The Frattini module

By

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1. Introduction. This note resulted from an effort to answer the following question of R. Baer. Suppose G is a finite group with a unique minimal normal subgroup N which is nonabelian. If p is a prime dividing $|N|$, does there exist a (finite) faithful $\mathbb{F}_p G$-module A admitting a group extension $A \rightarrowtail H \twoheadrightarrow G$ with $A \subseteq \Phi(H)$. This has been answered independently by W. Gaschütz and the authors; the answer is yes.

Throughout the paper, K denotes a field of characteristic p and G is a finite group whose order is divisible by p. All modules are finitely generated. If M is an irreducible (right) KG-module, then P_M is the (principal) indecomposable projective KG-module with M in the head, i.e. with $P_M/P_M J \cong M$ where $J = \Phi_{KG}(KG)$ is the Jacobson radical of KG. K is always regarded as trivial KG-module. $A_K(G)$ denotes the kernel of a projective cover $P_1 \twoheadrightarrow P_K J$; $A_K(G)$ is uniquely determined up to isomorphism.

We recall a theorem by Gaschütz.

Result (Gaschütz [4]). *Let $K = \mathbb{F}_p$ and $A = A_K(G)$. Then there exists a Frattini extension $A \rightarrowtail \hat{G} \twoheadrightarrow G$, i.e. with $A \subseteq \Phi(\hat{G})$. Any other Frattini extension of G by a (finite) KG-module is an epimorphic image over G of $\hat{G}$.*

The maximal Frattini extension $A \rightarrowtail \hat{G} \twoheadrightarrow G$ can be constructed via free presentations. We give the necessary details in section 3.

In view of Gaschütz's result, $A_K(G)$ is called the Frattini module of G (with respect to K). It is known that $A_K(G) \neq 0$ (since the characteristic p of K divides $|G|$; for a p'-group H, $A_K(H) = 0$) and that $A_K(G)$ is indecomposable (Lemma 1). An irreducible KG-module M has nonvanishing 1-cohomology if and only if M occurs in the socle of $A_K(G)$; $H^2(G, M) \neq 0$ if and only if M occurs in the largest completely reducible factor module ("head") of $A_K(G)$ (Lemma 3).

On the basis of these observations we establish first:

Theorem 1. *The centralizer in G of the socle of $A_K(G)$ is just the greatest p-nilpotent normal subgroup $O_{p'p}(G)$ of G.*

This already answers Professor Baer's question, because of Gaschütz's result.

The corresponding statement for the head of the Frattini module is false, in general. A counterexample is provided by the alternating group $G = \mathrm{Alt}(5)$ and $K = \mathbb{F}_2$. On the other hand, if G is p-solvable, $O_{p'p}(G)$ is also the centralizer of all irreducible KG-modules with nonvanishing 2-cohomology. This is a consequence of

Theorem 2. *If G is p-solvable, then* $\dim H^2(G, M) \geqq \dim H^1(G, M)$ *for any irreducible KG-module M.*

The proof of Theorem 2 depends upon the famous Fong-Swan theorem and a cohomological property of liftable modules (see section 5).

We are able to characterize the groups G for which $A_K(G)$ is faithful:

Theorem 3. *Let $A = A_K(G)$. Then either $C_G(A) = O_{p'}(G)$ or G is p-supersolvable with cyclic Sylow p-subgroups; the latter happens if and only if* $\dim A = 1$ *(and so $C_G(A) = O_{p'p}(G)$).*

In particular, $A_K(G)$ is faithful if and only if $O_{p'}(G) = 1$ and G is not metacyclic having a cyclic normal Sylow p-subgroup.

2. Minimal resolutions. It is convenient to discuss the cohomological aspects of the Frattini module within the context of minimal resolutions. Let (P_0, d_0) be a projective cover of the trivial KG-module K and, inductively for $n \geqq 1$, (P_n, d_n) a projective cover of the kernel $Y_{n-1} = \mathrm{Ker}\, d_{n-1}$. Then the exact sequence

$$\mathfrak{A}_K(G): \quad \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} K \to 0$$

is called a minimal (KG-projective) resolution of K. By uniqueness of projective covers all Y_n, P_n are uniquely determined (up to isomorphism). Clearly $P_0 = P_K$, $Y_0 = P_K J$, and $Y_1 = A_K(G)$. By construction all P_n belong to the principal p-block.

Lemma 1. *All Y_n are indecomposable, nonzero, and nonprojective.*

Note our permanent assumption that the characteristic p of K divides $|G|$. A proof of Lemma 1 is very accessible in Gruenberg [6] (see especially Theorem 2.9): $Y_0 \neq 0$ since K is not projective; Y_0 has no proper projective direct summand for this were a direct summand of P_0 contained in $P_0 J$ (projective KG-modules are injective; apply then Nakayama's lemma); Y_0 is indecomposable by a result of Heller [8], because K is indecomposable. Proceed by induction.

It is known that for a projective KG-module P socle and head are isomorphic. (It suffices to handle the case where P is indecomposable. A proof in this situation is outlined in Serre [10], Exercise 14.6.) We make use of this in proving

Lemma 2. *The socle* $\mathrm{soc}(Y_n)$ *is isomorphic to $P_n / P_n J$ ($n \geqq 0$).*

Proof. The claim is evident for $n = 0$; so let $n \geqq 1$. We have to show that any minimal submodule M of P_n is contained in Y_n. Let $P = P_M$ be the associated projective direct summand of P_n. Assume $M \nsubseteq Y_n$. Then $P \cap Y_n = 0$ since M is the unique minimal submodule of P. It follows that P is isomorphic to a submodule of

$$P_n / Y_n \cong Y_{n-1} \subseteqq P_{n-1} J .$$

Since P is injective, P is isomorphic to a direct summand of Y_{n-1}, contrary to Lemma 1. (Of course, one may also argue directly by means of Nakayama's lemma.)

Let M be a completely reducible KG-module. Then, because of $Y_{n-1} \subseteqq P_{n-1} J$, the map $\mathrm{Hom}_{KG}(P_{n-1}, M) \to \mathrm{Hom}_{KG}(Y_{n-1}, M)$ is zero. Since $P_n / P_n J \cong Y_{n-1} / Y_{n-1} J$, by definition of cohomology via projective resolutions we have thus, for $n \geqq 0$, K-isomorphisms

$$(*) \qquad H^n(G, M) \cong \mathrm{Hom}_{KG}(P_n, M) \cong \mathrm{Hom}_{KG}(P_n/P_n J, M)\,.$$

This is not new (see for instance [6], Lemma 2.11). Since $Y_1 \cong P_2/Y_2$ is the Frattini module $A = A_K(G)$, (*) together with Lemma 2 yields:

Lemma 3. *Let M be an irreducible KG-module and $D = \mathrm{Hom}_{KG}(M, M)$. Denote by r and s the multiplicities of M in $\mathrm{soc}(A)$ and A/AJ, respectively. Then there are D-isomorphisms $H^1(G, M) \cong D^{(r)}$, $H^2(G, M) \cong D^{(s)}$.*

Changing the field carries minimal resolutions to minimal resolutions. We include the basic argument.

Lemma 4. *Suppose L is an extension field of K. If M is an irreducible KG-module, then $M^L = M \otimes_K L$ is a completely reducible LG-module and*

$$\dim_L H^n(G, M^L) = \dim_K H^n(G, M) \quad (n \geqq 0)\,.$$

Proof. The existence of a (finite) splitting field (of characteristic p) shows that KG/J is a separable algebra. Observe also that $D = \mathrm{Hom}_{KG}(M, M)$ is a (commutative) field which is separable over K. We may conclude that $J^L = J \otimes_K L$ is the Jacobson radical of $LG = KG \otimes_K L$.

It is now clear that M^L is completely reducible. (In the important case that L/K is a finite Galois extension, M^L is a direct sum of irreducible LG-modules conjugate under the Galois group, each isomorphism type appearing with the same multiplicity.) Moreover, we see that

$$\mathfrak{A}_K(G)^L\colon \quad \cdots \to P_2^L \to P_1^L \to P_0^L \to L \to 0$$

is a minimal (LG-projective) resolution of L. Using the natural isomorphisms $\mathrm{Hom}_{LG}(P_n^L, M^L) = \mathrm{Hom}_{KG}(P_n, M)^L$ we have thus, in view of (*), isomorphisms of L-vector spaces

$$H^n(G, M^L) \cong \mathrm{Hom}_{LG}(P_n^L, M^L) \cong H^n(G, M) \otimes_K L$$

for $n \geqq 0$. This proves the lemma.

As $\mathfrak{A}_K(G)^L = \mathfrak{A}_L(G)$ is a minimal resolution of L, for any extension field L of K, we may infer that all kernels Y_n are indeed absolutely indecomposable. In particular, $A_K(G)$ is absolutely indecomposable. The (exact) scalar extension functor $- \otimes_K L$ takes $A_K(G)$ to $A_L(G)$, preserving the Loewy structures. Essentially we may restrict ourselves, therefore, to the study of the Frattini module of G over the prime field $K = \mathbb{F}_p$.

3. Relation modules. We fix a free presentation $R \rightarrowtail F \twoheadrightarrow G$ with F of finite rank $d(F)$. Let P be a maximal projective submodule of the KG-module $R^K = R/R' \otimes K$, and let $A = R^K/P$. Clearly $R^K \cong P \oplus A$, and A possesses no proper projective summand.

Using the above notation, we have the following

Lemma 5 (Gaschütz [4]). *A is isomorphic to $A_K(G)$, and*

$$P \oplus P_1 \cong P_K \oplus (KG)^{(d(F)-1)}$$

where P_1 is the projective cover of $P_K J$.

For a proof we refer also to Gruenberg [6], Theorem 2.9.

Gaschütz has handled the (crucial) case $K = \mathbb{F}_p$ only. Here we may identify $R^K = R/R' R^p$. Write bars for factor groups modulo $R' R^p$. If $M \rightarrowtail H \twoheadrightarrow G$ is a group extension with M a (finite) KG-module and $d(H) \leqq d(F)$, then there exists always an epimorphism $\varphi \colon \bar{F} \to H$ "over G", i.e. making the diagram

$$\begin{array}{ccccc} \bar{R} & \rightarrowtail & \bar{F} & \twoheadrightarrow & G \\ \downarrow & & \downarrow \varphi & & \| \\ M & \rightarrowtail & H & \twoheadrightarrow & G \end{array}$$

commutative ([6], Proposition 6.14). This provides an approach to Gaschütz's other result mentioned in the introduction. One verifies that $A \rightarrowtail \bar{F}/P \twoheadrightarrow G$ represents the maximal Frattini p-extension of G. This extension may also be described by taking any minimal supplement of $\bar{R}$ in $\bar{F}$.

Remarks. Gruenberg has extended the Gaschütz theory to more general coefficient rings. Consider for instance the $\mathbb{Z}G$-lattice R/R'. Let $R/R' = P \oplus A$ be a projective excision, i.e. P is $\mathbb{Z}G$-projective and A has no proper projective summand. (Note, however, that the Krull-Schmidt theorem does not hold for $\mathbb{Z}G$-lattices.) Then A is a faithful $\mathbb{Z}G$-module, provided G is not cyclic ([6], Proposition 5.12). This result has stimulated Theorem 3.

If $A = A_K(G)$ for K a field, as usual, $\dim H^2(G, A) = 1$ by a theorem of Tate (cf. [5], § 11.3), and any nonsplit extension $A \rightarrowtail \hat{G} \twoheadrightarrow G$ represents the (unique) maximal essential extension in the category $(\frac{KG}{})$ discussed in [5], § 11. "Essential" means that any supplement of A in $\hat{G}$ which intersects A in a KG-module coincides with $\hat{G}$.

Lemma 6. *Let H be a subgroup of G. Then, as a KH-module, $A_K(G) \cong A_K(H) \oplus Q$ for some projective KH-module Q.*

Proof. Consider the free presentation $R \rightarrowtail F \twoheadrightarrow G$; let F_0 denote the inverse image in F of H. By Schreier's theorem, $R \rightarrowtail F_0 \twoheadrightarrow H$ is a free presentation of H with F_0 of finite rank. From Lemma 5 it follows that there are a projective KG-module P and a projective KH-module P_0 such that

$$A_K(G) \oplus P \cong R/R' \otimes K \cong A_K(H) \oplus P_0$$

as KH-modules. Since P is also KH-projective and since $A_K(H)$ has no proper projective summand (Lemma 1), application of the Krull-Schmidt theorem gives the assertion.

If G is a p-group, the minimum number of generators $d(G) = \dim H^1(G, K)$ equals the dimension of the socle of $A = A_K(G)$, and $\dim A/AJ = \dim H^2(G, K)$ is the minimum number of KG-generators $d_{KG}(A)$ of A. In this case $P_K = KG$, and

$$\dim A = 1 + (d(G) - 1)\,|G|\,.$$

Proposition 1. *Let G_p be a Sylow p-subgroup of G. Then* $\dim A_K(G) = 1 + k\,|G_p|$ *for some integer* $k \geqq d(G_p) - 1$.

Proof. Straightforward from Lemma 6.

4. Socle and head of $A_K(G)$. By a theorem of Brauer [3], $O_{p'}(G)$ is the centralizer (kernel) of the principal p-block (ideal). Since the Frattini module $A = A_K(G)$ belongs to the principal p-block, we have thus

$$(**) \qquad O_{p'}(G) \subseteqq C_G(A).$$

It follows that $O_{p'p}(G)$ is contained in the centralizer of both $\operatorname{soc}(A)$ and A/AJ.

Theorem 1. $O_{p'p}(G) = C_G(\operatorname{soc}(A))$.

First Proof. Let $C = C_G(\operatorname{soc}(A))$. By Lemma 3 C is the centralizer of the irreducible KG-modules with nonvanishing 1-cohomology. We know $C \supseteqq O_{p'p}(G)$. Assume C is not p-nilpotent. Let $H = O^p(C)$ be the smallest normal subgroup of C with p-factor group. Since H is not a p'-group, application of Shapiro's lemma and the exact cohomology sequence gives the existence of an irreducible KH-module B with $H^1(H, B) \neq 0$. Since $H^1(H, K) = \operatorname{Hom}(H, K) = 0$, B is nontrivial.

The same argument yields a KG-composition factor M of the coinduced module $\operatorname{Hom}_{KH}(KG, B)$ with $H^1(G, M) \neq 0$. By Clifford theory, $C_H(M)$ is the intersection of some conjugates in G of $C_H(B)$. Since B is a nontrivial KH-module, this implies $C_G(M) \not\supseteqq H = O^p(C)$, a contradiction.

Second Proof. The following argument is more direct and avoids the use of Shapiro's lemma. Assume again that $H = O^p(C)$ is not a p'-group. Then $A_K(H) \neq 0$ by Lemma 1. By Lemma 6 $A_K(H)$ is isomorphic to a KH-submodule of $A = A_K(G)$. Let B be a minimal KH-submodule of A occuring in the socle of $A_K(H)$. Let $\bar{B} = \sum_{g \in G} Bg$. Since H is normal in G, Bg is a KH-module being G-conjugate to B ($g \in G$). (Compare with Clifford theory.)

Let M be a minimal KG-submodule of $\bar{B}$. Then M is centralized by H. On the other hand, M is a direct sum of irreducible KH-modules conjugate to B. Consequently $B \cong K$ is a trivial KH-module and so by Lemma 3 $\operatorname{Hom}(H, K) = H^1(H, K) \neq 0$, contradicting $O^p(H) = H$.

The situation for 2-cohomology, i.e. for the head of the Frattini module, is quite different. To illustrate this we give an example.

Example 1. Let $G = \operatorname{Alt}(5)$ be the alternating group of degree 5.

(a) $K = \mathbb{F}_2$.

There are 3 distinct types of irreducible KG-modules, say K, P, X. P is the reduction of the (absolutely) irreducible $\mathbb{Q}G$-module of dimension 4 coming from the permutation representation of G on 5 letters; P is the Steinberg module and thus projective (cf. [10], § 16.4). X decomposes over $\mathbb{F}_4$ into 2 distinct absolutely irreducible modules of dimension 2 which are conjugate under a field automorphism. (Note

$G \cong SL(2, 4)$; X is a composition factor of the indecomposable permutation module obtained from the 2-transitive representation of $G \cong PSL(2, 5)$ on 6 letters.) We have $KG = P_K \oplus P_X^{(2)} \oplus P^{(4)}$.

By direct calculation (or looking at the Cartan matrix given in [10], § 18.6) one gets the following Loewy sections for P_K, P_X:

$$P_K\colon \quad K, X, K \oplus K, X, K,$$

$$P_X\colon \quad X, K \oplus K, X, K \oplus K, X.$$

Hence $A = A_K(G)$ has dimension 5, and $A/AJ \cong K$, $\operatorname{soc}(A) \cong X$. In particular, X is the only irreducible KG-module with nonvanishing 1-cohomology, K the only one with nonvanishing 2-cohomology.

(b) $K = \mathbb{F}_3$.

Here $A_K(G)$ is (absolutely) irreducible of dimension 4.

(c) $K = \mathbb{F}_5$.

Besides the trivial module and the Steinberg module, there is only one irreducible KG-module Y ($\dim Y = 3$). Socle and head of $A_K(G)$ are isomorphic to Y, and $\dim A_K(G) = 6$.

Remark. Baer's problem as formulated at the very beginning can be settled easily if the (unique, nonabelian) minimal normal subgroup N of G has a nontrivial irreducible module X over $\mathbb{F}_p$ with $H^2(N, X) \neq 0$. In general, since N is not p-nilpotent, the Frobenius p-nilpotence criterion and Shapiro's lemma guarantee the existence of an $\mathbb{F}_p N$-module Y such that $H^2(N, Y) \neq 0$ and $H^0(N, Y) = 0$. The coinduced module $M = \operatorname{Hom}_N(\mathbb{F}_p G, Y)$ satisfies $H^n(G, M) \cong H^n(N, Y)$ for all $n \geqq 0$ (Shapiro). Let $M \rightarrowtail H \twoheadrightarrow G$ be any nonsplit extension. Then every minimal supplement of M in H represents a faithful Frattini extension of G, as desired.

If G is p-solvable (and K a field of characteristic p, as usual), there is always a nontrivial irreducible KG-module with nonvanishing 2-cohomology, provided G is not p-nilpotent. This will be shown in the next section. Here we handle the following special situation.

Proposition 2. *If the Sylow p-subgroups of G are cyclic, socle and head of $A = A_K(G)$ are irreducible, and $O_{p'p}(G) = C_G(A/AJ)$.*

Proof. Since the Sylow p-subgroups of G are cyclic, by a result of Alperin and Janusz [1] every projective KG-module P_n appearing in the minimal resolution $\mathfrak{A}_K(G)$ of K is indecomposable. Hence from Lemmas 1 and 3 it follows that $\operatorname{soc}(A) \cong P_1/P_1 J$ and $A/AJ \cong P_2/P_2 J$ are irreducible. ($A_K(G)$ is even uniserial here!)

Let $H = C_G(A/AJ)$. By (**) $H \supseteq O_{p'p}(G)$. Assume H is not p-nilpotent. Since the Sylow p-subgroups of H are cyclic, we know that socle and head of $A_K(H)$ are irreducible. From Theorem 1 it follows $H^1(H, K) = 0$, hence $O^p(H) = H$ and $\operatorname{Ext}(H/H', K) = 0$. Furthermore p does not divide the order of the Schur multiplier $H_2(H)$ of H. From the universal coefficient theorem for cohomology we may infer that $H^2(H, K) = 0$. But $A_K(H)$ is isomorphic to a factor module of A (Lemma 6).

Since H is a normal subgroup of G centralizing A/AJ, Lemma 3 yields the contradiction $H^2(H, K) \neq 0$. (Clearly one may also argue using Shapiro.)

5. p-solvable groups. By a p-adic field we mean a field of characteristic 0 which is complete with respect to a discrete valuation and whose residue class field is of characteristic p. Hasse and Schmidt have proved that there are p-adic fields with prescribed residue class fields (cf. [7], p. 10). We fix a p-adic field E with residue class field K. Let S be the valuation ring of E, $\mathfrak{p}$ its maximal ideal ($K = S/\mathfrak{p}$). A KG-module M is called liftable (to S or E) if there is an S-free SG-module $\tilde{M}$ whose reduction modulo $\mathfrak{p}$ is M ($M = \tilde{M} \otimes_S K$).

Proposition 3 (Scott [9]). *If M is an irreducible KG-module which can be lifted, then* $\dim H^2(G, M) \geqq \dim H^1(G, M)$.

Proof. Scott's proof deals with Brauer characters, assuming that E is a splitting field for G. We give an alternative approach avoiding characters. It is well known that the minimal resolution $\mathfrak{A}_K(G)$ of K (section 2) can be lifted to a minimal (SG-projective) resolution

$$\mathfrak{A}_S(G): \quad \cdots \to \tilde{P}_2 \xrightarrow{\tilde{d}_2} \tilde{P}_1 \xrightarrow{\tilde{d}_1} \tilde{P}_0 \xrightarrow{\tilde{d}_0} S \to 0$$

of S (see for instance [10], § 14.4). By hypothesis there is an S-free SG-module $\tilde{M}$ such that $M = \tilde{M} \otimes_S K$. Write bars for the corresponding EG-modules. Since $\mathrm{Hom}_{SG}(\tilde{P}_n, \tilde{M})$ is S-free, we have natural isomorphisms

$$\mathrm{Hom}_{SG}(\tilde{P}_n, \tilde{M}) \otimes_S E = \mathrm{Hom}_{EG}(\bar{P}_n, \bar{M})$$

and $$\mathrm{Hom}_{SG}(\tilde{P}_n, \tilde{M}) \otimes_S K = \mathrm{Hom}_{KG}(P_n, M)\,.$$

From (*) it follows $\dim_K H^n(G, M) = \dim_E \mathrm{Hom}_{EG}(\bar{P}_n, \bar{M})$ ($n \geqq 0$).

By Maschke's theorem $\bar{P}_n$ is completely reducible. Let a_n denote the multiplicity of the (irreducible) EG-module $\bar{M}$ in $\bar{P}_n$, b_n that in $\bar{Y}_n = \tilde{Y}_n \otimes_S E$ ($\tilde{Y}_n = \mathrm{Ker}\,\tilde{d}_n$), and let $c = \dim \mathrm{Hom}_{KG}(M, M)$. Then $\dim H^n(G, M) = c\,a_n$ and $b_n = a_{n+1} - b_{n+1}$ ($n \geqq 0$). If $M \cong K$ is a trivial module, $a_0 - b_0 = 1$ and $H^0(G, M) \cong K$. If M is nontrivial, $a_0 = b_0$ and $H^0(G, M) = 0$. Thus for any integer $r \geqq 1$

$$\sum_{n=1}^{r} (-1)^n \dim H^n(G, M) = (-1)^r c\, b_r\,.$$

Specializing to $r = 2$ gives $\dim H^2(G, M) - \dim H^1(G, M) = c\,b_2 \geqq 0$, as desired.

It is obvious that the trivial module $M = K$ can be lifted. In this case Proposition 3 may also be deduced from the universal coefficient theorem. One obtains more precisely

$$\dim H^2(G, K) = \dim H^1(G, K) + \dim H_2(G) \otimes K\,.$$

Theorem 2. *Assume G is p-solvable. Then the socle of the Frattini module $A = A_K(G)$ is isomorphic to a direct summand of A/AJ.*

Proof. According to Lemma 3, the claim of Theorem 2 is equivalent to the statement that $\dim H^2(G, M) \geqq \dim H^1(G, M)$ for every irreducible KG-module M. This will be derived from the preceding proposition.

By the classical result of Hasse-Schmidt there is a p-adic field E with residue class field K. Let F be the field obtained from E by adjoining the $|G|$-th roots of unity. F is again p-adic and its residue class field L is a finite extension of K. By Lemma 4 $M^L = M \otimes_K L$ is a completely reducible LG-module and $\dim_K H^n(G, M) = \dim_L H^n(G, M^L)$ for all $n \geqq 0$. By the Fong-Swan theorem (for a lucid proof see [10], § 17.6) every irreducible summand of M^L can be lifted. Now apply Proposition 3.

Corollary 1. *If G is p-solvable, $O_{p'p}(G) = C_G(A/AJ)$.*

Proof. Immediate from (**) and Theorems 1 and 2.

In case G is p-solvable, $\mathrm{soc}(A)$ can be described very elegantly. Denote by A_0 the direct sum of all complemented p-chief factors of G, counting multiplicities with regard to some fixed chief series of G. It is known that an irreducible $\mathbb{F}_pG$-module M has nonvanishing 1-cohomology here if and only if M is isomorphic (as G-module) to a complemented p-chief factor of G, and that $H^1(G, M) \cong \mathrm{Hom}_G(C_G(M), M)$. For a proof based on the Hochschild-Serre sequence we refer to [2], Lemmas 3 and 4. It follows that A_0 is isomorphic to the socle of the Frattini module of G over $\mathbb{F}_p$.

Applying Theorem 2 and Gaschütz's result stated in the introduction, we have therefore the following

Corollary 2. *Suppose G is p-solvable and A_0 is the direct sum of all complemented p-chief factors of G, as above. Then A_0 is isomorphic to the socle of $A_K(G)$ $(K = \mathbb{F}_p)$, and there is a Frattini extension $A_0 \rightarrowtail H \twoheadrightarrow G$ of G.*

6. p-supersolvable groups. Recall that G is p-supersolvable if and only if $G/O_{p'p}(G)$ is an abelian group whose exponent divides $p-1$. In particular, p-supersolvable groups are of p-length 1. Because of Brauer's theorem, G is p-supersolvable if and only if every irreducible module in the principal p-block is of dimension 1 (cf. [2]; it suffices to study the case $K = \mathbb{F}_p$).

Proposition 4. *Assume G is p-solvable of p-length 1. Then the restriction from G to a Sylow p-subgroup G_p of G takes $\mathfrak{A}_K(G)$ to $\mathfrak{A}_K(G_p)$, preserving lower and upper Loewy series of any projective P_n.*

Proof. Let M be a KG-module all of whose composition factors are in the principal block. We show that $MJ = [M, G_p]$. This means that M/MJ is also the largest completely reducible factor module of M viewed as a KG_p-module. The proof that $\mathrm{soc}(M) = H^0(G_p, M)$ is the group of fixed points under G_p is quite similar.

By hypothesis $O_{p'p}(G) = O_{p'}(G)\, G_p$. Applying Brauer's theorem therefore yields $[M, G_p] = [M, O_{p'p}(G)] \subseteqq MJ$. On the other hand, $M/[M, G_p]$ is a completely reducible $K[G/O_{p'p}(G)]$-module (Maschke). Consequently $MJ = [M, G_p]$, as desired.

It is now evident that $\mathfrak{A}_K(G)$, viewed as a sequence of KG_p-modules, is a minimal KG_p-projective resolution of K.

Corollary 1. *Let G be of p-length 1, and let $A = A_K(G)$ as usual. Then*

$$\dim A = 1 + (d(G_p) - 1)\,|G_p|, \quad \dim \operatorname{soc}(A) = d(G_p) \quad \text{and}$$
$$\dim A/AJ = \dim H^2(G_p, K).$$

Moreover, $\dim A/AJ > \frac{1}{4}(\dim \operatorname{soc}(A))^2$.

Proof. Only the final statement needs some comment. It follows from the Golod-Šafarevič theorem which says that

$$\dim H^2(G_p, K) > \frac{d(G_p)^2}{4}$$

(cf. [5], § 7.3).

Corollary 2 (Gaschütz). *If G is p-supersolvable, $d_{KG}(A) > \dfrac{d(G_p)^2}{4h}$ where h is the order of $G/O_{p'p}(G)$.*

Proof. Since the irreducibles in the principal p-block have dimension 1, by Nakayama's lemma and Lemma 3

$$d_{KG}(A) = d_{KG}(A/AJ) = \max\{\dim H^2(G, M)\},$$

where M runs through all irreducible modules occuring in A/AJ. By Brauer's theorem, the number of these modules does not exceed $h = |\operatorname{Hom}(G/O_{p'p}(G), \mathbb{F}_p^*)|$. (In fact, h is just the number of distinct irreducibles in the principal p-block.) Thus $d_{KG}(A) \geqq \frac{1}{h}\dim A/AJ$. Apply Corollary 1.

Corollary 3. $\dim A = 1$ *if and only if G is p-supersolvable with cyclic Sylow p-subgroups.*

Proof. If $\dim A = 1$, G has cyclic Sylow p-subgroups by Proposition 1. Furthermore, then $C_G(A) = O_{p'p}(G)$ according to Theorem 1, and $G/C_G(A)$ is cyclic of order dividing $p - 1$.

Conversely, if G is p-supersolvable having cyclic Sylow p-subgroups, $\dim A = 1$ by Corollary 1. (In the solvable case the result of Alperin-Janusz becomes obvious!)

We see that the Frattini module is trivial if and only if G is p-nilpotent with cyclic Sylow p-subgroups.

7. Faithful Frattini modules. Theorem 3 is a consequence of Theorem 1, the preceding Corollary 3, and the following

Proposition 5. *Suppose the dimension of $A = A_K(G)$ is at least 2. Then*

$$C_G(A) = O_{p'}(G).$$

Proof. By (**) $C_G(A) \supseteqq O_{p'}(G)$. Assume $C_G(A) \neq O_{p'}(G)$. Then there is a cyclic p-subgroup $H \neq 1$ in $C_G(A)$. Clearly $A_K(H) \cong K$. By Lemma 6, $A \cong K \oplus Q$ as a KH-module where Q is KH-projective (KH-free). Since $\dim A \geqq 2$, $Q \neq 0$. Now Q is a direct sum of some copies of the regular module KH and thus faithful for H. It follows $C_H(A) = 1$, a contradiction. We are done.

Corollary. *The Frattini module $A_K(G)$ is faithful if and only if $O_{p'}(G) = 1$ and G is not metacyclic having cyclic Sylow p-subgroups.*

Proof. Clear.

In particular, $A_K(G)$ is faithful when G is a noncyclic p-group. In this case one can construct an explicit Frattini extension of G by a faithful $\mathbb{F}_p G$-module as follows.

Example 2. Let $K = \mathbb{F}_p$ and G be a p-group with $d = d(G) > 1$. Let $\{x_1, \ldots, x_d\}$ be a minimal set of generators for G and $\{e_x \mid x \in G\}$ a K-basis of KG. G acts on KG by $e_x y = e_{xy}$. Consider the subgroup $H = \langle (e_{x_i}, x_i) \mid i = 1, \ldots, d\rangle$ of the semidirect product of KG and G (which is the regular wreath product $K \wr G$). Let $B = \{(z, 1) \mid (z, 1) \in H\}$. Then $H/B \cong G$, and from $d(H) = d(G)$ it follows $B \subseteqq \Phi(H)$. It is straightforward that B is indeed a faithful module for G.

By Gaschütz's result, B is an epimorphic image of $A = A_K(G)$. Note that $\dim B < |G| < \dim A$. One may ask for a faithful KG-module B of least dimension, $f(G)$, admitting a Frattini extension $B \rightarrowtail H \twoheadrightarrow G$. We give an upper bound for $f(G)$.

If G contains a noncyclic maximal subgroup N which is not a direct factor, $f(G) \leqq p \cdot f(N)$ by Shapiro's lemma. If no such N exists, G is either elementary abelian or a quaternion group of order 8. When G is elementary abelian of rank $r = d = 2$ and p odd, it is easy to show that $f(G) = 3$ by constructing an example H. For arbitrary r we get $f(G) \leqq 3^{[(r+1)/2]}$ by taking a subgroup of index 1 or p in the direct product of $\left[\frac{r+1}{2}\right]$ copies of that group H. In general, if G is a p-group, p odd, containing an elementary abelian subgroup of order p^r, then

$$f(G) \leqq p^{-r} |G| \cdot 3^{[(r+1)/2]}.$$

This is quite a bit less than $\dim A = 1 + |G| (d-1)$. In case $p = 2$, one obtains

$$f(G) \leqq 2^{-r} |G| \left(5^{[(r+1)/2]} - \delta\right),$$

where $\delta = 0$ if r is even and $\delta = 1$ otherwise.

Example 3. Let $\tilde{G} = SL(2, 5)$ and $K = \mathbb{F}_2$. By the above corollary and Gaschütz's result there is a faithful $K\tilde{G}$-module B of least dimension admitting a Frattini extension $B \rightarrowtail H \twoheadrightarrow \tilde{G}$. Of course, B is an epimorphic image of $A_K(\tilde{G})$. We claim that B is uniserial of dimension 9.

Let $Z(\tilde{G}) = \langle z \rangle$ and $G = \tilde{G}/\langle z \rangle$, the alternating group of degree 5. By minimality, $X = [B, z] = \operatorname{soc}(B)$ is an irreducible $K\tilde{G}$-module. Let $C = H^0(\langle z \rangle, B)$ and D be the inverse image in H of $\langle z \rangle$. Then $B/C \cong X$ and $D = \Phi(H)$. Since $\tilde{G}$ is the full covering group of G, B/C is a nontrivial KG-module. We may conclude that $X = D' \subseteqq \Phi(D) \subseteqq C$ and that $\Phi(D)/X$ is of order 1 or 2.

Now $D/\Phi(D) \rightarrowtail H/\Phi(D) \twoheadrightarrow G$ is a Frattini extension. Thus from Example 1 it follows that $\Phi(D) = C$, $D/C \cong A_K(G)$, and $X \cong B/C$ is the 4-dimensional KG-module with nonvanishing 1-cohomology.

Assume $C = X$. Then B is a free module for $\langle z \rangle$. But then $H^n(\langle z \rangle, B) = 0$ for all $n \geqq 1$. In particular, D splits over B and the complements are conjugate. If L is a complement to B in D, by a Frattini argument we get

$$H = N_H(L)\, B = N_H(L)\,,$$

because $B \subseteqq \Phi(H)$. However, this would imply that D is abelian. Consequently $C/X \cong K$ is of order 2. B is an uniserial $K\bar{G}$-module with Loewy sections X, K, X and $\dim B = 9$, as asserted.

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