Linear independence of root equations for $M/G/1$ type Markov chains

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There is a classical technique for determining the equilibrium probabilities of $M/G/1$ type Markov chains. After transforming the equilibrium balance equations of the chain, one obtains an equivalent system of equations in analytic functions to be solved. This method requires finding all singularities of a given matrix function in the unit disk and then using them to obtain a set of linear equations in the finite number of unknown boundary probabilities. The remaining probabilities and other measures of interest are then computed from the boundary probabilities. Under certain technical assumptions, the linear independence of the resulting equations is established by a direct argument involving only elementary results from matrix theory and complex analysis. Simple conditions for the ergodicity and nonergodicity of the chain are also given.

Keywords: Matrix analytic method, transform method, ergodicity.

1. Introduction

Transform techniques have proved useful in the study of Markov chains of $M/G/1$ type. With this approach, the balance equations of the Markov chain are transformed to obtain an expression for the generating function of the state probabilities, typically, queue occupancy probabilities. The expression thus obtained usually contains some unknown boundary probabilities, the determination of which involves the analyticity of the generating function in the open unit disk. As part of the procedure of solving for these unknown probabilities, one needs to determine the singularities of a given matrix function within the unit disk. These singularities are then used to obtain a set of linearly independent equations, the solution of which yields the required unknown boundary probabilities. This provides a characterization of the generating function of the equilibrium state probabilities.

The above procedure is well-known and has been applied extensively in the analysis of Markov chains of $M/G/1$ type. However, to the best of our knowledge, only Bailey [1] proved the independence of the linear equations obtained from the zeros for a certain subclass of chains (see also [3]). The goal of this paper is to establish the independence of the linear equations whose solution yields the unknown boundary probabilities for rather general $M/G/1$ type Markov chains. The algorithmic analysis of these chains has been pioneered by Neuts [8]. However, the work presented here is in the spirit of the early papers of Neuts [7] and Çinlar [2], in which a rigorous transform approach to these problems was carried out. We consider discrete time Markov chains, since the analysis of continuous time Markov chains of $M/G/1$ type is similar.

2. $M/G/1$ type Markov chains

For some positive integer M let the pair of integers (i,j), $i \geq 0$, $0 \leq j \leq M-1$, denote the states of an irreducible discrete time Markov chain of $M/G/1$ type. Typically, in queueing systems the integer i corresponds to the number of customers in the queue (the *level*) and can take arbitrary nonnegative values. The integer j corresponds to the *phase* or the *stage* of the system and is assumed to take finitely many values. For $j = 0, 1, \ldots, M-1$, $l = 0, 1, \ldots, M-1$, $k \geq 0$, and a positive integer N, the one step transition probabilities of the discrete time Markov chain are given as follows:

$$b_{i,k;j,l} = P[\text{transition from state } (i,j) \text{ to state } (k,l)], \quad 0 \leq i \leq N-1,$$
$$a_{k;j,l} = P[\text{transition from state } (i,j) \text{ to state } (k+i-N,l)], \quad i \geq N.$$

Thus the one step transitions from state (i,j) are homogeneous starting at level N, i.e., for $i \geq N$, transitions from (i, j) to (k,l) depend on i and k through the difference $k-i$. Naturally, we have $\Sigma_{k=0}^{\infty}\Sigma_{l=0}^{M-1} b_{i,k;j,l} = 1$ and $\Sigma_{k=0}^{\infty}\Sigma_{l=0}^{M-1} a_{k;j,l} = 1$ for all i,j.

Denote the irreducible one step transition probability matrix of the Markov chain by $\mathbf{P}$. When written in matrix form $\mathbf{P}$ is

$$\mathbf{P} \triangleq \begin{bmatrix} \mathbf{b}_{0,0} & \mathbf{b}_{0,1} & \cdots & \mathbf{b}_{0,N-1} & \mathbf{b}_{0,N} & \mathbf{b}_{0,N+1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ \mathbf{b}_{N-1,0} & \mathbf{b}_{N-1,1} & \cdots & \mathbf{b}_{N-1,N-1} & \mathbf{b}_{N-1,N} & \mathbf{b}_{N-1,N+1} & \cdots \\ \mathbf{a}_0 & \mathbf{a}_1 & \cdots & \mathbf{a}_{N-1} & \mathbf{a}_N & \mathbf{a}_{N+1} & \cdots \\ 0 & \mathbf{a}_0 & \cdots & \mathbf{a}_{N-2} & \mathbf{a}_{N-1} & \mathbf{a}_N & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1}$$

where

$$\mathbf{a}_k \triangleq \begin{bmatrix} a_{k;0,0} & a_{k;0,1} & \cdots & a_{k;0,M-1} \\ a_{k;1,0} & a_{k;1,1} & \cdots & a_{k;1,M-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k;M-1,0} & a_{k;M-1,1} & \cdots & a_{k;M-1,M-1} \end{bmatrix},$$

$$\mathbf{b}_{i,k} \triangleq \begin{bmatrix} b_{i,k;0,0} & b_{i,k;0,1} & \cdots & b_{i,k;0,M-1} \\ b_{i,k;1,0} & b_{i,k;1,1} & \cdots & b_{i,k;1,M-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{i,k;M-1,0} & b_{i,k;M-1,1} & \cdots & b_{i,k;M-1,M-1} \end{bmatrix}.$$

Our interest is in determining $\pi_{i,j}$, the equilibrium probability of state (i,j), when these probabilities exist. Let $\mathbf{\Pi}_i \triangleq [\pi_{i,0}, \pi_{i,1}, \ldots, \pi_{i,M-1}]$ be the $1 \times M$ vector of equilibrium probabilities associated with the ith level, and let $\mathbf{\Pi} = [\mathbf{\Pi}_0, \mathbf{\Pi}_1, \ldots]$. The equilibrium equations $\mathbf{\Pi} = \mathbf{\Pi}\mathbf{P}$ for the chain are

$$\mathbf{\Pi}_k = \sum_{i=0}^{N-1} \mathbf{\Pi}_i \mathbf{b}_{i,k} + \sum_{i=N}^{N+k} \mathbf{\Pi}_i \mathbf{a}_{N+k-i}, \quad k \geq 0. \tag{2}$$

Define the following generating functions:

$$B_{i;j,l}(z) \triangleq \sum_{k=0}^{\infty} b_{i,k;j,l} z^k, \quad j,l = 0,1,\ldots,M-1, \quad i = 0,1,\ldots,N-1,$$

$$A_{j,l}(z) \triangleq \sum_{k=0}^{\infty} a_{k;j,l} z^k, \quad j,l = 0,1,\ldots,M-1,$$

$$G_l(z) \triangleq \sum_{i=N}^{\infty} \pi_{i,l} z^{i-N}, \quad l = 0,1,\ldots,M-1.$$

Note that the functions $B_{i;j,l}(z)$, $A_{j,l}(z)$ are analytic in the open unit disk and continuous in the closure. This is also true of the functions $G_l(z)$ when the $\pi_{i,j}$ exist. Next define $\mathbf{G}(z) \triangleq [G_0(z), G_1(z), \ldots, G_{M-1}(z)]$, and let

$$\mathbf{A}(z) \triangleq \begin{bmatrix} A_{0,0}(z) & A_{0,1}(z) & \cdots & A_{0,M-1}(z) \\ A_{1,0}(z) & A_{1,1}(z) & \cdots & A_{1,M-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ A_{M-1,0}(z) & A_{M-1,1}(z) & \cdots & A_{M-1,M-1}(z) \end{bmatrix},$$

$$\mathbf{B}_i(z) \triangleq \begin{bmatrix} B_{i;0,0}(z) & B_{i;0,1}(z) & \cdots & B_{i;0,M-1}(z) \\ B_{i;1,0}(z) & B_{i;1,1}(z) & \cdots & B_{i;1,M-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ B_{i;M-1,0}(z) & B_{i;M-1,1}(z) & \cdots & B_{i;M-1,M-1}(z) \end{bmatrix}.$$

The functions $\mathbf{A}(z)$ and $\mathbf{B}_i(z)$ may be regarded as matrix generating functions, namely, $\mathbf{A}(z) = \Sigma_{k=0}^{\infty}\mathbf{a}_k z^k$ and $\mathbf{B}_i(z) = \Sigma_{k=0}^{\infty}\mathbf{b}_{i,k}z^k$. Transforming (2) we obtain

$$\mathbf{G}(z)[z^N\mathbf{I}_M - \mathbf{A}(z)] + \sum_{i=0}^{N-1} \mathbf{\Pi}_i[z^i\mathbf{I}_M - \mathbf{B}_i(z)] = 0, \tag{3}$$

where $\mathbf{I}_M$ is the $M \times M$ identity matrix. Note that $\mathbf{A}(1)$ and $\mathbf{B}_i(1)$, $0 \le i \le N-1$, are stochastic matrices. Equation (3) is the fundamental relation for the discrete time Markov chain of $M/G/1$ type. We discuss the determination of the probability vector $\mathbf{\Pi}$ in the next section.

Example: Bailey's bulk queue

Consider a discrete time queue with c servers. The service time of a customer is one slot, and the probability of k customers arriving at the system during a slot is α_k (customers that arrive in a slot can be served only in a subsequent slot). For this queueing system $M = 1$ and $N = c$. Let i correspond to the number of customers in the queue at slot boundaries. We have that $b_{i,k;0,0} = a_{k;0,0} = \alpha_k$, $0 \le i \le c-1$, $k \ge 0$. Therefore, $B_{i;0,0}(z) = A_{0,0}(z) = \Sigma_{k=0}^{\infty}\alpha_k z^k = \alpha(z)$, and

$$G_0(z)[z^c - \alpha(z)] + \sum_{i=0}^{c-1} \pi_{i,0}[z^i - \alpha(z)] = 0.$$

In [1] Bailey considered a discrete time Markov chain defined at the departure instants of a continuous time bulk queueing system. Customer arrivals to the bulk queue were assumed to be Poisson with rate λ, and the time between departure epochs was assumed to have mean $\bar{v}$ and Laplace transform $\beta(s)$. This embedded chain is a special case of the discrete time system described above with $\alpha(z) = \beta(\lambda - \lambda z)$. It was shown in [1] that if $\lambda\bar{v} < c$, then $z^c - \alpha(z)$ has c distinct simple zeros in the unit disk when $\beta(s) = [\mu/(s+\mu)]^p$. Furthermore, a direct proof that these zeros yield c linearly independent equations for the c unknowns $\pi_{i,0}$, $0 \le i \le c-1$, was also given in [1]. Note that in this scalar example the functions $B_{i;0,0}(z) = A_{0,0}(z)$ for all i. The simplified structure when all $\mathbf{B}_i(z) = \mathbf{A}(z)$ also appears in several nonscalar examples, such as the c server queue with constant service times and a versatile Markovian arrival process (see [8]).

Example: An ATM multiplexer

Consider a slotted system for which voice and data traffic is to be multiplexed over a single channel [6]. Time slots are aggregated into frames, with N slots constituting a frame. Voice connections become active or inactive at the beginning of a frame, and at most $K \leq N$ connections can be active during a frame. The number of active voice connections is governed by a $K+1$ state irreducible aperiodic Markov chain with transition matrix $Q = [q_{j,l}]$. Each active voice connection occupies one slot of the frame, and the remaining slots are allocated to data packets. Each data packet is one slot in length, and there is an infinite buffer for them. The arrival process of data packets is given by a generating function $R(z)$, and packets that arrive during a frame can only be transmitted in subsequent frames. Let $i = 0, 1, \ldots$ correspond to the number of data packets in the system at the beginning of a frame, and let $j = 0, 1, \ldots, K$ correspond to the number of active voice connections at the beginning of a frame. The resulting Markov chain is of $M/G/1$ type with

$$\mathbf{A}(z) = R(z)\begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0K} \\ zq_{10} & zq_{11} & \cdots & zq_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ z^K q_{K0} & z^K q_{K1} & \cdots & z^K q_{KK} \end{bmatrix}.$$

Also, $\mathbf{B}_i(z) = R(z)Q$ for $0 \leq i \leq N-K$, and for $i = N-K+1, \ldots, N-1$,

$$\mathbf{B}_i(z) = R(z)\begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0K} \\ \vdots & \vdots & \ddots & \vdots \\ zq_{N-i+1,0} & zq_{N-i+1,1} & \cdots & zq_{N-i+1,K} \\ \vdots & \vdots & \ddots & \vdots \\ z^{K-N+i} q_{K0} & z^{K-N+i} q_{K1} & \cdots & z^{K-N+i} q_{KK} \end{bmatrix}.$$

In this case we have

$$\mathbf{G}(z)[z^N \mathbf{I}_{K+1} - \mathbf{A}(z)] + \sum_{i=0}^{N-1} \mathbf{\Pi}_i [z^i \mathbf{I}_{K+1} - \mathbf{B}_i(z)] = 0.$$

Remark

For a general $M/G/1$ type Markov chain, the transition matrix has the expanded form (see [8])

$$\mathbf{P} \triangleq \begin{bmatrix} \mathbf{b}_{-1,-1} & \mathbf{b}_{-1,0} & \mathbf{b}_{-1,1} & \cdots & \mathbf{b}_{-1,N-1} & \mathbf{b}_{-1,N} & \mathbf{b}_{-1,N+1} & \cdots \\ \mathbf{b}_{0,-1} & \mathbf{b}_{0,0} & \mathbf{b}_{0,1} & \cdots & \mathbf{b}_{0,N-1} & \mathbf{b}_{0,N} & \mathbf{b}_{0,N+1} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ \mathbf{b}_{N-1,-1} & \mathbf{b}_{N-1,0} & \mathbf{b}_{N-1,1} & \cdots & \mathbf{b}_{N-1,N-1} & \mathbf{b}_{N-1,N} & \mathbf{b}_{N-1,N+1} & \cdots \\ 0 & \mathbf{a}_0 & \mathbf{a}_1 & \cdots & \mathbf{a}_{N-1} & \mathbf{a}_N & \mathbf{a}_{N+1} & \cdots \\ 0 & 0 & \mathbf{a}_0 & \cdots & \mathbf{a}_{N-2} & \mathbf{a}_{N-1} & \mathbf{a}_N & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{4}$$

where $\mathbf{b}_{-1,-1}$ is $K \times K$, $\mathbf{b}_{-1,k}$ is $K \times M$ for $k = 0, 1, \ldots$, and $\mathbf{b}_{i,-1}$ is $M \times K$ for $i = 0, \ldots, N-1$. That is, there are K additional boundary states that cannot be reached from the homogeneous part of the chain. However, the analysis of such chains reduces to the previous case ($K = 0$) in the following way.

Transforming the equations $\mathbf{\Pi} = \mathbf{\Pi P}$, with $\mathbf{\Pi} = [\mathbf{\Pi}_{-1}, \mathbf{\Pi}_0, \mathbf{\Pi}_1, \ldots]$, we obtain the set of equations

$$-\sum_{i=0}^{N-1} \mathbf{\Pi}_i \mathbf{b}_{i,-1} + \mathbf{\Pi}_{-1}[\mathbf{I}_K - \mathbf{b}_{-1,-1}] = 0, \tag{5}$$

$$\mathbf{G}(z)[z^N \mathbf{I}_M - \mathbf{A}(z)] + \sum_{i=0}^{N-1} \mathbf{\Pi}_i [z^i \mathbf{I}_M - \mathbf{B}_i(z)] - \mathbf{\Pi}_{-1}\mathbf{B}_{-1}(z) = 0, \tag{6}$$

with $\mathbf{B}_{-1}(z) = \Sigma_{k=0}^{\infty} \mathbf{b}_{-1,k} z^k$. Since $\mathbf{P}$ is irreducible, the spectral radius of $\mathbf{b}_{-1,-1}$ must be less than 1, which implies that $\mathbf{I}_K - \mathbf{b}_{-1,-1}$ is invertible. We thus may solve (5) for $\mathbf{\Pi}_{-1}$ as

$$\mathbf{\Pi}_{-1} = \sum_{i=0}^{N-1} \mathbf{\Pi}_i \mathbf{b}_{i,-1}[\mathbf{I}_K - \mathbf{b}_{-1,-1}]^{-1}.$$

Substituting this expression into eq. (6), we obtain

$$\mathbf{G}(z)[z^N I_M - \mathbf{A}(z)] + \sum_{i=0}^{N-1} \mathbf{\Pi}_i [z^i \mathbf{I}_M - \mathbf{B}_i^*(z)] = 0, \tag{7}$$

where for $i = 0, \ldots, N-1$

$$\mathbf{B}_i^*(z) = \mathbf{B}_i(z) + \mathbf{b}_{i,-1}[\mathbf{I}_K - \mathbf{b}_{-1,-1}]^{-1}\mathbf{B}_{-1}(z) \tag{8}$$

$$= \mathbf{B}_i(z) + \sum_{l=0}^{\infty} \mathbf{b}_{i,-1}[\mathbf{b}_{-1,-1}]^l \mathbf{B}_{-1}(z). \tag{9}$$

Thus (7) has the same form as eq. (3).

Clearly, $\mathbf{B}_i^*(z)$ has nonnegative power series coefficients. Also, it is easy to see that $\mathbf{B}_i^*(1)$ is a stochastic matrix. Simply use $\mathbf{B}_{-1}(1)\mathbf{1} = \mathbf{1} - \mathbf{b}_{-1,-1}\mathbf{1} = [\mathbf{I}_M - \mathbf{b}_{-1,-1}]\mathbf{1}$ and eq. (8) to obtain

$$\begin{aligned}\mathbf{B}_i^*(1)\mathbf{1} &= \mathbf{B}_i(1)\mathbf{1} + \mathbf{b}_{i,-1}[\mathbf{I}_K - \mathbf{b}_{-1,-1}]^{-1}\mathbf{B}_{-1}(1)\mathbf{1} \\ &= \mathbf{B}_i(1)\mathbf{1} + \mathbf{b}_{i,-1}\mathbf{1} \\ &= \mathbf{1}.\end{aligned}$$

The $M/G/1$ type Markov chain that is represented in eq. (7) is the embedded chain obtained by considering those times when the original chain is not in the K additional boundary states. This can be seen directly from the representation of $\mathbf{B}_i^*(z)$ given in (9). A transition from a boundary level $i = 0, \ldots, N-1$ to a level $k \geq 0$ may either occur in one step via the term $\mathbf{B}_i(z)$ or it may take place when the original process first enters the additional boundary states, remains there for l steps, and then enters level k. This embedded chain is irreducible, and the study of chains with the additional boundary states reduces to the study of chains of the form given in (1).

3. Assumptions and main results

In this section we introduce several assumptions about the matrices $\mathbf{A}(z)$ and $\mathbf{B}_i(z)$, $0 \leq i \leq M-1$. We then state and prove the main results of our paper. As explained in Section 2 the first step in the analysis of (3) is the determination of the singularities of the matrix function $\mathbf{\Delta}(z) = z^N\mathbf{I}_M - \mathbf{A}(z)$ within the unit disk. Recall that $\mathbf{A}(z)$ and $\mathbf{B}_i(z)$, $0 \leq i \leq M-1$, are analytic in the open unit disk and continuous in its closure. Throughout the paper we make the following assumptions:

(A1) All zeros of det $\mathbf{\Delta}(z)$ in the closed unit disk are simple.

(A2) The function det $\mathbf{\Delta}(z)$ does not vanish on the unit circle except at $z = 1$.

Under these assumptions, the number and location of the zeros of det $\mathbf{\Delta}(z)$ in the closed unit disk depend on the quantity

$$\gamma = \frac{d}{dz}\det \mathbf{\Delta}(z)\bigg|_{z=1}. \tag{10}$$

Note that $\gamma \neq 0$ since $z = 1$ is a simple zero of det $\mathbf{\Delta}(z)$. The following result is known [8].

LEMMA 1

Let assumptions (A1)-(A2) hold.

(i) If $\gamma > 0$, then $\det \mathbf{\Delta}(z)$ has exactly $MN - 1$ zeros in the open unit disk and a simple zero at $z = 1$.

(ii) If $\gamma < 0$, then $\det \mathbf{\Delta}(z)$ has exactly MN zeros in the open unit disk and a simple zero at $z = 1$.

Now we can exploit the analyticity of $G_l(z)$, $0 \le l \le M - 1$, in the open unit disk. We rewrite (3) as

$$\mathbf{G}(z) \det \mathbf{\Delta}(z) = \sum_{i=0}^{N-1} \mathbf{\Pi}_i [\mathbf{B}_i(z) - z^i \mathbf{I}_M] \operatorname{adj} \mathbf{\Delta}(z), \tag{11}$$

where $\operatorname{adj} \mathbf{\Delta}(z)$ is the classical adjoint of the matrix $\mathbf{\Delta}(z)$. Since $G_l(z)$, $0 \le l \le M - 1$ are analytic functions in the open unit disk, the vector on the righthand side of (11) must be zero whenever $\det \mathbf{\Delta}(z)$ vanishes there. Note that each zero of $\det \mathbf{\Delta}(z)$ yields M equations for the unknowns $\pi_{i,j}$, $0 \le j \le M - 1$, $0 \le i \le N - 1$. However, we will prove that it yields only one linearly independent equation. We first prove two lemmas that give known results about the adjoint of the matrix $\mathbf{\Delta}(\xi)$ when ξ is a simple zero of $\det \mathbf{\Delta}(\xi)$ (not necessarily in the unit disk) and the known structure of harmonic vectors $\mathbf{X}$ the entries of which satisfy $x_i \to 0$.

LEMMA 2

Suppose ξ is a simple zero of $\det \mathbf{\Delta}(z)$. Then the rank of the matrix $\operatorname{adj} \mathbf{\Delta}(\xi)$ is 1.

Proof

From the equation

$$\mathbf{\Delta}(\xi) \operatorname{adj} \mathbf{\Delta}(\xi) = \det \mathbf{\Delta}(\xi) \mathbf{I}_M = 0, \tag{12}$$

we see that the columns of $\operatorname{adj} \mathbf{\Delta}(\xi)$ are right null vectors of the matrix $\mathbf{\Delta}(\xi)$. Since the dimension of the right null space of $\mathbf{\Delta}(\xi)$ is 1 when ξ is a simple zero of $\det \mathbf{\Delta}(z)$ (see [4]), it follows that the columns of $\operatorname{adj} \mathbf{\Delta}(\xi)$ are all multiples of the same vector. □

LEMMA 3

Suppose $\mathbf{PX} = \mathbf{X}$ and the entries of $\mathbf{X}$ satisfy $\lim_{i\to\infty} x_i = 0$. Then $\mathbf{X} = 0$.

Proof

Since $x_i \to 0$, there is some entry of $\mathbf{X}$, say x_l, such that $|x_l| \ge |x_i|$ for all i.

Now $\mathbf{P}^n\mathbf{X} = \mathbf{X}$ for all n, so

$$x_l = \sum_{j=0}^{\infty} p_{l,j}^{(n)} x_j,$$

where $p_{l,j}^{(n)}$ is the (l,j) entry of $\mathbf{P}^n$. Using the fact that $\mathbf{P}^n$ is stochastic, we obtain

$$|x_l| \leq \sum_{j=0}^{\infty} p_{l,j}^{(n)} |x_j| \leq \sum_{j=0}^{\infty} p_{l,j}^{(n)} |x_l| = |x_l|.$$

This shows that $|x_j| = |x_l|$ for all j such that $p_{l,j}^{(n)} \neq 0$. But $\mathbf{P}$ is irreducible, so for every pair (l,j) there exists $n = n(l,j)$ such that $p_{l,j}^{(n)} \neq 0$. Thus all entries of $\mathbf{X}$ have the same modulus. Since $\lim_{i\to\infty} x_i = 0$, this common value must be zero, i.e. $\mathbf{X} = 0$. □

THEOREM 1

Suppose ξ is a simple zero of $\det \mathbf{\Delta}(z)$ in the open unit disk. Then there is exactly one linearly independent equation among the M equations

$$\sum_{i=0}^{N-1} \mathbf{\Pi}_i[\mathbf{B}_i(\xi) - \xi^i \mathbf{I}_M] \operatorname{adj} \mathbf{\Delta}(\xi) = 0.$$

Proof

There is at most one linearly independent equation among the M equations by lemma 2. The existence of at least one equation, i.e., not all of the coefficients $[\mathbf{B}_i(\xi) - \xi^i \mathbf{I}_M] \operatorname{adj} \mathbf{\Delta}(\xi)$ are zero, is proved as follows. We begin by defining the $MN \times M$ matrix function

$$\mathbf{B}(z) \triangleq \begin{bmatrix} \mathbf{I}_M - \mathbf{B}_0(z) \\ z\mathbf{I}_M - \mathbf{B}_1(z) \\ \vdots \\ z^{N-1}\mathbf{I}_M - \mathbf{B}_{N-1}(z) \end{bmatrix}.$$

Defining the $\infty \times M$ matrix $\mathbf{E}(z)$ as

$$\mathbf{E}(z) \triangleq \begin{bmatrix} \mathbf{I}_M \\ z\mathbf{I}_M \\ z^2\mathbf{I}_M \\ \vdots \end{bmatrix},$$

we have that ($\mathbf{I}$ is the $\infty \times \infty$ identity matrix)

$$[\mathbf{I}-\mathbf{P}]\mathbf{E}(z)=\begin{bmatrix} \mathbf{I}_M-\mathbf{B}_0(z) \\ z\mathbf{I}_M-\mathbf{B}_1(z) \\ \vdots \\ z^{N-1}\mathbf{I}_M-\mathbf{B}_{N-1}(z) \\ z^N\mathbf{I}_M-\mathbf{A}(z) \\ z^{N+1}\mathbf{I}_M-z\mathbf{A}(z) \\ \vdots \end{bmatrix}=\begin{bmatrix} \mathbf{B}(z) \\ \mathbf{\Delta}(z) \\ z\mathbf{\Delta}(z) \\ \vdots \end{bmatrix}. \tag{13}$$

Multiplying this equation by $\operatorname{adj}\mathbf{\Delta}(z)$ and using (12), we obtain that at a zero ξ of $\det\mathbf{\Delta}(z)$

$$[\mathbf{I}-\mathbf{P}]\mathbf{E}(\xi)\operatorname{adj}\mathbf{\Delta}(\xi)=\begin{bmatrix} \mathbf{B}(\xi)\operatorname{adj}\mathbf{\Delta}(\xi) \\ 0 \end{bmatrix}. \tag{14}$$

Therefore, if $\mathbf{B}(\xi)\operatorname{adj}\mathbf{\Delta}(\xi)=0$, we have $\mathbf{PX}=\mathbf{X}$ where $\mathbf{X}\triangleq\mathbf{E}(\xi)\operatorname{adj}\mathbf{\Delta}(\xi)$. But $\lim_{i\to\infty} x_i=0$ since $|\xi|<1$, so $\mathbf{X}=0$ by lemma 3. This is impossible, since $\operatorname{adj}\mathbf{\Delta}(\xi)\neq 0$ by lemma 2. □

Let $\mathbf{C}(\xi)$ be a nonzero column of $\operatorname{adj}\mathbf{\Delta}(\xi)$. The simple zero ξ of $\det\mathbf{\Delta}(z)$ yields the equation

$$\sum_{i=0}^{N-1}\mathbf{\Pi}_i[\mathbf{B}_i(\xi)-\xi^i\mathbf{I}_M]\mathbf{C}(\xi)=0 \tag{15}$$

for the unknowns $\pi_{i,j}$, $0\le j\le M-1$, $0\le i\le N-1$.

Suppose assumption (A1) holds, and enumerate all the (simple) zeros of $\det\mathbf{\Delta}(z)$ in the open unit disk as $\xi_1,\xi_2,\ldots,\xi_R$, where $R=MN-1$ if $\gamma>0$ and $R=MN$ if $\gamma<0$. The question we now address is whether the equations in (15) for the different zeros are linearly independent. We prove the following:

THEOREM 2

Let assumptions (A1)-(A2) hold. When $\gamma\neq 0$, the R linear equations

$$\sum_{i=0}^{N-1}\mathbf{\Pi}_i[\mathbf{B}_i(\xi_r)-\xi_r^i\mathbf{I}_M]\mathbf{C}(\xi_r)=0, \quad 1\le r\le R, \tag{16}$$

are linearly independent.

Proof

To prove the theorem we need to show that the $MN \times R$ matrix (recall that $R \leq MN$)

$$\mathbf{B}^* \triangleq [\mathbf{B}(\xi_1)\mathbf{C}(\xi_1)\ \mathbf{B}(\xi_2)\mathbf{C}(\xi_2) \cdots \mathbf{B}(\xi_R)\mathbf{C}(\xi_R)]$$

has linearly independent columns, i.e., $\mathbf{B}^*$ has rank R. Consider a linear combination of the columns which is equal to the zero vector. That is, let α_r, $1 \leq r \leq R$, be a set of (complex) scalars such that

$$\sum_{r=1}^{R} \alpha_r \mathbf{B}(\xi_r)\mathbf{C}(\xi_r) = 0. \tag{17}$$

We will show that $\alpha_r = 0$ for $1 \leq r \leq R$.

From the definition of $\mathbf{C}(\xi)$ and using (12), we obtain that at a zero ξ of $\det \mathbf{\Delta}(z)$

$$[\mathbf{I} - \mathbf{P}]\mathbf{E}(\xi)\mathbf{C}(\xi) = \begin{bmatrix} \mathbf{B}(\xi)\mathbf{C}(\xi) \\ 0 \end{bmatrix}. \tag{18}$$

Therefore, (17) yields

$$\sum_{r=1}^{R} \alpha_r [\mathbf{I} - \mathbf{P}]\mathbf{E}(\xi_r)\mathbf{C}(\xi_r) = 0. \tag{19}$$

Define the vector

$$\mathbf{X} \triangleq \sum_{r=1}^{R} \alpha_r \mathbf{E}(\xi_r)\mathbf{C}(\xi_r) = \sum_{r=1}^{R} \alpha_r \begin{bmatrix} \mathbf{C}(\xi_r) \\ \xi_r \mathbf{C}(\xi_r) \\ \xi_r^2 \mathbf{C}(\xi_r) \\ \vdots \end{bmatrix}. \tag{20}$$

Then from (19) we have $\mathbf{PX} = \mathbf{X}$, i.e. $\mathbf{X}$ is regular (harmonic) [5].

Since $|\xi_r| < 1$ for all r, we have that $\lim_{i \to \infty} x_i = 0$. By lemma 3 we have $\mathbf{X} = 0$. Now from (20) we therefore obtain

$$\sum_{r=1}^{R} \alpha_r \begin{bmatrix} \mathbf{C}(\xi_r) \\ \xi_r \mathbf{C}(\xi_r) \\ \xi_r^2 \mathbf{C}(\xi_r) \\ \vdots \end{bmatrix} = 0. \tag{21}$$

In particular we have that

$$\sum_{r=1}^{R} \alpha_r \begin{bmatrix} \mathbf{C}(\xi_r) \\ \xi_r \mathbf{C}(\xi_r) \\ \vdots \\ \xi_r^{R-1} \mathbf{C}(\xi_r) \end{bmatrix} = 0. \tag{22}$$

Define the $MR \times R$ matrix $\mathbf{U}$ as follows:

$$\mathbf{U} \triangleq \begin{bmatrix} \mathbf{C}(\xi_1) & \mathbf{C}(\xi_2) & \cdots & \mathbf{C}(\xi_R) \\ \xi_1 \mathbf{C}(\xi_1) & \xi_2 \mathbf{C}(\xi_2) & \cdots & \xi_R \mathbf{C}(\xi_R) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^{R-1} \mathbf{C}(\xi_1) & \xi_2^{R-1} \mathbf{C}(\xi_2) & \cdots & \xi_R^{R-1} \mathbf{C}(\xi_R) \end{bmatrix}.$$

We now prove the following lemma.

LEMMA 4

The rank of the matrix $\mathbf{U}$ is R.

Proof

We first write $\mathbf{U} = \mathbf{VW}$, where $\mathbf{V}$ is the $MR \times MR$ matrix

$$\mathbf{V} \triangleq \begin{bmatrix} \mathbf{I}_M & \mathbf{I}_M & \cdots & \mathbf{I}_M \\ \xi_1 \mathbf{I}_M & \xi_2 \mathbf{I}_M & \cdots & \xi_R \mathbf{I}_M \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^{R-1} \mathbf{I}_M & \xi_2^{R-1} \mathbf{I}_M & \cdots & \xi_R^{R-1} \mathbf{I}_M \end{bmatrix}$$

and $\mathbf{W}$ is the $MR \times R$ matrix

$$\mathbf{W} \triangleq \begin{bmatrix} \mathbf{C}(\xi_1) & 0 & \cdots & 0 \\ 0 & \mathbf{C}(\xi_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{C}(\xi_R) \end{bmatrix}.$$

Since each vector $\mathbf{C}(\xi_r) \neq 0$, $\mathbf{W}$ clearly has rank R. We next claim that the matrix $\mathbf{V}$ is nonsingular. To see this, note that by interchanging rows and columns $\mathbf{V}$ is equivalent to the block diagonal matrix

$$\mathbf{V}_1 = \begin{bmatrix} \mathbf{V}^* & 0 & \cdots & 0 \\ 0 & \mathbf{V}^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{V}^* \end{bmatrix},$$

where each of the M diagonal blocks is the $R \times R$ matrix

$$\mathbf{V}^* \triangleq \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_R \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^{R-1} & \xi_2^{R-1} & \cdots & \xi_R^{R-1} \end{bmatrix}.$$

Now $\mathbf{V}^*$ is a Vandermonde matrix [4] with all ξ_r distinct, and thus it is nonsingular. Therefore, $\mathbf{V}_1$, and hence $\mathbf{V}$, is nonsingular. This implies that rank $\mathbf{U} =$ rank $\mathbf{W} = R$, since multiplication by a nonsingular matrix preserves rank (see [4]).

It follows from lemma 4 that the columns of $\mathbf{U}$ are linearly independent. Since the linear combination of these columns in (22) is zero, we obtain that $\alpha_r = 0$ for $1 \le r \le R$. This shows that the columns of $\mathbf{B}^*$ are linearly independent and completes the proof of the theorem. □

When $\gamma < 0$, theorem 2 implies that the MN linear equations of (16) in the MN unknowns $\pi_{i,j}$, $0 \le j \le M-1$, $0 \le i \le N-1$ are linearly independent. Therefore, the unique solution of the set of equations in (16) is $\pi_{i,j} = 0$, $0 \le j \le M-1$, $0 \le i \le N-1$. It then follows from (2) that $\pi_{i,j} = 0$ for all i,j. Consequently, we conclude:

COROLLARY 1

Let assumptions (A1)-(A2) hold. When $\gamma < 0$, the Markov chain is not ergodic.

When $\gamma > 0$, theorem 2 implies that the $MN - 1$ linear equations of (16) in the MN unknowns $\pi_{i,j}$, $0 \le j \le M-1$, $0 \le i \le N-1$ are linearly independent. That is, the set of equations in (16) has a one dimensional solution space. When determining the equilibrium probabilities, an additional equation for the unknowns is obtained from the zero at $z = 1$. However, substituting $z = 1$ directly into (3) does not yield a new equation, since one obtains the identity $0 = 0$ from

$$\mathbf{G}(z) \det \mathbf{\Delta}(z)|_{z=1} = \sum_{i=0}^{N-1} \mathbf{\Pi}_i [\mathbf{B}_i(z) - z^i \mathbf{I}_M] \operatorname{adj} \mathbf{\Delta}(z)|_{z=1}.$$

To see this, first note that the left-hand side is zero, since $\det \mathbf{\Delta}(1) = 0$. Further, the right-hand side is also zero, since each $\mathbf{B}_i(1)$ is stochastic and it will be shown in lemma 5 that adj $\boldsymbol{\Delta}(1)$ has constant columns.

Therefore, we use L'Hospital's rule to obtain the equation

$$\mathbf{G}(1) = \frac{1}{\gamma}\left\{\sum_{i=0}^{N-1} \boldsymbol{\Pi}_i \frac{d}{dz}[\mathbf{B}_i(z) - z^i\mathbf{I}_M] \operatorname{adj} \boldsymbol{\Delta}(z)|_{z=1}\right\}. \tag{23}$$

Let

$$\mathbf{D}_i(z) \triangleq \frac{d}{dz}[z^i\mathbf{I}_M - \mathbf{B}_i(z)] \operatorname{adj} \boldsymbol{\Delta}(z), \tag{24}$$

and let $\mathbf{e}_M$ denote the $M \times 1$ vector with all entries 1, i.e., $\mathbf{e}_M = [1\,1\,\ldots\,1]^T$. Then multiplying (23) by $\mathbf{e}_M$, we obtain that

$$\gamma\left[1 - \sum_{i=0}^{N-1} \boldsymbol{\Pi}_i\mathbf{e}_M\right] = -\sum_{i=0}^{N-1} \boldsymbol{\Pi}_i\mathbf{D}_i(1)\mathbf{e}_M. \tag{25}$$

Here we used the fact that the probabilities sum to 1, i.e.,

$$\mathbf{G}(1)\mathbf{e}_M = \sum_{i=N}^{\infty} \boldsymbol{\Pi}_i\mathbf{e}_M = 1 - \sum_{i=0}^{N-1} \boldsymbol{\Pi}_i\mathbf{e}_M. \tag{26}$$

Equation (25) can be rewritten as

$$\sum_{i=0}^{N-1} \boldsymbol{\Pi}_i[\gamma\mathbf{I}_M - \mathbf{D}_i(1)]\mathbf{e}_M = \gamma, \tag{27}$$

which is an additional equation for the unknowns $\pi_{i,j}$, $0 \le j \le M-1$, $0 \le i \le N-1$.

For Bailey's bulk queue ($M = 1$, $N = c$), the c eqs. (16) and (27) take the form

$$\sum_{i=0}^{c-1} \pi_{i,0}[\xi_r^c - \xi_r^i] = 0, \quad 1 \le r \le c-1,$$

$$\sum_{i=0}^{c-1} \pi_{i,0}[c - i] = \gamma,$$

assuming $z^c - \alpha(z)$ has simple zeros. Since the structure of the matrix corresponding to these equations is essentially that of a Vandermonde matrix, it follows that the equations are linearly independent [1]. The question we now address is whether eq. (27) and the equations in (16) are linearly independent in the general case.

We first prove a known result about the structure of the adjoint of $\mathbf{\Delta}(1)$.

LEMMA 5

The matrix $\operatorname{adj}\mathbf{\Delta}(1)$ has constant columns, i.e.,

$$\operatorname{adj}\mathbf{\Delta}(1) = \begin{bmatrix} \delta_1 & \delta_2 & \cdots & \delta_M \\ \delta_1 & \delta_2 & \cdots & \delta_M \\ \vdots & \vdots & \ddots & \vdots \\ \delta_1 & \delta_2 & \cdots & \delta_M \end{bmatrix}. \tag{28}$$

Further, the δ_j are nonnegative, and $\Sigma_{j=1}^{M}\delta_j$ is positive.

Proof

Since $\mathbf{\Delta}(1)$ has zero row sums, its right null space is generated by the $M \times 1$ vector all entries of which are 1. Therefore, as in the proof of lemma 2, the adjoint has the form (28). To show that the $\delta_j \geq 0$, first observe that the jth diagonal entry of $\operatorname{adj}\mathbf{\Delta}(1)$ is $\delta_j = (-1)^{j+j}\det\mathbf{\Delta}_j$, where $\mathbf{\Delta}_j$ is the matrix obtained by removing the jth row and jth column from $\mathbf{\Delta}(1)$. Using $\mathbf{\Delta}(1) = \mathbf{I}_M - \mathbf{A}(1)$, it follows that $\delta_j = \det[\mathbf{I}_M - \mathbf{K}_j]$, where $\mathbf{K}_j$ is identical to $\mathbf{A}(1)$ except that its jth column is identically zero. For $0 \leq t < 1$, the matrix $I_M - t\mathbf{K}_j$ is invertible, because it is a strictly diagonally dominant matrix. Thus $\det[\mathbf{I}_M - t\mathbf{K}_j] > 0$, since this holds for $t = 0$ and $\det[\mathbf{I}_M - t\mathbf{K}_j]$ is never zero for $0 < t < 1$. Letting $t \to 1$, we have that $\delta_j = \det[\mathbf{I}_M - \mathbf{K}_j] \geq 0$. Finally, since the rank of $\operatorname{adj}\mathbf{\Delta}(1)$ is nonzero by lemma 2, at least one δ_j is positive, completing the proof. □

THEOREM 3

Let assumptions (A1)-(A2) hold. When $\gamma > 0$, the MN linear equations

$$\sum_{i=0}^{N-1} \mathbf{\Pi}_i[\mathbf{B}_i(\xi_r) - \xi_r^i\mathbf{I}_M]\mathbf{C}(\xi_r) = 0, \quad 1 \leq r \leq MN-1,$$

$$\sum_{i=0}^{N-1} \mathbf{\Pi}_i[\gamma\mathbf{I}_M - \mathbf{D}_i(1)]\mathbf{e}_M = \gamma$$

are linearly independent.

Proof

We begin by defining the $MN \times M$ matrix function

$$\mathbf{D}(z) \triangleq \begin{bmatrix} \mathbf{D}_0(z) - \gamma \mathbf{I}_M \\ \mathbf{D}_1(z) - \gamma \mathbf{I}_M \\ \vdots \\ \mathbf{D}_{N-1}(z) - \gamma \mathbf{I}_M \end{bmatrix},$$

where we recall that the definition of $\mathbf{D}_i(z)$ is given in (24). To prove the theorem we need to show that the $MN \times MN$ matrix

$$\hat{\mathbf{B}} = [\mathbf{B}(\xi_1)\mathbf{C}(\xi_1) \ \mathbf{B}(\xi_2)\mathbf{C}(\xi_2) \ \cdots \ \mathbf{B}(\xi_{MN-1})\mathbf{C}(\xi_{MN-1}) \ \mathbf{D}(1)\mathbf{e}_M]$$

has linearly independent columns, i.e., $\hat{\mathbf{B}}$ is nonsingular. Consider a linear combination of the columns which is equal to the zero vector. That is, let β_r, $1 \leq r \leq MN - 1$, and β be (complex) scalars such that

$$\sum_{r=1}^{MN-1} \beta_r \mathbf{B}(\xi_r)\mathbf{C}(\xi_r) + \beta \mathbf{D}(1)\mathbf{e}_M = 0. \tag{29}$$

Without loss of generality we may assume that $\beta \geq 0$. We will show that $\beta_r = 0$ for $1 \leq r \leq MN - 1$ and that $\beta = 0$.

From (13) we have that

$$\frac{d}{dz}\{[\mathbf{I} - \mathbf{P}]\mathbf{E}(z) \operatorname{adj} \boldsymbol{\Delta}(z)\} = \begin{bmatrix} \frac{d}{dz}\{[\mathbf{I}_M - \mathbf{B}_0(z)] \operatorname{adj} \boldsymbol{\Delta}(z)\} \\ \frac{d}{dz}\{[z\mathbf{I}_M - \mathbf{B}_1(z)] \operatorname{adj} \boldsymbol{\Delta}(z)\} \\ \vdots \\ \frac{d}{dz}\{[z^{N-1}\mathbf{I}_M - \mathbf{B}_{N-1}(z)] \operatorname{adj} \boldsymbol{\Delta}(z)\} \\ \frac{d}{dz}\{\det \boldsymbol{\Delta}(z)\mathbf{I}_M\} \\ \frac{d}{dz}\{ z\det \boldsymbol{\Delta}(z)\mathbf{I}_M\} \\ \vdots \end{bmatrix}.$$

Therefore,

$$\frac{d}{dz}\{[\mathbf{I}-\mathbf{P}]\mathbf{E}(z)\,\mathrm{adj}\,\mathbf{\Delta}(z)\}|_{z=1} = \begin{bmatrix} \mathbf{D}_0(1) \\ \mathbf{D}_1(1) \\ \vdots \\ \mathbf{D}_{N-1}(1) \\ \gamma\mathbf{I}_M \\ \gamma\mathbf{I}_M \\ \vdots \end{bmatrix},$$

and so

$$[\mathbf{I}-\mathbf{P}]\frac{d}{dz}\{\mathbf{E}(z)\,\mathrm{adj}\,\mathbf{\Delta}(z)\}|_{z=1} - \gamma\mathbf{E}(1) = \begin{bmatrix} \mathbf{D}(1) \\ 0 \end{bmatrix}.$$

Multiplying the above equation by $\mathbf{e}_M$, we obtain

$$[\mathbf{I}-\mathbf{P}]\frac{d}{dz}\{\mathbf{E}(z)\,\mathrm{adj}\,\mathbf{\Delta}(z)\mathbf{e}_M\}|_{z=1} - \gamma\mathbf{e} = \begin{bmatrix} \mathbf{D}(1)\mathbf{e}_M \\ 0 \end{bmatrix},$$

where $\mathbf{e}$ is an infinite vector with all entries 1. From (29) we have that

$$\sum_{r=1}^{MN-1} \beta_r[\mathbf{I}-\mathbf{P}]\mathbf{E}(\xi_r)\mathbf{C}(\xi_r) + \beta\{[\mathbf{I}-\mathbf{P}]\frac{d}{dz}\{\mathbf{E}(z)\,\mathrm{adj}\,\mathbf{\Delta}(z)\mathbf{e}_M\}|_{z=1} - \gamma\mathbf{e}\} = 0, \quad (30)$$

where we recall that $\beta \geq 0$. Define

$$\mathbf{Y} \triangleq \mathrm{Re}\left\{\sum_{r=1}^{MN-1} \beta_r\mathbf{E}(\xi_r)\mathbf{C}(\xi_r)\right\} + \beta\frac{d}{dz}\{\mathbf{E}(z)\,\mathrm{adj}\,\mathbf{\Delta}(z)\mathbf{e}_M\}|_{z=1}. \quad (31)$$

Taking real parts in (30) yields $[\mathbf{I}-\mathbf{P}]\mathbf{Y} = \beta\gamma\mathbf{e}$, since β is real.

We now claim that the entries of $\mathbf{Y}$ are bounded below. To see this, first note that

$$\frac{d}{dz}\{\mathbf{E}(z)\,\mathrm{adj}\,\mathbf{\Delta}(z)\mathbf{e}_M\}|_{z=1} = \begin{bmatrix} \mathbf{K} \\ \delta\mathbf{e}_M + \mathbf{K} \\ 2\delta\mathbf{e}_M + \mathbf{K} \\ \vdots \end{bmatrix},$$

where $\mathbf{K} = (\frac{d}{dz})\{\text{adj}\,\mathbf{\Delta}(z)\mathbf{e}_M\}|_{z=1}$, $\delta = \Sigma_{j=1}^{M}\delta_j$, and δ_j is the common value of the entries from the jth column of adj $\mathbf{\Delta}(1)$. From lemma 5 we know that $\delta > 0$. Since also $\beta \geq 0$, the vector

$$\beta\frac{d}{dz}\{\mathbf{E}(z)\,\text{adj}\,\mathbf{\Delta}(z)\mathbf{e}_M\}|_{z=1} = \begin{bmatrix} \beta\mathbf{K} \\ \beta\delta\mathbf{e}_M + \beta\mathbf{K} \\ 2\beta\delta\mathbf{e}_M + \beta\mathbf{K} \\ \vdots \end{bmatrix} \geq \begin{bmatrix} \beta\mathbf{K} \\ \beta\mathbf{K} \\ \beta\mathbf{K} \\ \vdots \end{bmatrix}$$

has entries that are bounded below. In addition, $\text{Re}\{\Sigma_{r=1}^{MN-1}\beta_r\mathbf{E}(\xi_r)\mathbf{C}(\xi_r)\}$ has bounded entries, since $|\xi_r| < 1$ for all r. Thus $\mathbf{Y}$ has entries that are bounded below, completing the proof of the claim.

This shows there is a constant $\nu \geq 0$ such that $\mathbf{Z} \triangleq \mathbf{Y} + \nu\mathbf{e} \geq 0$. Note also that $[\mathbf{I} - \mathbf{P}]\mathbf{Z} = [\mathbf{I} - \mathbf{P}]\mathbf{Y}$, so

$$[\mathbf{I} - \mathbf{P}]\mathbf{Z} = \beta\gamma\mathbf{e}.$$

Since $\beta \geq 0$ and $\gamma > 0$, we have $\mathbf{Z} \geq \mathbf{PZ}$, i.e., the vector $\mathbf{Z}$ is nonnegative superregular (superharmonic) [5]. Therefore,

$$\mathbf{Z} \geq \mathbf{PZ} \geq \cdots \geq \mathbf{P}^{n-1}\mathbf{Z} \geq \mathbf{P}^n\mathbf{Z} \geq \cdots \geq 0. \tag{32}$$

From (32) we see that $\mathbf{P}^n\mathbf{Z}$ is a decreasing sequence which is bounded below (by 0), and so $\mathbf{L} \triangleq \lim_{n\to\infty}\mathbf{P}^n\mathbf{Z}$ exists. Since $\mathbf{Z} = \mathbf{PZ} + \beta\gamma\mathbf{e}$ and $\mathbf{P}^n$ is stochastic, we obtain $\mathbf{P}^n\mathbf{Z} = \mathbf{P}^{n+1}\mathbf{Z} + \beta\gamma\mathbf{e}$ for all $n \geq 0$. Letting $n \to \infty$ yields $\mathbf{L} = \mathbf{L} + \beta\gamma\mathbf{e}$. Thus $\beta\gamma = 0$ since $\mathbf{L}$ is finite, and so $\beta = 0$ since $\gamma > 0$. Therefore, $\Sigma_{r=1}^{MN-1}\beta_r[\mathbf{I} - \mathbf{P}]\mathbf{E}(\xi_r)\mathbf{C}(\xi_r) = 0$ from (30), which is the same as (19). From previous arguments, it then follows that $\beta_r = 0$ for all r, completing the proof of the theorem. □

Note that since $\gamma \neq 0$, the (unique) solution of these equations is nonzero. It is well-known in this case that the solution is positive [5]. Thus, when $\gamma > 0$ we obtain the unknown boundary probabilities from (16) and (27). Consequently, we conclude:

COROLLARY 2

Let assumptions (A1)-(A2) hold. When $\gamma > 0$, the Markov chain is ergodic.

Although the quantity γ defined in eq. (10) may not seem to have probabilistic significance, it can be given an interesting interpretation as follows. Differentiate the equation $\{\text{adj}\,\mathbf{\Delta}(z)\}\mathbf{\Delta}(z) = \det\mathbf{\Delta}(z)\mathbf{I}_M$ with respect to z, evaluate the result at $z = 1$ and multiply on the right by $\mathbf{e}_M$. This yields

$$\{\operatorname{adj} \boldsymbol{\Delta}(1)\}\boldsymbol{\Delta}'(1)\mathbf{e}_M + \frac{d}{dz} \operatorname{adj} \boldsymbol{\Delta}(z)\bigg|_{z=1} \boldsymbol{\Delta}(1)\mathbf{e}_M = \gamma \mathbf{e}_M.$$

Using $\boldsymbol{\Delta}(1)\mathbf{e}_M = 0$ and evaluating $\boldsymbol{\Delta}'(1)$, we obtain

$$\{\operatorname{adj} \boldsymbol{\Delta}(1)\}[N\mathbf{I}_M - \mathbf{A}'(1)]\mathbf{e}_M = \gamma \mathbf{e}_M.$$

Since $\{\operatorname{adj} \boldsymbol{\Delta}(1)\}\boldsymbol{\Delta}(1) = 0$, the rows of the adjoint are multiples of the stationary probability vector $\boldsymbol{\eta}$ of $\mathbf{A}(1)$. Furthermore, from lemma 5, these multiples are all positive and identical, and so

$$\operatorname{adj} \boldsymbol{\Delta}(1) = c \begin{bmatrix} \boldsymbol{\eta} \\ \vdots \\ \boldsymbol{\eta} \end{bmatrix},$$

where $c > 0$, $\boldsymbol{\eta}\mathbf{e}_M = 1$, and $\boldsymbol{\eta} = \boldsymbol{\eta}\mathbf{A}(1)$. Thus

$$\gamma = c[N - \boldsymbol{\eta}\mathbf{A}'(1)\mathbf{e}_M],$$

so that $-\gamma/c$ is the one step drift in the homogeneous part of the chain.

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