# Short Communications

## Universal relations for instantaneous deformations of viscoelastic fluids

K. R. Rajagopal and A. S. Wineman

Department of Mechanical Engineering University of Pittsburgh, Pennsylvania, and Department of Mechanical Engineering and Applied Mechanics, The University of Michigan, Ann Arbor, Michigan, (U.S.A.)

*Abstract:* The note considers viscoelastic fluids which undergo an instantaneous homogeneous deformation consisting of shear superposed on triaxial extension. Two relations involving the stress and deformation components are presented, which are valid for all such fluids, and hence are termed "universal relations". The first contains the Lodge-Meissner relation as a special case; the second arises when a block is deformed by shear traction only. It relates dimensional changes to the amount of shear.

*Key words:* Viscoelastic fluids, instantaneous deformation, universal relation, combined shear and extension, Lodge-Meissner relation

### 1. Introduction

It is the purpose of this short note to point out two new universal relations which apply to viscoelastic fluids.

Consider an incompressible isotropic material which is initially at rest, subjected to an instantaneous arbitrary finite deformation at time $t = 0$, and then held in that state of deformation. Rivlin [1] observed that the response functional for the stress at any time $t > 0$ reduces to a function of the deformation gradient $\boldsymbol{F}$, which is constant, and the elapsed time $t$. He then showed that this is an isotropic function of $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$, with scalar coefficients which depend on the invariants of $\boldsymbol{B}$ and time $t$. Rivlin pointed out that, except for the dependence on time $t$, the constitutive equation has the same form as that for an isotropic incompressible nonlinear elastic material. As a consequence, many of the results for nonlinear elastic materials immediately apply to general isotropic materials undergoing instantaneous deformations. The results of interest in this note are universal relations, i.e., relations between the various physical quantities which hold for all materials of a particular class, independent of material parameters. Thus, universal relations can be used to determine whether a particular fluid can indeed be characterized by a constitutive expression which belongs to the class which satisfies the universal relations. Of special interest is the universal relation in simple shear in which the ratio of the difference of the normal stresses in the plane of the shear to the shear stress is independent of material properties and is linear in the amount of shear.

Later, Lodge and Meissner [2] considered incompressible viscoelastic fluids which were deformed from one state of rest to another by an instantaneous simple shear deformation. They assumed the stress to be bounded, and thus that the fluid exhibited instantaneous elasticity. For fluids for which the stress can be calculated from the configurational entropy of a Gaussian molecular network, they showed that the above mentioned universal relation in simple shear holds for times $t > 0$. This relation is referred to in the polymer fluid literature as the "Lodge-Meissner" relation. They then argued on heuristic grounds that the universal relation applies to any incompressible liquid with instantaneous elasticity. Lodge [3] subsequently provided a proof that the universal relation does indeed apply to the broader class of fluids. His proof is similar to that given by Rivlin, in that when the fluid goes from one state of rest to another by an instantaneous deformation at $t = 0$, the stress is given as an isotropic function of the elapsed time and $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$, where $\boldsymbol{F}$ is the deformation gradient for the instantaneous deformation. Since this is the same form as the constitutive equation for an isotropic incompressible nonlinear elastic solid, the universal relation for simple shear of isotropic elastic solids is extended to the broader class of fluids under consideration.

Using the approach of Rivlin and Lodge, we exhibit two new universal relations which also apply to viscoelastic fluids with instantaneous elastic response. Wineman and Gandhi [4] derived a new universal relation for isotropic nonlinear elastic materials that is more general than that for simple shear, and contains the latter as a special case. Thus, in the context of viscoelastic fluids with instantaneous elasticity, this new universal relation contains the Lodge-Meissner relation as a special case. Rajagopal and Wineman [5] used this new universal relation to derive another which is of a purely geometric nature within the context of isotropic nonlinear elasticity, and arises when shear tractions are applied to a block of material and normal tractions are absent.

## 2. Two new universal relations

With respect to a cartesian coordinate system, let $X_i$ denote either (a) the coordinate of a particle of an elastic body in its reference state, or (b) the coordinate of a body at time $t = 0^-$, just preceding the instantaneous deformation. Let $x_i$ denote either (a) the coordinate of the elastic body in its current state, or (b) the coordinate of the body at time $t = 0^+$, just after the instantaneous deformation. Let $F_{ij} = [\partial x_i / \partial X_j]$. Consider the homogeneous deformation:

$$x_1 = \lambda_1 X_1 + \varkappa \lambda_2 X_2, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3. \quad (1)$$

In this deformation, the material undergoes changes in length along each coordinate axis, followed by a simple shear in which plane $x_2 =$ constant displaces along the $x_1$ direction. Parameter $\varkappa$ represents the displacement along the $x_1$ axis per unit current length in the $x_2$ direction, and is interpreted as the amount of shear.

The constitutive equation for an isotropic elastic solid has the form

$$\boldsymbol{\sigma} = \alpha_1 \boldsymbol{I} + \alpha_2 \boldsymbol{B} + \alpha_3 \boldsymbol{B}^2, \quad (2)$$

where $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$, $\boldsymbol{I}$ denotes the identity, and $\alpha_1, \alpha_2, \alpha_3$ are scalars which depend on the invariants of $\boldsymbol{B}$. According to Rivlin and Lodge, this is also the constitutive equation for an isotropic material which goes from one state of rest to another by an instantaneous deformation, in which case $\alpha_1, \alpha_2, \alpha_3$ also depend on time $t$.

It is a straightforward calculation to establish, using eq. (1) and eq. (2), the following universal relation for simple shear superposed on triaxial extension

$$\sigma_{11} - \sigma_{22} = \left(\frac{\lambda_1^2 + \lambda_2^2 \varkappa^2 - \lambda_2^2}{\varkappa \lambda_2^2}\right) \sigma_{12}. \quad (3)$$

Since (3) is independent of $\alpha_1$ as well as $\alpha_2$ and $\alpha_3$, this universal relation is equally valid for compressible and incompressible materials.

Note that when $\lambda_1 = \lambda_2$, relation (3) reduces to the form

$$\sigma_{11} - \sigma_{22} = \varkappa \sigma_{12}. \quad (4)$$

When $\lambda_1 = \lambda_2 = 1$, this is known in the polymer fluid literature as the Lodge-Meissner relation. In the context of nonlinear elasticity, this relation has been attributed to Rivlin [6].

Now consider a block of material whose edges are parallel to the coordinate axes. Suppose that this block is subjected to the deformation (1) and that this deformation is produced by shear tractions only. Rajagopal and Wineman [5] have shown that the absence of applied normal tractions implies that the separation between the faces of the block and the amount of shear satisfy

$$\lambda_1^2 = \lambda_2^2 (1 + \varkappa^2). \quad (5)$$

This is a new universal relation which involves geometric quantities only. It holds for both incompressible and compressible isotropic materials.

For the specific fluid model considered by Lodge and Meissner in [2], when the response to an instantaneous arbitrary deformation is analogous to that of a neo-Hookean solid, it is found that the new dimensions of the block are related to the shear by

$$\lambda_1 = (1 + \varkappa^2)^{1/3}, \quad \lambda_2 = \lambda_3 = (1 + \varkappa^2)^{-1/6}. \quad (6)$$

## References


1. Rivlin RS (1956) Quarterly of Applied Mathematics XIV:83
2. Lodge AS, Meissner J (1972) Rheol Acta 11:351
3. Lodge AS (1975) Rheol Acta 14:664
4. Wineman A, Gandhi M (1984) J Elasticity 14:97
5. Rajagopal KR, Wineman AS (1987) J Elasticity 17:75
6. Rivlin RS, Proc 2nd Tech Conf (London, June 23–25, 1948), Cambridge: Heffer


(Received February 16, 1988)

Authors' addresses:

Prof. K. R. Rajagopal
Department of Mechanical Engineering
University of Pittsburgh
Pittsburgh, PA 15261 (U.S.A.)

Prof. A. S. Wineman
Department of Mechanical Engineering and Applied Mechanics
The University of Michigan
Ann Arbor, MI 48109 (U.S.A.)