')x^{(1)}(t'), \tag{10}$$

where the superscript (n) denotes differentiation n times with respect to t' inside the integral and with respect to t otherwise. Integrating (10) by parts and using (7), we obtain

$$\begin{aligned} I(t) &= \tau_e x^{(1)} - \tau_e \int_{-\infty}^{t} \mathrm{d}t'\, G(t-t')x^{(2)}(t') \\ &= \tau_e x^{(1)}(t) - \tau_e^2 \int_{-\infty}^{t} \mathrm{d}t'\, G^{(1)}x^{(2)}(t'). \end{aligned} \tag{11}$$

treating the integral in (11) analogous to that in (10), we next obtain

$$\begin{aligned} I(t) = \tau_e x^{(1)}(t) - \tau_e^2 x^{(2)}(t) + \tau_e^3 x^{(3)}(t) \\ - \tau_e^4 \int_{-\infty}^{t} \mathrm{d}t'\, G^{(1)}(t-t')x^{(4)}(t'). \end{aligned} \tag{12}$$

Substituting (12) in (4) and using (7) leads to

$$\begin{aligned} &M\left[x^{(2)} - \tau_e x^{(3)} + \tau_e \int_{-\infty}^{t} \mathrm{d}t'\, G(t-t')x^{(4)}(t')\right] + V'(x) \\ &\quad = Mx^{(2)} + M\sum_{n=3}^{\infty} (-1)^n \tau_e^{n-2} x^{(n)} + V'(x) \\ &\quad = F(t) + f(t). \end{aligned} \tag{13}$$

This is our desired quantum equation of motion. It might be argued that the derivation of (13) from (4) could be obtained more easily if we carried out a Taylor expansion in powers of τ. But this would defeat the whole purpose because all our results are exact. In general, (13) is *not* a perturbative series.

The corresponding classical equation is obtained by taking the mean value of (13). Then, since the mean value of $F(t)$ is zero, the classical equation is formally the same as (13) except that $F(t)$ is dropped and all the quantities on the left-hand side should be interpreted as mean values. The integral on the left-hand side of (13) or, alternatively, all terms with derivatives beyond $x^{(3)}$ constitute the corrections to the AL equation. It is clear that the latter equation is a good approximation only when $x^{(3)}$ is slowly varying on a τ_e time scale. In particular, the latter condition does not hold in the case of "runaway solutions." In other words, such "runaway solutions" to the AL equation occur in region where the equation is no longer valid. Finally, we note that our original equation (3) is clearly more elegant than the completely equivalent equation (13); in particular, it is much easier to solve. In addition, our equation for the radiating electron has the merit of reducing to Newton's equation in the absence of an external force (a feature missing in the AL equation); this is immediately apparent from (3) but not from (13).

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