Flow of a viscoelastic fluid over a stretching sheet

K. R. Rajagopal, T. Y. Na, and A. S. Gupta

Department of Mechanical Engineering, University of Pittsburgh,
Department of Mechanical Engineering, University of Michigan at Dearborn, and
Department of Mathematics, Indian Institute of Technology at Kharagpur

Abstract: This paper presents a study of the flow of an incompressible second-order fluid past a stretching sheet. The problem has a bearing on some polymer processing application such as the continuous extrusion of a polymer sheet from a die.

Key words: Second-order fluid, stretching sheet, similarity solution

1. Introduction

The flow in the boundary layer of an incompressible viscous fluid on moving solid surfaces has been investigated by Sakiadis [1]. Due to the entrainment of the ambient fluid, this boundary layer is quite different from that in Blasius flow past a flat plate. Erickson, Fan and Fox [2] extended this problem to the case in which the transverse velocity at the moving surface is non-zero, with heat and mass transfer in the boundary layer being taken into account. These investigations have a bearing on the problem of a polymer sheet extruded continuously from a die. It is often tacitly assumed that the sheet is inextensible, but situations may arise in the polymer industry in which it is necessary to deal with a stretching plastic sheet, as pointed out by McCormack and Crane [3]. Danberg and Fansler [4] investigated the non-similar solution for the flow in the boundary layer past a wall that is stretched with a velocity proportional to the distance along the wall, the free-stream velocity being constant. Gupta and Gupta [5] analysed the heat and mass transfer corresponding to the similarity solution for the boundary layer over a stretching sheet subjected to suction or blowing.

All the above investigations were restricted to flows of Newtonian fluids. However, of late non-Newtonian fluids have become more and more important industrially. The laminar boundary layer on an inextensible continuous flat surface moving with a constant velocity in its own plane in a non-Newtonian fluid characterized by a power-law model (Ostwald-de Waele fluid) was studied by Fox, Erickson and Fan [6] using both exact and approximate methods. Apart from the limitations of the above power-law model, which does not exhibit any elastic properties (such as normal-stress differences in shear flow), in certain polymer processing applications one deals with flows of a viscoelastic fluid over a stretching sheet. This provides the motivation for the present study in which the flow of a class of viscoelastic fluids past a stretching sheet is examined. These are the incompressible second-order fluids whose constitutive equation based on the postulate of gradually fading memory was given by Coleman and Noll [7] as

$$\boldsymbol{T} = -p\,\boldsymbol{l} + \mu\,\boldsymbol{A}_1 + \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2, \tag{1.1}$$

where $\boldsymbol{T}$ is the stress tensor, p the pressure, μ, α_1, α_2 are material constants with $\alpha_1 < 0$, and $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are defined as

$$\boldsymbol{A}_1 = (\text{grad}\ \boldsymbol{v}) + (\text{grad}\ \boldsymbol{v})^{\mathrm{T}}, \tag{1.2}$$

$$\boldsymbol{A}_2 = \frac{d}{dt}\boldsymbol{A}_1 + \boldsymbol{A}_1 \cdot \text{grad}\ \boldsymbol{v} + (\text{grad}\ \boldsymbol{v})^{\mathrm{T}} \cdot \boldsymbol{A}_1 . \tag{1.3}$$

Coleman and Noll showed that the model (1.1) exhibits normal-stress differences in shear flow and is an approximation to a simple fluid in the sense of retardation. This model is applicable to some dilute polymer solutions (such as the 5.4 percent solution of polyisobutylene in cetane reported by Markovitz and Coleman [8]) at low rates of shear.

2. Calculation

We consider the flow of a second-order fluid obeying (1.1) past a wall coinciding with the plane $y = 0$, the flow being confined to $y > 0$. Two equal and opposite forces are applied along the x-axis so that the wall is stretched whilst keeping the origin fixed (figure 1).

The steady two-dimensional boundary layer equations for this fluid were derived by Beard and Walters [9]. In usual notation these equations are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2}-k\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}+v\frac{\partial^3 u}{\partial y^3}\right], \tag{2.2}$$

where

$$\nu=\mu/\varrho, \quad k=-\alpha_1/\varrho. \tag{2.3}$$

In deriving these equations it was assumed that in addition to the usual boundary layer approximations the contribution due to the normal stresses is of the same order of magnitude as that due to the shear stresses. Thus both ν and k are $O(\delta^2)$, δ being the boundary layer thickness.

The boundary conditions are

$$u=cx, \quad v=0 \quad \text{at} \quad y=0; \qquad u\to 0 \quad \text{as} \quad y\to\infty, \quad c>0. \tag{2.4}$$

The flow is caused solely by the stretching of the wall, the free stream velocity being zero. Eqs. (2.1) and (2.2) have the similarity solution

$$u=cxf'(\eta), \quad v=-(\nu c)^{1/2}f(\eta), \tag{2.5}$$

where the similarity variable η is given by

$$\eta=(c/\nu)^{1/2}y. \tag{2.6}$$

Clearly u and v given in (2.5) satisfy (2.1), and substituting into (2.2) gives

$$f'^2-ff''=f'''-k_1[2f'f'''-(f'')^2-ff^{\text{iv}}], \tag{2.7}$$

where a prime denotes differentiation with respect to η

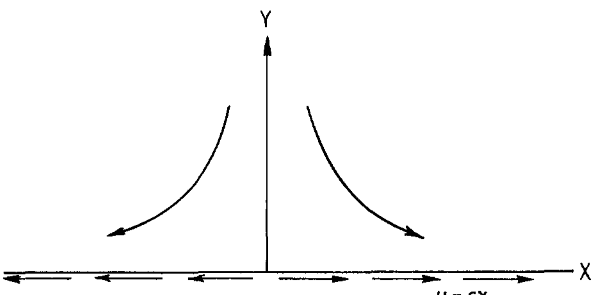

Fig. 1. A sketch of the physical problem

and

$$k_1=kc/\nu. \tag{2.8}$$

The boundary conditions (2.4) are transformed to

$$f'(0)=1, \quad f(0)=0, \quad f'(\infty)=0. \tag{2.9}$$

Eq. (2.7) is an equation of fourth order with the three boundary conditions (2.9). To overcome this difficulty (which essentially involves a singular perturbation problem) we follow Beard and Walters [9] and assume k_1 to be small. Since a second-order fluid obeying (1.1) represents the behaviour of fluids with short memory (Craik [10]) and the characteristic time scale associated with the motion is large compared with the time $|\alpha_1/\mu|$ representing the memory of the fluid, the assumption of small k_1 is valid, in particular, for dilute polymer solutions.

Hence, we expand f in a power series as

$$f=f_0+k_1f_1+k_1^2f_2+\dots. \tag{2.10}$$

Substituting (2.10) in (2.7) and equating the coefficients of k_1^0 and k_1 results in

$$f_0'^2-f_0f_0''=f_0''', \tag{2.11}$$

$$f_1'''+f_0f_1''+f_0''f_1-2f_0'f_1'=s, \tag{2.12}$$

with

$$s=2f_0'f_0'''-f_0''^2-f_0f_0^{\text{iv}}. \tag{2.13}$$

Using (2.9), the boundary conditions for f_0 and f_1 are

$$f_0'(0)=1, \quad f_0(0)=0, \quad f_0'(\infty)=0, \tag{2.14}$$

$$f_1'(0)=0, \quad f_1(0)=0, \quad f_1'(\infty)=0. \tag{2.15}$$

The solution of (2.11) satisfying (2.14) is

$$f_0(\eta)=1-e^{-\eta}. \tag{2.16}$$

For solving (2.12), which is linear in $f_1(\eta)$, we assume

$$f_1=f_A+\beta_1 f_B \tag{2.17}$$

such that

$$f_A'''+f_0f_A''-2f_0'f_A'+f_0''f_A=s, \tag{2.18}$$

$$f_B'''+f_0f_B''-2f_0'f_B'+f_0''f_B=0, \tag{2.19}$$

with the boundary conditions [see (2.15)]

$$f_A(0)=f_A'(0)=0, \quad f_A''(0)=0, \tag{2.20}$$

$$f_B(0)=f_B'(0)=0, \quad f_B''(0)=1. \tag{2.21}$$

Eqs. (2.18) and (2.19) are then integrated numerically using the Runge-Kutta method and the boundary conditions (2.20) and (2.21). To determine the parameter β_1 in (2.17), we use the third boundary condition in (2.15) which gives

$$\beta_1 = -\frac{f_A'(\infty)}{f_B'(\infty)}. \qquad (2.22)$$

Using (1.1), (2.5) and (2.9), the dimensionless shear stress τ at the wall is given by

$$\tau = x^{-1} \nu^{-1/2} c^{-3/2} (T_{xy})_{y=0} = (1 - k_1) f''(0). \qquad (2.23)$$

3. Results

The following table gives the values of τ for several values of k_1.

Table 1. Values of τ

k_1	τ
0.005	− 0.9975
0.01	− 0.9949
0.03	− 0.9846
0.05	− 0.9738

It is seen that the magnitude of the skin-friction coefficient decreases with increasing values of the elastic parameter k_1. For industrial applications, this result is of some importance since the power expenditure involved in stretching the sheet decreases with increasing k_1.

Table 2 gives the variation of $f'(\eta)$ with η for several values of k_1.

Table 2. Values of $f'(\eta)$

k_1 \ η	0.28	0.64	1.11	2.47	4.70	6.32	8.38
0.005	0.7535	0.5241	0.3290	0.0842	0.0090	0.0018	0.0002
0.01	0.7529	0.5232	0.3280	0.0837	0.0089	0.0017	0.0002
0.05	0.7487	0.5164	0.3205	0.0795	0.0080	0.0015	0.0001

Since u is proportional to $f'(\eta)$ [see (2.5)] it follows that u decreases as k_1 increases. Thus the boundary layer thickness increases as the value of the parameter k_1 increases, which might be regarded as a manifestation of the presence of normal stresses inside the boundary layer. In fact, the physical explanation of the thickening of the boundary layer may be attributed to tensile stresses in the layer which cause an axial contraction and hence thickening of the layer in the transverse direction.

It may be noticed that the cross-viscosity coefficient α_2 does not affect the velocity distribution since the flow is two-dimensional although α_2 affects the pressure distribution.

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Authors' addresses:

Dr. K. R. Rajagopal
Department of Mechanical Engineering
University of Pittsburgh
Pittsburgh, PA 15261 (USA)

Dr. T. Y. Na
Department of Mechanical Engineering
University of Michigan
Dearborn, MI 48128 (USA)

Prof. A. S. Gupta
Department of Mathematics
Indian Institute of Technology
Kharagpur 721302 (India)